A NON-COMPUTATIONAL APPROACH TO THE GRADINGS ON $\mathfrak{f}_4$

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Abstract. The fine group gradings on the exceptional Lie algebra $\mathfrak{f}_4$ have previously been determined by means of computational methods. A new argument is given to prove that there are just four fine gradings on $\mathfrak{f}_4$.

1. Introduction

There has been a lot of research around the gradings on simple Lie algebras during the last years. Probably one of the reasons of such activity is that fine gradings are closely related to the structure of the algebras. To be more precise, gradings on classical Lie algebras have been studied in [5], [3] and [17] and lately revised in [15] and [2] to obtain an irredundant list of nonequivalent fine gradings and nonisomorphic gradings respectively; gradings on $\mathfrak{g}_2$ appear in [9] and [4]; gradings on $\mathfrak{d}_4$ are in [12] and [15], jointly with some descriptions in [13]; and gradings on $\mathfrak{f}_4$ can be found in [11].

In fact, there are descriptions of fine gradings on $\mathfrak{f}_4$ also in [14] and [10], but these papers can not assure if the described gradings cover the whole list of fine gradings. The only proof of this fact appears in [11], and it is a computational-based proof, quite technical, which needs a precise knowledge of the coordinate matrices of automorphisms of $\mathfrak{f}_4$ extending the action of elements in the Weyl group.

It does not happen only in $\mathfrak{f}_4$, but in general, that it is not a difficult task to describe gradings (it only requires enough knowledge of the algebra) but it could be quite difficult to prove that every fine grading is equivalent to one grading of a determined list. The classical case was the first to be studied because the authors worked with associative techniques, taking advantage that these algebras live in matrix algebras. But in the exceptional case several different techniques have been tested until now. The computational proof in the case of $\mathfrak{f}_4$ is based in the fact that the subgroup of automorphisms producing the grading is contained in the normalizer of a maximal torus of the automorphism group, thus the authors worked with a precise matricial description of the elements in such normalizer (these matricial descriptions can be obtained in [24]).

Our objective in this paper is to provide an alternative proof of the fact that there are 4 fine gradings on $\mathfrak{f}_4$ up to equivalence. This proof will not use computational tools, but the result that the 2-groups of the automorphism group of $\mathfrak{f}_4$ live in Spin(9) and hence, after projection (Spin(9) is the universal covering of SO(9)) inside some maximal abelian diagonalizable group of SO(9). But all the gradings on the Lie algebra so(9) are elementary (induced by the natural module), and can be easily extracted from [3]. Therefore, in an indirect way, we will also use matrix methods.

The purpose is to make the paper as selfcontained as possible. It is organized as follows. We will work over an algebraically closed field $\mathbb{K}$ of characteristic zero,

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although this hypothesis could have been relaxed. Section 2 contains the interpretation of the gradings in terms of algebraic groups, in particular of the fine gradings by means of MAD-groups. There are also several useful results about the structure of the MAD-groups of an automorphism group of a semisimple Lie algebra, applicable not only to $\mathfrak{f}_4$. Probably the most interesting result in this part is that every MAD-group (different from the maximal torus) contains a nontoral p-group for certain prime p, which must be 2 or 3 in the $\mathfrak{f}_4$-case. Afterwards we exhibit in Section 3 some natural descriptions of the four fine gradings on $\mathfrak{f}_4$. The objective will be to prove, in Section 6, that these are all the fine gradings on $\mathfrak{f}_4$ up to equivalence. The machinery is developed in Section 4 and Section 5, devoted to 2-groups and 3-groups respectively. The key point is that if the MAD-group is not isomorphic to $\mathbb{Z}_3^3$, then it contains an order 2 automorphism fixing a subalgebra of type $\mathfrak{b}_4$ and hence it lives in its centralizer, which is the spin group. In order to compute the MAD-groups of Spin(9), we provide a concrete description of this spin group, then of the projections of some of its elements in the orthogonal group $\mathrm{O}(9) \cong \mathrm{aut}(\mathfrak{b}_4)$, which allows us to work with the MAD-groups of O(9). We also enclose the model of $\mathfrak{f}_4$ based on $\mathfrak{b}_4$ in order to get a precise description of the relationship between $\mathrm{aut}(\mathfrak{b}_4)$ and $\mathrm{aut}(\mathfrak{f}_4)$. A similar development, but less detailed, is realized in Section 5 to extract the information about the 3-groups from $\mathrm{SL}(3) \times_{\mathbb{Z}_3} \mathrm{SL}(3)$.

2. Generalities on gradings

2.1. **Basic definitions.** Let $\mathcal{L}$ be a finite-dimensional Lie $\mathbb{K}$-algebra. The term grading will always mean group grading, that is, a decomposition in vector subspaces $\mathcal{L} = \oplus_{g\in G}\mathcal{L}_g$ where G is a finitely generated abelian group and the homogeneous spaces verify $[\mathcal{L}_g, \mathcal{L}_h] \subset \mathcal{L}_{gh}$ for any $g, h \in G$ (denoting by juxtaposition the product in G). We also assume that G is generated by $\mathrm{Supp}(G) := \{g \in G \mid \mathcal{L}_g \neq 0\}$, called the *support* of the grading.

Given two gradings $\mathcal{L} = \oplus_{g\in G}U_g$ and $\mathcal{L}' = \oplus_{h\in H}V_h$, we shall say that they are *isomorphic* if there are a group isomorphism $\sigma\colon G \to H$ and an isomorphism $\varphi\colon \mathcal{L} \to \mathcal{L}'$ such that $\varphi(U_g) = V_{\sigma(g)}$ for any $g \in G$. The above two gradings are said to be *equivalent* if there are a bijection $\sigma\colon \mathrm{Supp}(G) \to \mathrm{Supp}(H)$ and an isomorphism $\varphi\colon \mathcal{L} \to \mathcal{L}'$ such that $\varphi(U_g) = V_{\sigma(g)}$ for any $g \in \mathrm{Supp}(G)$. The first grading is a *refinement* of the second one if there are a surjective map $\sigma\colon \mathrm{Supp}(G) \to \mathrm{Supp}(H)$ and an isomorphism $\varphi\colon \mathcal{L} \to \mathcal{L}'$ such that $\varphi(U_g) \subset V_{\sigma(g)}$ for any $g \in \mathrm{Supp}(G)$.

A grading is *fine* if its unique refinement is the given grading. Our objective will be to classify fine gradings on $\mathfrak{f}_4$ up to equivalence.

2.2. **Group techniques.** The gradings on $\mathcal{L}$ can be seen as the simultaneous diagonalizations relative to quasitori of the group of automorphisms $\mathrm{aut}(\mathcal{L})$. If $\mathcal{L} = \oplus_{g\in G}\mathcal{L}_g$ is a G-grading, the map $\psi\colon \mathfrak{X}(G) = \hom(G, \mathbb{K}^\times) \to \mathrm{aut}(\mathcal{L})$ mapping each $\alpha \in \mathfrak{X}(G)$ to the automorphism $\psi_\alpha\colon \mathcal{L} \to \mathcal{L}$ given by $\mathcal{L}_g \ni x \mapsto \psi_\alpha(x) := \alpha(g)x$ is a group homomorphism. Since G is finitely generated, then $\psi(\mathfrak{X}(G))$ is a quasitorus. And conversely, if Q is a quasitorus and $\psi\colon Q \to \mathrm{aut}(\mathcal{L})$ is a homomorphism, $\psi(Q)$ is formed by semisimple automorphisms and we have a $\mathfrak{X}(Q)$-grading $\mathcal{L} = \oplus_{g\in\mathfrak{X}(Q)}\mathcal{L}_g$ given by $\mathcal{L}_g = \{x \in V \mid \psi(q)(x) = g(q)x\ \forall q \in Q\}$, with $\mathfrak{X}(Q)$ a finitely generated abelian group.

A grading is fine if and only if the quasitorus producing the grading is a maximal abelian subgroup of semisimple elements, usually called a *MAD* ("maximal abelian diagonalizable")-group. It is convenient to observe that the number of conjugacy classes of MAD-groups of $\mathrm{aut}(\mathcal{L})$ agrees with the number of equivalence classes of fine gradings on $\mathcal{L}$.

We would like to dive a little bit in the structure of these MAD-groups, for purposes not only for this paper, but for other Lie algebras.

2.3. **Structure of a MAD-group.** We study now the MAD-groups of $\mathrm{aut}(\mathcal{L}) =: G$, for $\mathcal{L}$ a finite-dimensional semisimple Lie $\mathbb{K}$-algebra. Of course there is always at least one MAD-group, the maximal torus formed by the automorphisms fixing a Cartan subalgebra (all the maximal tori are conjugated). Any other MAD-group Q has to be nontoral (that is, not contained in a torus). Moreover, as any quasitorus, this Q is the direct product of a torus by a finite subgroup of G. The purpose of this section is to prove that Q contains a nontoral p-group for some prime p, that is, if we write the finite subgroup as a direct product of p_i-groups for different primes p_i, some of the factors are nontoral. We have the conjecture that all the factors are nontoral.

Recall first a pair of facts which help to check torality. We enclose the proofs for the seek of completeness.

Lemma 1. [1, Theorem 8.2.(3)] *Let $\mathcal{G}$ be a linear algebraic group over an algebraically closed field. Assume that $\mathcal{G}$ is a connected reductive group such that its commutator subgroup is simply connected. If Q is a subquasitorus of $\mathcal{G}$ generated by at most two elements, then Q is toral.*

Proof. Take $Q = \overline{\langle f_1, f_2\rangle}$, and consider Z the centralizer of f_1 in $\mathcal{G}$, which is connected by [7, Th 3.5.6, p. 93]. As any semisimple element in a connected group belongs to a torus, there is a maximal torus T of Z such that $f_2 \in Z$. But f_1 is in the center of Z and hence in all the maximal tori of Z, so that $\langle f_1, f_2\rangle \subset T$ and $Q \subset T$. □

Lemma 2. [12, Lemma 2] *If T is a torus of G and H is a toral subgroup of G commuting with T, then HT is toral.*

Proof. Let Z be the centralizer of H in $\mathrm{aut}(\mathcal{L}) = G$. As H is toral, there is a maximal torus T' of $\mathrm{aut}(\mathcal{L})$ such that $H \subset T'$. Hence $T' \subset Z$ and it is also a maximal torus of Z. But $T \subset Z$ so that there is $p \in Z$ such that $pTp^{-1} \subset T'$. Consequently $p(HT)p^{-1} = HpTp^{-1} \subset HT' \subset (T')^2 \subset T'$ and HT is contained in the torus $p^{-1}T'p$. □

It is very useful to recall the version in [25, Theorem 3.15, p. 92] of the Borel-Serre Theorem, which in particular implies that every quasitorus of G is contained in the normalizer of some maximal torus. But we will need a slightly improved version of this result (which also generalizes [11, Proposition 7]).

Lemma 3. *If H_1 is a toral subgroup of G and H_2 is a diagonalizable subgroup of G which commutes with H_1, then there is a maximal torus T of G such that $H_1 \subset T$ and H_2 is contained in the normalizer $\mathrm{N}(T)$.*

Proof. Let $Z = \mathrm{Cent}_G(H_1)$. As H_1 is toral, there is a torus T_1 of G such that $H_1 \subset T_1 \subset Z$. As T_1 is connected and it contains 1_G, then $T_1 \subset Z_0$, where Z_0 denotes the connected component of Z containing the unit. Now we apply the previously cited theorem [25, Theorem 3.15, p. 92] to H_1H_2, a diagonalizable subgroup of Z, so that there is a maximal torus T of Z such that $H_1H_2 \subset \mathrm{N}(T)$. In particular $H_2 \subset H_1H_2 \subset \mathrm{N}(T)$. Note also that $H_1 \subset T$ because H_1 is in the center of Z_0, and precisely the set of semisimple elements of Z_0 coincides with the intersection of all the maximal tori of Z_0 (one of them is our T), according to [6, Corollary 11.1]. □

Lemma 4. *If a prime p does not divide the order of the Weyl group of $\mathcal{L}$, then every abelian p-group $H \leq G$ is toral.*

Proof. The elements in H have order a power of p, so that they are semisimple and, as in the previous lemma, there is a maximal torus T such that $H \subset \mathrm{N}(T)$. Let us check that any $f \in H$ verifies that $f \in T$. Let us take $\pi\colon \mathrm{N}(T) \to \mathrm{N}(T)/T$ the projection onto the semidirect product of the Weyl group and the group of diagram automorphisms. The order of $\pi(f)$ must be a divisor of the order of f, certain p^k for some $k \in \mathbb{N}$. But also the order of $\pi(f)$ divides the order of the Weyl group, which is coprime to p^k. So $\pi(f) = 1$ and $f \in T$. □

We will use a pair of times the following trivial result.

Lemma 5. *If T is a torus and H_1 and H_2 are finite groups of coprime orders such that H_2 commutes with T and $H_1 \subset T \times H_2$, then H_1 is contained in T.*

Proof. It is clear, since the projection of H_1 in H_2 must be trivial. □

Lemma 6. *If T is a maximal torus of G, and $f \in \mathrm{N}(T)$ is an element of order $r \in \mathbb{N}$, then the set $T^{\langle f\rangle}$ of the elements in T commuting with f is equal to SH for some subtorus S of T and a subgroup $H \subset \{t \in T \mid t^r = 1_G\}$ such that $S \cap H = \{1_G\}$.*

Proof. Recall that we have an action $\mathrm{N}(T) \times T \to T, (g, t) \mapsto g \cdot t := gtg^{-1}$. Hence we can write $T^{\langle f\rangle} = \{t \in T \mid ft = tf\} = \{t \in T \mid f \cdot t = t\}$. As it is a diagonalizable group (a quasitorus), there are a subtorus S and a finite group H such that $T^{\langle f\rangle} = SH$ and $S \cap H = \{1_G\}$.

Note that the map $s\colon T \to T$ given by $s(t) = \pi_{i=0}^{r-1} f^i \cdot t$ is an algebraic group homomorphism, so that $s(T)$ is a subtorus of T. As $t(f \cdot s(t)) = \pi_{i=0}^{r} f^i \cdot t = s(t)(f^r \cdot t)$ and $f^r = 1_G$, we get that $s(t) \in T^{\langle f\rangle}$. Hence the torus $s(T)$ must be contained in the only maximal torus of $T^{\langle f\rangle}$, that is, S. Let us check now that if $t \in H$ then $t^r = 1_G$. Indeed, as $f \cdot t = t$, we have $s(t) = t^r$, so that $t^r \in H \cap s(T) \subset H \cap S = \{1_G\}$. □

Lemma 7. *If H_1 and H_2 are toral subgroups of G which commute, of coprime orders r and s respectively, then the group H_1H_2 is toral.*

Proof. As in Lemma 3, there is a torus T such that $H_1 \subset T$ and $H_2 \subset \mathrm{N}(T)$. We can take $H_2 = \langle\{f_1, \dots, f_m\}\rangle$ for certain generators f_i. Call s_i the order of f_i, which is a divisor of s. By Lemma 6 and the notations therein, $H_1 \subset T^{\langle f_1\rangle} = \{t \in T \mid tf_1 = f_1t\}$ and it coincides with T_1V_1 for some T_1 subtorus of T and a subgroup $V_1 \subset \{t \in T \mid t^{s_1} = 1\}$, with $T_1 \cap V_1 = \{1\}$. As the cardinal of V_1 divides ${s_1}^{\dim T}$, it also divides $s^{\dim T}$ and hence this cardinal is coprime to r (recall that $\gcd(\mathrm{r}, \mathrm{s}) = 1$). Then, by applying Lemma 5, we get that $H_1 \subset T_1$. Now T_1 is a torus and $\langle\{f_1\}\rangle$ is toral (it is contained in H_2) commuting with T_1, so, by Lemma 2 we get that $T_1\langle\{f_1\}\rangle$ is toral and hence $H_1\langle\{f_1\}\rangle$ is toral too. Now the process begins again. By Lemma 3, there is a torus T' such that $H_1 \cup \{f_1\} \subset T'$ and $H_2 \subset \mathrm{N}(T')$. Hence $H_1 \cup \{f_1\} \subset T^{\langle f_2\rangle} = \{t \in T \mid tf_2 = f_2t\}$, which, according to Lemma 6, coincides with T_2V_2 for some subtorus T_2 of T' and a subgroup $V_2 \subset \{t \in T \mid t^{s_2} = 1\}$ such that $T_2 \cap V_2 = \{1\}$. Taking into account that the order of H_1 is r, coprime to the cardinal of V_2 (which is a divisor of a power of s), we can apply Lemma 5 to conclude that $H_1 \subset T_2$. We get that $\langle T_2, f_1, f_2\rangle$ is toral by applying Lemma 2 to the torus T_2 and to the toral subgroup $\langle f_1, f_2\rangle$, which commutes with T_2. As $\langle H_1, f_1, f_2\rangle$ is contained in $\langle T_2, f_1, f_2\rangle$, it is also toral. The application of the lemmas 5, 3, 6 and 2 allows to conclude the torality of $\langle H_1 \cup \{f_j \mid j = 1, \dots, i\}\rangle$ from the one of $\langle H_1 \cup \{f_j \mid j = 1, \dots, i-1\}\rangle$, so an induction argument ends the proof. □

Hence,

Corollary 1. *If H_i is a finite toral p_i-subgroup of G for each $i \in \{1, \dots, s\}$, with p_i prime and $p_i \neq p_j$ if $i \neq j$, and the group generated by $H_1 \cup \dots \cup H_s$ is abelian, then such group is toral.*

Some immediate consequences are the following, for general Lie algebras and for our concrete case:

Corollary 2. *Any nontoral quasitorus of G contains a nontoral finite p-group for some prime p.*

Proof. Take into account that such quasitorus is a direct product $T \times H_1 \times \cdots \times H_s$ of a torus T and some finite abelian p_i-groups H_i such that $p_i \neq p_j$ if $i \neq j$. Now apply Lemma 2 and the previous corollary. $\square$

Remark 1. *We could think that every nontoral quasitorus of G contains a nontoral elementary p-group for some prime p. This result would be relevant for the study of the gradings on the remaining exceptional Lie algebras (type $\mathfrak{e}$), because there is a lot of information about elementary p-groups (the maximal ones are detailed in* [16] *and for $p = 3$ in* [1]*). But that conjecture is not true: take, for instance, the quasitorus $Q = \langle\{t_{-1,1,-1,1}, t_{1,-1,-1,1}, \widetilde{\sigma}_{105} t_{1,1,1,i}\}\rangle \cong \mathbb{Z}_2^2 \times \mathbb{Z}_4$ (notations as in* [11]*). It is nontoral, but every proper subquasitorus is toral, in particular those isomorphic to $\mathbb{Z}_2^3$.*

Corollary 3. *Any abelian p-subgroup of* $\operatorname{aut}(\mathfrak{f}_4)$ *is toral if $p > 3$. Any nontoral quasitorus of* $\operatorname{aut}(\mathfrak{f}_4)$ *contains either a finite nontoral 2-group or a finite nontoral 3-group.*

Proof. It is a consequence of Corollary 2 and Lemma 4, because the cardinal of the Weyl group is $1152 = 2^7 3^2$, with 2 and 3 the only prime divisors. $\square$

We will need to precise a little more for the $\mathfrak{f}_4$-case. Although we have not achieved to prove that any quasitorus of $\operatorname{aut}(\mathcal{L})$ is product of a torus times several p_i-nontoral groups, what is true is the next result.

Proposition 1. *If $Q = T \times P \times R$ is a MAD-group of $G = \operatorname{aut}(\mathcal{L})$, for $\mathcal{L}$ a finite-dimensional semisimple Lie algebra, with T a torus, R a finite nontoral p-group (p prime) and P a nontrivial toral group of order coprime to p, then R contains a proper nontoral subquasitorus.*

Proof. Take R' a maximal toral subquasitorus of R. By Lemma 7 and Lemma 2, the subquasitorus $T \times P \times R'$ of Q is also toral, and according to Lemma 3, there is a maximal torus T' of G such that $T \times P \times R' \subset T'$ and $R = \langle R' \dot{\cup} \{f_1, \ldots, f_r\}\rangle \subset \mathrm{N}(T')$ with $\langle R' \cup \{f_1, \ldots, f_i\}\rangle \subsetneq \langle R' \cup \{f_1, \ldots, f_{i+1}\}\rangle$ for all $i = 1, \ldots, r-1$. Note that the quasitorus generated by $R' \cup \{f_i\}$ is nontoral for all $i = 1, \ldots, r$. We have only to prove that $r \geq 2$. But if $r = 1$, the maximality of Q implies that $(T')^{\langle f_1 \rangle} = T \times P \times R'$, a contradiction with Lemma 6. $\square$

Corollary 4. *Any MAD-group of* $\operatorname{aut}(\mathfrak{f}_4)$ *which does not contain a nontoral 3-group is $T \times R_2 \times R$, where T is a torus, R is a finite toral group of odd order and R_2 is a finite nontoral 2-group, and either R is trivial or R_2 has at least four direct factors.*

Proof. It is enough to apply the previous proposition jointly with Corollary 3 and Lemma 1, since $\operatorname{aut}(\mathfrak{f}_4)$ is simply connected. $\square$

In the last section we will prove that the group R in Corollary 4 has necessarily to be trivial.

3. Description of gradings on $\mathfrak{f}_4$

There are four fine gradings on $\mathfrak{f}_4$ described in [11] and in [10]. We enclose here a description of each of them for the seek of completeness, since our main aim is to prove that they are essentially all the possible fine gradings on $\mathfrak{f}_4$. These descriptions would also work for arbitrary (algebraically closed) fields of characteristic different from 2 or 3. All the gradings on the symmetric composition algebras, as well as the different constructions used for $\mathfrak{f}_4$, can be found in detail in [14, Sections 4 and 5].

Given a symmetric composition algebra $(C, *, b)$ of dimension 8, consider the *orthogonal Lie algebra* $\mathrm{o}(C, b) = \{d \in \mathrm{End}_{\mathbb{K}}(C) \mid b(d(x), y) + b(x, d(y)) = 0 \; \forall x, y \in C\}$, and the subalgebra of $\mathrm{o}(C, b)^3$ (with componentwise multiplication) defined by

$$\mathfrak{tri}(C, *, b) = \{(d_0, d_1, d_2) \in \mathrm{o}(C, b)^3 \mid d_0(x * y) = d_1(x) * y + x * d_2(y) \; \forall x, y \in C\},$$

called the *triality algebra*. One can form the $\mathbb{Z}_2^2$-graded Lie algebra

$$\mathcal{L} = \mathfrak{tri}(C, *, b) \oplus \iota_0(C) \oplus \iota_1(C) \oplus \iota_2(C),$$

where the bracket is given by

- $\mathfrak{tri}(C, *, b)$ is a Lie subalgebra of $\mathcal{L}$,
- $[(d_0, d_1, d_2), \iota_i(x)] = \iota_i(d_i(x))$,
- $[\iota_i(x), \iota_{i+1}(y)] = \iota_{i+2}(x * y)$ (indices modulo 3),
- $[\iota_i(x), \iota_i(y)] = \theta^i(t_{x,y})$,

being $t_{x,y}$ the element in $\mathfrak{tri}(C, *, b)$ defined by

$$t_{x,y} = (\sigma_{x,y}, \frac{1}{2} q(x, y)\mathrm{id}_C - r_x l_y, \frac{1}{2} q(x, y)\mathrm{id}_C - l_x r_y),$$

with $\sigma_{x,y}(z) = b(x, z)y - b(y, z)x$, $r_x(z) = z * x$ and $l_x(z) = x * z$ for all $x, y, z \in C$; and where θ denotes the order 3 automorphism of $\mathfrak{tri}(C, *, b)$ given by $\theta(d_0, d_1, d_2) := (d_2, d_0, d_1)$. This algebra is of type $\mathfrak{f}_4$ independently of the considered 8-dimensional symmetric composition algebra C. There are two of such algebras up to isomorphism: the Okubo algebra Ok and the para-Hurwitz algebra pH. The algebra Ok has a natural $\mathbb{Z}_3^2$-grading (coming from the nontoral $\mathbb{Z}_3^2$-grading on the matrix algebra $\mathrm{Mat}_{3\times 3}(\mathbb{K})$) and the algebra pH has a natural $\mathbb{Z}_2^3$-grading (coming from the $\mathbb{Z}_2^3$-grading on the octonion algebra). So, we can consider on $\mathcal{L} \cong \mathfrak{f}_4$:

A $\mathbb{Z}^4$-grading: given by the root decomposition on $\mathcal{L}$ relative to a Cartan subalgebra.

A $\mathbb{Z}_3^3$-grading: obtained by combining the $\mathbb{Z}_3^2$-grading on Ok with the $\mathbb{Z}_3$-grading on $\mathcal{L}$ induced by the *triality automorphism* θ.

A $\mathbb{Z}_2^5$-grading: obtained by combining the $\mathbb{Z}_2^3$-grading on pH with the following $\mathbb{Z}_2^2$-grading on $\mathcal{L}$: $\mathcal{L}_{(\bar{0},\bar{0})} = \mathfrak{tri}(C, *, b)$, $\mathcal{L}_{(\bar{0},\bar{1})} = \iota_0(C)$, $\mathcal{L}_{(\bar{1},\bar{0})} = \iota_1(C)$ and $\mathcal{L}_{(\bar{1},\bar{1})} = \iota_2(C)$.

A $\mathbb{Z}_2^3 \times \mathbb{Z}$-grading: Consider the Albert algebra $\mathcal{J} = \mathbb{K}^3 \oplus \iota_0(C) \oplus \iota_1(C) \oplus \iota_2(C)$ with the product described in [14, Theorem 5.15.], where C is the para-Hurwitz algebra pH. Such algebra $\mathcal{J}$ has a $\mathbb{Z}$-grading produced as the simultaneous diagonalization relative to $2[r_{\iota_0(1)}, r_{(1,0,0)}] \in \mathrm{Der}(\mathcal{J})$ (r_x denotes the multiplication operator in $\mathcal{J}$). It is compatible with the $\mathbb{Z}_2^3$-grading on pH, and it induces the corresponding grading on $\mathrm{Der}(\mathcal{J}) \cong \mathfrak{f}_4$.

4. 2-groups of $\mathrm{aut}(\mathfrak{f}_4)$

Taking in mind Corollary 4, our programme will be: First we will try to obtain all the information about the 2-groups of $\mathrm{aut}(\mathfrak{f}_4)$ by means of the spin group, and afterwards we will extract the information about the 3-groups from the special linear groups.

4.1. **Spin group.** Let V be a 9-dimensional $\mathbb{K}$-vector space endowed with a nondegenerate quadratic form $q\colon V \to \mathbb{K}$. Let $b_q\colon V \times V \to \mathbb{K}$ be the associated symmetric bilinear form given by $b_q(x, y) = \frac{1}{2}(q(x + y) - q(x) - q(y))$. Recall that the orthogonal group is $\mathrm{O}(V, q) = \{f \in \mathrm{gl}(V) \mid b_q(x, y) = b_q(f(x), f(y)) \, \forall x, y \in V\}$ and the special orthogonal group is $\mathrm{SO}(V, q) = \{f \in \mathrm{O}(V, q) \mid \det(f) = 1\}$. It is well known that the spin group is the universal covering of the special orthogonal group. Let us concrete a description suitable for our purposes, following the notations in [26, Chapters 15, 16].

Let $T(V) = \sum_{n=0}^{\infty} V^{\otimes n}$ be the associative tensor algebra. Let I be the ideal of $T(V)$ generated by $\{v \otimes v - q(v)1 \mid v \in V\}$. The Clifford algebra is the (unital) associative algebra given by the quotient

$$\mathrm{Cl}(V,q) = T(V)/I$$

and $\mathrm{Cl}(V,q)^-$ is, as always, the same vector space endowed with the bracket $[x,y] = xy - yx$. Let

$$\mu\colon \mathrm{Cl}(V,q) \to \mathrm{Cl}(V,q)$$

be the automorphism which extends $\mu(v) = -v$ for $v \in V$. As μ is an order 2 automorphism, it induces a $\mathbb{Z}_2$-grading on the Clifford algebra, with even and odd parts denoted respectively by $\mathrm{Cl}(V,q)_{\bar{0}}$ and $\mathrm{Cl}(V,q)_{\bar{1}}$. If we denote by $\mathrm{Cl}(V,q)^{\times}$ the group of invertible elements in the Clifford algebra, the Clifford group is defined by

$$\Gamma(V,q) := \{x \in \mathrm{Cl}(V,q)^{\times} \mid \mu(x)Vx^{-1} \subset V\}.$$

Obviously we can consider the group homomorphism

$$\begin{array}{rcl} \rho\colon \Gamma(V,q) & \to & \mathrm{GL}(V) \\ x & \mapsto & \rho(x); \qquad \rho(x)(v) = \mu(x)vx^{-1}\ \forall v \in V. \end{array}$$

As $q(\mu(x)vx^{-1}) = q(v)$ for any $v \in V$, we actually have a representation $\rho\colon \Gamma(V,q) \to \mathrm{O}(V,q)$. Any $v \in V$ such that $q(v) \neq 0$ is invertible, and $-vwv^{-1} = (wv - 2b_q(v,w)1)v^{-1} = w - 2b_q(v,w)/b_q(v,v)v$, hence $v \in \Gamma(V,q)$ and $\rho(v)$ is the reflection relative to the hyperplane orthogonal to v. According to the Cartan-Diedonné Theorem, every isometry of V is composition of reflections relative to hyperplanes orthogonal to nonisotropic vectors, so that $\rho(\Gamma(V,q)) = \mathrm{O}(V,q)$, $\Gamma(V,q) = \{\lambda u_1 \dots u_r \mid \lambda \in \mathbb{K}^{\times}, u_i \in V, q(u_i) \neq 0, r \geq 0\}$ and $\ker(\rho) = \mathbb{K}^{\times} (= \mathbb{K} \setminus \{0\})$. As $\det(\rho(v)) = -1$, we also conclude that $\rho(\Gamma(V,q) \cap \mathrm{Cl}(V,q)_{\bar{0}}) = \mathrm{SO}(V,q)$ and $\Gamma(V,q) \cap \mathrm{Cl}(V,q)_{\bar{0}} = \{\lambda u_1 \dots u_{2r} \mid \lambda \in \mathbb{K}^{\times}, u_i \in V, q(u_i) \neq 0, r \geq 0\}$. Hence we have the following short exact sequences

$$\begin{array}{l} 1 \to \mathbb{K}^{\times} \to \Gamma(V,q) \to \mathrm{O}(V,q) \to 1, \\ 1 \to \mathbb{K}^{\times} \to \Gamma(V,q) \cap \mathrm{Cl}(V,q)_{\bar{0}} \to \mathrm{SO}(V,q) \to 1. \end{array}$$

The spin group lives inside the even part of the Clifford group. Take the spinor norm $N\colon \Gamma(V,q) \to \mathbb{K}^{\times}$ given by $N(x) = \mu(x^*)x$, where $*$ is the involution given by $v^* = v$ for any $v \in V$. In particular, $N(v) = -q(v)$. The spin group is defined as $\mathrm{Spin}(V,q) = \{x \in \Gamma(V,q) \cap \mathrm{Cl}(V,q)_{\bar{0}} \mid |N(x)| = 1\}$. As $N(\lambda u_1 \dots u_{2r}) = \lambda^2 \pi_{i=1}^{2r} q(u_i)$ and $\mathbb{K}$ is algebraically closed, we can scale to get

$$\mathrm{Spin}(V,q) = \{\pm \pi_{i=1}^{2r} u_i \mid u_i \in V,\ q(u_i) = 1\},$$

and now it is clear that $\rho|_{\mathrm{Spin}(V,q)}\colon \mathrm{Spin}(V,q) \to \mathrm{SO}(V,q)$ is still an epimorphism, with kernel $\{\pm 1\} \cong \mathbb{Z}_2$. From now on ρ will denote this restriction $\rho|_{\mathrm{Spin}(V,q)}$.

4.2. **Distinguished elements in the spin group.** Let us focus our attention on some remarkable elements in the Clifford and spin groups, which will be of special relevance for our description of the MAD-groups of $\mathrm{Spin}(V,q)$. Let

$$B := \{e_0, u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\}$$

be a $\mathbb{K}$-basis of V such that the matrix of b_q relative to B is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_4 \\ 0 & I_4 & 0 \end{pmatrix}$. We denote also by

$$\begin{array}{llll} e_1 = \frac{1}{\sqrt{2}}(u_1 + v_1), & e_3 = \frac{1}{\sqrt{2}}(u_2 + v_2), & e_5 = \frac{1}{\sqrt{2}}(u_3 + v_3), & e_7 = \frac{1}{\sqrt{2}}(u_4 + v_4), \\ e_2 = \frac{\mathrm{i}}{\sqrt{2}}(u_1 - v_1), & e_4 = \frac{\mathrm{i}}{\sqrt{2}}(u_2 - v_2), & e_6 = \frac{\mathrm{i}}{\sqrt{2}}(u_3 - v_3), & e_8 = \frac{\mathrm{i}}{\sqrt{2}}(u_4 - v_4), \end{array}$$

where $\mathfrak{i} \in \mathbb{K}$ is a primitive fourth root of the unit ($\mathfrak{i}^2 = -1$). Thus, the matrix of b_q relative to the basis

$$B' := \{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$$

is the identity matrix I_9. Observe first that $q(\frac{1}{\sqrt{2}}(\beta u_i + \frac{1}{\beta}v_i)) = 1$ for any $\beta \in \mathbb{K}^\times$, $i = 1, 2, 3, 4$ ($\{e_1, \dots, e_8\}$ are particular cases). If we denote by $[f]_{B'}$ the matrix associated to $f \in \mathrm{O}(V, q)$ with respect to the base B', when computing the matrix related to $\rho(\frac{1}{\sqrt{2}}(\beta u_i + \frac{1}{\beta}v_i))$, the block corresponding to $\{e_{2i-1}, e_{2i}\} \subset B'$ is

$$R_\beta := \frac{1}{2}\begin{pmatrix} -\beta^2 - \frac{1}{\beta^2} & \mathfrak{i}(\beta^2 - \frac{1}{\beta^2}) \\ \mathfrak{i}(\beta^2 - \frac{1}{\beta^2}) & \beta^2 + \frac{1}{\beta^2} \end{pmatrix}.$$

Hence the matrix related to the image of $(\frac{1}{\sqrt{2}}(\beta u_i + \frac{1}{\beta}v_i))(\frac{1}{\sqrt{2}}(u_i + v_i)) = \frac{1}{2}(\beta u_i v_i + \frac{1}{\beta}v_i u_i) \in \mathrm{Spin}(V, q)$ has a block of the form

$$S_\beta := R_\beta R_1 = \frac{1}{2}\begin{pmatrix} \beta^2 + \frac{1}{\beta^2} & \mathfrak{i}(\beta^2 - \frac{1}{\beta^2}) \\ -\mathfrak{i}(\beta^2 - \frac{1}{\beta^2}) & \beta^2 + \frac{1}{\beta^2} \end{pmatrix}.$$

Thus the element

$$s_{\alpha\beta\delta\epsilon} := \frac{1}{16}(\alpha u_1 v_1 + \frac{1}{\alpha}v_1 u_1)(\beta u_2 v_2 + \frac{1}{\beta}v_2 u_2)(\delta u_3 v_3 + \frac{1}{\delta}v_3 u_3)(\epsilon u_4 v_4 + \frac{1}{\epsilon}v_4 u_4)$$

belongs to $\mathrm{Spin}(V, q)$ and $[\rho(s_{\alpha\beta\delta\epsilon})]_{B'} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & S_\alpha & 0 & 0 & 0 \\ 0 & 0 & S_\beta & 0 & 0 \\ 0 & 0 & 0 & S_\delta & 0 \\ 0 & 0 & 0 & 0 & S_\epsilon \end{pmatrix}$. Moreover,

$$\mathcal{T} = \{s_{\alpha\beta\delta\epsilon} \mid \alpha, \beta, \delta, \epsilon \in \mathbb{K}^\times\} \tag{1}$$

is a torus of $\mathrm{Spin}(V, q)$, since $s_{\alpha\beta\delta\epsilon}s_{\alpha'\beta'\delta'\epsilon'} = s_{\alpha\alpha'\beta\beta'\delta\delta'\epsilon\epsilon'}$.

On the other hand, $\rho(e_i)(e_j) = (-1)^{\delta_{ij}}e_j$ for δ_{ij} the Kronecker symbol, so that

$$d_i := [\rho(e_i)]_{B'} = \mathrm{diag}\{(-1)^{\delta_{ij}}\}_{j=0,\dots,8} \tag{2}$$

is the diagonal matrix of size 9 whose entries in the diagonal are all 1's up to one -1 in the ith position. Hence $e_i e_j \in \mathrm{Spin}(V, q)$ and $[\rho(e_i e_j)]_{B'} = d_i d_j$.

4.3. **Model of $\mathfrak{f}_4$ based on $\mathfrak{b}_4$.** We describe in this subsection the $\mathbb{Z}_2$-grading on $\mathfrak{f}_4$ such that $\mathrm{Spin}(V, q)$ is precisely the subgroup of automorphisms preserving the grading. This kind of gradings on $\mathfrak{f}_4$ whose even part type is $\mathfrak{b}_4$ is well known, appearing for instance in [19].

With the notations in subsections 4.1 and 4.2, the orthogonal algebra

$$\mathrm{so}(V, q) = \{f \in \mathrm{gl}(V) \mid b_q(f(x), y) + b_q(x, f(y)) = 0 \ \forall x, y \in V\}$$

is a Lie algebra of type $\mathfrak{b}_4$. If we denote by $W = \mathrm{span}\langle u_1, \dots, u_4\rangle$, this is a totally isotropic subspace of V and consider the exterior algebra

$$S := \wedge W = \mathbb{K} \oplus W \oplus \wedge^2(W) \oplus \wedge^3(W) \oplus \wedge^4(W)$$

with the $\mathbb{Z}$-grading given by $|x| = n$ if $x \in \wedge^n(W)$. Thus $\mathrm{End}(S) =: E = \oplus_{n\in\mathbb{Z}} E_n$ is also $\mathbb{Z}$-graded, for $E_n = \{f \in \mathrm{End}(S) \mid f(\wedge^m(W)) \subset \wedge^{m+n}(W) \ \forall m \in \mathbb{N}\}$. Let us recall how $\mathrm{so}(V, q)$ acts on the 16-dimensional vector space S, following [23, §8.A.]. First consider the map

$$\gamma \colon V \to \mathrm{End}(\wedge W)$$

given by

$$\gamma(\lambda e_0 + u + v) = \lambda \tilde{I} + l_u + d_v$$

where $u \in W$, $v \in \mathrm{span}\langle v_1, \dots, v_4\rangle$ (which can be identified to W^* by means of $v \mapsto b_q(v, -)$), and

- $\tilde{I} \in E_0$ is the map producing the $\mathbb{Z}_2$-grading on S, that is, $\tilde{I}|_{\mathbb{K}\oplus\wedge^2(W)\oplus\wedge^4(W)} =$ id and $\tilde{I}|_{W\oplus\wedge^3(W)} = -\text{id}$.
- The map $l_u\colon \wedge W \to \wedge W$ is given by $l_u(w) = u\wedge w$ if $w \in S$. Thus $l_u \in E_1$.
- The map d_v is defined on $\wedge^n(W)$ by induction on the degree n: $d_v(1) = 0$, $d_v(w) = 2b_q(v,w)1$ for $w \in W$ and $d_v(x\wedge y) = d_v(x)\wedge y + (-1)^{|x|}x\wedge d_v(y)$ if $x, y \in \cup_{m=0}^{4} \wedge^m (W)$. In particular $d_v \in E_{-1}$.

It is clear that $\gamma(x)^2 = q(x)\text{id}_{\wedge W}$ for any $x \in V$, so that γ induces a homomorphism of associative algebras $\tilde{\gamma}$ from $\text{Cl}(V,q)$ to $\text{End}(\wedge W)$, and, in particular, a homomorphism of Lie algebras from $\text{Cl}(V,q)^-$ to $\text{gl}(\wedge W)$ (which turns out to be an isomorphism).

As we have a monomorphism $\iota\colon \text{so}(V,q) \to \text{Cl}(V,q)^-$ given by $b_q(a,-)c - b_q(c,-)a \mapsto -\frac{1}{4}[a,c]$, the composition

$$\tilde{\gamma}\iota\colon \text{so}(V,q) \to \text{gl}(\wedge W)$$

provides a representation of the Lie algebra $\text{so}(V,q)$. We know that this $\text{so}(V,q)$-module $\wedge W$ is the spin module, that is, it is irreducible with maximal weight λ_4 (λ_i the fundamental weights). Indeed, $\mathfrak{h} = \langle h_i \mid i = 1,\dots,4\rangle$ is a Cartan subalgebra of $\text{so}(V,q)$ for $h_i := b_q(v_i,-)u_i - b_q(u_i,-)v_i$. This element acts on $\wedge W$ as $\tilde{\gamma}(-\frac{1}{4}[v_i,u_i]) = \frac{1}{4}(l_{u_i}d_{v_i} - d_{v_i}l_{u_i})$, in other words

$$h_i\cdot(u_{j_1}\wedge\cdots\wedge u_{j_r}) = \begin{cases} \frac{1}{2}u_{j_1}\wedge\cdots\wedge u_{j_r} & \text{if } i \in \{j_1,\dots,j_r\} \\ -\frac{1}{2}u_{j_1}\wedge\cdots\wedge u_{j_r} & \text{if } i \notin \{j_1,\dots,j_r\}. \end{cases} \tag{3}$$

Note that a set of simple roots of $\mathfrak{b}_4$ relative to the Cartan subalgebra $\mathfrak{h}$ is given by $\alpha_1(h) = \omega_1-\omega_2$, $\alpha_2(h) = \omega_2-\omega_3$, $\alpha_3(h) = \omega_3-\omega_4$, $\alpha_4(h) = \omega_4$, if $h = \sum_{i=1}^4 \omega_i h_i$ is a generic element in $\mathfrak{h}$. Now a maximal vector in $\wedge W$ is $s = u_1\wedge u_2\wedge u_3\wedge u_4$, since it is annihilated by L_α for all $\alpha \in \Phi^+$. That means that the maximal weight λ is given by $h\cdot s = \lambda(h)s$, so that $\lambda = \sum_{i=1}^4 m_i\lambda_i$, where $m_i = h_{\alpha_i}\cdot s$ for $h_{\alpha_i} = h_i - h_{i+1}$ ($i \le 3$), $h_{\alpha_4} = 2h_4$. Equation (3) gives that such maximal weight is $\lambda = \lambda_4$.

Now we construct

$$\mathcal{L} = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}} = \text{so}(V,q) \oplus (\wedge W)$$

with the product given by

- $\text{so}(V,q)$ is a Lie subalgebra.
- If $f \in \text{so}(V,q)$ and $s \in \wedge W$, we define $[f,s] = \tilde{\gamma}\iota(f)(s)$, that is, $\mathcal{L}_{\bar{0}}$ acts in $\mathcal{L}_{\bar{1}}$ by means of the spin action.
- There is, up to scalar, an unique $\text{so}(V,q)$-invariant map $\wedge W\times\wedge W \to \text{so}(V,q)$ (there is only one module of type $V(\lambda_2)$ in the decomposition into irreducible submodules of $V(\lambda_4)\otimes V(\lambda_4)$). To fix a scalar, we have fixed an $\text{so}(V,q)$-invariant symmetric bilinear form $(\cdot|\cdot)\colon \wedge W\times\wedge W \to \mathbb{K}$ (also determined up to scalar) and we have taken the dualized action of the previous one: if $s, s' \in \wedge W$, we take $[s,s'] \in \text{so}(V,q)$ the only element satisfying $\text{tr}([s,s']\,f) = ([f,s]|s')$ for all $f \in \text{so}(V,q)$.

This $\mathbb{Z}_2$-graded Lie algebra $\mathcal{L}$ is simple of type $\mathfrak{f}_4$. We call φ the grading automorphism:

$$\varphi|_{\text{so}(V,q)} = \text{id}, \quad \varphi|_{\wedge W} = -\text{id}. \tag{4}$$

The aim of this subsection is to prove next that the centralizer of φ in the automorphism group of $\mathfrak{f}_4$ is just the group $\text{Spin}(V,q)$. As usual, $\text{Ad}\colon \text{SO}(V,q) \to \text{gl}(\text{so}(V,q))$ will denote the adjoint map given by $\text{Ad}\,A(f) = AfA^{-1}$ for any $A \in \text{SO}(V,q)$, $f \in \text{so}(V,q) \equiv \mathfrak{b}_4$.

Proposition 2. *If $x \in \mathrm{Spin}(V,q)$, the map $\psi_x\colon \mathfrak{f}_4 \to \mathfrak{f}_4$ given by $\psi_x|_{\mathrm{so}(V,q)} = \mathrm{Ad}\,\rho(x)$ and $\psi_x|_{\wedge W} = \tilde\gamma(x)$, is an automorphism of the Lie algebra $\mathfrak{f}_4$; and the map*

$$\psi\colon \mathrm{Spin}(V,q) \to \mathrm{Cent}_{\mathrm{aut}(\mathfrak{f}_4)}(\varphi)$$

given by $x \mapsto \psi_x$ is a group isomorphism.

Proof. Check first that ψ_x is an automorphism, so that ψ is well defined. Take $s, s' \in S$ and $f = b_q(a,-)b - b_q(b,-)a \in \mathfrak{b}_4$, for $a, b \in V$. As $[f,s] = \tilde\gamma\iota(f)(s) = \tilde\gamma(-\frac{1}{4}[a,b])(s)$, then $\psi_x[f,s] = \tilde\gamma(-\frac{1}{4}x[a,b])(s)$. But $\psi_x(f) = \rho(x)f\rho(x)^{-1}$, so that

$$[\psi_x(f), \psi_x(s)] = \tilde\gamma\iota(\rho(x)f\rho(x)^{-1})\tilde\gamma(x)(s) = \tilde\gamma\left(-\frac{1}{4}[\rho(x)a, \rho(x)b]x\right)(s).$$

Taking into account that $\rho(x)ax = \mu(x)ax^{-1}x = xa$ (since $\mu|_{\mathrm{Spin}(V,q)} = \mathrm{id}$), we get

$$\psi_x([f,s]) = [\psi_x(f), \psi_x(s)].$$

On the other hand, $\tilde\gamma(x)^{-1}[f, \tilde\gamma(x)(s)] = \tilde\gamma(x)^{-1}\tilde\gamma\iota(f)\tilde\gamma(x)(s) = \tilde\gamma(\frac{-1}{4}x^{-1}[a,b]x)(s) = [\rho(x)^{-1}f\rho(x), s]$, so that, as $(\cdot|\cdot)$ is $\mathrm{Spin}(V,q)$-invariant,

$$\begin{aligned}\mathrm{tr}([\psi_x(s), \psi_x(s')]f) &= ([f, \psi_x(s)]|\psi_x(s')) = (\tilde\gamma(x)^{-1}[f, \tilde\gamma(x)(s)]|s') \\ &= ([\rho(x)^{-1}f\rho(x), s]|s') = \mathrm{tr}([s,s']\rho(x)^{-1}f\rho(x)) \\ &= \mathrm{tr}(\rho(x)[s,s']\rho(x)^{-1}f) = \mathrm{tr}(\psi_x([s,s'])f)\end{aligned}$$

and, as f is arbitrary, consequently $[\psi_x(s), \psi_x(s')] = \psi_x([s,s'])$. We have proved, then, that $\psi_x \in \mathrm{aut}(\mathfrak{f}_4)$.

Now note that if $F \in \mathrm{Cent}_{\mathrm{aut}(\mathfrak{f}_4)}(\varphi)$ such that $F|_{\mathfrak{b}_4} = \mathrm{id}_{\mathfrak{b}_4}$, then $F \in \{\mathrm{id}_{\mathfrak{f}_4}, \varphi\}$. Indeed, $F|_S \in \hom_{\mathfrak{b}_4}(S,S) = \mathbb{K}\,\mathrm{id}_S$ by Schur's Lemma, so there is $\beta \in \mathbb{K}$ such that $F|_S = \beta\mathrm{id}_S$, but, as $[S,S] = \mathfrak{b}_4$, that scalar $\beta \in \{1,-1\}$ and so F is respectively $\{\mathrm{id}, \varphi\}$. Let us see the epimorphic character of ψ: if $F \in \mathrm{Cent}_{\mathrm{aut}(\mathfrak{f}_4)}(\varphi)$, we can find $x \in \mathrm{Spin}(V,q)$ such that $\psi_x = F$. Indeed, as F commutes with φ, it preserves the $\mathbb{Z}_2$-grading, so we can consider the restriction $F|_{\mathfrak{b}_4} \in \mathrm{aut}(\mathfrak{b}_4) = \mathrm{Ad}(\mathrm{SO}(V,q))$. Hence there is $A \in \mathrm{SO}(V,q)$ such that $\mathrm{Ad}\,A = F|_{\mathfrak{b}_4}$. Take $x \in \rho^{-1}(A)$, so that $\rho^{-1}(A) = \{\pm x\}$. Thus $F^{-1} \circ \psi_x|_{\mathfrak{b}_4} = \mathrm{id}_{\mathfrak{b}_4}$ and, as above, $F^{-1} \circ \psi_x \in \{\mathrm{id} = \psi_1, \varphi = \psi_{-1}\}$. Hence $F \in \{\psi_x, \psi_{-x}\}$.

Finally let us check that ψ is injective. If $\psi_x = \mathrm{id}_{\mathfrak{f}_4}$, then $\mathrm{Ad}\,\rho(x) = \mathrm{id}_{\mathfrak{b}_4}$, and $\rho(x)f = f\rho(x)$ for all $f \in \mathrm{so}(V,q)$. Thus $\rho(x) = \mathrm{id}_V$ and $x \in \ker(\rho) = \{\pm 1\}$. The possibility $x = -1$ does not occur since $\psi_{-1} = \varphi \neq \mathrm{id}_{\mathfrak{f}_4}$. □

4.4. **Every 2-group lives in** $\mathrm{Spin}(V,q)$**.** We would like to prove that every MAD-group with a nontoral 2-group contains some automorphism conjugated to the automorphism φ described in Equation (4), that is, some automorphism whose fixed subalgebra is of type $\mathfrak{b}_4$. I acknowledge A. Viruel for the communication of this result. For its proof, first recall a well known fact.

Lemma 8. [21, Lemma 3.1] *Fix a maximal torus $T \subset \mathrm{aut}(\mathcal{L})$ for $\mathcal{L}$ a semisimple Lie algebra, and an element $f \in T$. Let $W = \mathrm{N}_{\mathrm{aut}(\mathcal{L})}(T)/T$ and $W_f = \mathrm{N}_{\mathrm{Cent}(f)}(T)/T$ be the Weyl groups of $\mathrm{aut}(\mathcal{L})$ and of the centralizer $\mathrm{Cent}(f)$, respectively. Then the number of elements in T conjugate (in $\mathrm{aut}(\mathcal{L})$) to f is just the Weyl group index $[W : W_f]$.*

Proof. Recall that two elements in T are conjugate in $\mathrm{aut}(\mathcal{L})$ if and only if they are conjugate in $\mathrm{N}_{\mathrm{aut}(\mathcal{L})}(T)$. Thus the set of elements in T conjugate to $f \in T$ is just $\{\sigma f\sigma^{-1} \mid \sigma \in \mathrm{N}_{\mathrm{aut}(\mathcal{L})}(T)\}$, which is in bijective correspondence with the set of left classes $\{wW_f \mid w \in W\}$. Such bijection is given by $\sigma f\sigma^{-1} \mapsto (\sigma T)W_f$. Note that if two elements $\sigma, \sigma' \in \mathrm{N}_{\mathrm{aut}(\mathcal{L})}(T)$ verify $(\sigma T)W_f = (\sigma' T)W_f$, there is $c \in \mathrm{N}_{\mathrm{Cent}(f)}(T)$ such that $\sigma'^{-1}\sigma T = cT$, hence $\sigma'^{-1}\sigma \subset \mathrm{Cent}(f)T \subset \mathrm{Cent}(f)$ so that $\sigma f\sigma^{-1} = \sigma' f\sigma'^{-1}$. □

Thus, if $W_{\mathrm{aut}(\mathfrak{f}_4)} = \mathrm{N}_{\mathrm{aut}(\mathfrak{f}_4)}(\mathcal{T})/\mathcal{T}$ and $W_{\mathrm{Spin}(V,q)} = \mathrm{N}_{\mathrm{Spin}(V,q)}(\mathcal{T})/\mathcal{T}$, then the index $[W_{\mathrm{aut}(\mathfrak{f}_4)} : W_{\mathrm{Spin}(V,q)}]$ is computed easily by counting in any maximal torus of $\mathrm{aut}(\mathfrak{f}_4)$ how many elements are fixing a subalgebra of type $\mathfrak{b}_4$. Recall from [22] that there are two conjugacy classes of order 2 automorphisms in $\mathrm{aut}(\mathfrak{f}_4)$, characterized by fixing subalgebras of type $\mathfrak{b}_4$ and $\mathfrak{c}_3 \oplus \mathfrak{a}_1$, whose dimensions are 36 and 24 respectively. If $\mathfrak{h}$ is a Cartan subalgebra, $\mathfrak{f}_4 = \mathfrak{h} \oplus (\oplus_{\alpha\in\Phi} L_\alpha)$ denotes the decomposition in root spaces relative to $\mathfrak{h}$ and $\Delta = \{\alpha_i\}_{i=1}^4$ is a set of simple roots of Φ, a maximal torus can be described as $\{t_{x,y,z,u} \mid x, y, z, u \in \mathbb{K}^\times\}$, where $t = t_{x,y,z,u}$ is the automorphism determined by $t|_\mathfrak{h} = \mathrm{id}$, $t|_{L_{\alpha_1}} = x\,\mathrm{id}$, $t|_{L_{\alpha_2}} = y\,\mathrm{id}$, $t|_{L_{\alpha_3}} = z\,\mathrm{id}$ and $t|_{L_{\alpha_4}} = u\,\mathrm{id}$. As the eigenvalues are

$$(1,1,1,1) \cup (u, z, y, x, zu, yz, xy, xyz, yzu, yz^2, xyzu, yz^2u, xyz^2, xyz^2u, yz^2u^2,\\ xy^2z^2, xy^2z^2u, xyz^2u^2, xy^2z^3u, xy^2z^2u^2, xy^2z^3u^2, xy^2z^4u^2, xy^3z^4u^2, x^2y^3z^4u^2)^{\pm 1}$$

the only choices of $(x, y, z, u) \in \{\pm 1\}^4$ providing a list with 36 $1's$ and 16 -1's are $(1, 1, 1, -1), (1, 1, -1, 1)$ and $(1, 1, -1, -1)$. Hence, according to Lemma 8, the index of the Weyl group of $\mathrm{Spin}(V, q)$ in the Weyl group of $\mathrm{aut}(\mathfrak{f}_4)$ is 3 (of course this is known in the literature, see, for instance, [20, p. 248]). A consequence is the following.

Proposition 3. *If a quasitorus Q of $\mathrm{aut}(\mathfrak{f}_4)$ is the direct product of a torus T and a 2-group, then Q is conjugated to a subquasitorus of $\mathrm{Spin}(V, q)$.*

Proof. By Lemma 3, we can change Q by one of its conjugated quasitori such that $T \subset \mathcal{T}$ and $Q \subset \mathrm{N}_{\mathrm{aut}(\mathfrak{f}_4)}(\mathcal{T})$, where $\mathcal{T}$ is the maximal torus of $\mathrm{Spin}(V, q)$ defined in Equation (1), which is also a maximal torus of $\mathrm{aut}(\mathfrak{f}_4)$ through the map ψ defined in Proposition 2. Denote by $p\colon W_{\mathrm{aut}(\mathfrak{f}_4)} \to W_{\mathrm{aut}(\mathfrak{f}_4)}/W_{\mathrm{Spin}(V,q)}$ the projection onto the set of left classes. Let f be an element in Q. We can take $f = f_0 t$ with $t \in \mathcal{T}$ and $f_0 \in \mathrm{N}_{\mathrm{aut}(\mathfrak{f}_4)}(\mathcal{T})$ of order a power of 2. Thus $f_0\mathcal{T} \in W_{\mathrm{aut}(\mathfrak{f}_4)}$ and its projection $p(f_0\mathcal{T}) \in W_{\mathrm{aut}(\mathfrak{f}_4)}/W_{\mathrm{Spin}(V,q)}$ has order a power of 2. But $[W_{\mathrm{aut}(\mathfrak{f}_4)} : W_{\mathrm{Spin}(V,q)}] = 3$, so that $p(f_0\mathcal{T}) = 1$ and there is $f_1 \in \mathrm{N}_{\mathrm{Spin}(V,q)}(\mathcal{T})$ such that $f_0\mathcal{T} = f_1\mathcal{T}$. Hence $f \in f_0\mathcal{T} = f_1\mathcal{T} \subset \mathrm{Spin}(V, q)$. □

In other words, such $Q \leq \mathrm{aut}(\mathfrak{f}_4)$ commutes with an automorphism conjugated to φ which fixes a subalgebra of type $\mathfrak{b}_4$, and hence it is contained in a MAD-group of $\mathrm{Spin}(V, q)$. We will compute these MAD-groups by taking advantage of the knowledge of the MAD-groups of $\mathrm{SO}(V, q)$, since the map $\rho\colon \mathrm{Spin}(V, q) \to \mathrm{SO}(V, q)$ will allow us to use that information.

4.5. **MAD-groups of** $\mathrm{SO}(9)$**.** According to [3], every grading on the Lie algebra $\mathrm{so}(V, q) \cong \mathfrak{b}_4$ is elementary, which means induced by the natural module V. Let us explain a little bit more about this concept. If we choose an arbitrary (finitely generated and abelian) group G, and take a decomposition $V = \oplus_{g\in G} V_g$ as a sum of vector subspaces (possibly some of them zero), we have a G-grading induced on $\mathrm{gl}(V) = \mathcal{L} = \oplus_{g\in G}\mathcal{L}_g$ given by $\mathcal{L}_g = \{f \in \mathrm{gl}(V) \mid f(V_h) \subset V_{g+h}\ \forall h \in G\}$ (although G is not necessarily generated by the support). Such grading induces a G-grading on $\mathrm{so}(V, q) = \mathfrak{g}$ provided $\mathfrak{g} = \oplus_{g\in G}(\mathfrak{g} \cap \mathcal{L}_g)$. We will describe this kind of gradings simply by assigning a degree in G to each element in some convenient basis of V.

Following the arguments in [3] or [17], it is easy to conclude that there are five fine gradings on $\mathrm{so}(V, q)$, over the universal grading groups (see [9] for the definition and details) $\mathbb{Z}^4$, $\mathbb{Z}^3 \times \mathbb{Z}_2^2$, $\mathbb{Z}^2 \times \mathbb{Z}_2^4$, $\mathbb{Z} \times \mathbb{Z}_2^6$ and $\mathbb{Z}_2^8$, induced by the following choices of base and assignments of degree on the vector space V:

- The $\mathbb{Z}^4$-grading induced by

$$\begin{array}{llll} e_0 \mapsto (0000) & & & \\ u_1 \mapsto (1000) & u_2 \mapsto (0100) & u_3 \mapsto (0010) & u_4 \mapsto (0001) \\ v_1 \mapsto (-1000) & v_2 \mapsto (0\,{-1}00) & v_3 \mapsto (00\,{-1}0) & v_4 \mapsto (000\,{-1}). \end{array}$$

- The $\mathbb{Z}^3 \times \mathbb{Z}_2^2$-grading induced by

$$\begin{array}{lll} e_0 \mapsto (000\bar{1}\bar{1}) & e_1 \mapsto (000\bar{1}\bar{0}) & e_2 \mapsto (000\bar{0}\bar{1}) \\ u_2 \mapsto (100\bar{0}\bar{0}) & u_3 \mapsto (010\bar{0}\bar{0}) & u_4 \mapsto (001\bar{0}\bar{0}) \\ v_2 \mapsto (-100\bar{0}\bar{0}) & v_3 \mapsto (0\,{-1}0\bar{0}\bar{0}) & v_4 \mapsto (00\,{-1}\bar{0}\bar{0}). \end{array}$$

- The $\mathbb{Z}^2 \times \mathbb{Z}_2^4$-grading induced by

$$\begin{array}{llll} e_0 \mapsto (00\bar{1}\bar{1}\bar{1}\bar{1}) & & & \\ e_1 \mapsto (00\bar{1}\bar{0}\bar{0}\bar{0}) & e_2 \mapsto (00\bar{0}\bar{1}\bar{0}\bar{0}) & e_3 \mapsto (00\bar{0}\bar{0}\bar{1}\bar{0}) & e_4 \mapsto (00\bar{0}\bar{0}\bar{0}\bar{1}) \\ u_3 \mapsto (10\bar{0}\bar{0}\bar{0}\bar{0}) & v_3 \mapsto (-10\bar{0}\bar{0}\bar{0}\bar{0}) & u_4 \mapsto (01\bar{0}\bar{0}\bar{0}\bar{0}) & v_4 \mapsto (0\,{-1}\bar{0}\bar{0}\bar{0}\bar{0}). \end{array}$$

- The $\mathbb{Z} \times \mathbb{Z}_2^6$-grading induced by

$$\begin{array}{lll} e_0 \mapsto (0\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}) & e_1 \mapsto (0\bar{1}\bar{0}\bar{0}\bar{0}\bar{0}\bar{0}) & e_2 \mapsto (0\bar{0}\bar{1}\bar{0}\bar{0}\bar{0}\bar{0}) \\ e_3 \mapsto (0\bar{0}\bar{0}\bar{1}\bar{0}\bar{0}\bar{0}) & e_4 \mapsto (0\bar{0}\bar{0}\bar{0}\bar{1}\bar{0}\bar{0}) & e_5 \mapsto (0\bar{0}\bar{0}\bar{0}\bar{0}\bar{1}\bar{0}) \\ e_6 \mapsto (0\bar{0}\bar{0}\bar{0}\bar{0}\bar{0}\bar{1}) & u_4 \mapsto (1\bar{0}\bar{0}\bar{0}\bar{0}\bar{0}\bar{0}) & v_4 \mapsto (-1\bar{0}\bar{0}\bar{0}\bar{0}\bar{0}\bar{0}). \end{array}$$

- The $\mathbb{Z}_2^8$-grading induced by

$$\begin{array}{lll} e_0 \mapsto (\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}) & e_1 \mapsto (\bar{1}\bar{0}\bar{0}\bar{0}\bar{0}\bar{0}\bar{0}\bar{0}) & e_2 \mapsto (\bar{0}\bar{1}\bar{0}\bar{0}\bar{0}\bar{0}\bar{0}\bar{0}) \\ e_3 \mapsto (\bar{0}\bar{0}\bar{1}\bar{0}\bar{0}\bar{0}\bar{0}\bar{0}) & e_4 \mapsto (\bar{0}\bar{0}\bar{0}\bar{1}\bar{0}\bar{0}\bar{0}\bar{0}) & e_5 \mapsto (\bar{0}\bar{0}\bar{0}\bar{0}\bar{1}\bar{0}\bar{0}\bar{0}) \\ e_6 \mapsto (\bar{0}\bar{0}\bar{0}\bar{0}\bar{0}\bar{1}\bar{0}\bar{0}) & e_7 \mapsto (\bar{0}\bar{0}\bar{0}\bar{0}\bar{0}\bar{0}\bar{1}\bar{0}) & e_8 \mapsto (\bar{0}\bar{0}\bar{0}\bar{0}\bar{0}\bar{0}\bar{0}\bar{1}). \end{array}$$

The induced gradings on $\mathrm{so}(V,q)$ agree with the gradings produced as the simultaneous diagonalizations relative to the following MAD-groups of $\mathrm{SO}(V,q)$ (respectively), where we are identifying the elements in $\mathrm{SO}(V,q)$ with their matrices relative to the base B' (notations as in Subsection 4.2):

- $Q_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & S_\alpha & 0 & 0 & 0 \\ 0 & 0 & S_\beta & 0 & 0 \\ 0 & 0 & 0 & S_\delta & 0 \\ 0 & 0 & 0 & 0 & S_\epsilon \end{pmatrix} \mid \alpha, \beta, \delta, \epsilon \in \mathbb{K}^\times \right\}$,
- $Q_2 = \langle \left\{ \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & I_7 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & I_6 \end{pmatrix}, \begin{pmatrix} I_3 & 0 & 0 & 0 \\ 0 & S_\beta & 0 & 0 \\ 0 & 0 & S_\delta & 0 \\ 0 & 0 & 0 & S_\epsilon \end{pmatrix} \mid \beta, \delta, \epsilon \in \mathbb{K}^\times \right\} \rangle$,
- $Q_3 = \langle \left\{ d_0d_1, d_0d_2, d_0d_3, d_0d_4, \begin{pmatrix} I_5 & 0 & 0 \\ 0 & S_\delta & 0 \\ 0 & 0 & S_\epsilon \end{pmatrix} \mid \delta, \epsilon \in \mathbb{K}^\times \right\} \rangle$,
- $Q_4 = \langle \left\{ d_0d_1, d_0d_2, d_0d_3, d_0d_4, d_0d_5, d_0d_6, \begin{pmatrix} I_7 & 0 \\ 0 & S_\epsilon \end{pmatrix} \mid \epsilon \in \mathbb{K}^\times \right\} \rangle$,
- $Q_5 = \langle \{ d_0d_1, d_0d_2, d_0d_3, d_0d_4, d_0d_5, d_0d_6, d_0d_7, d_0d_8 \} \rangle$.

4.6. **MAD-groups of** $\mathrm{Spin}(9)$**.** If Q is a MAD-group of $\mathrm{Spin}(V,q)$, that is, a maximal abelian subgroup of semisimple elements, its image $\rho(Q)$ is also abelian and formed by semisimple elements, so that it lives in a MAD-group of $\mathrm{SO}(V,q)$ and there are $f \in \mathrm{SO}(V,q)$ and $i \in \{1,\dots,5\}$ such that $\rho(Q) \subset fQ_if^{-1}$. By replacing Q with $g^{-1}Qg$ for $g \in \rho^{-1}(f)$, we can assume without loss of generality that such $Q \subset \rho^{-1}(Q_i)$. But it is easy to have concrete descriptions of generators of the group $\rho^{-1}(Q_i)$, taking into account that $\rho(e_0e_i) = d_0d_i$, according to Equation (2):

- $\rho^{-1}(Q_1) = \{ s_{\alpha\beta\delta\epsilon} \mid \alpha, \beta, \delta, \epsilon \in \mathbb{K}^\times \} = \mathcal{T}$,
- $\rho^{-1}(Q_2) = \langle \{ \pm e_0e_1, \pm e_0e_2, s_{1\beta\delta\epsilon} \mid \beta, \delta, \epsilon \in \mathbb{K}^\times \} \rangle$,
- $\rho^{-1}(Q_3) = \langle \{ \pm e_0e_1, \pm e_0e_2, \pm e_0e_3, \pm e_0e_4, s_{11\delta\epsilon} \mid \delta, \epsilon \in \mathbb{K}^\times \} \rangle$,
- $\rho^{-1}(Q_4) = \langle \{ \pm e_0e_1, \pm e_0e_2, \pm e_0e_3, \pm e_0e_4, \pm e_0e_5, \pm e_0e_6, s_{111\epsilon} \mid \epsilon \in \mathbb{K}^\times \} \rangle$,
- $\rho^{-1}(Q_5) = \langle \{ \pm e_0e_1, \pm e_0e_2, \pm e_0e_3, \pm e_0e_4, \pm e_0e_5, \pm e_0e_6, \pm e_0e_7, \pm e_0e_8 \} \rangle$.

Note that these groups $\rho^{-1}(Q_i) \leq \mathrm{Spin}(V,q)$ are not abelian if $i \neq 1$, whereas $\rho^{-1}(Q_1)$ is a 4-dimensional maximal torus of $\mathrm{Spin}(V,q)$. The following considerations about some of their elements will be useful for us:

(i) The element e_ie_j has order 4 if $i \neq j$ ($(e_ie_j)^2 = -1$), and $e_ie_je_ke_l$ has order 2 if i, j, k, l are distinct.

(ii) If i,j,k are distinct indices, e_ie_j anticommutes with e_ie_k. More generally, $(e_{i_1}\dots e_{i_s})(e_{j_1}\dots e_{j_r}) = (-1)^m(e_{j_1}\dots e_{j_r})(e_{i_1}\dots e_{i_s})$ if m is the cardinal of the set $\{i_1,\dots,i_s\}\cap\{j_1,\dots,j_r\}$.

(iii) If some $s_{\alpha\beta\delta\epsilon}$ is in certain $\rho^{-1}(Q_i)$, then it belongs to the center of such $\rho^{-1}(Q_i)$. In particular $-1 = s_{-1111} = s_{1-111} = s_{11-11} = s_{111-1}$ belongs to the center of $\rho^{-1}(Q_i)$ for all i.

Lemma 9. *If σ is a permutation of $J=\{0,1,\dots,8\}$, there is $x\in\mathrm{Spin}(V,q)$ such that $xe_jx^{-1}\in\{\pm e_{\sigma(j)}\}$ for all $j\in J$.*

Proof. It is enough to check the result for one transposition. For $\sigma=(1,2)$, note that

$$\begin{aligned} s_{\xi111}e_1s_{\xi111}^{-1} &= e_2,\\ s_{\xi111}e_2s_{\xi111}^{-1} &= -e_1,\\ s_{\xi111}e_js_{\xi111}^{-1} &= e_j \end{aligned}$$

for any $j\in J\setminus\{1,2\}$, where ξ is a square root of $\mathfrak{i}$ ($\xi^8=1$). Observe also that $s_{\xi111}=\frac{(\xi+\xi^7)+(\xi^3+\xi^5)e_1e_2}{2}$, so that the element $\frac{(\xi+\xi^7)+(\xi^3+\xi^5)e_ie_j}{2}\in\mathrm{Spin}(V,q)$ works for interchanging an arbitrary pair of indices $\{i,j\}\subset J$. □

Thus,

Theorem 1. *If Q is a MAD-group of $\mathrm{Spin}(V,q)$, then it is conjugated to one of the following quasitori:*

(a) $P_1=\mathcal{T}$,

(b) $P_2=\langle\{e_1e_2e_3e_4, e_1e_2e_5e_6, e_0e_1e_3e_5, s_{111\epsilon}\mid \epsilon\in\mathbb{K}^\times\}\rangle\cong\mathbb{Z}_2^3\times\mathbb{K}^\times$,

(c) $P_3=\langle\{-1, e_1e_2e_3e_4, e_1e_2e_5e_6, e_1e_2e_7e_8, e_1e_3e_5e_7\}\rangle\cong\mathbb{Z}_2^5$.

Remark 2. *Note that these P_i's are actually MAD-groups of $\mathrm{Spin}(V,q)$. To be sure we have only to check that P_2 is not subconjugated to $P_1=\mathcal{T}$, that is, that $P_2'=\langle\{e_1e_2e_3e_4, e_1e_2e_5e_6, e_0e_1e_3e_5\}\rangle$ is a nontoral group isomorphic to $\mathbb{Z}_2^3$. This is equivalent to proving that $\rho(P_2')$ is a nontoral group of $\mathrm{SO}(V,q)$. Identifying the elements in $\mathrm{so}(V,q)$ and $\mathrm{SO}(V,q)$ with their matrices relative to B', a straightforward computation shows that the set of skewsymmetric matrices of size 9 which commute with $\langle\{d_1d_2d_3d_4, d_1d_2d_5d_6, d_0d_1d_3d_5\}\rangle$ is the 1-dimensional space $\{(a_{ij})_{i,j=0\dots8}\mid a_{78}=-a_{87},\ a_{ij}=0$ otherwise$\}$. Thus the fixed component by the diagonalization produced by $\rho(P_2')$ has dimension strictly less than 4 (precisely 1), so that it does not contain a Cartan subalgebra and the grading is nontoral (*[9, p. 94]*).*

Proof of Theorem 1. We can suppose that Q is an abelian subgroup of some $\rho^{-1}(Q_i)$. Note also that $-1\in Q$ by maximality of Q, since $\langle -1,Q\rangle$ is always abelian and diagonalizable.

If $i=1$, then $Q=\rho^{-1}(Q_1)=\mathcal{T}$ by maximality ($\mathcal{T}$ is already abelian).

If $i=2$, then $\{s_{1\beta\delta\epsilon}\mid\beta,\delta,\epsilon\in\mathbb{K}^\times\}\subsetneq Q\subset\{s_{1\beta\delta\epsilon}\}\cdot\{e_0e_1,e_0e_2,e_1e_2,1\}$. Necessarily there is an element $x\in\{e_0e_1,e_0e_2,e_1e_2\}$ belonging to Q. But no other element in that set commutes with x, hence $Q=\{s_{1\beta\delta\epsilon}\}\cdot\{1,x\}$. We can assume that $x=e_1e_2=s_{-\mathfrak{i}111}$, because of the previous lemma. But then $Q\subsetneq\mathcal{T}$, which contradicts the maximality of Q.

If $i=3$, then $\{s_{11\delta\epsilon}\mid\delta,\epsilon\in\mathbb{K}^\times\}\subsetneq Q\subset\{s_{11\delta\epsilon}\}\cdot\langle\{e_ie_j\mid i,j=0,1,\dots,4\}\rangle$. There is $\bar{x}=(x_1,\dots,x_r)$ with $x_i=a_{i,1}\dots a_{i,n_i}$, $a_{i,j}\in\{e_0,\dots,e_4\}$, n_i even, such that $Q=\{s_{11\delta\epsilon}\}\cdot\langle\{x_1,\dots,x_r\}\rangle$ and each $x_j\notin\{s_{11\delta\epsilon}\}\cdot\langle\{x_1,\dots,x_{j-1}\}\rangle$. Among the possible $\bar{x}$ verifying such conditions, choose one such that the attached $\bar{n}=(n_1,\dots,n_r)\in\mathbb{N}^r$ is minimum in $\cup_{s\in\mathbb{N}}\mathbb{N}^s$ with the lexicographical order. In particular $n_1\le\cdots\le n_r$, taking into account that for any permutation $\sigma\in S_r$, $\bar{x}^\sigma=(x_{\sigma(1)},\dots,x_{\sigma(r)})$ verifies the same conditions as $\bar{x}$. Indeed, assume that $x_{\sigma(j)} \in \{s_{11\delta\epsilon}\} \cdot \langle\{x_{\sigma(1)}, \dots, x_{\sigma(j-1)}\}\rangle$. Thus $x_{\sigma(j)} = s_{11de} x_{\sigma(1)}^{s_1} \dots x_{\sigma(j-1)}^{s_{j-1}}$ for some $d, e \in \mathbb{K}^\times$ and $s_i \in \{0,1\}$, since $x_i^2 = (-1)^{\frac{n_i}{2}-1} \in \{\pm 1\}$. Now we choose $k \in \{1, \dots, j-1\}$ such that $\sigma(k)$ is the greatest index with $s_k \neq 0$ (necessarily $\sigma(k) > \sigma(j)$ and $s_k = 1$) and then $x_{\sigma(k)} = \pm s_{11de} x_{\sigma(1)}^{s_1} \dots \hat{x}_{\sigma(k)}^{s_k} \dots x_{\sigma(j-1)}^{s_{j-1}} x_{\sigma(j)} \in \{s_{11\delta\epsilon}\} \cdot \langle\{x_1, \dots, x_{\sigma(k)-1}\}\rangle$, a contradiction. As $\mathrm{Spin}(V,q)$ is a simply connected group, Lemma 1 and Lemma 2 can be applied to get that $r \geq 3$. If $n_1 = 2$, then we can assume that $x_1 = e_1e_2$ by Lemma 9, because the element used for conjugating does not change $s_{11\delta\epsilon}$. In the same way we can assume that $x_2 = e_3e_4$ if $n_2 = 2$ (and $x_2 = e_1e_2e_3e_4$ if $n_2 = 4$, but then $\bar{n}$ would not be minimal). But now there is no possibility for x_3 (it should have an even -not 2- number of indices in common with $\{1,2\}$ and with $\{3,4\}$). If $n_1 = 4$, then we can assume that $x_1 = e_1e_2e_3e_4$ but there is no x_2 with the required conditions.

If $i = 4$, we have a similar situation: $\{s_{111\epsilon} \mid \epsilon \in \mathbb{K}^\times\} \subsetneq Q \subset \{s_{111\epsilon}\} \cdot \langle\{e_ie_j \mid i,j = 0,1,\dots,6\}\rangle$, so that we can take $Q = \{s_{111\epsilon}\} \cdot \langle\{x_1, \dots, x_r\}\rangle$ for certain $x_j \in \mathrm{Spin}(V,q) \setminus \{s_{111\epsilon}\} \cdot \langle\{x_1, \dots, x_{j-1}\}\rangle$ product of $n_j \in \{2,4,6\}$ elements in $\{e_0, \dots, e_6\}$. Again the $r \geq 3$ generators have been chosen such that $\bar{n} = (n_1, \dots, n_r)$ is minimum, and, in particular, $n_1 \leq \dots \leq n_r$. If $n_1 = n_2 = 2$, then we can assume that $x_1 = e_1e_2 = s_{-\mathrm{i}111}$ and that $x_2 = e_3e_4$, again by following Lemma 9. As the e_i's involved in x_3 are only e_0, e_5, e_6 (otherwise there would be another $\bar{x}'$ with $n_3' < n_3$ so that $\bar{n}' = (n_1, n_2, n_3', \dots)$ is lesser than $\bar{n}$), this implies that $n_3 = 2$, so that we can assume that $x_3 = e_5e_6$. But nothing more in $\rho^{-1}(Q_4)$ commutes with all these elements, hence $Q = \{s_{111\epsilon}\} \cdot \langle\{s_{-\mathrm{i}111}, s_{1-\mathrm{i}11}, s_{11-\mathrm{i}1}\}\rangle$, which is strictly contained in $\mathcal{T}$, a contradiction. If $n_1 = 2$ and $n_2 = 4$ we can assume that $x_1 = e_1e_2$ and that $x_2 = e_3e_4e_5e_6$. Now there is no x_3 satisfying the conditions (with at least four e_i's involved, then e_1 and e_2 would appear and we could lessen n_3 in $\bar{n}$). Neither there is any possibility with $n_1 = 2$ and $n_2 = 6$. Hence $n_1 = 4$. That forces $n_2 = 4 = n_3$ because if some $n_i = 6$, x_i would have four indices in common with x_1 (there are not enough elements for having only two in common) and x_1x_i would have only two involved elements (less $\bar{n}$ again). So we can assume that $x_1 = e_1e_2e_3e_4$, that $x_2 = e_1e_2e_5e_6$, and that x_3 has just two e_i's in common with x_1 and 2 with x_2. These elements cannot be e_1 and e_2 (there is only e_0 to add) so that there are in x_3 one element in $\{e_1, e_2\}$, one element in $\{e_3, e_4\}$ and one element in $\{e_5, e_6\}$, and consequently we can assume that $x_3 = e_0e_1e_3e_5$. Now $P_2 \subset Q$, but it is clear that not more elements in $\rho^{-1}(Q_4)$ commute with Q.

If $i = 5$, we can take similarly to the previous cases $Q = \langle\{-1, x_1, \dots, x_r\}\rangle$, where each x_j is a product of an even number $n_j \in \{2,4,6,8\}$ of elements in $\{e_0, \dots, e_8\}$, satisfying that $x_j \notin \langle\{-1, x_1, \dots, x_{j-1}\}\rangle$, $\bar{n} = (n_1, \dots, n_r)$ minimum, $n_1 \leq \dots \leq n_r$ and $r \geq 3$. If $\bar{n} = (2,2,2,\dots)$, then we can change the generators by $x_1 = e_1e_2$, $x_2 = e_3e_4$, $x_3 = e_5e_6$ and then necessarily $n_4 = 2$ and we can take $x_4 = e_7e_8$. Thus nothing more can be added and $Q \subset \mathcal{T}$. If $\bar{n} = (2,2,4,\dots)$, then we can change the generators into $x_1 = e_1e_2$, $x_2 = e_3e_4$, $x_3 = e_5e_6e_7e_8$ and again nothing more can be added and $Q \subset \mathcal{T}$. The choice $\bar{n} = (2,2,6,\dots)$ would not provide $\bar{n}$ minimal. If $\bar{n} = (2,4,\dots)$, then we can change the generators into $x_1 = e_1e_2$ and $x_2 = e_3e_4e_5e_6$. If $n_3 = 4$, we can take $x_3 = e_5e_6e_7e_8$, so that also $n_4 = 4$ and x_4 has two elements in $\{3,4,5,6\}$ and two in $\{5,6,7,8\}$ (none in $\{1,2\}$). Thus we can take $x_4 = e_0e_3e_5e_7$ and necessarily $Q = \langle\{-1, e_1e_2, e_3e_4e_5e_6, e_5e_6e_7e_8, e_0e_3e_5e_7\}\rangle$, which is not a MAD-group, because according to Lemma 9 the element $x \in \mathrm{Spin}(V,q)$ related to the permutation $\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 7 & 8 & 3 & 4 & 1 & 2 & 5 & 6 \end{pmatrix}$ verifies that xQx^{-1} is strictly contained in P_2. Of course the case $n_3 = 6$ is not possible and we conclude that $n_1 = 4$. Now we get that $n_1 = n_2 = n_3 = 4$ and modify the generators to be either $(x_1, x_2, x_3) = (e_1e_2e_3e_4, e_1e_2e_5e_6, e_1e_2e_7e_8)$ or $(e_1e_2e_3e_4, e_1e_2e_5e_6, e_1e_3e_5e_7)$.

The generated groups are different, even though both are isomorphic to $\mathbb{Z}_2^3$ as abstract groups, the first one is obviously toral (just contained in $\mathcal{T}$), but the second one is nontoral according to Remark 2 (we talked there about $e_0e_1e_3e_5$ as in case $i=4$, but we can map $e_0e_1e_3e_5$ into $\pm e_1e_3e_5e_7$ without moving $e_1,\dots,e_6$ by Lemma 9). In both cases there is a fourth element in $\rho^{-1}(Q_5)$ commuting with them: $x_4=e_1e_3e_5e_7$ and $x_4=e_1e_2e_7e_8$ respectively, which obviously leads us to the same Q. Now there is no possibility of adding anything more, so that $r=4$. □

Note that $\varphi=\psi_{-1}\in\psi(P_i)$ for all $i=1,2,3$. They are the only MAD-groups containing φ:

Corollary 5. *If Q is a MAD-group of* $\operatorname{aut}(\mathfrak{f}_4)$ *which contains φ, then Q is conjugated to*

(a) $\psi(P_1)\cong(\mathbb{K}^\times)^4$,
(b) $\psi(P_2)\cong\mathbb{Z}_2^3\times\mathbb{K}^\times$,
(c) $\psi(P_3)\cong\mathbb{Z}_2^5$.

Proof. As $\varphi\in Q$, then Q is contained in $\operatorname{Cent}_{\operatorname{aut}(\mathfrak{f}_4)}(\varphi)$, which coincides with $\psi(\operatorname{Spin}(V,q))$ according to Proposition 2. Taking into account that ψ is an isomorphism, Theorem 1 gives the result. □

5. 3-groups of $\operatorname{aut}(\mathfrak{f}_4)$

The objective here is to prove

Theorem 2. *There is an only nontoral* 3*-subgroup of* $\operatorname{aut}(\mathfrak{f}_4)$*. It is isomorphic to $\mathbb{Z}_3^3$ as abstract group. It is a MAD-group.*

There are several results in the literature related to this one, as

Proposition 4. *(*[27, Proposition 3.5.]*, more detailed in* [16, (7.4) THEOREM]*) There is an only nontoral elementary* 3*-group of* $\operatorname{aut}(\mathfrak{f}_4)$*. It is isomorphic to $\mathbb{Z}_3^3$ as abstract group. It is a MAD-group.*

The problem is that we cannot conclude Theorem 2 from this proposition, at least not directly, as we observed in Remark 1. On the other hand, the computational methods did not turn out to be difficult for this prime, but precisely our main aim is to avoid completely the usage of computer. Thus, we proceed as in the case of the prime 2, by following similar steps: a nontoral 3-group must contain some order 3 automorphism fixing a subalgebra of type $\mathfrak{a}_2\oplus\mathfrak{a}_2$ and hence it lives in the corresponding centralizer, certain quotient of $\mathrm{SL}(3)^2$. Then we try to use our knowledge of the gradings on matrix algebras.

5.1. **Model of $\mathfrak{f}_4$ based on $2\mathfrak{a}_2$.** Now let V and W be 3-dimensional vector spaces and take

$$\mathcal{L}=\mathcal{L}_{\bar 0}\oplus\mathcal{L}_{\bar 1}\oplus\mathcal{L}_{\bar 2}$$

the $\mathbb{Z}_3$-graded Lie algebra given by

$$\mathcal{L}_{\bar 0}=\mathrm{sl}(V)\oplus\mathrm{sl}(W),\qquad \mathcal{L}_{\bar 1}=V\otimes S^2(W),\qquad \mathcal{L}_{\bar 2}=V^*\otimes S^2(W^*),$$

where $S^2(U)$ denotes the symmetric tensors in $U\otimes U$ and the product is given in the following way:

- $\mathrm{sl}(V)\oplus\mathrm{sl}(W)$ is a Lie subalgebra.
- The actions of $\mathcal{L}_{\bar 0}$ on $V\otimes S^2(W)$ and on $V^*\otimes S^2(W^*)$ are the natural ones.

- We have fixed a nonzero trilinear alternating map $\det\colon V\times V\times V\to\mathbb{K}$ so that we identify $V\wedge V$ with V^* by means of $u\wedge v\mapsto\det(u,v,-)$. For $\det^*$ the dual map of det, we also identify $V^*\wedge V^*$ with V. Proceed similarly with W. Now for any $u,v\in V$, $w,x\in W$, $f,g\in V^*$, $h,j\in W^*$, and denoting by f_u the endomorphism $f(-)u\in\mathrm{gl}(V)$ and by $\pi f\equiv f-\frac{1}{3}\mathrm{tr}(f)\mathrm{id}$ the projection on the traceless endomorphisms,

$$\begin{aligned}
[f\otimes h\cdot h,u\otimes w\cdot w] &= h(w)^2\pi f_u+f(u)h(w)\pi h_w,\\
[u\otimes w\cdot w,v\otimes x\cdot x] &= (u\wedge v)\otimes(w\wedge x)\cdot(w\wedge x),\\
[f\otimes h\cdot h,g\otimes j\cdot j] &= (f\wedge g)\otimes(h\wedge j)\cdot(h\wedge j).
\end{aligned}$$

The so described algebra is simple of type $\mathfrak{f}_4$ (see [8] for details about this and other constructions of $\mathfrak{f}_4$). Call ϕ the order 3 grading automorphism. We compute its centralizer. Note that now the adjoint map denotes $\mathrm{Ad}\colon\mathrm{SL}(V)\to\mathrm{gl}(\mathrm{sl}(V))$ given by $\mathrm{Ad}\,x(f)=xfx^{-1}$ for any $x\in\mathrm{SL}(V)$, $f\in\mathrm{sl}(V)\equiv\mathfrak{a}_2$, and similarly for W. For $x\in\mathrm{SL}(V)$, $y\in\mathrm{SL}(W)$, consider the map $\Psi_{x,y}\colon\mathfrak{f}_4\to\mathfrak{f}_4$ given by $\Psi_{x,y}|_{\mathrm{sl}(V)}=\mathrm{Ad}\,x$, $\Psi_{x,y}|_{\mathrm{sl}(W)}=\mathrm{Ad}\,y$, $\Psi_{x,y}(v\otimes w_1\cdot w_2)=(x\cdot v)\otimes(y\cdot w_1)\cdot(y\cdot w_2)$ for any $v\in V$ and $w_1,w_2\in W$, and $\Psi_{x,y}(f\otimes g_1\cdot g_2)=(x\cdot f)\otimes(y\cdot g_1)\cdot(y\cdot g_2)$ for any $f\in V^*$ and $g_1,g_2\in W^*$, where $\cdot$ denotes the symmetric product as well as the action of SL on its natural and dual representations.

Proposition 5. *The map $\Psi_{x,y}$ is an automorphism of the Lie algebra $\mathcal{L}=\mathfrak{f}_4$ for all $x\in\mathrm{SL}(V)$ and $y\in\mathrm{SL}(W)$; and the map*

$$\Psi\colon\mathrm{SL}(V)\times\mathrm{SL}(W)\to\mathrm{Cent}_{\mathrm{aut}(\mathfrak{f}_4)}(\phi)$$

given by $(x,y)\mapsto\Psi_{x,y}$ is a group epimorphism with kernel $\{(\omega^n\mathrm{id}_V,\omega^n\mathrm{id}_W)\mid n=0,1,2\}\cong\mathbb{Z}_3$, for ω a primitive cubic root of the unit.

Proof. Proceed as in the proof of Proposition 2 to check that this is a well defined surjective map, and of course a group homomorphism. Let us compute the kernel. If $\Psi_{x,y}=\mathrm{id}_{\mathfrak{f}_4}$, the element x commutes with $\mathrm{sl}(V)$, and hence there is $\alpha\in\mathbb{K}$ such that $x=\alpha\,\mathrm{id}_V$. But, as $\det(x)=1$, necessarily $\alpha^3=1$. In the same way, $y=\beta\,\mathrm{id}_W$ with $\beta^3=1$. Now $\Psi_{x,y}|_{\mathcal{L}_{\bar 1}}=\alpha\beta^2\,\mathrm{id}$, so that $\alpha\beta^2$ must be equal to 1 and hence $\alpha=\beta$. □

5.2. **Every 3-group lives in** $\mathrm{SL}(3)^2/\mathbb{Z}_3$**.** We would like to prove that every nontoral 3-group contains some automorphism conjugated to ϕ.

According to [20, p. 248], the index of the Weyl group of $\mathrm{Cent}_{\mathrm{aut}(\mathfrak{f}_4)}(\phi)$ in the Weyl group of $\mathrm{aut}(\mathfrak{f}_4)$ is 32 (this number can also be easily computed with the trick described in Lemma 8), coprime to 3. Again this fact implies that

Proposition 6. *If Q is a 3-group of $\mathrm{aut}(\mathfrak{f}_4)$, then Q is conjugated to a subquasitorus of $\mathrm{Cent}_{\mathrm{aut}(\mathfrak{f}_4)}(\phi)$.*

Which can be proved analogously to Proposition 3.

5.3. **MAD-groups of** $\mathrm{SL}(3)$**.**

Proposition 7. *There are four fine gradings on the algebra $\mathrm{sl}(3)$. Their grading groups are*

$$\mathbb{Z}^2,\qquad\mathbb{Z}\times\mathbb{Z}_2,\qquad\mathbb{Z}_2^3,\qquad\mathbb{Z}_3^2.$$

Equivalently, up to conjugation there are four MAD-groups of $\mathrm{aut}(\mathrm{sl}(3))\cong\mathrm{PSL}(3)\rtimes\mathbb{Z}_2$.

This result can be concluded from [5], but the gradings are explicitly computed in [18]. We do not really need a concrete description of all the gradings, it is enough for our purposes to recall which is the $\mathbb{Z}_3^2$-nontoral grading. If we denote by

$$b := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \qquad c := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

we can observe that b and c are elements of order 3 in SL(3) that do not commute: $bc = \omega cb$. On the contrary, their classes in $\mathrm{PSL}(3) = \mathrm{SL}(3)/\langle \omega I_3 \rangle$ do commute, and $\langle \{\bar{b}, \bar{c}\} \rangle \cong \mathbb{Z}_3^2$ is a MAD-group of PSL(3), where $\bar{x}$ denotes the class of the element $x \in \mathrm{SL}(3)$ modulo $\langle \omega I_3 \rangle$.

Identify $\mathrm{SL}(V)$ and $\mathrm{SL}(W)$ with SL(3) by means of their matrices relative to some fixed bases and also identify

$$\mathrm{Cent}_{\mathrm{aut}(\mathfrak{f}_4)}(\phi) = \Psi \left(\frac{\mathrm{SL}(V) \times \mathrm{SL}(W)}{\langle (\omega \mathrm{id}_V, \omega \mathrm{id}_W) \rangle} \right) \equiv \frac{\mathrm{SL}(3) \times \mathrm{SL}(3)}{\langle (\omega I_3, \omega I_3) \rangle}.$$

Now consider the projections

$$\pi_i \colon \frac{\mathrm{SL}(3) \times \mathrm{SL}(3)}{\langle (\omega I_3, \omega I_3) \rangle} \to \mathrm{PSL}(3) = \frac{\mathrm{SL}(3)}{\langle \omega I_3 \rangle}$$

given by $\pi_1(\Psi_{[x;y]}) = \bar{x}$ and $\pi_2(\Psi_{[x;y]}) = \bar{y}$, where $[x; y]$ denotes the class of the element $(x, y) \in \mathrm{SL}(3) \times \mathrm{SL}(3)$ modulo $\langle (\omega I_3, \omega I_3) \rangle$. Note that they are well defined because $\pi_i(\Psi_{[\omega I_3; \omega I_3]}) = \overline{\omega I_3} = \bar{I}_3$.

Proof of Theorem 2. Take Q a nontoral 3-group, which can be assumed contained in $\mathrm{Cent}_{\mathrm{aut}(\mathfrak{f}_4)}(\phi)$, so that each $\pi_i(Q)$ is a subquasitorus of aut(sl(3)) which lives in $(\mathbb{K}^\times)^2$, $(\mathbb{K}^\times) \times \mathbb{Z}_2$, $\mathbb{Z}_2^3$ or $\mathbb{Z}_3^2$ by Proposition 7. It is clear, as in Lemma 5, that $\pi_i(Q)$ is contained in $(\mathbb{K}^\times)^2$, $\mathbb{K}^\times$, id or $\mathbb{Z}_3^2$. But π_i maps nontoral groups into nontoral groups, so that we can also assume that $\pi_i(Q) = \langle \{\bar{b}, \bar{c}\} \rangle \cong \mathbb{Z}_3^2$. Now, an arbitrary element in Q is $\Psi_{[x;y]}$ with $x, y \in \{\omega^{n_1} b^{n_2} c^{n_3} \mid n_i = 0, 1, 2\} =: P$. Hence $x^3 = y^3 = I_3$, the element $\Psi_{[x;y]}$ has order 3 and Q is elementary, so that we could apply Proposition 4 to finish our proof. But, again for selfcontainedness, we are going to prove that

$$Q \cong \langle \{\Psi_{[I_3; \omega I_3]}, \Psi_{[b;b]}, \Psi_{[c;c]}\} \rangle =: Q'.$$

Take some elements $\Psi_{[b;y_1]} \in \pi_1^{-1}(\bar{b})$ and $\Psi_{[c;y_2]} \in \pi_1^{-1}(\bar{c})$. They commute, so that $[bc = \omega cb; y_1 y_2] = [cb; y_2 y_1]$ and $y_1 y_2 = \omega y_2 y_1$. In particular, $y_1 \notin \{I_3, \omega I_3, \omega^2 I_3\}$. As $\Psi_{[I_3;y]} \Psi_{[b;y_1]} \Psi_{[I_3;y]}^{-1} = \Psi_{[b; y y_1 y^{-1}]}$, we can replace y_1 by b (the 26 order 3 elements in P are conjugated in SL(3)). This implies that $y_2 = \omega^{n_1} b^{n_2} c$. As $\langle \{\Psi_{[b;b]}, \Psi_{[c;y_2]}\} \rangle$ is toral (arguments as in Lemma 1), we can find $\Psi_{[x_3;y_3]} \in Q \setminus \langle \{\Psi_{[b;b]}, \Psi_{[c;y_2]}\} \rangle$. We can assume that $x_3 = I_3$ (if $x_3 = b$, replace it by $\Psi_{[x_3;y_3]} \Psi_{[b;b]}^2$, and do the same for any of the other possibilities for x_3). Now, the commutativity condition forces y_3 to commute with b and $b^{n_2} c$, hence $y_3 \in \{1, \omega, \omega^2\} I_3$. But $y_3 \neq I_3$, so $\Psi_{[I_3; \omega I_3]} \in Q$. Thus $\langle \{\Psi_{[I_3; \omega I_3]}, \Psi_{[b;b]}, \Psi_{[c; b^{n_2} c]}\} \rangle \subset Q$. Note now that the diagonal matrix $p = \mathrm{diag}\{1, \omega^2, 1\} \in \mathrm{SL}(3)$ verifies that $pbp^{-1} = b$ and $pcp^{-1} = bc$, so that Q' is contained in a quasitorus conjugated to Q (by means of $\Psi_{[I_3;p]}$ or $\Psi_{[I_3;p^2]}$), but Q' is its own centralizer and we are done. □

6. MAD-groups of $\mathrm{aut}(\mathfrak{f}_4)$

Lemma 10. *The automorphisms $\psi_{\pm e_1 e_2 e_3 e_4 e_5 e_6 e_7 e_8}$ are conjugated to φ.*

Proof. Recall that any order 2 automorphism in $\mathrm{aut}(\mathfrak{f}_4)$ fixes a subalgebra of type either $\mathfrak{b}_4$ or $\mathfrak{c}_3 \oplus \mathfrak{a}_1$, so that the conjugacy class is determined by the dimension of the fixed part of any representative in the class (36 and 24 respectively). Thus we have only to check that $\dim \mathrm{Fix}(\varphi_i) = 36$ for $\varphi_1 = \psi_{e_1e_2e_3e_4e_5e_6e_7e_8}$ and $\varphi_2 = \psi_{-e_1e_2e_3e_4e_5e_6e_7e_8} = \varphi_1\varphi$.

First note that the restriction to the even part $\varphi_i|_{\mathrm{so}(V,q)} = \mathrm{Ad}\,\rho(e_1e_2e_3e_4e_5e_6e_7e_8)$ fixes the subalgebra $\mathrm{so}(V', q)$ for $V' = \mathrm{span}\langle\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}\rangle$, which is a Lie algebra of type $\mathfrak{d}_4$ and dimension 28.

In order to compute the fixed part of $\varphi_i|_{\wedge W}$, note that $\tilde{\gamma}(e_1e_2) = \tilde{\gamma}(\frac{\mathrm{i}}{2}[v_1, u_1]) = -2\mathrm{i}\tilde{\gamma}\iota(h_1)$ and, taking into account Equation (3), if $s = u_{j_1} \wedge \cdots \wedge u_{j_r}$,

$$\varphi_1(s) = -\varphi_2(s) = (2\mathrm{i})^4 h_1 \cdot (h_2 \cdot (h_3 \cdot (h_4 \cdot s))) = (-1)^{n_1+n_2+n_3+n_4} s$$

where $n_i = 0$ if $i \in \{j_1, \ldots, j_r\}$ and $n_i = 1$ otherwise. Hence $\sum_{i=1}^4 n_i = 4 - r$ and $\varphi_1(s) = s$ just when r is even. This means that $\mathrm{Fix}\,\varphi_1 = \wedge_{\bar{0}} W$ and $\mathrm{Fix}\,\varphi_2 = \wedge_{\bar{1}} W$, so that $\dim \mathrm{Fix}\,\varphi_i|_{\wedge W} = 8$ and $\dim \mathrm{Fix}\,\varphi_i = 36$, as desired. □

Theorem 3. *The fine gradings on* $\mathfrak{f}_4$ *are, up to equivalence, the four fine gradings described in Section* 3.

Proof. Take Q a MAD-group of $\mathrm{aut}(\mathfrak{f}_4)$ different from the maximal torus. If Q contains a nontoral 3-group R_3, then R_3 is itself a MAD-group by Theorem 2, and hence $Q = R_3$. Otherwise Q contains R_2 a nontoral 2-group by Corollary 3. Let us show that in this case Q is conjugated to either $\psi(P_2)$ or $\psi(P_3)$, where P_2 and P_3 are described in Theorem 1. According to Corollary 4, we are in the following situation: $Q = T \times R_2 \times R$ with T a torus, R_2 a nontoral 2-group and R a finite group of odd order. Now, by Proposition 3, we can assume that $T \times R_2 \subset \psi(\mathrm{Spin}(V, q))$ and, by Theorem 1, that $T \times R_2$ is contained in either $\psi(P_2) \cong \mathbb{Z}_2^3 \times \mathbb{K}^\times$ or $\psi(P_3) \cong \mathbb{Z}_2^5$. If R is trivial, then $Q = T \times R_2 \subset \psi(\mathrm{Spin}(V, q))$ and we have finished by Corollary 5. We are also done if $\varphi = \psi(-1) \in Q$, since then $Q \subset \mathrm{Cent}_{\mathrm{aut}(\mathfrak{f}_4)}(\varphi) = \psi(\mathrm{Spin}(V, q))$ (of course in this case R turns out to be trivial). If R is not trivial, by Corollary 4 the 2-group R_2 has at least 4 factors. If $R_2 \subset \psi(P_2)$, there is $\epsilon \in \mathbb{K}^\times$ of order a power of 2 (root of the unit) such that $\psi^{-1}(R_2) = \langle\{e_1e_2e_3e_4, e_1e_2e_5e_6, e_0e_1e_3e_5, s_{111\epsilon}\}\rangle$. Thus we have the contradiction $-1 = s_{111-1} \in \psi^{-1}(R_2)$. The other possibility is that R_2 is contained in $\psi(P_3)$. As $\psi(-1) \notin R_2$, the existence of the four factors forces R_2 to be the image under ψ of $\langle\{\alpha_1e_1e_2e_3e_4, \alpha_2e_1e_2e_5e_6, \alpha_3e_1e_2e_7e_8, \alpha_4e_1e_3e_5e_7\}\rangle$ for certain scalars $\alpha_j \in \{\pm1\}$. Hence there is $\alpha \in \{\pm1\}$ such that $\psi(\alpha e_1e_2e_3e_4e_5e_6e_7e_8) \equiv \varphi'$ belongs to R_2. According to the previous lemma, φ' is conjugated to φ and this finishes the proof. □

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