

SIMPLE $(-1, -1)$ BALANCED FREUDENTHAL KANTOR TRIPLE SYSTEMS

Alberto Elduque*

Departamento de Matemáticas, Facultad de Ciencias,
Universidad de Zaragoza, 50009 Zaragoza, Spain
e-mail address: elduque@posta.unizar.es

Noriaki Kamiya**

Center for Mathematical Sciences, The University of Aizu
965-8580 Aizu-Wakamatsu, Japan
e-mail address: kamiya@u-aizu.ac.jp

Susumu Okubo***

Department of Physics and Astronomy, University of Rochester,
Rochester, NY 14627, USA
e-mail address: okubo@pas.rochester.edu

Abstract: The simple finite dimensional $(-1, -1)$ balanced Freudenthal Kantor triple systems over fields of characteristic zero are classified.

1. Introduction.

In 1954, H. Freudenthal [FR] constructed the exceptional simple Lie algebras of types E_7 and E_8 by means of the exceptional simple Jordan algebras. The construction of E_8 has been extended in several ways to give 5-graded Lie algebras

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

starting with some nonassociative algebras or triple systems, which appear as the component \mathfrak{g}_1 .

The concept of (ε, δ) -Freudenthal Kantor triple system covers many of these systems:

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Definition 1.1 [YO] Let $\varepsilon, \delta = \pm 1$. A vector space V over a field F , endowed with a trilinear operation $V \times V \times V \rightarrow V$, $(x, y, z) \mapsto xyz$, is said to be a (ε, δ) -Freudenthal Kantor triple system $((\varepsilon, \delta)$ -FKTS for short) if the following two conditions are satisfied

$$(i) \quad [l_{a,b}, l_{c,d}] = l_{a,bc,d} + \varepsilon l_{c,lb,a,d},$$

$$(ii) \quad l_{d,c}k_{a,b} - \varepsilon k_{a,b}l_{c,d} = k_{k_{a,b}c,d}$$

for any $a, b, c, d \in V$, where $l_{a,b}, k_{a,b} : V \rightarrow V$ are given by $l_{a,b}c = abc$, $k_{a,b}c = acb - \delta bca$.

Thus a $(-1, 1)$ -FKTS is exactly a generalized Jordan triple system of second order in the sense of Kantor [K] (if $k = 0$ this is just a Jordan triple system), while a $(1, -1)$ -FKTS with $k = 0$ is an anti-Jordan triple system (see [FF1] for the definition of anti-Jordan pair (U^+, U^-) ; when $U^+ = U^-$ one gets an anti-Jordan triple system).

An (ε, δ) -FKTS V is said to be *balanced* $((\varepsilon, \delta)$ -BFKTS for short) if there exists a nonzero bilinear form $(|) : V \times V \rightarrow F$ such that $k_{a,b} = (a | b)Id$ for any $a, b \in V$. Since $k_{a,b} = -\delta k_{b,a}$ by its own definition, $(|)$ is either symmetric ($\delta = -1$) or skew-symmetric ($\delta = 1$). On the other hand, condition (ii) in Definition 1.1 gives here that $(|)$ is either symmetric or skew-symmetric according to ε being -1 or 1 , so that $\varepsilon = \delta$ in case V is balanced.

Any $(1, 1)$ -BFKTS becomes, by means of minor modifications of its triple product, a symplectic ternary algebra [FF2], a symplectic triple system [YA] or a Freudenthal triple system [M], and conversely. The simple finite dimensional Freudenthal triple systems were classified by [M], with some restrictions which are satisfied if the ground field is algebraically closed, and this amounts to a classification of the simple $(1, 1)$ -BFKTS (and of the symplectic ternary algebras [FF2]). The related 5-graded Lie algebras satisfy that $\mathfrak{g}_{\pm 2}$ is one dimensional.

Further properties of (ε, δ) -FKTS's can be found in [KA1-6, KAO, OKA1] and the references therein.

Our aim in this paper is to obtain the classification of the finite dimensional simple $(-1, -1)$ -BFKTS's over fields of characteristic 0. To achieve this, the classification [KAC] of the finite dimensional simple Lie superalgebras over algebraically closed fields of characteristic 0 will be used, but we will have to look at the known relationship between $(-1, -1)$ -FKTS's and 5-graded Lie superalgebras [Y] in a different way, suitable to our needs. This will be done in Section 2. The relevant examples of $(-1, -1)$ -BFKTS's will be given in Section 3 and, finally, Section 4 will provide the promised classification (Theorem 4.3), which asserts that the simple finite dimensional $(-1, -1)$ -BFKTS's fall into six classes, three of them with arbitrarily large dimension: orthogonal, unitarian and symplectic types; and another three classes of four dimensional (D_μ -type), seven dimensional (G-type) and eight dimensional systems (F-type).

Using Definition 1.1, the defining relations for a $(-1, -1)$ -BFKTS are

$$ab(xyz) = (abx)yz - x(bay)z + xy(abz), \quad (1.1)$$

$$abx + bax = (a | b)x = axb + bxa, \quad (1.2)$$

for any $a, b, x, y, z \in V$, where $(|)$ is a nonzero symmetric bilinear form. Over fields of characteristic $\neq 2$, put $\langle | \rangle = \frac{1}{2}(|)$ and then (1.2) is equivalent to

$$xxy = \langle x | x \rangle y = xyx \quad (1.3)$$

for any $x, y \in V$.

The main motivation for the classification of the simple $(-1, -1)$ -BFKTS's was provided by the recent paper [OKA2] by two of the authors, where the exceptional simple classical Lie superalgebras were constructed by using the last three classes mentioned above (D, G and F types). These triple systems are closely related to quaternion and octonion algebras. A different construction of the exceptional simple classical Lie superalgebras has been given in [BE] by means of a generalized Tits' construction (which also uses quaternion and octonion algebras).

2. $(-1, -1)$ balanced Freudenthal Kantor triple systems and Lie superalgebras.

The relationship between $(-1, -1)$ -BFKTS and Lie superalgebras has been studied in [OKA2]. A more useful approach for us is obtained as indicated by the next Theorem.

Theorem 2.1 *Let \mathfrak{g} be a finite dimensional Lie superalgebra over a field F of characteristic $\neq 2$ such that $\mathfrak{g}_{\bar{0}} = \mathfrak{sl}_2(F) \oplus \mathfrak{d}$ (direct sum of ideals) and $\mathfrak{g}_{\bar{1}} = U \otimes_F V$, where U is the two dimensional module for $\mathfrak{sl}_2(F)$ and V is a module for \mathfrak{d} . Let φ be a nonzero skew symmetric form on U , so that we may identify $\mathfrak{sl}_2(F) = \mathfrak{sp}(U, \varphi)$ and for any $a, b \in U$ consider the map $\varphi_{a,b} \in \mathfrak{sl}_2(F)$ given by*

$$\varphi_{a,b}(c) = \varphi(c, a)b + \varphi(c, b)a$$

for any $c \in U$. Then the product of odd elements in \mathfrak{g} is given by

$$[a \otimes u, b \otimes v] = \langle u | v \rangle \varphi_{a,b} + \varphi(a, b) d_{u,v} \quad (2.1)$$

for any $a, b \in U$ and $u, v \in V$, where $\langle | \rangle$ is a symmetric bilinear form and $d : V \times V \rightarrow \mathfrak{d}$ is a skew symmetric bilinear map that satisfy

$$\langle d(x) | y \rangle + \langle x | d(y) \rangle = 0, \quad (2.2a)$$

$$[d, d_{x,y}] = d_{d(x),y} + d_{x,d(y)}, \quad (2.2b)$$

$$d_{x,y}(y) = \langle y | x \rangle y - \langle y | y \rangle x, \quad (2.2c)$$

for any $x, y \in V$ and $d \in \mathfrak{d}$.

Conversely, let V be a vector space endowed with a symmetric bilinear form $\langle | \rangle : V \times V \rightarrow F$ and a skew symmetric bilinear map $d : V \times V \rightarrow \text{End}_F(V)$ ($(u, v) \mapsto d_{u,v}$). Assume that:

$$\langle d_{u,v}(x) | y \rangle + \langle x | d_{u,v}(y) \rangle = 0, \quad (2.3a)$$

$$[d_{u,v}, d_{x,y}] = d_{d_{u,v}(x),y} + d_{x,d_{u,v}(y)}, \quad (2.3b)$$

$$d_{x,y}(y) = \langle y | x \rangle y - \langle y | y \rangle x, \quad (2.3c)$$

for any $u, v, x, y \in V$. Let \mathfrak{d} be $\text{span}\{d_{u,v} : u, v \in V\}$ (a Lie subalgebra of $\text{End}_F(V)$ by (2.3b)) and let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the superalgebra where \mathfrak{g}_0 is the Lie algebra $\mathfrak{sl}_2(F) \oplus \mathfrak{d} = \mathfrak{sp}(U, \varphi) \oplus \mathfrak{d}$, \mathfrak{g}_1 is the \mathfrak{g}_0 -module $U \otimes_F V$ and where the product of odd elements is given by (2.1). Then \mathfrak{g} is a Lie superalgebra.

Proof: Since $\text{Hom}_{\mathfrak{sp}(U, \varphi)}(U \otimes_F U, F)$ is spanned by the form φ and $\text{Hom}_{\mathfrak{sp}(U, \varphi)}(U \otimes_F U, \mathfrak{sp}(U, \varphi))$ is spanned by the symmetric map $a \otimes b \mapsto \varphi_{a,b}$, formula (2.1) follows. Formulae (2.2a) and (2.2b) follow from the Jacobi superidentity applied to the elements $d \in \mathfrak{d}$ and $a \otimes x, b \otimes y \in U \otimes_F V$ and (2.2c) follows from the Jacobi superidentity applied to three odd elements.

The converse is a straightforward computation. \square

With $V, d : V \times V \rightarrow \text{End}_F(V)$ and $\langle | \rangle$ as before, consider the triple product in V given by

$$xyz = d_{x,y}z + \langle x | y \rangle z \quad (2.4)$$

for any $x, y, z \in V$. Conditions (2.3a–c) translate into:

$$xxy = \langle x | x \rangle y = xyx \quad (2.5a)$$

$$uv(xyz) = (uvx)yz - x(vuy)z + xy(uvz) \quad (2.5b)$$

$$\langle uvx | y \rangle = \langle x | vuy \rangle \quad (2.5c)$$

for any $u, v, x, y, z \in V$. Let us check (2.5b) for instance. For this, denote by $l_{x,y}$ the map $z \mapsto xyz$ for any $x, y, z \in V$, then for any $u, v, x, y \in V$

$$\begin{aligned} [l_{u,v}, l_{x,y}] &= [d_{u,v}, d_{x,y}] \quad (\text{since } l_{u,v} - d_{u,v} \text{ is scalar}) \\ &= d_{d_{u,v}(x),y} + d_{x,d_{u,v}(y)} \\ &= l_{d_{u,v}(x),y} - \langle d_{u,v}(x) | y \rangle + l_{x,d_{u,v}(y)} - \langle x | d_{u,v}(y) \rangle \\ &= l_{d_{u,v}(x),y} - l_{x,d_{v,u}(y)} \\ &= l_{uvx,y} - \langle u | v \rangle l_{x,y} - l_{x,vuy} + \langle v | u \rangle l_{x,y} \\ &= l_{uvx,y} - l_{x,vuy} \end{aligned}$$

and this is equivalent to (2.5b). Conversely, conditions (2.5a–c) give conditions (2.3a–c), if (2.4) is used to define $d_{x,y}$ for $x, y \in V$.

Conditions (2.5a) and (2.5b) are just the defining conditions (1.3) and (1.1) of a $(-1, -1)$ -BFKTS, while condition (2.5c) is a consequence of (2.5a-b) [KA2]. We include a proof of this fact by completeness:

Take $x = y$ in (2.5b) and use (2.5a) to get

$$\begin{aligned} \langle x | x \rangle uvz &= (uvx)xz - x(vux)z + \langle x | x \rangle uvz \\ &= (uvx)xz + \left((vux)xz - 2\langle x | vux \rangle z \right) + \langle x | x \rangle uvz \\ &= 2\langle u | v \rangle xxz - 2\langle x | vux \rangle z + \langle x | x \rangle uvz \end{aligned}$$

and this shows that $\langle x | vux \rangle = \langle u | v \rangle \langle x | x \rangle$ for any $x, u, v \in V$. Linearizing this one obtains that $\langle x | vuy \rangle + \langle y | vux \rangle = 2\langle u | v \rangle \langle x | y \rangle$ for any $x, y, u, v \in V$, whence

$$\langle x | vuy \rangle = \langle 2\langle u | v \rangle x - vux | y \rangle = \langle uvx | y \rangle,$$

as desired. In the same way, (2.3a) follows from (2.3b) and (2.3c).

Because of (2.3a-b), $\mathfrak{d} = d_{V,V}$ is a Lie algebra of derivations of the $(-1, -1)$ -BFKTS, which will be said to be the Lie algebra of inner derivations of V .

Given a vector space V endowed with a nonzero symmetric bilinear form $\langle | \rangle$ and a skew symmetric map $d : V \times V \rightarrow \text{End}_F(V)$, $(x, y) \mapsto d_{x,y}$ for any $x, y \in V$, satisfying conditions (2.3), denote by $\mathfrak{g}(V)$ the Lie superalgebra constructed in Theorem 2.1. Also, consider the triple product xyz defined on V by (2.4) and the triple product obtained directly from d : $\{xyz\} = d_{x,y}(z)$ for any $x, y, z \in V$.

Theorem 2.2 *Under the hypotheses above, the following conditions are equivalent:*

- (i) $\langle | \rangle$ is nondegenerate,
- (ii) $(V, \{xyz\})$ is a simple triple system,
- (iii) (V, xyz) is a simple triple system,
- (iv) $\mathfrak{g}(V)$ is a simple Lie superalgebra.

Proof: Assume that (i) is satisfied and let I be a nonzero ideal of the triple system $(V, \{xyz\})$. Then for any $x \in I$ and $y \in V$, $\{xyy\} = d_{x,y}(y) = -\langle y | y \rangle x + \langle x | y \rangle y \in I$, by (2.3c), and hence $\langle x | y \rangle y \in I$ for any $y \in I$. Since $\langle | \rangle$ is nondegenerate, there is a basis of V formed by elements y with $\langle x | y \rangle \neq 0$ and this shows that $I = V$. Conversely, $V^\perp = \{x \in V : \langle x | V \rangle = 0\}$ is an ideal of $(V, \{xyz\})$ because of (2.3a) and the linearization of (2.3c). Hence (ii) implies (i).

Similarly, condition (i) and the linearization of (2.5a) imply (iii), and conversely (iii) implies (i) since V^\perp is an ideal of (V, xyz) because of (2.5a) and (2.5c).

Now assume that (i) is satisfied and that $0 \neq I = I_{\bar{0}} \oplus I_{\bar{1}}$ is an ideal of the Lie superalgebra $\mathfrak{g}(V)$. By $\mathfrak{sl}_2(F)$ -invariance, $I_{\bar{1}} = U \otimes_F W$ for a subspace W of V . Let $x \in V$ and $y \in W$ with $\langle x | y \rangle \neq 0$, then for any $a \in U$, $[a \otimes x, a \otimes y] = -\langle x | y \rangle \varphi_{a,a}$, so $\varphi_{a,a} \in I_{\bar{0}}$ for any a and $\mathfrak{sl}_2(F) \subseteq I_{\bar{0}}$. But then $\mathfrak{g}_{\bar{1}} = [\mathfrak{sl}_2(F), \mathfrak{g}_{\bar{1}}] \subseteq I$ and

$\mathfrak{g}_0 = [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \subseteq I$, so $I = \mathfrak{g}$. Otherwise $W = 0$, so $I_{\bar{1}} = 0$, but then it is easy to show that $I = 0$.

Conversely, the graded subspace $d_{V, V^\perp} \oplus (U \otimes_F V^\perp)$ is an ideal of $\mathfrak{g}(V)$, so (iv) implies (i). \square

Since the nondegeneracy of a bilinear form is preserved under scalar extensions, it immediately follows that:

Corollary 2.3 *With the same notation as above, if $\langle | \rangle$ is nondegenerate, then $(V, \{xyz\})$, (V, xyz) and $\mathfrak{g}(V)$ are central simple.*

3. Examples.

This section is devoted to constructing the examples of simple $(-1, -1)$ balanced Freudenthal Kantor triple systems that will appear in the classification. Throughout this section, the ground field F will be assumed of characteristic $\neq 2$.

3.1 Hermitian type:

Let R be a unital separable associative algebra over F of degree ≤ 2 . Therefore, R is, up to isomorphism, either the ground field F , $F \times F$, a quadratic separable field extension K of F or a quaternion algebra Q over F . In any case, R is endowed with an involution of the first kind, $x \mapsto \bar{x}$, such that $x + \bar{x}, x\bar{x} = \bar{x}x \in F$ for any $x \in R$. Let V be a left module over R endowed with a nondegenerate hermitian form $h : V \times V \rightarrow R$. That is, h is F -bilinear and satisfies for any $x, y \in V$ and $r \in R$:

$$\begin{aligned} h(rx, y) &= rh(x, y), \\ h(x, y) &= \overline{h(y, x)}, \\ h(x, V) &= 0 \text{ if and only if } x = 0. \end{aligned} \tag{3.1}$$

Then the symmetric bilinear form $V \times V \rightarrow F$ defined by means of:

$$\langle x | y \rangle = \frac{1}{2} \left(h(x, y) + h(y, x) \right) \tag{3.2}$$

for any $x, y \in V$, is nondegenerate and determines h .

Define now the triple product on V by means of

$$xyz = h(z, x)y - h(z, y)x + h(x, y)z \tag{3.3}$$

for any $x, y, z \in V$.

It is clear that $xyx = h(x, x)y = \langle x | x \rangle y = xyx$ for any $x, y \in V$ and a straightforward computation shows that this triple product satisfies (2.5b) too. Therefore

V is a $(-1, -1)$ -BFKTS which will be said to be of *hermitian type*. Depending on $\dim_F R$ being either 1, 2 or 4, V will be said to be of *orthogonal*, *unitarian* or *symplectic type*, respectively, for reasons that will become clear later on.

Let us compute the Lie algebra $\mathfrak{d} = d_{V,V}$ in this case. Assume first that $R = F$, the ground field, then $d_{x,y} = \langle - | x \rangle y - \langle - | y \rangle x =: \sigma_{x,y}$ for any $x, y \in V$, and these maps span the orthogonal Lie algebra $\mathfrak{o}(V)$. From the construction in [KAC, Supplement to 2.1.2], $\mathfrak{g}(V)$ is the orthosymplectic Lie superalgebra $\mathfrak{osp}(V \oplus U)$. A word of caution is needed here: the multiplication of odd elements in [KAC, Supplement to 2.1.2] should read

$$[a \otimes c, b \otimes d] = -(a, b)_0 c \circ d + (c, d)_1 a \wedge b.$$

(A minus sign has been added.)

Now, in case R is a quadratic étale algebra, that is, either $K = F \times F$ or K is a quadratic field extension of F , then for any $x, y \in V$,

$$d_{x,y} = h_{x,y} + h_0(x, y)id, \quad (3.4)$$

where

$$h_{x,y} = h(-, x)y - h(-, y)x \quad (3.5)$$

and

$$h_0(x, y) = h(x, y) - \langle x | y \rangle = \frac{1}{2}(h(x, y) - h(y, x)). \quad (3.6)$$

Note that

$$h_{x,y} \in \mathfrak{u}(V, h) = \{f \in \text{End}_K(V) : h(f(x), y) + h(x, f(y)) = 0 \text{ for any } x, y \in V\}.$$

Since $\overline{h_0(x, y)} = -h_0(x, y)$, it follows that $\mathfrak{d} \subseteq \mathfrak{u}(V, h)$.

In the split case: $K = F \times F = Fe_1 \oplus Fe_2$, for orthogonal idempotents e_1 and e_2 ($\bar{e}_1 = e_2$), let $W = e_1V$ and $\tilde{W} = e_2V$. Then $h(W, W) = 0 = h(\tilde{W}, \tilde{W})$ and for any $a \in W$ and $u \in \tilde{W}$, $h(a, u) \in Fe_1$. Hence there is a bilinear nondegenerate form $(|) : W \times \tilde{W} \rightarrow F$, such that $h(a, u) = (a | u)e_1$ for any $a \in W$ and $u \in \tilde{W}$. This bilinear form determines h and allows us to identify \tilde{W} with the dual W^* . Therefore we may assume that $V = W \times W^*$, with the natural structure of module over $K = F \times F$, and with

$$h((a, \alpha), (b, \beta)) = (\beta(a), \alpha(b))$$

for any $a, b \in W$ and $\alpha, \beta \in W^*$. Moreover, in this case $\mathfrak{u}(V, h)$ is isomorphic to $\mathfrak{gl}(W)$ by means of the isomorphism that takes any $f \in \text{End}_F(W) = \mathfrak{gl}(W)$ to the endomorphism of $V = W \times W^*$ given by $(a, \alpha) \mapsto (f(a), -\alpha \circ f)$. Through this isomorphism, $h_{(a,0),(0,\alpha)}$ corresponds to the endomorphism of W given by $c \mapsto -\alpha(c)a$, and hence $d_{(a,0),(0,\alpha)}$ corresponds to $c \mapsto -\alpha(c)a + \frac{1}{2}\alpha(a)c$. If the dimension

of W is not 2, this shows that $\mathfrak{d} = \mathfrak{u}(V, h) \cong \mathfrak{gl}(W)$, while if the dimension is 2, $\mathfrak{d} = \mathfrak{su}(V, h) \cong \mathfrak{sl}(W)$.

By scalar extension, we have that $\mathfrak{d} = \mathfrak{u}(V, h)$ if $\dim_K V \neq 2$ ($\dim_F V \neq 4$) and $\mathfrak{d} = \mathfrak{su}(V, h)$ if $\dim_K V = 2$.

Finally, assume that R is a quaternion algebra Q . Again $d_{x,y} = h_{x,y} + h_0(x,y)id$, but now $h_{x,y}$ is Q -linear, while $h_0(x,y)id$ is not in general, since the center of Q is F . It is easily checked here that

$$[h_{x,y}, h_{u,v}] = h_{h_{x,y}(u),v} + h_{u,h_{x,y}(v)}$$

for any $x, y, u, v \in V$, and thus $h_{V,V} = \text{span}\{h_{x,y} : x, y \in V\}$ is a Lie algebra contained in

$$\mathfrak{sp}(V, h) = \{f \in \text{End}_Q(V) : h(f(x), y) + h(x, f(y)) = 0 \text{ for any } x, y \in V\},$$

and $\mathfrak{d} = d_{V,V}$ is contained in $\mathfrak{sp}(V, h) \oplus Q_0 id$, where $Q_0 = [Q, Q]$ is the set of skew symmetric elements in Q relative to its involution, which form a three dimensional simple Lie algebra.

Again, consider the split case: $Q = \text{End}_F(U)$ for a two dimensional vector space U endowed with a nonzero skew symmetric bilinear map φ which induces the involution in Q . Standard arguments of complete reducibility as a module over Q show that $V = U \otimes_F W$ for some vector space W over F . For any $q \in Q_0 = \mathfrak{sl}(U) = \mathfrak{sp}(U, \varphi)$ and for any $x, y \in V$,

$$\begin{aligned} \langle qx | y \rangle &= \frac{1}{2}(h(qx, y) + h(y, qx)) \\ &= \frac{1}{2}(qh(x, y) + \overline{qh(x, y)}) \\ &= \frac{1}{2}(h(x, y)q + \overline{h(x, y)q}) \\ &= -\langle x | qy \rangle, \end{aligned}$$

so Q_0 embeds into the orthogonal Lie algebra $\mathfrak{o}(V, \langle | \rangle)$ and, therefore, $\langle | \rangle$ is invariant under the action of $\mathfrak{sl}(U) = \mathfrak{sp}(U, \varphi)$. But, up to scalars, φ is the unique bilinear form on U which is $\mathfrak{sp}(U, \varphi)$ -invariant, so

$$\langle a \otimes u | b \otimes v \rangle = \frac{1}{2}\varphi(a, b)\psi(u, v)$$

for any $a, b \in U$, $u, v \in W$, for a skew-symmetric nondegenerate bilinear form $\psi : W \times W \rightarrow F$.

Since the hermitian form h is completely determined by $\langle | \rangle$, it turns out that $h : V \times V \rightarrow Q = \text{End}_F(U)$ is given by

$$h(a \otimes u, b \otimes v) = \psi(u, v)\varphi(-, b)a$$

for any $a, b \in U$ and $u, v \in W$. Note that h thus defined is hermitian and

$$\frac{1}{2}(h(a \otimes u, b \otimes v) + h(b \otimes v, a \otimes u)) = \frac{1}{2}\psi(u, v)(\varphi(-, b)a - \varphi(-, a)b).$$

But $\varphi(a, b)c + \varphi(b, c)a + \varphi(c, a)b = 0$ for any $a, b, c \in U$, so

$$\frac{1}{2}(h(a \otimes u, b \otimes v) + h(b \otimes v, a \otimes u)) = \frac{1}{2}\psi(u, v)\varphi(a, b)id.$$

Hence, for any $a, b \in U$ and $u, v \in W$:

$$\begin{aligned} h_0(a \otimes u, b \otimes v) &= \frac{1}{2}(h(a \otimes u, b \otimes v) - h(b \otimes v, a \otimes u)) \quad (\text{see (3.6)}) \\ &= \frac{1}{2}\psi(u, v)(\varphi(-, b)a + \varphi(-, a)b) \\ &= \frac{1}{2}\psi(u, v)\varphi_{a,b}, \end{aligned}$$

and thus, for any $a, b, c \in U$ and $u, v, w \in W$:

$$\begin{aligned} h_{a \otimes u, b \otimes v}(c \otimes w) &= \psi(w, u)\varphi(b, a)c \otimes v - \psi(w, v)\varphi(a, b)c \otimes u \\ &= -\varphi(a, b)c \otimes (\psi(w, u)v + \psi(w, v)u) \\ &= -\varphi(a, b)c \otimes \psi_{u,v}(w). \end{aligned}$$

Therefore, $h_{V,V} = \mathfrak{sp}(V, h) := \{f \in \text{End}_Q(V) : h(f(x), y) + h(x, f(y)) = 0 \text{ for any } x, y \in V\} \cong \mathfrak{sp}(W, \psi)$ (which acts on $V = U \otimes_F W$ in a natural way: on the second factor). Moreover, from (3.4),

$$\begin{aligned} d_{a \otimes u, b \otimes v} &= h_{a \otimes u, b \otimes v} + h_0(a \otimes u, b \otimes v)id \\ &= \frac{1}{2}\varphi_{a,b} \otimes \psi(u, v)id - \varphi(a, b)id \otimes \psi_{u,v}, \end{aligned}$$

so $\mathfrak{d} = d_{V,V} = \mathfrak{sp}(U, \varphi) \oplus \mathfrak{sp}(W, \psi) = \mathfrak{sl}(U) \oplus \mathfrak{sp}(W, \psi)$.

For general Q , again extending scalars we arrive at $h_{V,V} = \mathfrak{sp}(V, h)$ (which is a simple Lie algebra of type C) and \mathfrak{d} is the direct sum of the three dimensional simple Lie algebra Q_0 and of the simple Lie algebra $\mathfrak{sp}(V, h)$.

Summarizing the above discussion:

Proposition 3.1. *Let R be a unital separable associative algebra of degree ≤ 2 over a field F of characteristic $\neq 2$, and let V be a left module over R endowed with a nondegenerate hermitian form $h : V \times V \rightarrow R$. Endow V with the structure of a simple $(-1, -1)$ -BFKTS of hermitian type (with associated symmetric bilinear form given by $\langle x | y \rangle = \frac{1}{2}(h(x, y) + h(y, x))$ for any $x, y \in V$) and let $\mathfrak{d} = d_{V,V}$ be the associated Lie algebra of inner derivations. Then:*

- (i) If $R = F$, then $\mathfrak{d} = \mathfrak{o}(V, \langle | \rangle)$.
- (ii) If $R = K$ is a quadratic étale algebra, then $\mathfrak{d} = \mathfrak{u}(V, h)$ unless $\dim_F(V) = 4$. In this latter case, $\mathfrak{d} = \mathfrak{su}(V, h)$.
- (iii) If R is a quaternion algebra Q , then $\mathfrak{d} \cong Q_0 \oplus \mathfrak{sp}(V, h)$, where $\mathfrak{sp}(V, h)$ acts naturally on V , and the simple three dimensional Lie algebra Q_0 acts by left multiplication on the Q module V .

3.2 D_μ -type

Let V be a four dimensional vector space, endowed with a nondegenerate symmetric bilinear form $\langle | \rangle$. Let Φ be a nonzero skew symmetric multilinear form: $\Phi : V \times V \times V \times V \rightarrow F$. Define a skew symmetric triple product $[xyz]$ on V by means of:

$$\Phi(x, y, z, t) = \langle [xyz] | t \rangle, \quad (3.7)$$

for any $x, y, z, t \in V$.

Lemma 3.2. *With the hypotheses above, there exists a nonzero scalar $\mu \in F$ such that*

$$\langle [a_1 a_2 a_3] | [b_1 b_2 b_3] \rangle = \mu \det(\langle a_i | b_j \rangle), \quad (3.8)$$

for any $a_i, b_i \in V$ ($i = 1, 2, 3$).

Proof: By extending scalars to an algebraic closure it may be assumed that F is algebraically closed. Then take an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of V . By skew symmetry of Φ , $[e_1 e_2 e_3] = \nu e_4$ for some $0 \neq \nu \in F$. Let $\mu = \nu^2$. Scale Φ by ν^{-1} so that we may assume $[e_1 e_2 e_3] = e_4$. Then it is enough to prove that (3.8) is satisfied with $\mu = 1$.

The skew symmetry of Φ forces $[e_{\sigma(1)} e_{\sigma(2)} e_{\sigma(3)}] = (-1)^\sigma e_{\sigma(4)}$ for any permutation σ , where $(-1)^\sigma$ denotes the corresponding signature. But then, for any permutations σ and τ ,

$$\begin{aligned} \langle [e_{\sigma(1)} e_{\sigma(2)} e_{\sigma(3)}] | [e_{\tau(1)} e_{\tau(2)} e_{\tau(3)}] \rangle &= (-1)^\sigma (-1)^\tau \langle e_{\sigma(4)} | e_{\tau(4)} \rangle \\ &= \det(\langle e_{\sigma(i)} | e_{\tau(j)} \rangle)_{i,j=1,2,3}, \end{aligned}$$

as required. \square

Now, for any such V and Φ , and for any $x, y \in V$, consider the endomorphism $d_{x,y} \in \text{End}_F(V)$ by means of

$$d_{x,y} z = [xyz] + \langle z | x \rangle y - \langle z | y \rangle x. \quad (3.9)$$

As shown in [O1, §5], conditions (2.3a-b) are satisfied, so if the triple product xyz on V is defined by means of

$$xyz = [xyz] + \langle z | x \rangle y - \langle z | y \rangle x + \langle x | y \rangle z. \quad (3.10)$$

for any $x, y, z \in V$, then V becomes a $(-1, -1)$ -BFKTS, which will be said to be of D_μ -type.

Assume for a while that the scalar μ in (3.8) is a square, $\mu = \nu^2$, $0 \neq \nu \in F$, and that $\langle | \rangle$ represents 1. Then, by [E2, Theorem 2], V is endowed with a binary multiplication that makes it a quaternion algebra Q over F , with involution $x \mapsto \bar{x}$ such that $x\bar{x} = \langle x | x \rangle$ for any $x \in V$, satisfying

$$\nu^{-1}[xyz] = x\bar{y}z - \langle x | y \rangle z + \langle z | x \rangle y - \langle z | y \rangle x$$

for any $x, y, z \in V$. Therefore, for any $x, y, z \in V$, (3.9) shows that:

$$\begin{aligned} d_{x,y}(z) &= \nu x\bar{y}z + (1 + \nu)(\langle z | x \rangle y - \langle z | y \rangle x) - \nu \langle x | y \rangle z \\ &= \nu x\bar{y}z + \frac{1 + \nu}{2}((x\bar{z} + z\bar{x})y - x(\bar{y}z + \bar{z}y)) - \frac{\nu}{2}(x\bar{y} + y\bar{x})z \\ &= \left(\nu x\bar{y} - \frac{1 + \nu}{2}x\bar{y} - \frac{\nu}{2}(x\bar{y} + y\bar{x}) \right) z + \frac{1 + \nu}{2}z\bar{x}y \\ &= \left(-\frac{1}{2}x\bar{y} - \frac{\nu}{2}y\bar{x} \right) z + \frac{1 + \nu}{2}z\bar{x}y \\ &= \frac{\nu - 1}{4}(x\bar{y} - y\bar{x})z + \frac{1 + \nu}{4}z(\bar{x}y - \bar{y}x), \end{aligned}$$

because $\bar{x}y + \bar{y}x = x\bar{y} + y\bar{x} = 2\langle x | y \rangle \in F$, so $\bar{x}y - \bar{y}x = 2\bar{x}y - (x\bar{y} + y\bar{x})$. Hence, for any $x, y \in V$,

$$d_{x,y} = L_p - R_q, \quad \text{with } p = \frac{\nu - 1}{4}(x\bar{y} - y\bar{x}), \quad q = -\frac{\nu + 1}{4}(\bar{x}y - \bar{y}x) \in Q_0,$$

where L and R denote left and right multiplications in $V = Q$. Therefore, if $\mu = 1$ ($\nu = \pm 1$), $\mathfrak{d} = d_{V,V}$ is isomorphic to the three dimensional simple Lie algebra Q_0 . However, if $\mu \neq 0, 1$ ($\nu \neq 0, \pm 1$), then $\mathfrak{d} = L_{Q_0} \oplus R_{Q_0}$, a direct sum of two copies of the three dimensional Lie algebra Q_0 , which coincides with the orthogonal Lie algebra $\mathfrak{o}(V, \langle | \rangle)$. Moreover, in this latter case, [BE, Lemma 3.1 and its proof], $\mathfrak{g}(V)$ is a form of the simple Lie superalgebra $\Gamma(-\frac{1}{2}, \frac{1-\nu}{4}, \frac{1+\nu}{4})$ (notation as in [SC, pp. 16-17]). That is, it is a form of $D(2, 1; \frac{\nu-1}{2})$ (see [OKA2]).

Simply by extending scalars, we obtain:

Proposition 3.3. *Let V be a four dimensional vector space over a field F of characteristic $\neq 2$ with a nondegenerate symmetric bilinear form $\langle | \rangle$. Let Φ be a nonzero skew symmetric 4-linear form and let the triple product $[xyz]$ be defined by means of $\langle [xyz] | t \rangle = \Phi(x, y, z, t)$ for any $x, y, z, t \in V$. Let $0 \neq \mu \in F$ be given by (3.8). Endow V with the structure of a simple $(-1, -1)$ -BFKTS by means of (3.10) and let $\mathfrak{d} = d_{V,V}$ be the corresponding Lie algebra of inner derivations. Then:*

- (i) *If $\mu = 1$, then \mathfrak{d} is a three dimensional simple ideal of the orthogonal Lie algebra $\mathfrak{o}(V, \langle | \rangle)$.*

(ii) If $\mu \neq 0, 1$, then \mathfrak{d} coincides with the orthogonal Lie algebra $\mathfrak{o}(V, \langle | \rangle)$.

There is some overlapping in the types considered up to now.

To begin with, let V be any four dimensional simple $(-1, -1)$ -BFKTS and let $[xyz]$ be defined by

$$[xyz] = xyz - \langle z | x \rangle y + \langle z | y \rangle x - \langle x | y \rangle z$$

for any $x, y, z \in V$. Because of (2.5a), $[xyz]$ is skew symmetric on its arguments. In case $[xyz]$ is identically zero, we are in presence of a system of orthogonal type. Otherwise, this is a system of D-type. This means that the systems of hermitian type with $R = K$ or Q and with $\dim_F V = 4$ are systems of D-type. Let us check which μ 's are involved in these cases. To do so, it is enough to consider the split cases.

Assume $K = F \times F$ and $V = W \times W^*$ with $h((a, \alpha), (b, \beta)) = (\beta(a), \alpha(b))$ for any $a, b \in W$ and $\alpha, \beta \in W$ and with $\dim_F W = 2$. Take $a, b \in W$ and $\alpha, \beta \in W^*$ with $\alpha(a) = 1 = \beta(b)$, $\alpha(b) = 0 = \beta(a)$. Then with $(a_1, \alpha_1) = (a, 0)$, $(a_2, \alpha_2) = (0, \alpha)$ and $(a_3, \alpha_3) = (b, \beta)$,

$$\det(\langle (a_i, \alpha_i) | (a_j, \alpha_j) \rangle) = \begin{vmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\frac{1}{4},$$

while $[(a, 0)(0, \alpha)(b, \beta)] = \frac{1}{2}(b, \beta)$ and

$$\langle [(a, 0)(0, \alpha)(b, \beta)] | [(a, 0)(0, \alpha)(b, \beta)] \rangle = \frac{1}{4} \langle (b, -\beta) | (b, -\beta) \rangle = -\frac{1}{4}.$$

Hence, $\mu = 1$ in this case. (This can also be deduced directly from the size of the Lie algebras \mathfrak{d} .)

Assume now that $R = Q$ is a quaternion algebra and $\dim_F V = 4$, then V is a free Q -module of rank 1 and hence we may assume that $V = Q$ and that $h(x, y) = \alpha x \bar{y}$ for any $x, y \in Q$, where $0 \neq \alpha = h(1, 1) \in F$. Then for any $x_1, x_2, x_3 \in Q$,

$$[x_1 x_2 x_3] = h_0(x_3, x_1)x_2 - h_0(x_3, x_2)x_1 + h_0(x_1, x_2)x_3$$

where $h_0(x, y) = \frac{1}{2}(h(x, y) - h(y, x)) = \alpha(x\bar{y} - y\bar{x}) \in Q_0$. By skew symmetry of h_0 ,

$$[x_1 x_2 x_3] = \frac{1}{2} \sum_{\sigma} h_0(x_{\sigma(1)}, x_{\sigma(2)})x_{\sigma(3)} = \frac{\alpha}{2} (-1)^{\sigma} x_{\sigma(1)} \bar{x}_{\sigma(2)} x_{\sigma(3)}$$

where the sum is over all the permutations of 1, 2, 3. Take $x_1 = 1$, x_2 , and x_3 mutually orthogonal to get $\langle (1 | 1) = h(1, 1) = \alpha$:

$$\det(\langle x_i | x_j \rangle) = \alpha \langle x_2 | x_2 \rangle \langle x_3 | x_3 \rangle,$$

while

$$[x_1x_2x_3] = -3\alpha x_2x_3$$

since $x_2x_3 = -x_3x_2$ and $\bar{x}_i = -x_i$ for $i = 2, 3$ while $\bar{1} = 1$. Thus,

$$\langle [x_1x_2x_3] \mid [x_1x_2x_3] \rangle = 9\alpha^3(x_2x_3)\overline{(x_2x_3)} = 9\alpha\langle x_2 \mid x_2 \rangle \langle x_3 \mid x_3 \rangle.$$

Thus, $\mu = 9$ in this case.

A final overlap occurs if V is of hermitian type with $R = K$ quadratic and with $\dim_F V = 2$. As above,

$$[x_1x_2x_3] = \frac{1}{2} \sum_{\sigma} h_0(x_{\sigma(1)}, x_{\sigma(2)})x_{\sigma(3)}$$

for any $x_1, x_2, x_3 \in V$. By skew symmetry and dimension, this is zero, and therefore we are in the situation of $R = F$. We summarize the above arguments in the following remark, whose last part follows from the structure of the Lie algebras of inner derivations.

Remark 3.4:

- The simple $(-1, -1)$ -BFKTS V of unitarian type and $\dim_F V = 2$ are also of orthogonal type.
- The simple $(-1, -1)$ -BFKTS V of unitarian type and $\dim_F V = 4$ are of D_1 -type.
- The simple $(-1, -1)$ -BFKTS V of symplectic type and $\dim_F V = 4$ are of D_9 -type.
- There are no more overlaps among different types.

3.3 G-type:

Let C be an eight-dimensional Cayley-Dickson (or octonion) algebra over F with norm n and trace t . Let C_0 be the set of trace zero elements. For any $x, y \in C$, the linear map

$$D_{x,y} = [L_x, L_y] + [L_x, R_y] + [R_x, R_y] \tag{3.11}$$

(where L_x and R_x denote the left and right multiplication by x) is known to be a derivation of C [S, Ch. III.8], and hence it leaves C_0 invariant. Consider then, for any $0 \neq \alpha \in F$, the nondegenerate symmetric bilinear form and the triple product on $V = C_0$ given by

$$\begin{aligned} \langle x \mid y \rangle &= -2\alpha t(xy) \\ xyz &= \alpha (D_{x,y}(z) - 2t(xy)z) \end{aligned}$$

for any $x, y, z \in V$. Since $D_{x,y}$ is a derivation and

$$\begin{aligned} D_{x,y}(y) &= xy^2 - y(xy) + xy^2 - (xy)y + y^2x - (yx)y, \\ &= 4y^2x - 2(xy + yx)y, \\ &= -4n(y)x - 2t(xy)y = -\langle y \mid y \rangle x + \langle x \mid y \rangle y, \end{aligned}$$

where we have used that $x^2 = -n(x)1 = -\frac{1}{2}t(x^2)1$ for any $x \in V = C_0$; it follows from (2.3) that V is a $(-1, -1)$ -BFKTS (see also [OKA2], which will be said to be of G-type. It is clear here that the Lie algebra \mathfrak{d} is the span of the $D_{x,y}$'s, which is precisely the Lie algebra of derivations of the Cayley-Dickson algebra C in case the characteristic is $\neq 3$ [S, Ch. III.8], a simple Lie algebra of type G_2 . (If the characteristic is 3, then this is a seven dimensional simple Lie algebra which is a form of $\mathfrak{psl}(7)$ [AEMN].)

3.4 F-type:

Let X be a 3-fold vector cross product on a vector space V of dimension 8, endowed with a nondegenerate symmetric bilinear form $\langle | \rangle$. That is, X is a three linear map $X : V \times V \times V \rightarrow V$, $(a, b, c) \mapsto X(a, b, c)$, satisfying (see [E2], [O2, Ch. 8] and the references therein):

$$\begin{aligned} \langle X(a_1, a_2, a_3) | a_i \rangle &= 0 \text{ for any } i = 1, 2, 3, \\ \langle X(a_1, a_2, a_3) | X(a_1, a_2, a_3) \rangle &= \det(\langle a_i | a_j \rangle), \end{aligned} \quad (3.12)$$

for any $a_1, a_2, a_3 \in V$.

It is known that (3.12) implies the skew symmetry of X . Moreover, X satisfies:

$$\begin{aligned} \langle X(a_1, a_2, a_3) | X(b_1, b_2, b_3) \rangle \\ = \det(\langle a_i | b_j \rangle) + \epsilon \sum_{\sigma \text{ even}} \sum_{\tau \text{ even}} \langle a_{\sigma(1)} | b_{\tau(1)} \rangle \Phi(a_{\sigma(2)}, a_{\sigma(3)}, b_{\tau(2)}, b_{\tau(3)}) \end{aligned} \quad (3.13)$$

for any $a_i, b_i \in V$ ($i = 1, 2, 3$), where $\Phi(a, b, c, d) = \langle a | X(b, c, d) \rangle$ for any $a, b, c, d \in V$, and $\epsilon = \pm 1$. In case $\epsilon = 1$ (resp. -1), X is said to be of type I (resp. II). Also, if $\dim_F V = 8$ and X is of type I, then $-X$ is of type II, and conversely.

Assume now that the characteristic of the ground field F is $\neq 2, 3$. Given a 3-fold vector cross product X of type I, define $d_{x,y} \in \text{End}_F(V)$, $x, y \in V$, by means of:

$$d_{x,y}z = \frac{1}{3}X(x, y, z) + \langle z | x \rangle y - \langle z | y \rangle x. \quad (3.14)$$

As shown in [O1, §5], condition (2.3b) is satisfied, so if the triple product xyz on V is defined by means of

$$xyz = \frac{1}{3}X(x, y, z) + \langle z | x \rangle y - \langle z | y \rangle x + \langle x | y \rangle z. \quad (3.15)$$

for any $x, y, z \in V$, then V becomes a $(-1, -1)$ -BFKTS, which will be said to be of F-type.

Since $d_{x,y}$ is a derivation of the triple system and it is skew symmetric relative to $\langle | \rangle$, it follows that $d_{x,y}$ is a derivation of the 3-fold vector cross product X . According to [E2, Theorem 12], if e is an element of V with $\langle e | e \rangle \neq 0$, $W = \{v \in V : \langle e | x \rangle = 0\}$, and q is the nondegenerate quadratic form on V defined by

$q(v) = -\langle e | e \rangle^{-1} \langle v | v \rangle$, then the Lie algebra of derivations of X is isomorphic to the orthogonal Lie algebra $\mathfrak{o}(W, q)$. Actually, V has the structure of an eight dimensional Cayley-Dickson algebra C with unit $1 = e$, so that there is an scalar $0 \neq \alpha \in F$ such that for any $a, b, c \in V = C$:

$$\begin{aligned} X(a, b, c) &= \alpha ((\bar{a}b)c + (a | c)b - (b | c)a - (a | b)c) \\ \langle a | b \rangle &= \alpha(a | b) \end{aligned}$$

where $x \mapsto \bar{x}$ denotes the involution and $(a | a) = a\bar{a}$ is the norm of C . Actually, $\alpha = \langle e | e \rangle$. Hence for any $x, y \in V$, $d_{x,y}$ is a derivation of the $3C$ -product given by $(\bar{a}b)c$ (see [E2]). But for any $x, y, z \in V = C$,

$$\frac{3}{\alpha} d_{x,y}(z) = (x\bar{y})z + 4(z | x)y - 4(z | x)x - (x | y)z,$$

in particular, for a traceless x ($\bar{x} = -x$):

$$\frac{3}{\alpha} d_{e,x}(y) = -xy + 2x(y + \bar{y}) - 2(x\bar{y} - yx) = xy + 2yx = (L + 2R)_x(y),$$

that is, $d_{e,x} = (L + 2R)_x$, where L and R denote the left and right multiplications in C . But these operators generate the Lie algebra of derivations of the triple product given by $(\bar{a}b)c$ [EM,E2] (see also [E3]), so we conclude that \mathfrak{d} is isomorphic to $\mathfrak{o}(W, q)$.

Note that in [OKA2] it is already proved that, after scalar extension, \mathfrak{d} is isomorphic to $\mathfrak{o}(7)$, by an explicit computation.

4. Classification.

Given a $(-1, -1)$ -BFKTS V over a field of characteristic $\neq 2$, in Section 2 a simple Lie superalgebra $\mathfrak{g} = \mathfrak{g}(V)$ has been defined that contains a copy $\mathfrak{s} = \mathfrak{s}(V)$ of $\mathfrak{sl}_2(F)$, which is an ideal of \mathfrak{g}_0 that is complemented by the ideal $\mathfrak{d} = \mathfrak{d}(V) = d_{V,V}$. In this situation $\mathfrak{d} = \{d \in \mathfrak{g}_0 : [d, \mathfrak{s}] = 0\}$ is completely determined by \mathfrak{g} and \mathfrak{s} . Moreover, as a module for \mathfrak{g}_0 , \mathfrak{g}_1 is the tensor product of the two dimensional irreducible module for \mathfrak{s} and the module V for \mathfrak{d} .

Consider a ground field F of characteristic $\neq 2$ and the pairs $(\mathfrak{g}, \mathfrak{s})$, where \mathfrak{g} is a Lie superalgebra over F and \mathfrak{s} is a complemented ideal of \mathfrak{g}_0 isomorphic to $\mathfrak{sl}_2(F)$. Two such pairs $(\mathfrak{g}^1, \mathfrak{s}^1)$, $(\mathfrak{g}^2, \mathfrak{s}^2)$ are said to be isomorphic if there is an isomorphism of Lie superalgebras $\phi : \mathfrak{g}^1 \rightarrow \mathfrak{g}^2$ such that $\phi(\mathfrak{s}^1) = \mathfrak{s}^2$.

Given a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and a nonzero scalar α , the new Lie superalgebra defined over \mathfrak{g} with the new product $[\ ,]_\alpha$ given, for homogeneous elements, by

$$\begin{cases} [x, y]_\alpha = \alpha[x, y] & \text{if both } x \text{ and } y \text{ are odd} \\ [x, y]_\alpha = [x, y] & \text{otherwise} \end{cases}$$

will be denoted by \mathfrak{g}_α . Also, given a $(-1, -1)$ -BFKTS V , we will denote by V_α the new $(-1, -1)$ -BFKTS defined on V but with the new product given by $(xyz)_\alpha = \alpha xyz$, and new symmetric bilinear form given by $\langle x | y \rangle_\alpha = \alpha \langle x | y \rangle$, for any $x, y, z \in V$. From the definitions, it is clear that $\mathfrak{g}(V_\alpha) = \mathfrak{g}(V)_\alpha$. Two $(-1, -1)$ -BFKTS V^1 and V^2 will be said to be *equivalent* in case there is a nonzero scalar α such that V^1 and V_α^2 are isomorphic.

Theorem 4.1 *Let V^1 and V^2 be two $(-1, -1)$ -BFKTS's. Then V^1 is equivalent to V^2 if and only if $(\mathfrak{g}(V^1), \mathfrak{s}(V^1))$ is isomorphic to $(\mathfrak{g}(V^2), \mathfrak{s}(V^2))$.*

Proof: Let $\mathfrak{g}^i = \mathfrak{g}(V^i)$ and $\mathfrak{d}^i = \mathfrak{d}(V^i) = d_{V^i, V^i}$ for $i = 1, 2$. Also, $\mathfrak{s}(V^1) = \mathfrak{s}(V^2) = \mathfrak{sp}(U, \varphi)$ as in Section 2. Thus

$$\begin{aligned}\mathfrak{g}_{\bar{0}}^i &= \mathfrak{sp}(U, \varphi) \oplus \mathfrak{d}^i, \\ \mathfrak{g}_{\bar{1}}^i &= U \otimes_F V^i,\end{aligned}$$

for $i = 1, 2$. Let $\Phi : \mathfrak{g}^1 \rightarrow \mathfrak{g}^2$ be an isomorphism such that it restricts to an automorphism of $\mathfrak{sp}(U, \varphi)$. But any automorphism ξ of $\mathfrak{sp}(U, \varphi)$ can be extended as in [E4, proof of Lemma 2.1] to an isomorphism from \mathfrak{g}^2 onto \mathfrak{g}_α^2 for some nonzero scalar α and, therefore, we may (and will) assume that Φ is the identity on $\mathfrak{sp}(U, \varphi)$. Since \mathfrak{d}^i is the centralizer of $\mathfrak{sp}(U, \varphi)$ in $\mathfrak{g}_{\bar{0}}^i$, $i = 1, 2$, Φ restricts to an isomorphism $\Psi : \mathfrak{d}^1 \rightarrow \mathfrak{d}^2$. Also, Φ restricts then to an isomorphism of $\mathfrak{sp}(U, \varphi)$ -modules $\Phi_{\bar{1}} : U \otimes_F V^1 \rightarrow U \otimes_F V^2$. Since U is absolutely irreducible as a module for $\mathfrak{sp}(U, \varphi)$, there is an isomorphism of vector spaces $\psi : V^1 \rightarrow V^2$ such that $\Phi(a \otimes x) = a \otimes \psi(x)$ for any $a \in U$ and $x \in V^1$.

Now, for any $x, y, z \in V^1$ and any $a \in U$:

$$\begin{aligned}a \otimes \psi(d_{x,y}(z)) &= \Phi([d_{x,y}, a \otimes z]) \\ &= [\Psi(d_{x,y}), a \otimes \psi(z)] \\ &= a \otimes \Psi(d_{x,y})(\psi(z))\end{aligned}$$

so

$$\psi(d_{x,y}(z)) = \Psi(d_{x,y})(\psi(z)), \quad (4.1)$$

for any $x, y, z \in V^1$. Also, for any $a, b \in U$ and $x, y \in V^1$:

$$\begin{aligned}\Phi([a \otimes x, b \otimes y]) &= [a \otimes \psi(x), b \otimes \psi(y)] \\ &= \langle \psi(x) | \psi(y) \rangle \varphi_{a,b} + \varphi(a, b) d_{\psi(x), \psi(y)},\end{aligned}$$

but also

$$\begin{aligned}\Phi([a \otimes x, b \otimes y]) &= \Phi(\langle x | y \rangle \varphi_{a,b} + \varphi(a, b) d_{x,y}) \\ &= \langle x | y \rangle \varphi_{a,b} + \varphi(a, b) \Psi(d_{x,y});\end{aligned}$$

so

$$\begin{cases} \Psi(d_{x,y}) = d_{\psi(x), \psi(y)}, \\ \langle \psi(x) | \psi(y) \rangle = \langle x | y \rangle, \end{cases} \quad (4.2)$$

for any $x, y \in V^1$, which, together with (4.1), shows that ψ is an isomorphism between the triple systems V^1 and V^2 .

For the converse, if V^1 and V^2 are equivalent, there is a $0 \neq \alpha \in F$ such that V^1 and V_α^2 are isomorphic. From here it is easy to deduce that the pairs $(\mathfrak{g}(V^1), \mathfrak{s}(V^1))$ and $(\mathfrak{g}(V_\alpha^2), \mathfrak{s}(V^2))$ are isomorphic. But $\mathfrak{g}(V_\alpha^2)$ is isomorphic to $\mathfrak{g}(V^2)$ by means of an isomorphism taking $\mathfrak{s}(V^2)$ into itself (see [E4, proof of Lemma 2.1]). \square

In order to classify the simple $(-1, -1)$ -BFKTS of finite dimension over a field of characteristic zero, we will first assume that the ground field F is algebraically closed. Following Theorems 2.1, 2.2 and 4.1, we will determine, up to isomorphism, the pairs $(\mathfrak{g}, \mathfrak{s})$, where \mathfrak{g} is a simple finite dimensional Lie superalgebra and \mathfrak{s} is an ideal of \mathfrak{g}_0 isomorphic to $\mathfrak{sl}(2)$:

Theorem 4.2 *Let F be an algebraically closed field of characteristic zero. The following list exhausts, up to isomorphism, the pairs $(\mathfrak{g}, \mathfrak{s})$, where \mathfrak{g} is a simple finite dimensional Lie superalgebra over F ($\mathfrak{g}_1 \neq 0$) and \mathfrak{s} is a three dimensional simple ideal of \mathfrak{g}_0 :*

- (i) $\mathfrak{g} = \mathfrak{sl}(m, 2)$, $m \geq 3$, and \mathfrak{s} is the (unique) ideal of \mathfrak{g}_0 isomorphic to $\mathfrak{sl}(2)$.
- (ii) $\mathfrak{g} = \mathfrak{psl}(2, 2)$ and \mathfrak{s} is any of the two simple ideals of \mathfrak{g}_0 .
- (iii) $\mathfrak{g} = \mathfrak{osp}(m, 2)$, $m \geq 1$, $m \neq 4$, so that $\mathfrak{g}_0 = \mathfrak{o}(m) \oplus \mathfrak{sp}(2)$, and \mathfrak{s} is the copy of $\mathfrak{sp}(2)$.
- (iv) $\mathfrak{g} = \mathfrak{osp}(4, 2r)$, $r \geq 2$, so that $\mathfrak{g} = \mathfrak{o}(4) \oplus \mathfrak{sp}(2r)$ and \mathfrak{s} is any of the two simple simple ideals of $\mathfrak{o}(4) \cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$.
- (v) $\mathfrak{g} = D(2, 1; \alpha)$, $\alpha \neq 0, -1$, so that $\mathfrak{g}_0 = \mathfrak{sp}(U, \varphi) \oplus \mathfrak{sp}(U, \varphi) \oplus \mathfrak{sp}(U, \varphi)$, U being a two dimensional vector space and φ a nonzero skew symmetric bilinear form on U , $\mathfrak{g}_1 = U \otimes_F U \otimes_F U$, with the natural multiplication in \mathfrak{g}_0 and natural action of \mathfrak{g}_0 on \mathfrak{g}_1 in which the i^{th} copy of $\mathfrak{sp}(U, \varphi)$ acts on the i^{th} factor of U , and with the multiplication of odd elements given by:

$$\begin{aligned} [u_1 \otimes u_2 \otimes u_3, v_1 \otimes v_2 \otimes v_3] &= \varphi(u_2, v_2)\varphi(u_3, v_3)\varphi_{u_1, v_1} \\ &\quad + \alpha\varphi(u_1, v_1)\varphi(u_3, v_3)\varphi_{u_2, v_2} - (1 + \alpha)\varphi(u_1, v_1)\varphi(u_2, v_2)\varphi_{u_3, v_3} \end{aligned}$$

for any $u_i, v_i \in U$, $i = 1, 2, 3$. Here \mathfrak{s} is the first copy of $\mathfrak{sp}(U, \varphi)$.

- (vi) $\mathfrak{g} = G(3)$ and \mathfrak{s} is the (unique) ideal of \mathfrak{g}_0 isomorphic to $\mathfrak{sl}(2)$.
- (vii) $\mathfrak{g} = F(4)$ and \mathfrak{s} is the (unique) ideal of \mathfrak{g}_0 isomorphic to $\mathfrak{sl}(2)$.
- (viii) $\mathfrak{g} = \mathfrak{osp}(3, 2r)$, $r \geq 1$, and \mathfrak{s} is the copy of $\mathfrak{o}(3)$ in \mathfrak{g}_0 .

Moreover, different election of the simple ideal \mathfrak{s} in (ii) or (iv) give isomorphic pairs and two pairs in (v) corresponding to the values α_1 and α_2 are isomorphic if and only if either $\alpha_1 = \alpha_2$ or $\alpha_1 + \alpha_2 = -1$.

Proof: A careful look at the list of simple Lie superalgebras in [KAC, Theorem 5] shows that the semisimple part of $\mathbf{W}(n)_0$ ($n \geq 2$), of $\mathbf{S}(n)$ ($n \geq 3$) and of $\tilde{\mathbf{S}}(n)$

($n \geq 3$), is isomorphic to $\mathfrak{sl}(n)$ [KAC, Propositions 3.1.1 and 3.3.1], while $\mathbf{W}(2)$ is isomorphic to $\mathfrak{sl}(1, 2)$. Also, the semisimple part of $\mathbf{H}(n)$ ($n \geq 4$) is isomorphic to $\mathfrak{o}(n)$ [KAC, Proposition 3.3.6], while $\mathbf{H}(4)$ is isomorphic to $\mathfrak{psl}(2, 2)$. Hence, it is enough to deal with the classical algebras. One checks easily that the simple classical Lie superalgebras with \mathfrak{g}_0 containing a three dimensional simple ideal are those listed above. Since $\mathfrak{osp}(4, 2)$ is isomorphic to $D(2, 1; 1)$, this has been excluded from (iii) and included in (v), and since $\mathfrak{sl}(1, 2)$ is isomorphic to $\mathfrak{osp}(2, 2)$, this has been included in (iii).

The last assertion about cases (ii) and (iv) is clear. Also, the Lie algebras in (v) are the ones denoted by $\Gamma(1, \alpha, -(1 + \alpha))$ in [SC, pp. 16-17]. Here we have three copies of $\mathfrak{sl}(2)$ in \mathfrak{g}_0 , but there are isomorphisms preserving the three copies from $\Gamma(\sigma_1, \sigma_2, \sigma_3)$ ($\sigma_1 + \sigma_2 + \sigma_3 = 0$) onto $\Gamma(\eta\sigma_1, \eta\sigma_2, \eta\sigma_3)$ for any $0 \neq \eta \in F$, and also natural isomorphisms permuting the three copies of $\mathfrak{sl}(2)$ (and the corresponding σ_i 's). Therefore, the distinguished copy of $\mathfrak{sl}(2)$ can always be taken to be the first one. Finally, if there is an isomorphism from $\Gamma(1, \alpha, -1 - \alpha)$ onto $\Gamma(1, \beta, -1 - \beta)$ that takes the first copy of $\mathfrak{sl}(2)$ in $\Gamma(1, \alpha, -1 - \alpha)$ to the first copy of $\mathfrak{sl}(2)$ in $\Gamma(1, \beta, -1 - \beta)$, then it takes the second copy of $\mathfrak{sl}(2)$ in $\Gamma(1, \alpha, -1 - \alpha)$ to either the second or the third copy of $\mathfrak{sl}(2)$ in $\Gamma(1, \beta, -1 - \beta)$, whence the last assertion of the Theorem. \square

Now we are ready for our main Theorem, it asserts that the examples in Section 3 exhaust all the simple $(-1, -1)$ -BFKTS's:

Theorem 4.3 *Let V be a finite dimensional simple $(-1, -1)$ -BFKTS over a field F of characteristic zero with associated symmetric bilinear form $\langle | \rangle$. Then either:*

(i) *The multiplication in V is given by*

$$xyz = \langle z | x \rangle y - \langle z | y \rangle x + \langle x | y \rangle z$$

for any $x, y, z \in V$ (orthogonal type).

(ii) *There is a quadratic étale algebra K over F such that V is a free K -module of rank ≥ 3 , endowed with a hermitian form $h : V \times V \rightarrow K$ such that*

$$\begin{cases} \langle x | y \rangle = \frac{1}{2}(h(x, y) + h(y, x)) \\ xyz = h(z, x)y - h(z, y)x + h(x, y)z \end{cases}$$

for any $x, y, z \in V$ (unitarian type).

(iii) *There is a quaternion algebra Q over F such that V is a free left Q -module of rank ≥ 2 , endowed with a hermitian form $h : V \times V \rightarrow Q$ such that*

$$\begin{cases} \langle x | y \rangle = \frac{1}{2}(h(x, y) + h(y, x)) \\ xyz = h(z, x)y - h(z, y)x + h(x, y)z \end{cases}$$

for any $x, y, z \in V$ (symplectic type).

- (iv) $\dim_F V = 4$ and there is a nonzero skew symmetric multilinear form $\Phi : V \times V \times V \times V \rightarrow F$ such that for any $x, y, z \in V$:

$$xyz = [xyz] + \langle z | x \rangle y - \langle z | y \rangle x + \langle x | y \rangle z,$$

where $[xyz]$ is defined by means of $\Phi(x, y, z, t) = \langle [xyz] | t \rangle$ for any $x, y, z, t \in V$. In this case, there is a nonzero scalar $\mu \in F$ such that (3.8) holds (D_μ -type).

- (v) $\dim_F V = 7$ and there is an eight dimensional Cayley-dickson algebra C over F with trace t and a nonzero scalar $\alpha \in F$ such that $V = C_0 = \{x \in C : t(x) = 0\}$, and for any $x, y, z \in V$:

$$\begin{cases} \langle x | y \rangle = -2\alpha t(xy) \\ xyz = \alpha(D_{x,y}(z) - 2t(xy)z) \end{cases}$$

where $D_{x,y}$ is the inner derivation of C given by (3.11) (G -type).

- (vi) $\dim_F V = 8$ and $(V, \langle | \rangle)$ is endowed with a 3-fold vector cross product X of type I such that

$$xyz = \frac{1}{3}X(x, y, z) + \langle z | x \rangle y - \langle z | y \rangle x + \langle x | y \rangle z$$

for any $x, y, z \in V$. (F -type.)

Moreover, two triple systems in different items cannot be isomorphic and:

- (i') Two triple systems of orthogonal type are isomorphic if and only if the corresponding symmetric bilinear forms are isometric.
- (ii') Two triple systems of unitarian type V_1 and V_2 , with associated quadratic étale algebras K_1 and K_2 and hermitian forms h_1 and h_2 , are isomorphic if and only if the hermitian pairs (V_1, h_1) and (V_2, h_2) are isomorphic; that is, there is an isomorphism of F -algebras $\sigma : K_1 \rightarrow K_2$ and a linear bijection $\varphi : V_1 \rightarrow V_2$ such that $h_2(\varphi(x), \varphi(y)) = \sigma(h_1(x, y))$ for any $x, y \in V_1$.
- (iii') Two triple systems of symplectic type V_1 and V_2 , with associated quaternion algebras Q_1 and Q_2 and hermitian forms h_1 and h_2 , are isomorphic if and only if the hermitian pairs (V_1, h_1) and (V_2, h_2) are isomorphic.
- (iv') Two triple systems of D_μ -type, with associated scalars μ_1 and μ_2 , are isomorphic if and only if the corresponding symmetric bilinear forms are isometric and $\mu_1 = \mu_2$.
- (v') Two triple systems of G -type, with associated Cayley-Dickson algebras C_1 and C_2 and scalars α_1 and α_2 , are isomorphic if and only if so are C_1 and C_2 and $\alpha_1 = \alpha_2 \gamma^2$ for some $0 \neq \gamma \in F$.
- (vi') Two triple systems of F -type V_1 and V_2 , with associated type I 3-fold vector cross products X_1 and X_2 , are isomorphic if and only if so are the triple systems (V_1, X_1) and (V_2, X_2) .

Proof: First, the new triple product defined on V by $[xyz] = xyz - \langle z | x \rangle y + \langle z | y \rangle x - \langle x | y \rangle z$ for any $x, y, z \in V$ is skew symmetric because of (2.5a). If this is

identically zero, V is of orthogonal type. Otherwise, if the dimension of V is 4, V is of D_μ -type.

Hence, in what follows, assume that $\dim_F V \neq 4$. Then, after extending scalars to an algebraic closure \bar{F} of F , if $\bar{V} = \bar{F} \otimes_F V$, $(\mathfrak{g}(\bar{V}), \mathfrak{s}(\bar{V}))$ is one of the pairs considered in cases (i), (iii), (iv), (vi) or (vii) in Theorem 4.2. Note that case (viii) does not appear since there $\mathfrak{g}_{\bar{1}}$ is a direct sum of adjoint modules for \mathfrak{s} instead of a direct sum of two dimensional irreducible modules.

Because of Theorem 4.1 and the computations in Section 3, and since the classical Lie superalgebras other than $D(2, 1; \alpha)$'s are determined by its even part and the structure of $\mathfrak{g}_{\bar{1}}$ as a $\mathfrak{g}_{\bar{0}}$ -module [KAC, Proposition 2.1.4], it follows that case (i) in Theorem 4.2 corresponds to the unitarian type with $\bar{K} = \bar{F} \times \bar{F}$ and $\dim_F V \geq 6$, case (iii) in 4.2 corresponds to the orthogonal type, case (iv) to the symplectic type and $\dim_F V \geq 8$ and cases (vi) and (vii) to G and F types.

Therefore, it is enough to deal with the forms over F of the simple $(-1, -1)$ -BFKTS's over \bar{F} considered in Section 3 with dimension $\neq 4$.

It is clear that if \bar{V} is of orthogonal type, so is V . If \bar{V} is of unitarian type with $\dim_F V \geq 6$, then since $\bar{K} = \text{End}_{\bar{\mathfrak{d}}}(\bar{V}) = \bar{F} \otimes_F \text{End}_{\mathfrak{d}}(V)$, $K = \text{End}_{\mathfrak{d}}(V)$ is a quadratic étale algebra over F ; besides, there is a \bar{K} -hermitian form $\bar{h} : \bar{V} \times \bar{V} \rightarrow \bar{K}$ such that $xyz = \bar{h}(z, x)y - \bar{h}(z, y)x + \bar{h}(x, y)z$ for any $x, y, z \in \bar{V}$. But if $\{1, i\}$ is an F -basis of K with $i^2 = \alpha \in F$, then $\bar{h}(x, y) = \langle x | y \rangle - \alpha^{-1} \langle x | iy \rangle i$ for any $x, y \in \bar{V}$. Since both $\langle x | y \rangle$ and $\langle x | iy \rangle$ are in F in case $x, y \in V$, it follows that \bar{h} restricts to an hermitian form $h : V \times V \rightarrow K$ and V is the corresponding simple $(-1, -1)$ -BFKTS of unitarian type. A similar argument works in case \bar{V} is of symplectic type and $\dim_F V \geq 8$. In this case $\bar{\mathfrak{d}} = \bar{\mathfrak{b}} \oplus \bar{\mathfrak{s}}$ with $\bar{\mathfrak{s}} \cong \mathfrak{sl}(2, \bar{F}) \not\cong \bar{\mathfrak{b}}$, so that $\mathfrak{d} = \mathfrak{b} \oplus \mathfrak{s}$ for a suitable unique ideal \mathfrak{b} and $\bar{Q} = \text{End}_{\bar{\mathfrak{b}}}(\bar{V}) = \bar{F} \otimes_F \text{End}_{\mathfrak{b}}(V)$. Hence $\text{End}_{\mathfrak{b}}(V) = Q$ is a quaternion algebra and V is a free Q -module. Now one takes a suitable F -basis $\{1, i, j, k\}$ of Q and argues as above.

If \bar{V} is of G-type, then \mathfrak{d} is a form of G_2 , so there is an eight-dimensional Cayley-Dickson algebra C over F such that $\mathfrak{d} \cong \text{Der } C$ and V is, up to isomorphism, its seven dimensional irreducible module for \mathfrak{d} , that is C_0 , the set of traceless elements in C . Since $\text{Hom}_{\mathfrak{d}}(V \otimes_F V, \mathfrak{d})$ is one-dimensional, after identifying V with C_0 there exists a nonzero $\alpha \in F$ such that $d_{x,y} = \alpha D_{x,y}$ for any $x, y \in C_0 = V$. From here, using (2.3c), it follows that V is of G-type.

Finally, if \bar{V} is of F-type, define $X : V \times V \times V \rightarrow F$ by

$$X(x, y, z) = 3 \left(xyz - \langle z | x \rangle y + \langle z | y \rangle x - \langle x | y \rangle z \right)$$

for any $x, y, z \in V$. Then X is a 3-fold vector cross product of type I (because it is so after extending scalars) and hence V is of F-type.

Moreover, two simple $(-1, -1)$ -BFKTS's of different types cannot be isomorphic because the corresponding Lie algebras of inner derivations are not. Also note that, because of (2.5a), any isomorphism among two $(-1, -1)$ -BFKTS's is an isometry of the corresponding symmetric bilinear forms. Now (i') is clear and (ii') (respectively

(iii') follows from the fact that K_1 and K_2 (resp. Q_1 and Q_2) are determined as centralizers of the action of a suitable ideal of the Lie algebra of inner derivations.

Let us check (iv'), so let $(V_i, (xyz)_i)$ be two simple $(-1, -1)$ -BFKTS's of D_{μ_i} -type ($i = 1, 2$). If $\varphi : V_1 \rightarrow V_2$ is an isomorphism, then it is an isometry and thus $\varphi([xyz]_1) = [\varphi(x)\varphi(y)\varphi(z)]_2$ for any $x, y, z \in V_1$. Hence

$$\begin{aligned} \langle \varphi([x_1x_2x_3]_1) \mid \varphi([x_1x_2x_3]_1) \rangle_2 &= \langle [x_1x_2x_3]_1 \mid [x_1x_2x_3]_1 \rangle_1 \\ &= \mu_1 \det(\langle x_i \mid x_j \rangle_1), \end{aligned}$$

but also

$$\begin{aligned} \langle \varphi([x_1x_2x_3]_1) \mid \varphi([x_1x_2x_3]_1) \rangle_2 &= \langle [\varphi(x_1)\varphi(x_2)\varphi(x_3)]_2 \mid [\varphi(x_1)\varphi(x_2)\varphi(x_3)]_2 \rangle_2 \\ &= \mu_2 \det(\langle \varphi(x_i) \mid \varphi(x_j) \rangle_2) \\ &= \mu_2 \det(\langle x_i \mid x_j \rangle_1). \end{aligned}$$

Therefore, $\mu_1 = \mu_2$. Conversely, assume that $\varphi : V_1 \rightarrow V_2$ is an isometry and that $\mu_1 = \mu_2 = \mu$. Consider $\Phi_i : V_i^4 \rightarrow F$ ($i = 1, 2$) given by $\Phi_i(x_1, x_2, x_3, x_4) = \langle [x_1x_2x_3]_i \mid x_4 \rangle_i$. Also, let $\tilde{\Phi}_1 : V_1^4 \rightarrow F$ be defined by

$$\tilde{\Phi}_1(x_1, x_2, x_3, x_4) = \Phi_2(\varphi(x_1), \varphi(x_2), \varphi(x_3), \varphi(x_4))$$

for any $x_1, x_2, x_3, x_4 \in F$. Since $\dim_F V_1 = 4$ and both Φ_1 and $\tilde{\Phi}_1$ are skew symmetric, they are proportional, and hence there is a nonzero scalar $\alpha \in F$ such that $\tilde{\Phi}_1 = \alpha\Phi_1$. For any $x_1, x_2, x_3, y_1, y_2, y_3 \in F$:

$$\begin{aligned} \Phi_2(\varphi(x_1), \varphi(x_2), \varphi(x_3), \varphi([y_1y_2y_3]_1)) &= \tilde{\Phi}_1(x_1, x_2, x_3, [y_1y_2y_3]_1) \\ &= \alpha\Phi_1(x_1, x_2, x_3, [y_1y_2y_3]_1) \\ &= \alpha\mu \det(\langle x_i \mid y_j \rangle_1) \\ &= \alpha\mu \det(\langle \varphi(x_i) \mid \varphi(y_j) \rangle_2) \\ &= \Phi_2(\varphi(x_1), \varphi(x_2), \varphi(x_3), \alpha[\varphi(y_1)\varphi(y_2)\varphi(y_3)]_2), \end{aligned}$$

where we have used (3.8) and the fact that φ is an isometry. Thus $\varphi([y_1y_2y_3]_1) = \alpha[\varphi(y_1)\varphi(y_2)\varphi(y_3)]_2$ for any $y_1, y_2, y_3 \in V_1$. But now, again by (3.8), this shows that $\mu \det(\langle y_i \mid y_j \rangle_1) = \alpha^2 \mu \det(\langle y_i \mid y_j \rangle_1)$ for any y_i 's, so that $\alpha^2 = 1$. If $\alpha = 1$ we are done, otherwise $\alpha = -1$. In this latter case, choose any isometry σ of $\langle \mid \rangle_1$ with $\det \sigma = -1$ and consider $\hat{\varphi} = \varphi\sigma : V_1 \rightarrow V_2$. Then if $\hat{\Phi}_1(x_1, x_2, x_3, x_4) = \Phi_2(\hat{\varphi}(x_1), \hat{\varphi}(x_2), \hat{\varphi}(x_3), \hat{\varphi}(x_4))$ for any $x_i \in V_1$ ($i = 1, 2, 3, 4$), we have:

$$\begin{aligned} \hat{\Phi}_1(x_1, x_2, x_3, x_4) &= \Phi_2(\hat{\varphi}(x_1), \hat{\varphi}(x_2), \hat{\varphi}(x_3), \hat{\varphi}(x_4)) \\ \tilde{\Phi}_1(\sigma(x_1), \sigma(x_2), \sigma(x_3), \sigma(x_4)) &= \alpha(\det \sigma)\Phi_1(x_1, x_2, x_3, x_4) \\ &= \Phi_1(x_1, x_2, x_3, x_4), \end{aligned}$$

because $\alpha = -1 = \det \sigma$ and Φ_1 is multilinear and alternating. The same argument as above, with $\tilde{\Phi}_1$ replaced by $\hat{\Phi}_1$ shows that $\hat{\varphi}$ is an isomorphism between the two triple systems.

With regard to (v'), if $\varphi : V^1 \rightarrow V^2$ is an isomorphism of two triple systems of G-type with associated Cayley-Dickson algebras C^1 and C^2 and scalars α_1 and α_2 , then φ is an isometry of the associated symmetric bilinear forms and for any $x, y, z \in V_1$

$$\varphi(d_{x,y}z) = d_{\varphi(x),\varphi(y)}\varphi(z). \quad (4.2)$$

Also, $\phi : \mathfrak{d}^1 = d_{V^1,V^1} \rightarrow \mathfrak{d}^2: d \mapsto \varphi d \varphi^{-1}$ is an isomorphism of Lie algebras and φ becomes an isomorphism of \mathfrak{d}^1 -modules, where V^2 is a \mathfrak{d}^1 -module through ϕ . Since $\text{Hom}_{\mathfrak{d}^1}(\Lambda^2(V^1), V^1)$ is spanned by $x \wedge y \mapsto [x, y] = xy - yx$ (multiplication in C^1), there is a nonzero scalar $\mu \in F$ such that

$$\varphi([x, y]) = \mu[\varphi(x), \varphi(y)] \quad (4.3)$$

for any $x, y \in V^1 = C_0^1 = \{z \in C^1 : t(z) = 0\}$. In particular, $\mu\varphi : (C_0^1, [,]) \rightarrow (C_0^2, [,])$ is an isomorphism of Malcev algebras and hence C^1 and C^2 are isomorphic (see, for instance, [E1, (3.1)]). But the associator $(x, y, z) = (xy)z - x(yz)$ in C^1 is skew symmetric on its arguments, so for any $x, y, z \in C^1$, $(x, y, z) = -(x, z, y) = (z, x, y) = (y, z, x)$, so that

$$L_{xy} - L_x L_y = [L_x, R_y] = R_y R_x - R_{xy} = [R_x, L_y],$$

hence $ad_{xy} - L_x L_y + R_y R_x = 2[L_x, R_y]$ for any $x, y \in C^1$, where $ad_x y = [x, y] = (L_x - R_x)(y)$. Permuting x and y and subtracting we get

$$ad_{[x,y]} = [L_x, L_y] + [R_x, R_y] + 4[L_x, R_y] = D_{x,y} + 3[L_x, R_y].$$

On the other hand,

$$\begin{aligned} [ad_x, ad_y] &= [L_x - R_x, L_y - R_y] = [L_x, L_y] + [R_x, R_y] - 2[L_x, R_y] \\ &= D_{x,y} - 3[L_x, R_y], \end{aligned}$$

and from here we conclude that

$$2D_{x,y} = ad_{[x,y]} + [ad_x, ad_y]$$

for any $x, y \in C^1$. Since $d_{x,y} = \alpha_1 D_{x,y}$ and $d_{\varphi(x),\varphi(y)} = \alpha_2 D_{\varphi(x),\varphi(y)}$ for any $x, y \in V^1$, equation (4.3) gives:

$$\begin{aligned} \varphi d_{x,y} &= \frac{\alpha_1}{2} \varphi(ad_{[x,y]} + [ad_x, ad_y]) \\ &= \frac{\alpha_1}{2} \mu^2 \left(ad_{[\varphi(x),\varphi(y)]} + [ad_{\varphi(x)}, ad_{\varphi(y)}] \right) \varphi \\ &= \frac{\alpha_1}{2} \mu^2 D_{\varphi(x),\varphi(y)} \varphi, \\ d_{\varphi(x),\varphi(y)} &= \frac{\alpha_2}{2} D_{\varphi(x),\varphi(y)}, \end{aligned}$$

for any $x, y \in V^1$, so that equation (4.2) gives $\alpha_1\mu^2 = \alpha_2$, as desired. Conversely, if $\psi : C^1 \rightarrow C^2$ is an isomorphism and $\alpha_2 = \alpha_1\mu^2$, then the map $\varphi : V^1 = C_0^1 \rightarrow V^2 = C_0^2$, given by $\varphi(x) = \mu^{-1}\psi(x)$ for any $x \in V^1$, is an isomorphism of triple systems.

We are left with the isomorphism problem for the $(-1, -1)$ -BFKTS's of F-type. For these we need some preliminaries, which have their own independent interest:

Lemma 4.4 *Let X be a 3-fold vector cross product of type I on an eight dimensional vector space V over a field F of characteristic $\neq 2$, and let $\langle | \rangle$ be the associated nondegenerate symmetric bilinear form (so that (3.11) is satisfied). Then $\langle | \rangle$ is determined by X .*

Proof: Because of (3.12), for any $a, b, c, d \in V$:

$$\begin{aligned} -\langle d | X(a, b, X(a, b, c)) \rangle &= \langle X(a, b, c) | X(a, b, d) \rangle \\ &= \det \begin{vmatrix} \langle a | a \rangle & \langle a | b \rangle & \langle a | d \rangle \\ \langle b | a \rangle & \langle b | b \rangle & \langle b | d \rangle \\ \langle c | a \rangle & \langle c | b \rangle & \langle c | d \rangle \end{vmatrix} \\ &= \left\langle \langle a \wedge b | a \wedge b \rangle c - \langle a \wedge b | a \wedge c \rangle b + \langle a \wedge b | b \wedge c \rangle a \mid d \right\rangle \end{aligned}$$

where $\langle a \wedge b | u \wedge v \rangle = \begin{vmatrix} \langle a | u \rangle & \langle a | v \rangle \\ \langle b | u \rangle & \langle b | v \rangle \end{vmatrix}$ for any $a, b, u, v \in V$. By nondegeneracy of $\langle | \rangle$, this gives:

$$X(a, b, X(a, b, c)) = \langle a \wedge b | c \wedge b \rangle a + \langle a \wedge b | a \wedge c \rangle b - \langle a \wedge b | a \wedge b \rangle c. \quad (4.4)$$

Hence, for any $a, b, c \in V$, if $d = X(a, b, c)$, then $X(a, b, d) \in Fa + Fb + Fc$ and, similarly (since $d = X(b, c, a) = X(c, a, b)$), $X(b, c, d), X(a, c, d) \in Fa + Fb + Fc$, so that $W = Fa + Fb + Fc + Fd$ is closed under X . Let us prove now that for any $0 \neq v \in V$:

$$X(v, V, V) = \{x \in V : \langle v | x \rangle = 0\}. \quad (4.5)$$

Because of (3.12), $X(v, V, V) \subseteq \{x \in V : \langle x | v \rangle = 0\}$. Now, take $a = v$ and let $b \in V$ linearly independent with a and such that $\langle | \rangle$ is nondegenerate on $W_b = Fa + Fb$. By (4.4) $c \in X(a, b, V) \subseteq X(v, V, V)$ for any $c \in W_b^\perp = \{x \in V : \langle x | a \rangle = 0 = \langle x | b \rangle\}$. Take any two such b 's with different W_b 's, then the sum of the W_b^\perp 's is $\{x \in V : \langle x | v \rangle = 0\}$, so (4.5) follows.

Thus, assume that X is also a 3-fold vector cross product of type I relative to another nondegenerate symmetric bilinear form $(|)$ on V . Then for any $0 \neq u, v \in V$, if $\langle u | v \rangle = 0$, then $u \in X(v, V, V)$ by (4.5), so by (3.12), also $(u | v) = 0$. The only possibility then is that $(|) = \alpha \langle | \rangle$ for some nonzero scalar $\alpha \in F$. But then (3.12) implies that $\alpha^3 = \alpha$, so $\alpha = \pm 1$ and (3.13) that $\alpha = 1$. \square

Note that if X is a 3-fold vector cross product of type I on an eight dimensional vector space V relative to the nondegenerate symmetric bilinear form $\langle | \rangle$, then X is a 3-fold vector cross product of type II relative to $-\langle | \rangle$. Also note that $\langle | \rangle$ does not determine X , since not every orthogonal transformation relative to $\langle | \rangle$ is an automorphism of X ([E2]).

Corollary 4.5 *Let X_i be a 3-fold vector cross product on an eight dimensional vector space V_i over a field F of characteristic $\neq 2$ with associated nondegenerate symmetric bilinear form $\langle | \rangle_i$ ($i = 1, 2$). Then if $\varphi : (V_1, X_1) \rightarrow (V_2, X_2)$ is an isomorphism, then it is also an isometry $\varphi : (V_1, \langle | \rangle_1) \rightarrow (V_2, \langle | \rangle_2)$.*

Now, the proof of item (vi) in Theorem 4.3 follows immediately from the Corollary above and this finishes its proof.

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