DERIVATIONS OF ORTHOSYMPECTIC LIE SUPERALGEBRAS

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ABSTRACT

In this paper we describe the derivations of orthosymplectic Lie superalgebras over a superring. In particular, we derive sufficient conditions under which the derivations can be expressed as a semidirect product of inner and outer derivations. We then present some examples for which these conditions hold.

INTRODUCTION

The original motivation for this paper came from the study of extended affine Lie algebras ([AABGP]). In particular we were interested in describing the derivations of Lie algebras over fields of characteristic 0 graded by root systems of type B_I and D_I and developing conditions under which we could write the derivation algebra as a semidirect product of the inner and the outer derivations. Such a decomposition is very useful in the construction of extended affine Lie algebras of type B and D from their centreless core ([AG]). In light of the recent activity in developing a theory of root-graded Lie superalgebras ([BeE], [GN]), it is of interest to consider Lie superalgebras. Our methods will allow us to describe derivations of various types of orthosymplectic Lie superalgebra

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over superrings containing $\frac{1}{2}$, including some examples which occur in the recent paper [CW].

Some work on derivations of Lie superalgebras was done by Kac in [K2] and Scheunert in [S]. In particular, they described the derivations of simple finite-dimensional Lie superalgebras over algebraically closed fields of characteristic 0. For orthosymplectic Lie superalgebras, their result is a special case of Corollary 4.13 presented in this paper. Significant work on derivations was recently done by Benkart in [Be]. She described the derivations of Lie algebras over fields of characteristic zero which are graded by finite root systems using the derivations of the coordinate algebras. The result described in Corollary 4.7 is a generalization of Benkart's result when we consider the Lie algebras which are graded by root systems of type B_I, $|I| \geq 3$ and D_I, $|I| \geq 4$.

We assume that K is a supercommutative and unital superring containing $\frac{1}{2}$ and A a superextension of K. We consider subalgebras of the orthosymplectic Lie superalgebra $\mathrm{osp}(q)$ where q is an A-quadratic form on an A-supermodule $\mathcal{M}$. The superalgebra $\mathrm{osp}(q)$ is the Lie superalgebra of all A-endomorphisms x of $\mathcal{M}$ such that $q(x(m), n) + (-1)^{|m||n|}q(x(n), m) = 0$ for all $m, n \in \mathcal{M}$. It has an ideal, $\mathrm{eosp}(q)$, which is defined to be the $\mathbb{Z}$-span of the maps $\mathbf{E}_{m,n}$ for homogeneous $m, n \in \mathcal{M}$ where $\mathbf{E}_{m,n}(p) = mq(n, p) - (-1)^{|n||p|}q(m, p)n$ for $p \in \mathcal{M}$. We show that if q_∞ is the orthogonal sum of the hyperbolic superplane and another quadratic form q on an A-supermodule $\mathcal{M}$, then the K-derivations of any subalgebra $\mathcal{E}_\infty$ of $\mathrm{osp}(q_\infty)$ containing $\mathrm{eosp}(q_\infty)$ can be described as a sum of the inner derivations and a Lie superalgebra $\mathcal{S} \oplus \mathcal{T}$ for certain $\mathcal{S}, \mathcal{T} \subset \mathrm{End}_K \mathcal{M}$. We also determine the intersection of the inner derivations and the superalgebra $\mathcal{S} \oplus \mathcal{T}$ and we determine the conditions under which we can write the algebra of derivations as a semidirect product of the inner derivations and a certain subalgebra. Finally, we describe some examples where we do get the splitting of K-derivations of $\mathcal{E}_\infty$, $\mathrm{Der}_K(\mathcal{E}_\infty)$, into a semidirect product of the inner derivations and a subalgebra of $\mathcal{S} \oplus \mathcal{T}$. In particular, when q is an almost diagonalizable A-quadratic form on a free A-supermodule of dimension greater than 2 (i.e., there exists a homogeneous basis $\{m_i \mid i \in I\}$ of $\mathcal{M}$ such that for each $i \in I$ there exists $\underset{\cdot}{i} \in I$ such that $q(m_i, m_j)$ is 0 if $i \neq \underset{\cdot}{i}$ and is invertible otherwise), we get the semidirect splitting of the derivations for a number of subalgebras $\mathcal{E}_\infty$. These include the centreless core L of an extended affine Lie algebra over $\mathbb{C}$ of type B_I or D_I and nullity ν and, in this case in particular, we get $\mathrm{Der}_{\mathbb{C}}(L/Z(L)) \cong \mathbf{ad}(L/Z(L)) \rtimes \mathrm{Der}_{\mathbb{C}}\, \mathbb{C}[t_1^{\pm 1}, \ldots, t_\nu^{\pm 1}]$ for I a finite set.

1 SUPERALGEBRAS

In this section we will describe some fundamental concepts of superstructures. The reader can find more extensive coverage of this material in [K2], [S] and [GN].

Let $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ be the field of two elements. A ring K is called a *superring* if K is a $\mathbb{Z}_2$-graded ring, i.e., $K = K_{\bar{0}} \oplus K_{\bar{1}}$ such that $K_\alpha K_\beta \in K_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{Z}_2$. We say that a superring K is *supercommutative* if $ab = (-1)^{\alpha\beta} ba$ for all $a \in K_\alpha, b \in K_\beta$ for all $\alpha, \beta \in \mathbb{Z}_2$. Throughout this paper K shall denote a unital and supercommutative superring and we will assume that K contains $\frac{1}{2}$. Note that $1 \in K_{\bar{0}}$.

A K-bimodule $\mathcal{M}$ is called a *K-supermodule* if $\mathcal{M} = \mathcal{M}_{\bar{0}} \oplus \mathcal{M}_{\bar{1}}$ for some $K_{\bar{0}}$-submodules $\mathcal{M}_{\bar{0}}$ and $\mathcal{M}_{\bar{1}}$ of $\mathcal{M}$ such that $K_\alpha \mathcal{M}_\beta \subset \mathcal{M}_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{Z}_2$ and the K-module action satisfies

$$am = (-1)^{\alpha\beta} ma \tag{1.1}$$

for all $a \in K_\alpha$, $m \in \mathcal{M}_\beta$. Throughout this paper, any time we talk about a $\mathbb{Z}_2$-graded structure $Z = Z_{\bar{0}} \oplus Z_{\bar{1}}$ we will call an element $z \in Z$ *homogeneous* if $z \in Z_\alpha$ for some $\alpha \in \mathbb{Z}_2$ and we will say that z is of *degree* α, denoted by $|z| = \alpha$; in this case we will assume throughout that when $|z|$ occurs in an expression, then it is assumed that z is homogeneous, and that the expression extends to the other elements by linearity. When we refer to a submodule of a supermodule we assume that the submodule is also $\mathbb{Z}_2$-graded.

If $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n$ are K-supermodules then the direct sum $\mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots \oplus \mathcal{M}_n$ is a K-supermodule via the action

$$a(m_1, m_2, \cdots, m_n) = (am_1, am_2, \cdots am_n)$$

for all $a \in K$, $m_i \in \mathcal{M}_i$, $i = 1, 2, \cdots, n$. A map $\phi : \mathcal{M}_1 \times \mathcal{M}_2 \times \ldots \times \mathcal{M}_n \to \mathcal{N}$ where $\mathcal{N}$ is a K-supermodule is said to have *degree* α ($\alpha \in \mathbb{Z}_2$) if

$$\phi(m_1, \ldots, m_n) \in \mathcal{N}_{\alpha+|m_1|+\cdots+|m_n|}$$

for all homogeneous $m_i \in \mathcal{M}_i$, $1 \le i \le n$. Given K-supermodules $\mathcal{M}$ and $\mathcal{N}$, we say that a map $\phi : \mathcal{M} \to \mathcal{N}$ is K-linear if ϕ is additive and

$$\phi(ma) = \phi(m)a \text{ for all } a \in K,\ m \in \mathcal{M}.$$

A K-linear map between K-supermodules $\mathcal{M}$ and $\mathcal{N}$ is called a *K-supermodule homomorphism from $\mathcal{M}$ to $\mathcal{N}$*. The set of all K-supermodule homomorphisms from $\mathcal{M}$ to $\mathcal{N}$ is denoted by $\mathrm{Hom}_K(\mathcal{M}, \mathcal{N})$ and is a K-supermodule where the K-action is given by $(af)(m) := a(f(m))$ and $(fa)(m) := f(am)$ for all $a \in K$, $f \in \mathrm{Hom}_K(\mathcal{M}, \mathcal{N})$. The $\mathbb{Z}_2$-grading is given by

$$\mathrm{Hom}_K(\mathcal{M}, \mathcal{N}) = (\mathrm{Hom}_K(\mathcal{M}, \mathcal{N}))_{\bar{0}} \oplus (\mathrm{Hom}_K(\mathcal{M}, \mathcal{N}))_{\bar{1}}$$

where $\mathrm{Hom}_K(\mathcal{M}, \mathcal{N})_\alpha = \{f \in \mathrm{Hom}_K(\mathcal{M}, \mathcal{N}) \mid \text{degree of } f \text{ is } \alpha\}$. In case $\mathcal{M} = \mathcal{N}$ we write $\mathrm{End}_K \mathcal{M}$ instead of $\mathrm{Hom}_K(\mathcal{M}, \mathcal{M})$. Given a K-supermodule $\mathcal{P}$, we say that a map $\phi : \mathcal{M} \times \mathcal{N} \to \mathcal{P}$ is *K-bilinear* if

$$\begin{aligned} \phi(m + m', n + n') &= \phi(m, n) + \phi(m, n') + \phi(m, n') + \phi(m', n'); && (1.2) \\ \phi(ma, n) &= \phi(m, an); \text{ and} && (1.3) \\ \phi(m, na) &= \phi(m, n)a && (1.4) \end{aligned}$$

for all $a \in K$, $m \in \mathcal{M}$, $n \in \mathcal{N}$. The set of all K-bilinear maps $\phi : \mathcal{M} \times \mathcal{N} \to \mathcal{P}$ forms a K-supermodule via the action

$$\begin{aligned}(a\phi)(m,n) &:= a(\phi(m,n)); \text{ and}\\ (\phi a)(m,n) &:= \phi(am,n)\end{aligned}$$

for all $a \in K$, $m \in \mathcal{M}$, $n \in \mathcal{N}$.

A K-bilinear map $\phi : \mathcal{M} \times \mathcal{M} \to \mathcal{N}$ is *supersymmetric* if

$$\phi(m,n) = (-1)^{|m||n|}\phi(n,m)$$

for all $m, n \in \mathcal{M}$. Note that if ϕ is supersymmetric, then $\phi|_{\mathcal{M}_{\bar{0}} \times \mathcal{M}_{\bar{0}}}$ is symmetric and $\phi|_{\mathcal{M}_{\bar{1}} \times \mathcal{M}_{\bar{1}}}$ is skewsymmetric. The *radical* $\mathcal{R}$ *of* a supersymmetric ϕ is defined by

$$\mathcal{R} = \{m \in \mathcal{M} \mid \phi(m,n) = 0 \ \forall n \in \mathcal{M}\}.$$

We say that ϕ is *nondegenerate* if $\mathcal{R} = \{0\}$.

A *K-quadratic map* between K-supermodules $\mathcal{M}$ and $\mathcal{N}$ is a K-supersymmetric, K-bilinear map $q : \mathcal{M} \times \mathcal{M} \to \mathcal{N}$ of degree $\bar{0}$. Given a K-supermodule $\mathcal{M}$, a K-quadratic map q from $\mathcal{M}$ to K is called a *K-quadratic form on* $\mathcal{M}$. Given two K-supermodules $\mathcal{M}_1$ and $\mathcal{M}_2$ with K-quadratic forms q_1 and q_2 respectively, we define their *orthogonal sum* $q = q_1 \oplus q_2$ to be the K-quadratic form on the K-supermodule $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ given by

$$\begin{aligned}q(m_1 \oplus m_2, n_1 \oplus n_2) &= (q_1 \oplus q_2)(m_1 \oplus m_2, n_1 \oplus n_2)\\ &= q(m_1, n_1) + q(m_2, n_2)\end{aligned}$$

for $m_1, n_1 \in \mathcal{M}_1$, $m_2, n_2 \in \mathcal{M}_2$.

Example 1.1 Let I be an arbitrary set. We define $H(I,K)$ to be the direct sum of free K-supermodules $Kh_{\pm i}$ for $i \in I$ where $h_{\pm i}$ are of even degree. In other words,

$$H(I) = H(I,K) = \oplus_{i \in I}(Kh_i \oplus Kh_{-i}).$$

where the elements of the basis $\{h_{\pm i}\}_{i \in I}$ are all of even degree. Then we have that $H(I) = H(I)_{\bar{0}} \oplus H(I)_{\bar{1}}$ is a K-supermodule where

$$H(I)_\alpha = \oplus_{i \in I}(K_\alpha h_i \oplus K_\alpha h_{-i})$$

for $\alpha \in \mathbb{Z}_2$. We call $H(I,K)$ the *hyperbolic K-superspace*. In the case when $|I| = 1$, we call $H(I,K)$ the *hyperbolic K-superplane*. We define the K-quadratic form q_I on the hyperbolic K-superspace $H(I)$ by setting

$$q_I(h_{\sigma i}, h_{-\mu j}) = \delta_{\sigma,\mu}\delta_{i,j}$$

for all $i, j \in I$, $\sigma, \mu \in \{\pm\}$ and extending q_I bilinearly over K. Note that q_I is the orthogonal sum of all $q_{\{i\}}$, $i \in I$.

A *superalgebra A over K* is a K-supermodule with a K-bilinear map $\cdot : A \times A \to A$ of degree $\bar{0}$. A *subalgebra B of the superalgebra A* is a K-submodule B of A such that B is closed under $\cdot|_{B\times B}$. A subalgebra B of A is an *ideal* of A if $A \cdot B \subset B$. A unital, associative, supercommutative superalgebra A over K is called a *superextension of K*.

Example 1.2 Let $\mathcal{G}$ be the associative K-algebra generated by $\{\xi_i\}_{i\in\mathbb{N}}$ subject to the relation $\xi_i\xi_j + \xi_j\xi_i = 0$ for all $i, j \in \mathbb{N}$. Since $\frac{1}{2} \in K$, we have in particular that $\xi_i^2 = 0$ for all $i \in \mathbb{N}$. The algebra $\mathcal{G}$ is called the *exterior algebra over K on a countable number of generators ξ_i, $i \in \mathbb{N}$*, or simply the *Grassmann algebra*. It is easy to check that the Grassmann algebra is a superextension of K.

Let $\mathcal{M}$ be a K-supermodule. Then $\operatorname{End}_K \mathcal{M}$ is a K-superalgebra via the usual composition of maps.

Given two K-superalgebras A and B, a K-algebra homomorphism $\phi : A \to B$ of degree $\bar{0}$ is called a *K-superalgebra homomorphism*. If, in addition, ϕ is bijective, we say that ϕ is an *isomorphism* and we write $A \overset{\phi}{\cong} B$, or simply $A \cong B$.

Definition 1.3 Let A be a K-superalgebra in a variety $\mathfrak{V}$ (later $\mathfrak{V}$ will be the variety of Jordan or Lie superalgebras). Let $\mathcal{U}$ be a K-supermodule equipped with a pair of K-bilinear mappings $(a, u) \mapsto au$, $(a, u) \mapsto ua$, $a \in A$, $u \in \mathcal{U}$, of $A \times \mathcal{U}$ into $\mathcal{U}$ of degree 0. Then $X = A \oplus \mathcal{U}$ is a K-supermodule on which we define a multiplication by

$$(a + u)(b + v) = ab + av + ub \tag{1.5}$$

for all $a, b \in A$, $u, v \in \mathcal{U}$. Since this product is K-bilinear, X is a superalgebra over K, called the *split null extension* of $\mathcal{A}$ determined by the bilinear mappings of A and $\mathcal{U}$ (see [J, Chap. II, Sect., 5] for the classical case). If X is a superalgebra in the variety $\mathfrak{V}$ then we say that $\mathcal{U}$ is a *$\mathfrak{V}$-module* for A. In this case, for each $\alpha \in \mathbb{Z}_2$ let $(\operatorname{Der}_K(A,\mathcal{U}))_\alpha$ be the space of all homogeneous K-module homomorphisms $d \in \operatorname{Hom}_K(A,\mathcal{U})_\alpha$ satisfying for all homogeneous $x \in A$ and all $y \in A$

$$d(xy) = d(x)y + (-1)^{\alpha|x|}xd(y).$$

We define

$$\operatorname{Der}_K(A,\mathcal{U}) = (\operatorname{Der}_K(A,\mathcal{U}))_{\bar{0}} \oplus (\operatorname{Der}_K(A,\mathcal{U}))_{\bar{1}}.$$

Then it is easy to see that $\operatorname{Der}_K(A,\mathcal{U})$ is a submodule of $\operatorname{End}_K \mathcal{M}$ and hence a K-supermodule. The elements of $\operatorname{Der}_K(A,\mathcal{U})$ are called the *K-derivations (from A to $\mathcal{U}$)*. If $A = \mathcal{U}$, we simply write $\operatorname{Der}_K A$.

A K-supermodule L is called a *Lie superalgebra* if it is equipped with a K-bilinear bracket multiplication $[\cdot,\cdot] : L \times L \to L$ such that:

(SL1) For all $x, y \in L$,

$$[x, y] = -(-1)^{|x||y|}[y, x];$$

(SL2) For all $x, y, z \in L$,

$$(-1)^{|x||z|}[[x, y], z] + (-1)^{|y||x|}[[y, z], x] + (-1)^{|z||y|}[[z, x], y] = 0.$$

The property (SL2) is called the *Jacobi identity* and is equivalent to

$$[[x, y], z] = [x, [y, z]] - (-1)^{|x||y|}[y, [x, z]] \tag{1.6}$$

for all $x, y, z \in L$.

For example, given an associative K-superalgebra A, we can define a Lie superalgebra structure on A, denoted by $A^{(-)}$, by defining the bracket operation via $[x, y] = xy - (-1)^{|x||y|}yx$ for all $x, y \in A$. In particular, one can easily check that $\mathrm{Der}_K A$ is a subalgebra of $(\mathrm{End}_K A)^{(-)}$.

If L is a Lie superalgebra, we define the *centre of* L by

$$Z(L) = \{x \in L \mid [x, y] = 0 \text{ for all } y \in L\}.$$

We have that $Z(L)$ is an ideal of L. A Lie superalgebra L is called a *central extension* of the Lie superalgebra $\tilde{L}$ if there exists a central ideal C of L (i.e., $C \subset Z(L)$) such that $L/C \cong \tilde{L}$.

For any element x of L, we can define the map $\mathbf{ad}(x) : L \to L$ by $\mathbf{ad}(x)(y) := [x, y]$. The K-supermodule $\mathbf{ad}(L)$ forms an ideal of $\mathrm{Der}_K L$ called the *inner derivation algebra of* L and denoted by $\mathbf{ad}\, L$. For example, any finite-dimensional Lie superalgebra L over a field k, whose Killing form is nondegenerate, has no outer derivations, i.e. $\mathrm{Der}_k L = \mathbf{ad}\, L$ (see [K2, Prop. 2.3.4]).

We recall the following definition (see [N1], for an equivalent definition see [LN]). A subset R of a real vector space X with a scalar product $(\cdot, \cdot)$ is called a *root system (in X)* if R has the following three properties:

(RS1) R generates X as a vector space and $0 \notin R$,

(RS2) for each $\alpha \in R$ we have $s_\alpha(R) = R$ where $s_\alpha(x) = x - 2\frac{(x,\alpha)}{(\alpha,\alpha)}\alpha$ for $x \in X$,

(RS3) for all $\alpha, \beta \in R$, we have $\langle \alpha, \beta \rangle \in \mathbb{Z}$, where for $x \in X,\ \alpha \in R$,

$$\langle x, \alpha \rangle := 2\frac{(x, \alpha)}{(\alpha, \alpha)}.$$

A root system R is *irreducible* if $R \neq 0$ and if R is not an orthogonal sum of two non-empty root systems. Every root system is an orthogonal sum of irreducible root systems which are uniquely determined and are called the *irreducible components of* R. Note that in the finite-dimensional case, these definitions conform with the traditional definition of (finite) root systems. An irreducible root system is isomorphic to a finite root system or to one of the infinite analogues of the root systems of type $A-D$ and BC ([LN]).

Let R be a root system. We say that a Lie superalgebra over K is *R-graded* ([GN]) if there exist supermodules $L^\alpha,\ \alpha \in R\cup\{0\}$ of L, such that

(SG1) $L = \oplus_{\alpha\in R\cup\{0\}} L^\alpha$;

(SG2) for all $\alpha, \beta \in R\cup\{0\}$,

$$[L^\alpha, L^\beta] \subset \begin{cases} L^{\alpha+\beta} & \text{if } \alpha+\beta \in R\cup\{0\} \\ \{0\} & \text{if } \alpha+\beta \notin R\cup\{0\} \end{cases} ;$$

(SG3) as a Lie superalgebra, L is generated by $\cup_{\alpha\in R} L^\alpha$;

(SG4) for every $\alpha \in R$ there exists $0 \neq X_\alpha \in (L^\alpha)_{\bar 0}$ such that $H_\alpha := [X_{-\alpha}, X_\alpha]$ operates on L^β ($\beta \in R\cup\{0\}$) by

$$[H_\alpha, z_\beta] = \langle \beta, \alpha\rangle z_\beta \quad (z_\beta \in L^\beta)$$

where

$$\langle \beta, \alpha\rangle = 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}.$$

This is a generalization of R-graded Lie algebras (see [BM] and [BeZ]) in the following sense: Every R-graded Lie algebra is an R-graded Lie superalgebra (by setting $L_{\bar 0} = L$ and $L_{\bar 1} = 0$). However, if L is an R-graded Lie superalgebra, $L_{\bar 0}$ need not be an R-graded Lie algebra (see Example 2.1).

A *Jordan superalgebra* ([K3]) $J = J_{\bar 0} \oplus J_{\bar 1}$ over K is a K-superalgebra such that for all $a, b, c, d \in J$,

(JSA1) $ab = (-1)^{|a||b|} ba$

(JSA2) $((ab)c)d + (-1)^{|d||c|} a((bd)c) + (-1)^{|b||a|+|d||c|} b((ad)c) =$

$$(ab)(cd) + (-1)^{|c||b|}(ac)(bd) + (-1)^{|d|(|b|+|c|)}(ad)(bc)$$

(JSA3) $(a^2c)a = a^2(ca)$.

Example 1.4 Let $\mathcal{M}$ be a K-supermodule with a K-quadratic form q. Then the algebra $J = K \oplus \mathcal{M}$ with multiplication $\circ$ given by

$$(a \oplus m) \circ (b \oplus n) = (ab + q(m,n)) \oplus (an + mb)$$

forms a Jordan superalgebra, called the Jordan superalgebra associated with the quadratic form q.

One can check that given a Jordan algebra J over K, a K-module $\mathcal{M}$ is a Jordan superalgebra module (or a *Jordan supermodule*) according to Definition 1.3 if $\mathcal{M}$ is a K-supermodule satisfying:

(JSAM1) $am = (-1)^{|a||m|}ma$

(JSAM2) $0 = (-1)^{|b||c|}[R_{ab}, R_c] + (-1)^{|a||b|}[R_{ca}, R_b] + (-1)^{|a||c|}[R_{bc}, R_a]$

(JSAM3) $R_cR_bR_a + (-1)^{|a||b|+|a||c|+|b||c|}R_aR_bR_c + (-1)^{|b||c|}R_{(ac)b} =$

$$(-1)^{(|a|+|b|)|c|}R_{ab}R_c + (-1)^{|a||b|}R_{ac}R_b + (-1)^{|b||c|}R_{cb}R_a$$

for all $a, b, c \in J$, $m \in \mathcal{M}$ where R_d denotes the action on the right of J on X, i.e., $R_dx = xd$.

2 ORTHOSYMPLECTIC LIE SUPERALGEBRAS

We will denote by A a superextension of K and by $\mathcal{M}$ an A-supermodule with an A-quadratic form q .

Set

$$\mathrm{osp}(q) = \{x \in \mathrm{End}_A \mathcal{M} \mid q(x(m), n) + (-1)^{|m||n|}q(x(n), m) = 0 \text{ for all } m, n \in \mathcal{M}\}.$$

We define, for homogeneous $m, n, p \in \mathcal{M}$,

$$\mathbf{E}_{m,n}(p) \quad = \quad mq(n, p) - (-1)^{|n||p|}q(m, p)n. \tag{2.1}$$

and we extend $\mathbf{E} : \mathcal{M} \times \mathcal{M} \to \mathrm{End}_A \mathcal{M}$ linearly. Let

$$\mathrm{eosp}(q) = \mathrm{span}_{\mathbb{Z}}\{\mathbf{E}_{m,n} \mid m, n \in \mathcal{M}\}.$$

We call $\mathrm{osp}(q)$ the *orthosymplectic Lie superalgebra (of q)* and $\mathrm{eosp}(q)$ the *elementary orthosymplectic Lie superalgebra (of q).*

The techniques developed in [N2] to describe root-graded Lie algebras can be adapted to the supercase, see [N2, Section 5.4] and [GN]. In particular, let I be an arbitrary set and q_I the quadratic form on $H(I; K)$ defined as in Example 1.1.

(B_I) Assume that $\frac{1}{6} \in K$ and $|I| \geq 3$. A Lie superalgebra L over K is B_I-graded if and only if there exists a superextension A of K, an A-supermodule $\mathcal{M}$ with an A-quadratic form q and an element $m \in \mathcal{M}$ satisfying $q(m, m) = 1$ such that L is a central extension of the Lie superalgebra $\mathrm{eosp}(q_I \oplus q)$.

(D_I) Assume that $\frac{1}{6} \in K$ and $|I| \geq 4$. A Lie superalgebra L over K is D_I-graded if and only if there exists a superextension A of K such that L is a central extension of the Lie superalgebra $\mathrm{eosp}(q_I)$.

Example 2.1 Let K be a commutative and unital ring containing $\frac{1}{6}$. We consider K as a superring with $K_{\bar{1}} = (0)$. Let I and J be arbitrary index sets ($|I| \geq 3$ and $J \neq \emptyset$) and let $\mathcal{M} = \oplus_{j\in J} Km_j$ be a free K-supermodule with a basis consisting of odd elements and with a nonzero K-quadratic form $q_{\mathcal{M}}$. Consider $L = \mathrm{eosp}(q_I \oplus q_{\mathcal{M}} \oplus q_0)$ where q_0 is the quadratic form on the free module Kx_0 given by $q_0(x_0, x_0) = 1$. Then by *(B_I)*, L is a B_I-graded Lie superalgebra over K. It can be shown in particular that $\mathrm{eosp}(q_{\mathcal{M}})(\hookrightarrow \mathrm{eosp}(q_I \oplus q_{\mathcal{M}} \oplus q_0))$ is contained in $(L^0)_{\bar{0}}$ but that it is not generated by $(L^\alpha)_{\bar{0}},\ \alpha \in B_I$. Hence $L_{\bar{0}}$ is not a B_I-graded Lie algebra.

If $\underline{x} = (x_i)_{i\in I}$ is a family in $\mathrm{eosp}(q)$ we say that $\underline{x}$ is *summable* if for every $m \in \mathcal{M},\ (x_i(m))_{i\in I}$ has only finitely many nonzero terms, and in this case we define $x = \sum_{i\in I} x_i \in \mathrm{End}_A \mathcal{M}$ by setting

$$\left(\sum_{i\in I} x_i\right)(m) = \sum_{i\in I_m} x_i(m)$$

for $I_m = \{i \in I \mid x_i(m) \neq 0\}$ (I_m is finite). We will say that $\sum_{i\in I} x_i$ is summable to indicate that $(x_i)_{i\in I}$ is summable. Also, given a summable $x = \sum_{i\in I} x_i$ we will throughout denote by I_m the finite subset of I such that for $i \in I,\ x_i(m) \neq 0$. The set of all summable $\sum_{i\in I} x_i,\ x_i \in \mathrm{eosp}(q)$ is denoted by $\overline{\mathrm{eosp}(q)}$ and is an A-submodule of $\mathrm{End}_A \mathcal{M}$ with the natural $\mathbb{Z}_2$-grading. Summable homomorphisms are discussed in a more general setting in [D]. In this section we will investigate some of the properties and the structures of $\mathrm{eosp}(q)$, $\overline{\mathrm{eosp}(q)}$ and $\mathrm{osp}(q)$.

Lemma 2.2 *Let $\mathcal{M}$ be an A-supermodule with an A-quadratic form q. Then*

1) *for all $m, n \in \mathcal{M}$ and for all $a \in A$, $\mathbf{E}_{m,n} \in \mathrm{osp}(q)$. Moreover, the map $\mathbf{E} : \mathcal{M} \times \mathcal{M} \to \mathrm{End}_A \mathcal{M} : (m, n) \mapsto \mathbf{E}_{m,n}$ is A-bilinear,*

$$\mathbf{E}_{m,n} = -(-1)^{|m||n|}\mathbf{E}_{n,m};\ \textit{and}$$

$$q(\mathbf{E}_{m,n}(p), r) = (-1)^{(|m|+|n|)(|p|+|r|)} q(\mathbf{E}_{p,r}(m), n)$$

for all $m, n, p, r \in \mathcal{M}$;

2) $\mathrm{osp}(q)$ *is a subalgebra of the Lie superalgebra* $(\mathrm{End}_A \mathcal{M})^{(-)}$*;*

3) *for all $m, n \in \mathcal{M}$ and $x \in \mathrm{osp}(q)$,*

$$[x, \mathbf{E}_{m,n}] = \mathbf{E}_{x(m),n} + (-1)^{|x||m|}\mathbf{E}_{m,x(n)};\ \textit{and hence} \qquad (2.2)$$

4) $\mathrm{eosp}(q)$ *and* $\overline{\mathrm{eosp}(q)}$ *are ideals of* $\mathrm{osp}(q)$.

5) *If A is a field of characteristic 0, q is nondegenerate and $\mathcal{M}$ is a free A-module of finite rank, then* $\mathrm{osp}(q) = \overline{\mathrm{eosp}(q)} = \mathrm{eosp}(q)$.

The easy proof of this lemma is left to the reader. However, we note that to prove that $\overline{\mathrm{eosp}(q)}$ is an ideal of $\mathrm{osp}(q)$ one uses the formula

$$\left[\sum_{i\in I} x_i, y\right] = \sum_{i\in I}[x_i, y] \in \overline{\mathrm{eosp}(q)}, \qquad \sum_{i\in I} x_i \in \overline{\mathrm{eosp}(q)},\ y \in X \tag{2.3}$$

for any subalgebra X of $\mathrm{End}_K \mathcal{M}$ of which $\mathrm{eosp}(q)$ is an ideal.

In what follows we will denote by $\mathcal{E}$ a subalgebra of $\mathrm{osp}(q)$ containing $\mathrm{eosp}(q)$. We set

$$\mathcal{E}_\infty = \{(a, m, n, x) \mid a \in A,\ m, n \in \mathcal{M},\ x \in \mathcal{E}\}$$

to be the Lie superalgebra with multiplication

$$\begin{aligned} &[(a, m, n, x), (a', m', n', x')] = (q(m, n') - q(n, m'), \\ &am' + xm' - ma' - (-1)^{|x'||m|}x'm, -an' + xn' + na' - (-1)^{|x'||n|}x'n, \\ &\mathbf{E}_{m,n'} + \mathbf{E}_{n,m'} + [x, x']) \end{aligned} \tag{2.4}$$

for all $a, a' \in A,\ m, m', n, n' \in \mathcal{M},\ x, x' \in \mathcal{E}$. Then $L = \mathcal{E}_\infty = L_+ \oplus L_0 \oplus L_-$ is a 3-graded Lie superalgebra (i.e., $L_\sigma L_\mu \subset L_{\sigma+\mu},\ \sigma, \mu \in \{\pm, 0\}$) where

$$\begin{aligned} L_0 &= \{(a, 0, 0, x) \mid a \in A,\ x \in \mathrm{eosp}(q)\} \\ L_+ &= \{(0, m, 0, 0) \mid m \in \mathcal{M}\} \\ L_- &= \{(0, 0, m, 0) \mid m \in \mathcal{M}\}. \end{aligned}$$

If we set $q_\infty := q_{\{\infty\}} \oplus q$ where $q_{\{\infty\}}$ is the A-quadratic form on the hyperbolic superplane $H(\{\infty\}, A)$ then $\mathrm{eosp}(q)_\infty = \mathrm{eosp}(q_\infty)$, $\mathrm{osp}(q)_\infty = \mathrm{osp}(q_\infty)$ and so $\mathrm{eosp}(q_\infty) \subset \mathcal{E}_\infty \subset \mathrm{osp}(q_\infty)$. In particular, we see from (B_I) and (D_I) that B_I-graded and D_I-graded Lie superalgebras ($|I| \geq 3$ and $|I| \geq 4$ respectively) over K containing $\frac{1}{6}$ are central extensions of $\mathrm{eosp}(q)_\infty$ for a suitable superextension A of K, an A-supermodule $\mathcal{M}$ and an A-quadratic form q.

Let $\mathcal{M}$ be a K-supermodule with a K-quadratic form q. Let $J = K \oplus \mathcal{M}$ be the Jordan superalgebra associated with the quadratic form q (see Example 1.4). Consider the action of J on the K-supermodule $X = \mathcal{E} \oplus \mathcal{M}$ given by

$$\begin{aligned} (x \oplus p)(a \oplus m) &= (xa + \mathbf{E}_{p,m}) \oplus (x(m) + pa) \\ &= (-1)^{|a\oplus m||x\oplus p|}(a \oplus m)(x \oplus p) \end{aligned} \tag{2.5}$$

for $a \in K$, $m, p \in \mathcal{M}$ and $x \in \mathcal{E}$. Then one can easily check that (JSAM1)-(JSAM3) hold and so X is a Jordan superalgebra module for J.

The following can be easily shown and will be used in the next section:

Proposition 2.3 *Let $\mathcal{M}$ be an A-supermodule with an A-quadratic form q. Let $L = \mathcal{E}_\infty$. Then $Z(L) = (\mathrm{Ann}_A \mathcal{M}, 0, 0, 0)$ where $Ann_A\mathcal{M} = \{a \in A \mid am = 0$ for all $m \in \mathcal{M}\}$.*

3 DERIVATIONS

In this section we will study the derivations of $\mathcal{E}_\infty$ where $\mathcal{E}$ is a subalgebra of $\mathrm{osp}(q)$ containing $\mathrm{eosp}(q)$ for an A-supermodule $\mathcal{M}$ with an A-quadratic form q. We will denote $\mathcal{E}_\infty$ by L throughout. Recall that $L = \{(a,m,n,x) \mid a \in A,\ m,n \in \mathcal{M},\ x \in \mathcal{E}\}$ with multiplication given by (2.4).

Proposition 3.1 *We have that*

$$\mathrm{Der}_K L = \mathbf{ad}\, L + \{d \in \mathrm{Der}_K L \mid d(1,0,0,0) \in Z(L)\}.$$

Moreover,

$$\begin{aligned}(\mathrm{Der}_K L)_0 &:= \{d \in \mathrm{Der}_K L \mid d(L_\sigma) \in L_\sigma, \sigma = 0, \pm\} \\ &= \{d \in \mathrm{Der}_K L \mid d(1,0,0,0) \in Z(L)\}.\end{aligned}$$

In particular, $\{d \in \mathrm{Der}_K L \mid d(1,0,0,0) \in Z(L)\}$ *is a subalgebra of* $\mathrm{Der}_K L$ *and*

$$\mathbf{ad}\, L \cap \{d \in \mathrm{Der}_K L \mid d(1,0,0,0) = 0\} = \mathbf{ad}\, L_0.$$

Proof: It is clear that $\mathbf{ad}\, L + (\mathrm{Der}_K L)_0 \subset \mathrm{Der}_K L$. For $x \in L$ we will write $x = x_- + x_0 + x_+$ where $x_\mu \in L_\mu,\ \mu = 0, \pm$ and we will denote $(1,0,0,0)$ simply by 1. Let $d \in \mathrm{Der}_K L$ and set $d' = d + \mathbf{ad}(d(1)_+ - d(1)_-)$. Then for all $x \in L$

$$\begin{aligned} d'(x_+) &= d(x_+) + [d(1)_+ - d(1)_-, x_+] = d[1, x_+] - [d(1)_-, x_+] \\ &= [d(1), x_+] + [1, d(x_+)] - [d(1)_-, x_+] \\ &= \underbrace{[(d(1))_0, x_+]}_{\in L_+} + \underbrace{[1, d(x_+)_+]}_{\in L_+} \in L_+ \end{aligned}$$

since

$$d(x_+)_- = (d[1, x_+])_- = \underbrace{[d(1), x_+]_-}_{=0} + [1, d(x_+)]_- = -d(x_+)_-$$

and so $d(x_+)_- = 0$. Similarly, $d'(x_-) \in L_-$. In addition, $[1, x_0] = 0$ implies $0 = [d(1), x_0] + [1, d(x_0)]$ which in turn implies $d(x_0)_\pm = \mp[d(1)_\pm, x_0]$ and hence $d'(x_0) = d(x_0)_0 \in L_0$. Therefore $\mathrm{Der}_K L = \mathbf{ad}\, L + (\mathrm{Der}_K L)_0$. Now, let $d \in (\mathrm{Der}_K L)_0$ and $x \in L$. Then

$$d(x_+) = d[1, x_+] = [d(1), x_+] + [1, \underbrace{d(x_+)}_{\in L_+}] = [d(1), x_+] + d(x_+)$$

which implies that $[d(1), x_+] = 0$. Similarly, $[d(1), x_-] = 0$. Moreover, since $[1, x_0] = 0$, we have $0 = [d(1), x_0] + [1, d(x_0)]$ and so since $d(x_0) \in L_0$, $[d(1), x_0] = 0$. Hence $(\mathrm{Der}_K L)_0 \subset \{d \in \mathrm{Der}_K L \mid d(1) \in Z(L)\}$. Conversely, if $d \in \mathrm{Der}_K L$ is such that $d(1) \in Z(L)$ then for all $x \in L$, $d(x_\pm) = \pm d[1, x_\pm]$ implies $d(L_\pm) \subset L_\pm$ and $[1, x_0] = 0$ implies $[1, d(x_0)] = -[d(1), x_0] = 0$ which in turn implies $d(L_0) \subset L_0$. Therefore $(\mathrm{Der}_K L)_0 = \{d \in \mathrm{Der}_K L \mid d(1) \in Z(L)\}$. It is easily verified that $(\mathrm{Der}_K L)_0$ is a subalgebra of $\mathrm{Der}_K L$ and so all of the statements of the proposition hold. ■

We remark that the proposition above holds for so called *toral 3-graded Lie superalgebras*, i.e., for the Lie superalgebras of the form $L = L_{-1} \oplus L_0 \oplus L_1$ with $L_\sigma L_\mu \subset L_{\sigma+\mu}$ for $\sigma, \mu \in \{0, \pm 1\}$ and such that there exists $\zeta \in L_0$ satisfying $[\zeta, x_\sigma] = \sigma x_\sigma$ for all $x_\sigma \in L_\sigma,\ \sigma \in \{0, \pm\}$.

In what follows, we will assume that $\operatorname{Ann}_A \mathcal{M} = 0$ and so by Proposition 2.3, $Z(L) = 0$. In addition we have the superalgebra monomorphism $A \to \operatorname{End}_A \mathcal{M} :$ $a \mapsto a \cdot Id|_{\mathcal{M}}$.

We make the following definitions which will be instrumental in describing the derivations of $\mathcal{E}_\infty$.

Definition 3.2 For each $\alpha \in \mathbb{Z}_2$, let $(\mathcal{S}_\mathcal{E})_\alpha$ be the set of all K-linear endomorphisms S of $\mathcal{M}$ of degree α satisfying

(S1) $[S, A \cdot Id] \subset A \cdot Id$;

(S2) $[S, q(m,n) \cdot Id] = (q(S(m), n) + (-1)^{|m||n|} q(S(n), m)) \cdot Id$ for all homogeneous $m, n \in \mathcal{M}$;

(S3) $[S, \mathcal{E}] \subset \mathcal{E}$.

Definition 3.3 For each $\alpha \in \mathbb{Z}_2$, let $(\mathcal{T}_\mathcal{E})_\alpha$ be the set of all K-linear endomorphisms T of $\mathcal{M}$ of degree α satisfying

(T1) $[T, A \cdot Id] \subset \mathcal{E}$;

(T2) $[T, q(m,n) \cdot Id] = \mathbf{E}_{T(m),n} + (-1)^{|m||n|} \mathbf{E}_{T(n),m}$ for all homogeneous $m, n \in \mathcal{M}$;

(T3) $[T, \mathcal{E}] \subset A \cdot Id$.

Set $\mathcal{S}_\mathcal{E} = (\mathcal{S}_\mathcal{E})_{\bar{0}} \oplus (\mathcal{S}_\mathcal{E})_{\bar{1}}$ and $\mathcal{T}_\mathcal{E} = (\mathcal{T}_\mathcal{E})_{\bar{0}} \oplus (\mathcal{T}_\mathcal{E})_{\bar{1}}$. We see immediately that $\mathcal{S}_\mathcal{E}$ and $\mathcal{T}_\mathcal{E}$ are submodules of the K-supermodule $\operatorname{End}_K(\mathcal{M})$. Moreover, $\mathcal{E} \subset \mathcal{S}_\mathcal{E}$ and $A \cdot Id \subset \mathcal{T}_\mathcal{E}$. We will write $\mathcal{S}$ and $\mathcal{T}$ for $\mathcal{S}_\mathcal{E}$ and $\mathcal{T}_\mathcal{E}$ respectively when the context is clear. In fact, if we define, for $N \in \{1, 2, 3\}$ and $\alpha \in \mathbb{Z}_2$,

$$\mathcal{S}_\alpha^{(N)} = \{S \in (\operatorname{End}_K \mathcal{M})_\alpha \mid S \text{ satisfies (SN)}\}$$

and

$$\mathcal{T}_\alpha^{(N)} = \{T \in (\operatorname{End}_K \mathcal{M})_\alpha \mid T \text{ satisfies (TN)}\}$$

and set $\mathcal{S}^{(N)} = \mathcal{S}_{\bar{0}}^{(N)} \oplus \mathcal{S}_{\bar{1}}^{(N)},\ \mathcal{T}^{(N)} = \mathcal{T}_{\bar{0}}^{(N)} \oplus \mathcal{T}_{\bar{1}}^{(N)}$ then

$$\mathcal{S}_\mathcal{E} = \bigcap_{N=1}^{3} \mathcal{S}^{(N)} \quad \text{and} \quad \mathcal{T}_\mathcal{E} = \bigcap_{N=1}^{3} \mathcal{T}^{(N)}.$$

Remark 3.4 Note that the $(\ )^{(N)}$ notation is traditionally used to denote the derived algebras of a Lie algebra. However in what follows we will not be using derived algebras at all so there should be no confusion resulting from the above definition.

We have the following properties of $\mathcal{S}$ and $\mathcal{T}$:

Proposition 3.5

(1) Properties of $\mathcal{S}_{\mathcal{E}}$:

(i) For all $S \in \mathcal{S}^{(1)}$, the map $\Delta : A \to A$ given by $\Delta(a)\cdot Id = [S, a\cdot Id]$ is a K-derivation of A. Moreover, $[\operatorname{End}_A \mathcal{M}, S] \subset \operatorname{End}_A \mathcal{M}$.

(ii) For all $S \in \mathcal{S}^{(2)}$ and for all $m, n \in \mathcal{M}$,

$$[S, \mathbf{E}_{m,n}] = \mathbf{E}_{S(m),n} - (-1)^{|m||n|}\mathbf{E}_{S(n),m}. \tag{3.1}$$

(iii) Let $X = \operatorname{eosp}(q)$, $\overline{\operatorname{eosp}(q)}$ or $\operatorname{osp}(q)$. Then $[\mathcal{S}^{(2)}, X] \subset X$. In particular, if $\mathcal{E} = \operatorname{eosp}(q)$, $\overline{\operatorname{eosp}(q)}$ or $\operatorname{osp}(q)$ then $\mathcal{S}^{(2)} \subset \mathcal{S}^{(3)}$.

(iv) $\operatorname{osp}(q)$ is an ideal of $\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$ and so

$$\operatorname{osp}(q) \cap \mathcal{S}^{(3)} \lhd \mathcal{S}_{\mathcal{E}}. \tag{3.2}$$

In particular, $\mathcal{E} \lhd \mathcal{S}_{\mathcal{E}}$. Moreover, $\operatorname{osp}(q) \lhd \mathcal{S}_{\mathcal{E}}$ for any ideal $\mathcal{E}$ of $\operatorname{osp}(q)$ which is true in particular for $\mathcal{E} = \operatorname{eosp}(q)$, $\overline{\operatorname{eosp}(q)}$ and $\operatorname{osp}(q)$.

(2) Properties of $\mathcal{T}_{\mathcal{E}}$:

(i) For all $T \in \mathcal{T}^{(1)}$, the map $\varphi : A \to \mathcal{E}$ given by $\varphi(a) = [T, a\cdot Id]$ is a K-derivation from A to $\mathcal{E}$.

(ii) For all $T \in \mathcal{T}^{(2)}$ and for all $m, n \in \mathcal{M}$,

$$[T, \mathbf{E}_{m,n}] = (q(T(m), n) - (-1)^{|m||n|}q(T(n), m))\cdot Id. \tag{3.3}$$

(iii) Suppose that $\mathcal{E} \subset \overline{\operatorname{eosp}(q)}$ and $\operatorname{Ann}_A(m) = 0$ for some $m \in \mathcal{M}$ if $\mathcal{E} \neq \operatorname{eosp}(q)$. Then $\mathcal{T}^{(2)} \subset \mathcal{T}^{(3)}$.

(iv) $[\mathcal{E}, [\mathcal{T}^{(3)}, A \cdot Id]] = (0)$ and $[\mathcal{T}^{(3)}, [\mathcal{E}, \mathcal{E}]] = (0)$.

Remark 3.6 We should note that the assertion (1iii) in the above proposition does not hold for general $\operatorname{eosp}(q) \subset \mathcal{E} \subset \operatorname{osp}(q)$, for an example see Corollary 4.10. Also, a slightly more general version of (2iii) is proved in [D].

Proof: We leave it to the reader to check the above statements. They follow from the properties (S1)-(S3) and (T1)-(T3), except for the following two cases: in (1iii) one needs to verify that $[\mathcal{S}^{(2)}, \operatorname{eosp}(q)] \subset \operatorname{eosp}(q)$ implies $[\mathcal{S}^{(2)}, \overline{\operatorname{eosp}(q)}] \subset \overline{\operatorname{eosp}(q)}$ for which one uses (2.3), and in (2iii) one needs to check that for $T \in \mathcal{T}^{(2)}$ and $x \in \overline{\operatorname{eosp}(q)}$, $[T, x] = \sum_i a_i \cdot Id$ for some index set I where for each $n \in \mathcal{M}$

there are only finitely many $i \in I$ such that $a_i n \neq 0$; this, together with the fact that $\mathrm{Ann}_A(m) = 0$ for some $m \in \mathcal{M}$, implies that in fact only finitely many $a_i,\ i \in I$ are nonzero and so $[\mathcal{T}^{(2)}, \overline{\mathrm{eosp}(q)}] \subset A \cdot Id$. ■

In fact, $\mathcal{S}$ and $\mathcal{T}$ give rise to a Lie superalgebra structure in the following way:

Proposition 3.7 *The K-supermodule $\mathcal{S} \oplus \mathcal{T} \subset \mathrm{End}_K \mathcal{M} \oplus \mathrm{End}_K \mathcal{M}$ is a Lie superalgebra with multiplication*

$$[S \oplus T, S' \oplus T']^{\tilde{}} = ([S, S'] + [T, T']) \oplus ([S, T'] + [T, S']) \tag{3.4}$$

where $[\ ,\ ]$ is the usual bracket multiplication in $(\mathrm{End}_K \mathcal{M})^{(-)}$.

Proof: By Proposition 3.5, the bracket $[\ ,\ ]^{\tilde{}}$ is well-defined. It is easy to check that $\mathcal{S} \oplus \mathcal{T} = (\mathcal{S}_{\bar{0}} \oplus \mathcal{T}_{\bar{0}}) \oplus (\mathcal{S}_{\bar{1}} \oplus \mathcal{T}_{\bar{1}})$ defines a $\mathbb{Z}_2$-grading which gives $\mathcal{S} \oplus \mathcal{T}$ a K-superalgebra structure under the multiplication given by (3.4). The fact that $(\mathrm{End}_K \mathcal{M})^{(-)}$ is a Lie superalgebra now implies the assertion. ■

Remark 3.8 It is important to note that $\mathcal{S} \oplus \mathcal{T}$ does not in general naturally imbed in $\mathrm{End}_K \mathcal{M}$. For example, if $q \equiv 0$ then $\mathrm{End}_A \mathcal{M} \subset \mathcal{S} \cap \mathcal{T}$ and hence $\mathcal{S} \cap \mathcal{T} \neq 0$.

Definition 3.9 Let $\mathcal{D}_{\mathcal{M}}$ be the set of all K-endomorphisms $d_{S,T}$ of L for which there exist $S \in \mathcal{S},\ T \in \mathcal{T}$ satisfying

$$\begin{aligned} d_{S,T}(a, m, n, x) \quad &= \quad ([S, a \cdot Id] + [T, x], (S + T)(m), \\ &\qquad (S - T)(n), [T, a \cdot Id] + [S, x]). \end{aligned} \tag{3.5}$$

Remark 3.10 Note that for every $d \in \mathcal{D}_{\mathcal{M}}$, $d = d_{S,T}$ for unique $S \in \mathcal{S},\ T \in \mathcal{T}$.

We will often use the following notation:

$$\begin{array}{lcll} a & := & (a, 0, 0, 0) & \text{for all } a \in A; \\ m^+ & := & (0, m, 0, 0) & \text{for all } m \in \mathcal{M}; \\ m^- & := & (0, 0, m, 0) & \text{for all } m \in \mathcal{M}; \\ x & := & (0, 0, 0, x) & \text{for all } x \in \mathrm{eosp}(q); \end{array} \tag{3.6}$$

the meaning will be clear from the context.

Theorem 3.11
1) We have $\mathcal{D}_{\mathcal{M}} = (\mathrm{Der}_K L)_0$. In particular, $\mathcal{D}_{\mathcal{M}}$ is a subalgebra of $\mathrm{Der}_K L$ and the map $\phi : \mathcal{S} \oplus \mathcal{T} \to \mathcal{D}_{\mathcal{M}} : S \oplus T \mapsto d_{S,T}$ is a Lie superalgebra isomorphism.
2) Moreover, $\mathrm{Der}_K L = \mathbf{ad}\, L + \mathcal{D}_{\mathcal{M}}$ and
3) we have that $\mathbf{ad}\, L \cap \mathcal{D}_{\mathcal{M}} = \{d_{S,T} \in \mathcal{D}_{\mathcal{M}} \mid S \in \mathcal{E},\ T \in A \cdot Id\} = \mathbf{ad}\, L_0$.
4) If $\mathcal{S} = \mathcal{E} \oplus \mathcal{S}_0$ and $\mathcal{T} = A \cdot Id \oplus \mathcal{T}_0$ where $\mathcal{S}_0 \oplus \mathcal{T}_0$ is a subalgebra of $\mathcal{S} \oplus \mathcal{T}$, then

$$\mathrm{Der}_K L = \mathbf{ad}\, L \rtimes \mathcal{D}_{\mathcal{S}_0, \mathcal{T}_0}.$$

Proof:

1) It is clear that $\mathcal{D}_{\mathcal{M}} \subset (\mathrm{Der}_K L)_0$. On the other hand, let $d = d_{\bar{0}} \oplus d_{\bar{1}} \in (\mathrm{Der}_K L)_0$ and so there exist $M, N \in \mathrm{End}_K \mathcal{M}$, $\varphi_A \in \mathrm{End}_K A$, $\varphi_{\mathcal{E}} \in \mathrm{Hom}_K(A, \mathcal{E})$ and $\psi_A \in \mathrm{Hom}_K(\mathcal{E}, A)$, $\psi_{\mathcal{E}} \in \mathrm{End}_K \mathcal{E}$ such that

$$d(a,m,n,x) = (\varphi_A(a) + \psi_A(x), M(m), N(n), \varphi_{\mathcal{E}}(a) + \psi_{\mathcal{E}}(x)).$$

Since d is a derivation, for all $a, b \in A$ and $x, y \in \mathcal{E}$ we have

$$\begin{aligned}(\psi_A([x,y]),&0,0,\psi_{\mathcal{E}}([x,y])) = d([(a,0,0,x),(b,0,0,y)]\\ &= [(\varphi_A(a)+\psi_A(x),0,0,\varphi_{\mathcal{E}}(a)+\psi_{\mathcal{E}}(x)),(b,0,0,y)]\\ &\quad + (-1)^{|(a,0,0,x)||(b,0,0,y)|}[(a,0,0,x),(\varphi_A(b)+\psi_A(y),0,0,\varphi_{\mathcal{E}}(b)+\psi_{\mathcal{E}}(y))\\ &= (0,0,0,[\varphi_{\mathcal{E}}(a)+\psi_{\mathcal{E}}(x),y] + (-1)^{|(a,0,0,x)||(b,0,0,y)|}[x,\varphi_{\mathcal{E}}(b)+\psi_{\mathcal{E}}(y)])\end{aligned}$$

which implies that

$$\psi_A[x,y] = 0,\ \varphi_{\mathcal{E}}(A) \subset Z(\mathcal{E}) \text{ and } \psi_{\mathcal{E}}(\mathcal{E}) \subset \mathrm{Der}_K \mathcal{E}.$$

Moreover, for all $a \in A$, $x \in \mathcal{E}$ and $m \in \mathcal{M}$,

$$\begin{aligned}(0, M(am+xm),0,0) = d(0,am+xm,0,0) &= d([(a,0,0,x),(0,m,0,0)])\\ &= [(\varphi_A(a)+\psi_A(x),0,0,\varphi_{\mathcal{E}}(a)+\psi_{\mathcal{E}}(x),(0,m,0,0)]\\ &\quad + (-1)^{|d||a|}[(a,0,0,x),(0,M(m),0,0)]\\ &= (0,(\varphi_A(a)+\psi_A(x)+\varphi_{\mathcal{E}}(a)+\psi_{\mathcal{E}}(x))(m),0,0)\\ &\quad + (-1)^{|d||a+x|}(0,aM(m)+xM(m),0,0).\end{aligned}$$

Hence

$$[M, a\cdot Id] = \varphi_A(a)\cdot Id + \varphi_{\mathcal{E}}(a) \text{ and } [M,x] = \psi_A(x)\cdot Id + \psi_{\mathcal{E}}(x)$$

for all $a \in A$ and $x \in \mathcal{E}$. Similarly,

$$[N, a\cdot Id] = -\varphi_A(a)\cdot Id + \varphi_{\mathcal{E}}(a) \text{ and } [N,x] = -\psi_A(x)\cdot Id + \psi_{\mathcal{E}}(x)$$

for all $a \in A$ and $x \in \mathcal{E}$. Let $S, T \in \mathrm{End}_K \mathcal{M}$ be given by

$$\begin{aligned} S &= \frac{1}{2}(M+N);\\ T &= \frac{1}{2}(M-N).\end{aligned}$$

Then $[S, a\cdot Id] = \varphi_A(a)\cdot Id \in A\cdot Id$ and $[T, a\cdot Id] = \varphi_{\mathcal{E}}(a) \in \mathcal{E}$. Moreover, $[S,x] = \psi_{\mathcal{E}}(x) \in \mathcal{E}$ and $[T,x] = \psi_A(x)\cdot Id \in A\cdot Id$. Hence S satisfies (S1) and (S3) and T satisfies (T1) and (T3). Moreover, we have that

$$d(a,m,n,x) = ([S,a\cdot Id] + [T,x], (S+T)m, (S-T)n, [S,x] + [T,a\cdot Id])$$

and so using the fact that $\mathbf{E}_{m,n} = [m^+, n^-] - q(m,n) = -(-1)^{|m||n|}\mathbf{E}_{n,m}$, we can show that

$$\begin{aligned}(q(S(m),n) + (-1)^{|m||n|}q(S(n),m))\cdot Id &= [S, q(m,n)\cdot Id]; \text{ and}\\ \mathbf{E}_{T(m),n} + (-1)^{|m||n|}\mathbf{E}_{T(n),m} &= [T, q(m,n)\cdot Id])\end{aligned}$$

and so S and T satisfy (S2) and (T2) respectively. Hence $S \oplus T \in \mathcal{S} \oplus \mathcal{T}$ and so $d \in \mathcal{D}_{\mathcal{E}}$. Therefore we have $\mathcal{D}_{\mathcal{E}} = (\mathrm{Der}_K L)_0$ and so $\mathcal{D}_{\mathcal{E}}$ is a subalgebra of $\mathrm{Der}_K \mathcal{E}_\infty$. and so by Proposition 3.1, $\mathcal{D}_{\mathcal{E}}$ is a subalgebra of $\mathrm{Der}_K L$. Let $\phi : \mathcal{S} \oplus \mathcal{T} \longrightarrow \mathcal{D}_{\mathcal{E}}$ be defined by $\phi(S \oplus T) = d_{S,T}$. Clearly this map is well defined and is K-linear since S and T are K-linear. Injectivity follows from Remark 3.10 and surjectivity from the definition of $\mathcal{D}_{\mathcal{E}}$. It is easy to check that ϕ is a Lie superalgebra homomorphism and hence isomorphism.
2) Follows from Proposition 3.1 and 1).
3) It is straightforward to verify that $d_{\mathcal{E},AId} \subset \mathbf{ad}\, L$. Conversely, let $d \in \mathbf{ad}\, L \cap \mathcal{D}_{\mathcal{E}}$. Then $d = d_{S,T} = \mathbf{ad}(a_0, m_0, n_0, x_0)$ for some $S \in \mathcal{S},\ T \in \mathcal{T},\ (a_0, m_0, n_0, x_0) \in L$. Then using (3.5) and the fact that $\frac{1}{2} \in K$, we have that $S = x_0 \in \mathcal{E}$ and $T = a_0\cdot Id \in A\cdot Id$. Hence the assertion follows.
4) Assume $\mathcal{S} = \mathcal{E} \oplus \mathcal{S}_0$ and $\mathcal{T} = A\cdot Id \oplus \mathcal{T}_0$ where $\mathcal{S}_0 \oplus \mathcal{T}_0$ is a subalgebra of $\mathcal{S} \oplus \mathcal{T}$. Then by Proposition 3.7, $\mathcal{S} \oplus \mathcal{T} = (\mathcal{E} \oplus A\cdot Id) \rtimes (\mathcal{S}_0 \oplus \mathcal{T}_0)$. Hence by 1), $\{d_{S,T} \mid S \in \mathcal{S}_0,\ T \in \mathcal{T}_0\}$ is a subalgebra of $\mathcal{D}_{\mathcal{E}}$ and

$$\begin{aligned}\mathcal{D}_{\mathcal{E}} &= \{d_{S,T} \mid S \in \mathcal{E},\ T \in A\cdot Id\} \rtimes \{d_{S,T} \mid S \in \mathcal{S}_0,\ T \in \mathcal{T}_0\}\\ &= \mathbf{ad}(L_0) \rtimes \{d_{S,T} \mid S \in \mathcal{S}_0,\ T \in \mathcal{T}_0\}.\end{aligned}$$

By 3), we have $\mathbf{ad}\, L \cap \mathcal{D}_{\mathcal{E}} = \mathbf{ad}(L_0)$. Hence by 2),

$$\mathrm{Der}_K L = \mathbf{ad}\, L + \mathcal{D}_{\mathcal{E}} = \mathbf{ad}\, L \oplus \{d_{S,T} \mid S \in \mathcal{S}_0, T \in \mathcal{T}_0\}.$$

The assertion now follows from 1). ∎

In fact, the A-supermodules $\mathcal{S}$ and $\mathcal{T}$ can be described in the following terms: Recall the Jordan superalgebra $J = A \oplus \mathcal{M}$ of Example 1.4. Given a subalgebra $\mathcal{E}$ of $\mathrm{osp}(q)$ containing $\mathrm{eosp}(q)$, recall the J-supermodule $X = \mathcal{E} \oplus \mathcal{M}$ with action of J on X given by (2.5). Let

$$\begin{aligned}\mathrm{Der}_*(J) &= \{d \in \mathrm{Der}_K(J) \mid d(A) \subset A \text{ and } d(\mathcal{M}) \subset \mathcal{M}\}; \text{ and}\\ \mathrm{Der}_*(J,X) &= \{d \in \mathrm{Der}_K(J,X) \mid d(A) \subset \mathcal{E} \text{ and } d(\mathcal{M}) \subset \mathcal{M}\}.\end{aligned}$$

Proposition 3.12 *In the above setting we have that*

(1) the map $\mathrm{Der}_*(J) \to \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)} : d \mapsto d|_{\mathcal{M}}$ *is a Lie superalgebra isomorphism; and*

(2) the map $\mathrm{Der}_*(J,X) \to \mathcal{T}^{(1)} \cap \mathcal{T}^{(2)} : d \mapsto d|_{\mathcal{M}}$ *is a K-supermodule isomorphism.*

Proof: (1) Let $d \in \mathrm{End}_K \mathcal{M}$ such that $d(A) \subset A$ and $d(\mathcal{M}) \subset \mathcal{M}$. Then from the definition of $\mathrm{Der}(J)$ and Proposition 3.5.1.(i) it follows that $d \in \mathrm{Der}_*(J)$ if and only if $d|_{\mathcal{M}}$ satisfies (S1) and (S2).
(2) Let $d \in \mathrm{Hom}_K(A \oplus \mathcal{M}, \mathcal{E} \oplus \mathcal{M})$ such that $d(A) \subset \mathcal{E}$ and $d(\mathcal{M}) \subset \mathcal{M}$. Then from the definition of $\mathrm{Der}(J, X)$ and Proposition 3.5.2.(i) it follows that $d \in \mathrm{Der}_*(J, X)$ if and only if $d|_{\mathcal{M}}$ satisfies (T1) and (T2). ■

4 DERIVATIONS AS A SEMIDIRECT PRODUCT

Here we will present some examples of subalgebras of $\mathrm{osp}(q)$ containing $\mathrm{eosp}(q)$ where we get a direct splitting of the algebra of derivations into inner and outer derivations. We keep the setting of Section 3.

We let $A^\times$ to be the set of all invertible homogeneous elements of A. Note that, since $\frac{1}{2} \in A$, $A^\times \subset A_{\bar{0}}$.

Given a free A-supermodule $\mathcal{M} = \oplus_{i \in I} A m_i$ with a quadratic form q, we say that q is *almost diagonalizable* if for each $i \in I$ there exists $\underset{\sim}{i} \in I$ such that

$$\begin{aligned} &q(m_i, m_{\underset{\sim}{i}}) \in A^\times; \text{ and} \\ &q(m_i, m_j) = 0 \text{ for all } j \in I,\, j \neq \underset{\sim}{i}. \end{aligned}$$

If q is almost diagonalizable then for each $i \in I$, $\underset{\sim}{i}$ is unique, $\underset{\sim}{\underset{\sim}{i}} = i$ and $|m_i| = |m_{\underset{\sim}{i}}|$. In addition, if q is *almost diagonalizable* with respect to $\{m_i \mid i \in I\}$ where $i = \underset{\sim}{i}$ then we say that q is *invertibly diagonalizable*. Note that if q is invertibly diagonalizable with respect to $\{m_i \mid i \in I\}$ then the basis elements m_i are even.

Proposition 4.1 *Let $\mathcal{P}$ be an A-supermodule with an A-quadratic form $q_{\mathcal{P}}$ and an element $p \in \mathcal{P}$ such that $\mathrm{Ann}_A(p) = (0)$. Let $\mathcal{N} = \oplus_{i=\pm 1} A n_i$ be a free A-supermodule with an invertibly diagonalizable A-quadratic form $q_{\mathcal{N}}$. Set $\mathcal{M} = \mathcal{N} \oplus \mathcal{P}$ with $q = q_{\mathcal{N}} \oplus q_{\mathcal{P}}$ and let $\mathcal{E}$ be a subalgebra of $\mathrm{osp}(q)$ containing $\mathrm{eosp}(q)$. Then*

$$\mathcal{T}_{\mathcal{E}} = \mathcal{T}^{(2)} = A \cdot Id.$$

Proof: Recall that $A \cdot Id \subset \mathcal{T} \subset \mathcal{T}^{(2)}$. Let $T \in \mathcal{T}^{(2)}$ and write

$$T = \begin{pmatrix} T_{\mathcal{N},\mathcal{N}} & T_{\mathcal{P},\mathcal{N}} \\ T_{\mathcal{N},\mathcal{P}} & T_{\mathcal{P},\mathcal{P}} \end{pmatrix}$$

with respect to $\begin{pmatrix} \mathcal{N} \\ \mathcal{P} \end{pmatrix}$, so $T_{\mathcal{N},\mathcal{P}} \in \mathrm{Hom}_K(\mathcal{N}, \mathcal{P})$, etc. Let $p \in \mathcal{P}$ and let $i, k \in \{\pm 1\}$, $k \neq i$. Then by (T2),

$$\begin{aligned} 0 &= [T, q(an_i, p) \cdot Id](n_k) = \mathbf{E}_{T(an_i),p}(n_k) + (-1)^{|a||p|} \mathbf{E}_{T(p),an_i}(n_k) \\ &= -q(T(an_i), n_k)p - (-1)^{|a||p|} q(T(p), n_k) a n_i. \end{aligned} \tag{4.1}$$

Since $\mathcal{M} = \mathcal{N} \oplus \mathcal{P}$ and $q(\mathcal{N}, \mathcal{P}) = q(\mathcal{P}, \mathcal{N}) = 0$ we have that if $p \in \mathcal{P}$ is such that $\mathrm{Ann}_A(p) = (0)$, then

$$0 = q(T(an_i), n_k) = q_{\mathcal{N}}(T_{\mathcal{N},\mathcal{N}}(an_i), n_k) \text{ for all } i, k \in I,\ k \neq i.$$

We write $T_{\mathcal{N},\mathcal{N}}(an_i) = t_{i,1}(a)n_1 + t_{i,-1}(a)n_{-1}$ where $t_i \in \mathrm{End}_K A$ for $i \in \{\pm 1\}$ and we obtain

$$0 = t_{i,k}(a)q(n_k, n_k).$$

Since $q(n_k, n_k) \in A^\times$, we therefore have that $t_{i,k}(a) = 0$ for $i \neq k$ and so $T_{\mathcal{N},\mathcal{N}}(an_i) = t_{ii}(a)n_i$. Moreover, since $p \in \mathcal{P}$ and $i \in \{\pm 1\}$ were arbitrary in (4.1) we have that

$$0 = q(T(p), n) = q_{\mathcal{N}}(T_{\mathcal{P},\mathcal{N}}(p), n) \text{ for all } n \in \mathcal{N},\ p \in \mathcal{P}$$

and so since $q_{\mathcal{N}}$ is nondegenerate, we have that $T_{\mathcal{P},\mathcal{N}} = 0$. Now, let $i \in \{\pm 1\}$. Then for all $p \in \mathcal{P}$,

$$\begin{aligned} 0 &= [T, q(n_i, p)\cdot Id](n_i) = \mathbf{E}_{T(n_i),p}(n_i) + \mathbf{E}_{T(p),n_i}(n_i) \\ &= -t_{ii}(1)q(n_i, n_i)p - T(p)q(n_i, n_i) \end{aligned}$$

and so we have that $T(p) = t_{ii}(1)p,\ i \in \{\pm 1\}$. Set $t := t_{ii}(1)$.

Let $i, k \in \{\pm 1\}$, $i \neq k$ and let $a \in A$. Then

$$\begin{aligned} 0 &= [T, q(n_i, an_k)\cdot Id](n_i) = (\mathbf{E}_{T(n_i),an_k} + \mathbf{E}_{T(an_k),n_i})(n_i) \\ &= -tq(n_i, n_i)an_k - t_{kk}(a)n_k q(n_i, n_i) - T_{\mathcal{N},\mathcal{P}}(an_k)q(n_i, n_i). \end{aligned}$$

Since $q(n_i, n_i) \in A^\times$, $T_{\mathcal{N},\mathcal{P}} = 0$ and $t_{kk}(a) = ta$. Hence $T = t \cdot Id$ and so $\mathcal{T} = A\cdot Id$. ■

Remark 4.2 For I and $\mathcal{P}$ different from the above the situation seems to be more complicated.

One can easily show that if $\mathcal{N}$ and $\mathcal{P}$ are A-supermodules with A-quadratic forms $q_{\mathcal{N}}$ and $q_{\mathcal{P}}$ respectively, then for $\mathcal{M} = \mathcal{N} \oplus \mathcal{P}$ with $q = q_{\mathcal{N}} \oplus q_{\mathcal{P}}$,

$$\mathrm{eosp}(q) = \mathrm{eosp}(q_{\mathcal{N}}) \oplus \mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}}) \oplus \mathrm{eosp}(q_{\mathcal{P}})$$

where $\mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}}) = \mathrm{span}_A\{\mathbf{E}_{n,p} \mid n \in \mathcal{N},\ p \in \mathcal{P}\}$ and

$$\overline{\mathrm{eosp}(q)} = \overline{\mathrm{eosp}(q_{\mathcal{N}})} \oplus \overline{\mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \oplus \overline{\mathrm{eosp}(q_{\mathcal{P}})}.$$

Moreover, if $q_{\mathcal{N}}$ is invertibly diagonalizable then

$$\mathrm{osp}(q) = \overline{\mathrm{eosp}(q_{\mathcal{N}})} \oplus \overline{\mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \oplus \mathrm{osp}(q_{\mathcal{P}}).$$

Given $S \in \mathcal{S}^{(1)}$ we denote by $\mathbf{ad}_A S$ the derivation $\Delta \in \mathrm{Der}_K A$ which satisfies $[S, a\cdot Id] = \Delta(a)\cdot Id$ for all $a \in A$. We define

$$\mathcal{S}_{\mathcal{N}}^{\mathcal{P}} = \{S_{\mathcal{N}} \oplus S_{\mathcal{P}} \in (\mathcal{S}_{\mathcal{N}}^{(1)} \cap \mathcal{S}_{\mathcal{N}}^{(2)}) \oplus (\mathcal{S}_{\mathcal{P}}^{(1)} \cap \mathcal{S}_{\mathcal{P}}^{(2)}) \mid \mathbf{ad}_A S_{\mathcal{N}} = \mathbf{ad}_A S_{\mathcal{P}}\}.$$

Proposition 4.3 *Let $\mathcal{M} = \mathcal{N} \oplus \mathcal{P}$ where $\mathcal{N}$ and $\mathcal{P}$ are A-supermodules with A-quadratic forms $q_{\mathcal{N}}$ and $q_{\mathcal{P}}$ respectively. Assume that $q_{\mathcal{N}}$ is almost diagonalizable with respect to some basis $\{n_i \mid i \in I\}$. Then*

$$\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)} \quad = \quad \overline{\mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \oplus \mathcal{S}_{\mathcal{N}}^{\mathcal{P}}$$

and

$$\mathcal{S} \quad = \quad \{S \in \overline{\mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \oplus \mathcal{S}_{\mathcal{N}}^{\mathcal{P}} \mid [S, \mathcal{E}] \subset \mathcal{E}\}.$$

Proof: Let $S \in \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$. By (S1), $[S, a \cdot Id] = \Delta(a) \cdot Id$ for some $\Delta \in \mathrm{Der}_K A$. Write

$$S = \begin{pmatrix} S_{\mathcal{N},\mathcal{N}} & S_{\mathcal{N},\mathcal{P}} \\ S_{\mathcal{P},\mathcal{N}} & S_{\mathcal{P},\mathcal{P}} \end{pmatrix}$$

where $S_{\mathcal{N},\mathcal{P}} \in \mathrm{Hom}_K(\mathcal{N}, \mathcal{P})$, etc. Then using (S1), $[S_{\mathcal{N},\mathcal{N}} \oplus S_{\mathcal{P},\mathcal{P}}, a Id] = \Delta(a) Id$ and $S_{\mathcal{N},\mathcal{P}} \oplus S_{\mathcal{P},\mathcal{N}} \in \mathrm{Hom}_A \mathcal{M}$. In addition, using the orthogonality of $\mathcal{N}$ and $\mathcal{P}$ and (S1) applied to $\mathcal{N}$ and $\mathcal{P}$ separately we have $S_{\mathcal{N},\mathcal{N}} \oplus S_{\mathcal{P},\mathcal{P}} \in \mathcal{S}_{\mathcal{N}}^{\mathcal{P}}$. Moreover, using (S2) we can show that

$$q((S_{\mathcal{N},\mathcal{P}} \oplus S_{\mathcal{P},\mathcal{N}})(m), m') + (-1)^{|m||m'|} q((S_{\mathcal{N},\mathcal{P}} \oplus S_{\mathcal{P},\mathcal{N}})(m'), m) = 0$$

and so $S_{\mathcal{N},\mathcal{P}} \oplus S_{\mathcal{P},\mathcal{N}} \in \mathrm{osp}(q) = \overline{\mathrm{eosp}(q_{\mathcal{N}})} \oplus \overline{\mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \oplus \mathrm{osp}(q_{\mathcal{P}})$. Hence clearly $S_{\mathcal{N},\mathcal{P}} \oplus S_{\mathcal{P},\mathcal{N}} \in \overline{\mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})}$ and so $\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)} \subset \overline{\mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \oplus \mathcal{S}_{\mathcal{N}}^{\mathcal{P}}$. Conversely, to show that $\overline{\mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \oplus \mathcal{S}_{\mathcal{N}}^{\mathcal{P}} \subset \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$, by Proposition 3.5.1.4, we only need to show that $\mathcal{S}_{\mathcal{N}}^{\mathcal{P}} \subset \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$. Let $S_{\mathcal{N}} \oplus S_{\mathcal{P}} \in \mathcal{S}_{\mathcal{N}}^{\mathcal{P}}$ and let $\Delta \in \mathrm{Der}_K A$ such that $[S_{\mathcal{N}} \oplus S_{\mathcal{P}}, a \cdot Id] = \Delta(a) \cdot Id$. Hence clearly $S_{\mathcal{N}} \oplus S_{\mathcal{P}} \in \mathcal{S}^{(1)}$. One can easily show that $S_{\mathcal{N}} \oplus S_{\mathcal{P}} \in \mathcal{S}^{(2)}$ and so the rest of the assertion follows. ■

Proposition 4.4 *Let $\mathcal{N}$ and $\mathcal{R}$ be free with homogeneous bases $\{n_i \mid i \in I\}$ and $\{r_j \mid j \in J\}$ and A-quadratic forms $q_{\mathcal{N}}$ and $q_{\mathcal{R}}$ respectively where $q_{\mathcal{N}}$ is almost diagonalizable with respect to $\{n_i \mid i \in I\}$ and $q_{\mathcal{R}} \equiv 0$. Set $\mathcal{M} = \mathcal{N} \oplus \mathcal{R}$ with the A-quadratic form $q = q_{\mathcal{N}} \oplus q_{\mathcal{R}}$. For $\Delta \in \mathrm{Der}_K A$ define $\Delta_{\mathcal{M}} \in \mathrm{End}_A \mathcal{M}$ by setting*

$$\Delta_{\mathcal{M}} \left(\sum_{i \in I} a_i n_i + \sum_{j \in J} b_j r_j \right) = \sum_{i \in I} \left(\frac{1}{2} \Delta(q(n_i, n_i)) q(n_i, n_i)^{-1} a_i + \Delta(a_i) \right) n_i + \sum_{j \in J} \Delta(b_j) r_j. \tag{4.2}$$

Set $(\mathrm{Der}_K A)_{\mathcal{M}} = \{\Delta_{\mathcal{M}} \mid \Delta \in \mathrm{Der}_K A\}$. *Then the map* $\mathrm{Der}_K A \to (\mathrm{Der}_K A)_{\mathcal{M}} : \Delta \mapsto \Delta_{\mathcal{M}}$ *is a Lie superalgebra isomorphism and*

$$\begin{aligned} \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)} &= \mathrm{osp}(q) \rtimes (\mathrm{Der}_K A)_{\mathcal{M}} \\ &= \overline{\mathrm{eosp}(q)} \rtimes (\mathrm{End}_A \mathcal{R} \rtimes (\mathrm{Der}_K A)_{\mathcal{M}}). \end{aligned}$$

Hence

$$\mathcal{S}_{\mathcal{E}} = \{S \in \mathrm{osp}(q) \rtimes (\mathrm{Der}_K A)_{\mathcal{M}} \mid [S, \mathcal{E}] \subset \mathcal{E}\}$$

and in particular, if $\mathcal{E} = \mathrm{eosp}(q)$, $\overline{\mathrm{eosp}(q)}$ *or* $\mathrm{osp}(q)$ *then*

$$\begin{aligned} \mathcal{S}_{\mathcal{E}} &= \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)} = \mathrm{osp}(q) \rtimes (\mathrm{Der}_K\, A)_{\mathcal{M}} \\ &= \overline{\mathrm{eosp}(q)} \rtimes (\mathrm{End}_A\, \mathcal{R} \rtimes (\mathrm{Der}_K\, A)_{\mathcal{M}}). \end{aligned}$$

Proof: Let $S \in \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$. By (S1), $[S, a \cdot Id] = \Delta(a) \cdot Id$ for some $\Delta \in \mathrm{Der}_K\, A$. Then for each $m \in \mathcal{M}$ and $a \in A$, $S(am) = (-1)^{|S||a|} a S(m) + \Delta(a) m$. Set $S_0 \in \mathrm{End}_A\, \mathcal{M}$ by

$$S_0(n_i) = S(n_i) - \frac{1}{2}\Delta(q(n_i, n_i)) q(n_i, n_i)^{-1} n_i \quad \text{and} \quad S_0(r_j) = S(r_j)$$

for $i \in I,\ j \in J$. Then for each $i, k \in I$, using (S2), $q(S_0(n_i), n_k) + (-1)^{|n_i||n_k|} q(S_0(n_k), n_i) = 0$. Moreover, for each $i \in I,\ j \in J$,

$$q(S_0(r_j), n_i) + (-1)^{|r_j||n_i|} q(S_0(n_i), r_j) = q(S(r_j), n_i) \overset{\text{(S2)}}{=} [S, q(r_j, n_i) \cdot Id] = 0$$

and for all $j, l \in J$,

$$q(S_0(r_j), r_l) + (-1)^{|r_j||r_l|} q(S_0(r_l), r_j) = 0.$$

Hence since $S_0 \in \mathrm{End}_A\, \mathcal{M}$, $S_0 \in \mathrm{osp}(q)$ and so by a direct calculation we get that $S = S_0 + \Delta_{\mathcal{M}}$. Therefore $\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)} \subset \mathrm{osp}(q) + (\mathrm{Der}_K\, A)_{\mathcal{M}}$. Conversely, to show that $\mathrm{osp}(q) + (\mathrm{Der}_K\, A)_{\mathcal{M}} \subset \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$, by Proposition 3.5.1.4, we only need to show that $(\mathrm{Der}_K\, A)_{\mathcal{M}} \subset \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$. This again follows from a direct calculation. Since $(\mathrm{Der}_K\, A)_{\mathcal{M}} \cap \mathrm{End}_A\, \mathcal{M} = (0)$ and $\mathrm{osp}(q) \in \mathrm{End}_A\, \mathcal{M}$, $\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)} = \mathrm{osp}(q) \oplus (\mathrm{Der}_K\, A)_{\mathcal{M}}$. One can check that the map $\mathrm{Der}_K\, A \to (\mathrm{Der}_K\, A)_{\mathcal{M}} : \Delta \mapsto \Delta_{\mathcal{M}}$ is a Lie superalgebra isomorphism. Hence $(\mathrm{Der}_K\, A)_{\mathcal{M}}$ is a subalgebra of $\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$ and so since by Proposition 3.5.1.(iv) $\mathrm{osp}(q)$ is an ideal of $\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$, it follows that $\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)} = \mathrm{osp}(q) \rtimes (\mathrm{Der}_K\, A)_{\mathcal{M}}$. Since $q_{\mathcal{R}} = 0$, $\mathrm{osp}(q) = \overline{\mathrm{eosp}(q)} \rtimes \mathrm{End}_K\, \mathcal{R}$ and so

$$\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)} = (\overline{\mathrm{eosp}(q)} \rtimes \mathrm{End}_K\, \mathcal{R}) \rtimes (\mathrm{Der}_K\, A)_{\mathcal{M}} = \overline{\mathrm{eosp}(q)} \oplus (\mathrm{End}_K\, \mathcal{R} \oplus (\mathrm{Der}_K\, A)_{\mathcal{M}})$$

Since $(\mathrm{Der}_K\, A)_{\mathcal{M}}$ and $\mathrm{End}_A\, \mathcal{R}$ are subalgebras of $\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$ to show that $\mathrm{End}_A\, \mathcal{R} \oplus (\mathrm{Der}_K\, A)_{\mathcal{M}}$ is a subalgebra of $\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$, we only need to show that $[(\mathrm{Der}_K\, A)_{\mathcal{M}}, \mathrm{End}_A\, \mathcal{R}] \subset (\mathrm{Der}_K\, A)_{\mathcal{M}} \oplus \mathrm{End}_A\, \mathcal{R}$. In fact, we will show that

$$[(\mathrm{Der}_K\, A)_{\mathcal{M}}, \mathrm{End}_A\, \mathcal{R}] \subset \mathrm{End}_A\, \mathcal{R}.$$

Indeed, let $\Delta \in \mathrm{Der}_K\, A$ and $S \in \mathrm{End}_A\, \mathcal{R}$. Then with respect to $\mathcal{M} = \begin{pmatrix} \mathcal{N} \\ \mathcal{R} \end{pmatrix}$, we have

$$[\Delta_{\mathcal{M}}, S] = \left[\begin{pmatrix} \Delta|_{\mathcal{N}} & 0 \\ 0 & \Delta|_{\mathcal{R}} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 \\ 0 & [\Delta|_{\mathcal{R}}, S] \end{pmatrix} \in \mathrm{Hom}_K(\mathcal{M}, \mathcal{R})$$

and so $[(\mathrm{Der}_K\, A)_{\mathcal{M}}, \mathrm{End}_A\, \mathcal{R}] \subset \mathrm{Hom}_K(\mathcal{M}, \mathcal{R})$. Moreover, if $S(r_j) = \sum_{i \in J} s_{ji} r_i$ for some $s_{ji} \in A$, $i, j \in J$ then $[[\Delta_{\mathcal{M}}, S], aId](br_j) = 0$ and so $[(\mathrm{Der}_K\, A)_{\mathcal{M}}, \mathrm{End}_A\, \mathcal{R}] \subset$

$0 \oplus \operatorname{End}_A \mathcal{R} = \operatorname{End}_A \mathcal{R}$. Hence $[(\operatorname{Der}_K A)_{\mathcal{M}} \oplus \operatorname{End}_A \mathcal{R}, (\operatorname{Der}_K A)_{\mathcal{M}} \oplus \operatorname{End}_A \mathcal{R}] \subset (\operatorname{Der}_K A)_{\mathcal{M}} \oplus \operatorname{End}_A \mathcal{R}$ and so $(\operatorname{Der}_K A)_{\mathcal{M}} \oplus \operatorname{End}_A \mathcal{R}$ is a subalgebra of $\mathcal{S}$. The rest of the assertion now follows from the definition of $\mathcal{S}$ and from Proposition 3.5.1.(iii). ■

For what follows we need the following:

Definition 4.5 We define the *locally inner derivations of* L to be the derivations $d \in \operatorname{Der}_K L$ such that for any finite subset $\{x_1, \ldots, x_n\}$ of L there exists $x \in L$ satisfying $d(x_i) = [x, x_i]$ (see [DZ]). We denote the locally inner derivations of L by $\mathbf{ad_{loc}}\, L$.

Remark 4.6 One can easily show that $\mathbf{ad_{loc}}\, L$ is an ideal of $\operatorname{Der}_K L$ and moreover that $\overline{\operatorname{eosp}(q)} \subset \mathbf{ad_{loc}} \operatorname{eosp}(q_\infty)$ (one uses the fact that $[x, \mathbf{E}_{m,n}] = \mathbf{E}_{x(m),n} + (-1)^{|x||m|} \mathbf{E}_{m,x(n)}$ for $x \in \operatorname{osp}(q)$ and $m, n \in \mathcal{M}$).

We will, given $S \in \mathcal{S}^{(1)}$, for simplicity write $[S, A \cdot Id]$ for the map $\Delta \in \operatorname{Der}_K A$ satisfying $[S, a \cdot Id] = \Delta(a) \cdot Id$.

Corollary 4.7 *Let $\mathcal{N}$ and $\mathcal{P}$ be A-supermodules with A-quadratic forms $q_{\mathcal{N}}$ and $q_{\mathcal{P}}$ respectively where $q_{\mathcal{N}}$ is almost diagonalizable with respect to some basis $\{n_i \mid i \in I\}$ of $\mathcal{N}$. Then*

$$\begin{array}{rcl} \overline{\operatorname{eosp}(q_{\mathcal{N}})} \oplus \overline{\operatorname{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \oplus (\mathcal{S}^{(1)}_{\mathcal{P}} \cap \mathcal{S}^{(2)}_{\mathcal{P}}) & \rightarrow & \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)} \\ x \oplus S & \mapsto & x \oplus (\mathbf{ad}_A S)_{\mathcal{N}} \oplus S \end{array}$$

is a Lie superalgebra isomorphism. If $\mathcal{E} = \operatorname{eosp}(q)$, $\overline{\operatorname{eosp}(q)}$ or $\operatorname{osp}(q)$ then

$$\mathcal{S} \cong \overline{\operatorname{eosp}(q_{\mathcal{N}})} \oplus \overline{\operatorname{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \oplus \mathcal{S}_{\mathcal{P}}$$

and in particular, if q is the orthogonal sum of an almost diagonalizable quadratic form and an invertibly diagonalizable A-quadratic form of rank 2, then

$$\operatorname{Der}_K \mathcal{E}_\infty \cong \begin{cases} \mathbf{ad_{loc}}(\mathcal{E}_\infty) + \mathcal{S}_{\mathcal{P}}, & \text{if } \mathcal{E} = \operatorname{eosp}(q); \\ \mathbf{ad}(\mathcal{E}_\infty) + \mathcal{S}_{\mathcal{P}}, & \text{if } \mathcal{E} = \overline{\operatorname{eosp}(q)},\ \operatorname{osp}(q). \end{cases}$$

Proof: By Proposition 4.3, $\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)} = \overline{\operatorname{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \oplus \mathcal{S}^{\mathcal{P}}_{\mathcal{N}}$. By Proposition 4.4, $\mathcal{S}_{\mathcal{N}} = \overline{\operatorname{eosp}(q_{\mathcal{N}})} \rtimes (\operatorname{Der}_K A)_{\mathcal{N}}$. We claim that

$$\mathcal{S}^{\mathcal{P}}_{\mathcal{N}} = (\overline{\operatorname{eosp}(q_{\mathcal{N}})} \oplus 0) \oplus \{(\mathbf{ad}_A S_{\mathcal{P}})_{\mathcal{N}} \oplus S_{\mathcal{P}} \mid S_{\mathcal{P}} \in \mathcal{S}_{\mathcal{P}}\}. \tag{4.3}$$

Indeed, if $S_{\mathcal{N}} \oplus S_{\mathcal{P}} \in \mathcal{S}^{\mathcal{P}}_{\mathcal{N}}$ let $\Delta = \mathbf{ad}_A S_{\mathcal{N}} = \mathbf{ad}_A S_{\mathcal{P}}$. Then $S_{\mathcal{N}} = x + \Delta_{\mathcal{N}} \in \overline{\operatorname{eosp}(q_{\mathcal{N}})} \oplus (\operatorname{Der}_K A)_{\mathcal{N}}$ and so $S_{\mathcal{N}} \oplus S_{\mathcal{P}} = (x \oplus 0) + (\Delta_{\mathcal{N}} \oplus S_{\mathcal{P}}) = (x \oplus 0) + (([S_{\mathcal{P}}, A \cdot Id])_{\mathcal{N}} \oplus S_{\mathcal{P}})$ which is the element of the right-hand side of (4.3). Conversely, since $\overline{\operatorname{eosp}(q_{\mathcal{N}})} \subset \operatorname{End}_A \mathcal{M}$ we have $\overline{\operatorname{eosp}(q_{\mathcal{N}})} \subset \mathcal{S}_{\mathcal{N}}$ with $\mathbf{ad}_A S_{\mathcal{N}} = 0$. Moreover, given $S_{\mathcal{P}} \in \mathcal{S}_{\mathcal{P}}$ with $\mathbf{ad}_A S_{\mathcal{P}} = \Delta \in \operatorname{Der}_K A$ we have $\Delta_{\mathcal{N}} \oplus S_{\mathcal{P}} \in \mathcal{S}^{\mathcal{P}}_{\mathcal{N}}$ since

$\Delta_{\mathcal{N}} \in S_{\mathcal{N}}$ by Proposition 4.4. If $0 = (x \oplus 0) + ((\mathbf{ad}_A S_{\mathcal{P}})_{\mathcal{N}} \oplus S_{\mathcal{P}})$ then $S_{\mathcal{P}} = 0$ hence $(\mathbf{ad}_A S_{\mathcal{P}})_{\mathcal{N}} = 0$ and so $x = 0$. Therefore

$$\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)} = \overline{\mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \oplus \overline{\mathrm{eosp}(q_{\mathcal{N}})} \oplus \{(\mathbf{ad}_A S_{\mathcal{P}})_{\mathcal{N}} \oplus S_{\mathcal{P}} \mid S_{\mathcal{P}} \in \mathcal{S}_{\mathcal{P}}\}.$$

It is easy to see that $\mathcal{S}_{\mathcal{P}} \to \{(\mathbf{ad}_A S_{\mathcal{P}})_{\mathcal{N}} \oplus S_{\mathcal{P}} \mid S_{\mathcal{P}} \in \mathcal{S}_{\mathcal{P}}\} : S_{\mathcal{P}} \mapsto (\mathbf{ad}_A S_{\mathcal{P}})_{\mathcal{N}} \oplus S_{\mathcal{P}}$ is a Lie superalgebra isomorphism and so the first assertion of the Corollary holds. The second assertion follows from 3.5.1.(iii). If $\mathcal{E} = \mathrm{eosp}(q)$, $\overline{\mathrm{eosp}(q)}$ or $\mathrm{osp}(q)$ then by Proposition 4.1, $\mathcal{S} = \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$. The rest of the Corollary now follows from Theorem 3.11 using the fact that $\overline{\mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \oplus \overline{\mathrm{eosp}(q_{\mathcal{N}})} \subset \overline{\mathrm{eosp}(q)}$ and that $\overline{\mathrm{eosp}(q)} \subset \mathbf{ad}_{\mathbf{loc}}(\mathrm{eosp}(q_\infty))$. ■

Remark 4.8 In [Be], Benkart described the derivations of root-graded Lie algebras over a field $\mathbb{F}$ of characteristic 0. In particular, for a centreless Lie algebra L graded by finite root systems B or D she proves that

$$\mathrm{Der}_{\mathbb{F}} L = \mathbf{ad}\, L + \mathrm{Der}_*(J)$$

for a certain Jordan algebra J. Since L is isomorphic to $\mathcal{E} = \mathrm{eosp}(q_I \oplus q_{x_0} \oplus q_{\mathcal{P}})$ for some finite index set I, base point x_0 and an A-module $\mathcal{P}$ where A is an extension of $\mathbb{F}$ and $q_I \oplus q_{x_0}$ is almost diagonalizable. In addition, since I is finite, one can show that $\overline{\mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \oplus \overline{\mathrm{eosp}(q_{\mathcal{N}})} = \mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}}) \oplus \mathrm{eosp}(q_{\mathcal{N}})$ and so using the above corollary, $\mathrm{Der}_K \mathcal{E}_\infty \cong \mathbf{ad}(\mathcal{E}_\infty) + \mathcal{S}_{\mathcal{P}}$. However, by Proposition 3.12, $\mathcal{S}_{\mathcal{P}} \cong \mathrm{Der}_*(A \oplus \mathcal{P})$ where $A \oplus \mathcal{P}$ is a Jordan algebra and this algebra is isomorphic to the Jordan algebra which appears in Benkart's description of B-graded Lie algebras.

Theorem 4.9 *Let $\mathcal{N}$ and $\mathcal{R}$ be free A-supermodules with* $\mathrm{rank}(\mathcal{N} \oplus \mathcal{R}) > 2$ *and A-quadratic forms $q_{\mathcal{N}}$ and $q_{\mathcal{R}}$ respectively such that $q_{\mathcal{R}} \equiv 0$ and $q_{\mathcal{N}}$ is an orthogonal sum of an almost diagonalizable A-quadratic form and an invertibly diagonalizable A-quadratic form on a free A-supermodule of rank 2. Let $\mathcal{M} = \mathcal{N} \oplus \mathcal{R}$ with the A-quadratic form $q = q_{\mathcal{N}} \oplus q_{\mathcal{R}}$ and let $\mathcal{E}$ be a subalgebra of* $\mathrm{osp}(q)$ *containing* $\mathrm{eosp}(q)$. *For $\Delta \in \mathrm{Der}_K A$ define $\Delta_{\mathcal{M}} \in \mathrm{End}_K \mathcal{M}$ using (4.2). Let the map $\{S \in \mathrm{osp}(q) \oplus (\mathrm{Der}_K A)_{\mathcal{M}} \mid [S, \mathcal{E}] \subset \mathcal{E}\} \to \mathcal{D}_{\mathcal{E}} : S \mapsto d_{S,0}$ be the Lie superalgebra monomorphism given by (3.5). Then* $\mathrm{Der}_K A \to (\mathrm{Der}_K A)_{\mathcal{M}}$: $\Delta \mapsto \Delta_{\mathcal{M}}$ *is a Lie superalgebra isomorphism and*

$$\mathrm{Der}_K \mathcal{E}_\infty = \mathbf{ad}\, \mathcal{E}_\infty + \{d_{S,0} \in \mathcal{D}_{\mathcal{E}} \mid S \in \mathrm{osp}(q) \oplus (\mathrm{Der}_K A)_{\mathcal{M}} \text{ such that } [S, \mathcal{E}] \subset \mathcal{E}\}.$$

with

$$\mathbf{ad}\, \mathcal{E}_\infty \cap \{d_{S,0} \in \mathcal{D}_{\mathcal{E}} \mid S \in \mathrm{osp}(q) \text{ such that } [S, \mathcal{E}] \subset \mathcal{E}\} = \mathbf{ad}\, \mathcal{E} = \{d_{S,0} \mid S \in \mathcal{E}\}.$$

Hence, whenever $\mathcal{S} = \mathcal{E} \rtimes \mathcal{S}_0$,

$$\mathrm{Der}_K \mathcal{E}_\infty = \mathbf{ad}(\mathcal{E}_\infty) \rtimes \{d_{S,0} \in \mathcal{D}_{\mathcal{E}} \mid S \in \mathcal{S}_0\}.$$

In particular:

(1) *If* $\mathcal{E} = \mathrm{eosp}(q)$ *then*

$$\mathrm{Der}_K \mathcal{E}_\infty = \mathbf{ad_{loc}}\, \mathcal{E}_\infty \rtimes \{d_{S,0} \in \mathcal{D} \mid S \in \mathrm{End}_A \mathcal{R} \oplus (\mathrm{Der}_K A)_{\mathcal{M}}\}.$$

Moreover, if $\mathcal{M}$ *is free of finite rank then*

$$\mathrm{Der}_K \mathcal{E}_\infty = \mathbf{ad}(\mathcal{E}_\infty) \rtimes \{d_{S,0} \in \mathcal{D} \mid S \in \mathrm{End}_A \mathcal{R} \oplus (\mathrm{Der}_K A)_{\mathcal{M}}\}.$$

(2) *If* $\mathcal{E} = \overline{\mathrm{eosp}(q)}$ *then*

$$\mathrm{Der}_K \mathcal{E}_\infty = \mathbf{ad}(\mathcal{E}_\infty) \rtimes \{d_{S,0} \in \mathcal{D} \mid S \in \mathrm{End}_A \mathcal{R} \oplus (\mathrm{Der}_K A)_{\mathcal{M}}\}.$$

(3) *If* $\mathcal{E} = \mathrm{osp}(q)$ *then*

$$\mathrm{Der}_K \mathcal{E}_\infty = \mathbf{ad}\, \mathcal{E}_\infty \rtimes \{d_{S,0} \in \mathcal{D} \mid S \in (\mathrm{Der}_K A)_{\mathcal{M}}\}.$$

Proof: By Theorem 3.11.2 we have that the map $\mathcal{S} \oplus \mathcal{T} \to \mathcal{D}_{\mathcal{E}} : (S,T) \mapsto d_{S,T}$ is a Lie superalgebra isomorphism with the induced bracket operation and $\mathrm{Der}_K \mathcal{E}_\infty = \mathbf{ad}(\mathcal{E}_\infty) + \mathcal{D}_{\mathcal{E}}$. By Proposition 4.4, $\mathcal{S} = \{S \in \mathrm{osp}(q) \rtimes (\mathrm{Der}_K A)_{\mathcal{M}} \mid [S, \mathcal{E}] \subset \mathcal{E}\}$ where the map $\mathrm{Der}_K A \to (\mathrm{Der}_K A)_{\mathcal{M}} : \Delta \mapsto \Delta_{\mathcal{M}}$ is a Lie superalgebra isomorphism. Moreover, by Proposition 4.1, $\mathcal{T} = A{\cdot}Id$. The main assertion now follows from Theorems 3.11.3 and 3.11.4. and so (3) follows immediately from this and (2) follows from the fact that $\mathrm{osp}(q) = \overline{\mathrm{eosp}(q)} \rtimes \mathrm{End}_A \mathcal{R}$ in this setting. Let $\mathcal{E} = \mathrm{eosp}(q)$. By Proposition 4.4,

$$\mathcal{S} = \mathrm{osp}(q) \rtimes (\mathrm{Der}_K A)_{\mathcal{M}} = \overline{\mathrm{eosp}(q)} \rtimes (\mathrm{End}_A \mathcal{R} \oplus (\mathrm{Der}_K A)_{\mathcal{M}})$$

and so

$$\mathrm{Der}_K \mathcal{E}_\infty = \mathbf{ad}\, \mathcal{E}_\infty + (\overline{\mathrm{eosp}(q)} \oplus \{d_{S,0} \in \mathcal{D}_{\mathcal{E}} \mid S \in \mathrm{End}_A \mathcal{R} \oplus (\mathrm{Der}_K A)_{\mathcal{M}}\}.$$

We will show that $\mathbf{ad_{loc}}(\mathcal{E}_\infty) = \mathbf{ad}(\mathcal{E}_\infty) + \overline{\mathrm{eosp}(q)}$. The inclusion $\supset$ is clear. Since $\mathbf{ad_{loc}}(\mathcal{E}_\infty) \subset \mathrm{End}_A \mathcal{E}_\infty$, from (3.5) we have $\mathbf{ad_{loc}}(\mathcal{E}_\infty) \cap d_{\mathrm{End}_A \mathcal{R} \oplus (\mathrm{Der}_K A)_{\mathcal{M}},0} = \mathbf{ad_{loc}}(\mathcal{E}_\infty) \cap d_{\mathrm{End}_A \mathcal{R},0}$. Suppose that $d_{S,0} \in \mathbf{ad_{loc}}(\mathrm{eosp}(q_\infty))$ for some $S \in \mathrm{End}_A \mathcal{R}$. Then for each $j \in R$ there exists $(a_0, m_0, n_0, x_0) \in \mathrm{eosp}(q_\infty)$ such that

$$\begin{aligned}
d_{S,0}(0, r_j, 0, 0) &= (0, S(r_j), 0, 0) = [(a_0, m_0, n_0, x_0), (0, r_j, 0, 0)] = (0, a_0 r_j, 0, \mathbf{E}_{n_0, r_j}) \\
d_{S,0}(0, 0, r_j, 0) &= (0, 0, S(r_j), 0) = [(a_0, m_0, n_0, x_0), (0, 0, r_j, 0)] = (0, 0, -a_0 r_j, \mathbf{E}_{m_0, r_j})
\end{aligned}$$

and so $S(r_j) = a_0 r_j$ and $S(r_j) = -a_0 r_j$ hence $S(r_j) = 0$. Since $j \in R$ was arbitrary, $S = 0$ and so $\mathbf{ad_{loc}}(\mathcal{E}_\infty) \cap d_{\mathrm{End}_A \mathcal{R},0} = (0)$. Hence since clearly $\mathbf{ad}\, \mathcal{E}_\infty \subset \mathbf{ad_{loc}}\, \mathcal{E}_\infty$ and since $\mathbf{ad_{loc}}(\mathcal{E}_\infty)$ is an ideal of $\mathcal{E}_\infty$,

$$\mathrm{Der}_K(\mathcal{E}_\infty) = \mathbf{ad_{loc}}(\mathcal{E}_\infty) \rtimes d_{\mathrm{End}_A \mathcal{R} \oplus (\mathrm{Der}_K A)_{\mathcal{M}},0}.$$

Now, if $\mathcal{M}$ is of finite rank then one can show that $\overline{\mathrm{eosp}(q)} = \mathrm{eosp}(q)$ (see [D]) and so $\mathbf{ad_{loc}}\, \mathcal{E}_\infty = \mathbf{ad}\, \mathcal{E}_\infty$ hence the assertion follows. ∎

The setting of Theorem 4.9 also holds in the following example: Let $\mathcal{M} = \oplus_{i\in I} Am_i$ be a free supermodule where $I = \mathbb{Z}$ or $I = \frac{1}{2}\mathbb{Z}$ and let q be the quadratic form on $\mathcal{M}$ defined by

$$\begin{aligned} q(m_i, m_j) &= (-1)^i \delta_{i,-j}, \quad i, j \in \mathbb{Z}; \\ q(m_i, m_j) &= (-1)^{i+\frac{1}{2}} \delta_{i,-j}, \quad i, j \in \frac{1}{2} + \mathbb{Z}; \\ q(m_i, m_j) &= 0, \quad i \in \mathbb{Z},\, j \in \frac{1}{2} + \mathbb{Z}. \end{aligned}$$

Define $\mathrm{osp}_{\,\mathrm{fd}}(q)$ to be the set of all A-endomorphisms (a_{ij}) in $\mathrm{osp}(q)$ of $\mathcal{M}$ which have finitely many nonzero diagonals. For $K = A = \mathbb{C}$, the algebra $\mathrm{osp}_{\,\mathrm{fd}}(q)$ has been studied by Kac ([K1]) for $I = \mathbb{Z}$ and by Cheng-Wang ([CW]) for $I = \frac{1}{2}\mathbb{Z}$.

Corollary 4.10 *If the elements of $\mathbb{Z}$ are invertible in A then $\mathrm{osp}_{\,\mathrm{fd}}(q)$ is self-normalizing in $\mathrm{osp}(q)$ and*

$$\mathrm{Der}_K\, \mathcal{E}_\infty = \mathbf{ad}(\mathcal{E}_\infty) \rtimes \{d_{S,0} \in \mathcal{D}_{\mathcal{E}} \mid S \in (\mathrm{Der}_K\, A)_{\mathcal{M}}\}.$$

Proof: By Proposition 4.4,

$$\mathcal{S} = \{S \in \mathrm{osp}(q) \oplus (\mathrm{Der}_K\, A)_{\mathcal{M}} \mid [S, \mathcal{E}] \subset \mathcal{E}\}.$$

Since the elements of $(\mathrm{Der}_K\, A)_{\mathcal{M}}$ satisfy (S1), by Proposition 3.5.1.1, $[(\mathrm{Der}_K\, A)_{\mathcal{M}}, \mathrm{End}_A\, \mathcal{M}] \subset \mathrm{End}_A\, \mathcal{M}$. Hence since the elements of $(\mathrm{Der}_K\, A)_{\mathcal{M}}$ are diagonal, $[(\mathrm{Der}_K\, A)_{\mathcal{M}}, \mathcal{E}] \subset \mathcal{E}$. Let $\{v_i \mid i \in I \cup J\}$ denote the standard basis of $\mathcal{M}$. Let $(a_{ij})_{i,j\in I\cup J} \in \mathrm{osp}(q)$. Since $\mathcal{M}$ is almost diagonalizable, one can show that $a_{i,j} = -a_{-j,-i} q(v_i, v_{-i}) q(v_j, v_{-j})$ for all $i, j \in I \cup J$. Define $(b_{ij})_{i,j\in I\cup J} \in \mathrm{osp}_{\,\mathrm{fd}}(q)$ by $b_{k,k} = k,\ k \neq 0$ and $b_{0,0} = 0$. Then $[(a_{ij}), (b_{ij})] \in \mathrm{osp}_{\,\mathrm{fd}}\, q$ implies that $(a_{ij}) \in \mathrm{osp}_{\,\mathrm{fd}}(q)$ and so $\mathrm{osp}_{\,\mathrm{fd}}(q)$ is self-normalizing. Hence

$$\begin{aligned} \mathcal{S} &= \{S \in \mathrm{osp}(q) \rtimes (\mathrm{Der}_K\, A)_{\mathcal{M}} \mid [S, \mathcal{E}] \subset \mathcal{E}\} = \{S \in \mathrm{osp}(q) \mid [S, \mathcal{E}] \subset \mathcal{E}\} \rtimes (\mathrm{Der}_K\, A)_{\mathcal{M}} \\ &= \mathrm{osp}_{\,\mathrm{fd}}(q) \rtimes (\mathrm{Der}_K\, A)_{\mathcal{M}} \end{aligned}$$

and so $\mathrm{Der}_K\, \mathcal{E}_\infty = \mathbf{ad}(\mathcal{E}_\infty) + \{d_{S,0} \in \mathcal{D}_{\mathcal{E}} \mid S \in \mathrm{osp}_{\,\mathrm{fd}}(q) \rtimes (\mathrm{Der}_K\, A)_{\mathcal{M}}\}$. The assertion now follows from Theorem 3.11.4. ∎

The following corollary is a generalization of Benkart's result for Lie algebras of type B and D ([Be, Theorem 3.6]).

Corollary 4.11 *Suppose $\mathcal{M} = H(I; A) \oplus \mathcal{P}$ where $|I| \geq 2$ or $|I| \geq 1$ and $\mathrm{Ann}_A(p) = (0)$ for some $p \in \mathcal{P}$. Let $q_{\mathcal{P}}$ be an A-quadratic form on $\mathcal{P}$ and set $q = q_I \oplus q_{\mathcal{P}}$. Let $L = \mathrm{eosp}(q_\infty)$ and $\mathcal{E} = \mathrm{eosp}(q)$ (e.g. L is a centreless D_J- or B_J-graded Lie superalgebra for $J = I \dot{\cup} \{\infty\}$, [GN]). Then*

$$\mathrm{Der}_K\, L = \mathbf{ad}\, L + \{d_{S,0} \in \mathcal{D}_{\mathcal{E}} \mid S \in \mathcal{S}_{\mathcal{E}}\}$$

and

$$\mathbf{ad}\, L \cap \{d_{S,0} \in \mathcal{D}_{\mathcal{E}} \mid S \in \mathcal{S}_{\mathcal{E}}\} = \mathbf{ad}\, \mathcal{E}.$$

Hence if $\mathcal{S} = \mathcal{E} \rtimes \mathcal{S}_0$ *then*

$$\operatorname{Der}_K L = \mathbf{ad}\, L \rtimes \{d_{S,0} \in \mathcal{D}_{\mathcal{E}} \mid S \in \mathcal{S}_0\}.$$

In particular, if $\mathcal{P} = \mathcal{N} \oplus \mathcal{R}$ *with the* A*-quadratic form* $q = q_{\mathcal{N}} \oplus q_{\mathcal{R}}$ *where* $\mathcal{N}$ *and* $\mathcal{R}$ *are free* A*-supermodules with* A*-quadratic forms* $q_{\mathcal{N}}$ *and* $q_{\mathcal{R}}$ *respectively such that* $q_{\mathcal{N}}$ *is almost diagonalizable and* $q_{\mathcal{R}} \equiv 0$ *then*

$$\operatorname{Der}_K L = \mathbf{ad_{loc}}\, L \rtimes \{d_{S,0} \in \mathcal{D}_{\mathcal{E}} \mid S \in \operatorname{End}_A \mathcal{R} \oplus (\operatorname{Der}_K A)_{\mathcal{M}})$$

where $d_{(\operatorname{Der}_K A)_{\mathcal{M}},0}$ *is isomorphic (as a Lie superalgebra) to* $\operatorname{Der}_K A$*. In particular, if* L *is of finite rank,*

$$\operatorname{Der}_K L = \mathbf{ad}\, L \rtimes \{d_{S,0} \in \mathcal{D}_{\mathcal{E}} \mid S \in \operatorname{End}_A \mathcal{R} \oplus (\operatorname{Der}_K A)_{\mathcal{M}}\}.$$

Proof: Since $|I| \geq 1$, $H(I; A)$ contains a two-dimensional invertibly diagonalizable free A-supermodule as a direct summand and so the assertion holds by Proposition 4.1. The rest follows from Theorem 4.9. ■

We can apply Theorem 4.9 to extended affine Lie algebras of type B and D:

Corollary 4.12 *Let* $\mathcal{K}$ *be the centreless core of an extended affine Lie algebra of type* B_l *or* D_l $(l \geq 3,\ l \geq 4$ *respectively) with nullity* ν*. Then*

$$\operatorname{Der}_{\mathbb{C}} \mathcal{K} \cong \mathbf{ad}\, \mathcal{K} \rtimes \operatorname{Der}_{\mathbb{C}} \mathbb{C}[t_1^{\pm 1}, \ldots, t_\nu^{\pm 1}].$$

The last assertion of the following was proved in [K2, Proposition 2.3.4] and [S, Proposition 3.1.2.3] and is a special case of Theorem 4.9.

Corollary 4.13 *Let* K *be an algebraically closed field of characteristic 0.*

(1) Let $m, n \in \mathbb{Z}^+$*,* $q = q_{\{1,\ldots,m\}} \oplus \tilde{q}_{\{1,\ldots,n\}}$ *over* K *and let* $q_0 : Km_0 \times Km_0 \to K$ *be the* K*-quadratic form on a free supermodule* Km_0 *given by* $q(m_0, m_0) = 1$*. Then* $\operatorname{eosp}(q) = \operatorname{osp}(2m, 2n)$ *and* $\operatorname{eosp}(q \oplus q_0) = \operatorname{osp}(2m+1, 2n)$*.*

(2) Let A *be a superextension of* K *and let* $L = \operatorname{osp}(m, 2n)$*,* $m, n \in \mathbb{Z}^+$*. Then* $Der_K(L \otimes A) \cong \mathbf{ad}\, L \rtimes \operatorname{Der}_K A$*. In particular, for* $A = K$*,* L *has no outer derivations.*

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