DEEP MATRICES AND THEIR FRANKENSTEIN ACTIONS

KEVIN MCCRIMMON

Abstract. The *algebra of deep matrices* $\mathcal{E}(X, \mathcal{A})$ is spanned over a coordinate algebra $\mathcal{A}$ by "deep matrix units" $E_{\mathbf{h}}^{\mathbf{k}}$ parameterized, not by single natural numbers like the standard matrix units E_i^j, but by all "deep indices" or "heads" $\mathbf{h},\mathbf{k}$ (*finite strings* of natural numbers or some other infinite set X). This algebra has a natural *Frankenstein action* on the free right $\mathcal{A}$-module $V(X, \mathcal{A})$ with basis of all "bodies" $\mathbf{b}$ (*infinite* sequences or strings), where $E_{\mathbf{h}}^{\mathbf{k}}$ chops off head $\mathbf{k}$ from the body $\mathbf{b}$ and sews on a new head $\mathbf{h}$ (replaces an initial string $\mathbf{k}$ of $\mathbf{b}$ by $\mathbf{h}$): $E_{\mathbf{h}}^{\mathbf{k}}(\mathbf{kd}) = \mathbf{hd}, E_{\mathbf{h}}^{\mathbf{k}}(\mathbf{b}) = 0$ if $\mathbf{b}$ doesn't begin with the string $\mathbf{k}$.

As with ordinary matrix algebras, the center and the ideals of the deep matrix algebra are just those of the coordinate algebra, because each nonzero element A is only "distance 1" away from a scalar: there exist a coordinate a and deep matrix units E, F such that $EAF = a1$. In particular, over a simple coordinate algebra $\mathcal{A}$ the deep matrices form a simple unital algebra which acts irreducibly on each *tail subspace* of $V(X, \mathcal{A})$, spanned by all $\mathbf{b}$ having the same "tail," where two strings $\mathbf{b},\mathbf{b'}$ have the same tail if they become the same after chopping off suitable heads (of perhaps different sizes).

Key Words: simple unital algebra, irreducible actions.
2000 MSC: Primary 16D30, Secondary 16D60

1. Prolegomenon

Deep matrices were born of musings on the difficulty of creating ideals in quadratic Jordan algebras, where the ideal generated by an element a consists of all finite sums of finite quadratic products of a by elements of the algebra. The number of summands and factors in such an expression could be considered a measure of its complexity. This is much clearer in an associative algebra: we can define the *algebraic distance* from a to b to be the length of the shortest expression $b = \sum_{i=1}^{n} x_i a y_i$ for b in terms of a (or ∞, if no such expression exists). The *diameter* of an algebra would be the supremum of all distances between nonzero elements. An algebra is simple precisely when every two nonzero elements are a finite distance apart, and P.M. Cohn showed that an algebra has finite diameter precisely when it is simple *and all its ultrapowers remain simple.*

We write $d_{\mathcal{A}}(a, b)$ if there is any ambiguity about the algebra in which we are computing distance. Distance increases (generating power decreases) under multiplication of a and decreases (reachability increases) under multiplication of b by elements $\hat{x}, \hat{y}$ of the unital hull; distances shrink in homomorphic images and grow in subalgebras (but remain the same in Peirce subalgebras):

$$d(\hat{x}a\hat{y}, b) \geq d(a, b) \geq d(a, \hat{x}b\hat{y}),$$

$$d(a, b + c) \leq d(a, b) + d(a, c), \quad d(a, c) \leq d(a, b)d(b, c),$$

$$d_{\overline{\mathcal{A}}}(\bar{a}, \bar{b}) \leq d_{\mathcal{A}}(a, b), \quad d(\overline{\mathcal{A}}) \leq d(\mathcal{A}), \quad d_{\mathcal{B}}(a, b) \geq d_{\mathcal{A}}(a, b),$$

$$d_{\mathcal{B}}(a, b) = d_{\mathcal{A}}(a, b), \quad d(\mathcal{B}) \leq d(\mathcal{A}) \text{ for } \mathcal{B} = e\mathcal{A}e \text{ for an idempotent } e \in \mathcal{A}.$$

For the subalgebra $\mathcal{B} = \Phi E_{11} + \Phi E_{12}$ of the algebra $\mathcal{A}$ of 2×2 matrices over Φ, $a = E_{12}, b = E_{11}$ have $b = E_{11}aE_{21}$ in $\mathcal{A}$, so $d_{\mathcal{A}}(a, b) = 1$ but $d_{\mathcal{B}}(a, c) = \infty$ for all $c \neq 0$ in $\mathcal{B}$. If $\mathcal{A}$ is a dense algebra of linear transformations on an infinite-dimensional right vector space V over a division algebra Δ, $\mathcal{A}$ can still retain finite diameter, but only with difficulty: by Litoff's Theorem, $\mathcal{A}$ contains for each finite n a subalgebra $\mathcal{A}_n$ having quotient $\overline{\mathcal{A}_n}$ isomorphic to $M(n, \Delta)$ and hence $d(\mathcal{A}_n) \geq d(\overline{\mathcal{A}_n}) = n$. Despite having these subalgebras of large diameter, $\mathcal{A}$ itself may have finite diameter (even diameter 1, as in the case of Deep Matrices), since the diameter of subalgebras not of the form $e\mathcal{A}e$ may exceed the diameter of $\mathcal{A}$.

The notion of distance rapidly loses significance in commutative algebras: then $b = \sum x_i a y_i = (\sum x_i y_i)a1$, so $d(a, b)$ is either ∞ or 1, and $\mathcal{A}$ has finite diameter $(= 1)$ iff $\mathcal{A}$ is simple (= a field). But for noncommutative algebras, distance and diameter do give an algebraic notion of "size". It is easy to see that the algebra $\mathcal{M}(n, \Delta)$ of $n \times n$ matrices over a division ring Δ (equivalently, the algebra $\mathcal{E}nd(V_\Delta)$ of linear transformations on an n-dimensional right vector space V over Δ) has diameter n. In particular, every division algebra has diameter 1. But the converse turns out to be false: just because any two nonzero elements are a distance 1 apart (each $a \neq 0$ has two friends x, y such that $xay = 1$) does not imply the algebra is a division algebra.

Algebras of diameter 1 have been constructed by L.A. Bokut [1], using transfinite induction and free algebras to show that every simple algebra without zero divisors imbeds in an algebra of diameter 1 (indeed, in an algebra $\mathcal{A}$ with the property that for every $a \neq 0, b, c, d, e, f, g \in \mathcal{A}, \alpha, \beta \in \Phi$, one can solve the equation $xay + ybx + \alpha xy + \beta yx + cx + xd + ey + yf + g = 0$ for x, y, not merely the equation $xay = 1$). Prof. Ken Goodearl suggests the following quick argument that *every* algebra $\mathcal{B}$ over a field Φ imbeds in one of diameter 1. We may assume the algebra $\mathcal{B}$ is unital, and let $\mathcal{E} = End_\Phi(V)$ be the ring of Φ-linear transformations of a free $\mathcal{B}$-module $V = \bigoplus \mathcal{B}_a$ over an index set of infinite cardinality $\aleph \geq dim_\Phi(\mathcal{B})$, so that $dim_\Phi(V) = \aleph dim_\Phi(\mathcal{B}) = \aleph$. $\mathcal{B}$ imbeds via the left regular action in $\mathcal{A} := \mathcal{E}/\mathcal{M}$ for the maximal ideal $\mathcal{M} = \{x \in \mathcal{E} \mid \text{rank}(x) < \aleph\}$, since each left multiplication $L_b \notin \mathcal{M}$ if $b \neq 0$. [Note that it has rank $dim_\Phi\big(\bigoplus_a b\mathcal{B}_a\big) \geq \sum_a 1 = \aleph$ since $b\mathcal{B}_a \neq 0$ for $b \neq 0$ and $\mathcal{B}$ unital]. $\mathcal{A}$ has diameter 1 since for any endomorphism $a \in \mathcal{E} \setminus \mathcal{M}$ we have $V = \ker(a) \oplus W = U \oplus im(a)$ with a an isomorphism of W on im (a), thus dim $(W) =$ dim (im $(a)) =$ rank $(a) = \aleph =$ dim (V) gives rise to a Φ-isomorphism $y : V \to W$, hence $xay = 1_V$ for

$$x : V \xrightarrow{proj} \text{im}(a) \xrightarrow{a^{-1}} W \xrightarrow{y^{-1}} V.$$

Thus in $\mathcal{A} = \overline{\mathcal{E}/\mathcal{M}}$ we have $\bar{x}\,\bar{a}\,\bar{y} = \bar{1}$, and $\mathcal{A}$ has diameter 1.

This example is fairly universal. Whenever 1 is a finite distance $n < \infty$ away from an element $a \in End_\Delta(V)$ with $\aleph = \dim_\Delta(V)$ infinite, then a must have rank

$r = \aleph$. Indeed, if $r = \dim_\Delta(a(V)) < \aleph$ were strictly smaller, the dimension of $V = 1(V) = \sum_{i=1}^n x_i a y_i(V) \subseteq \sum_{i=1}^n x_i a(V)$ would be $\leq \sum_{i=1}^n \dim_\Delta(x_i(a(V))) \leq \sum_{i=1}^n \dim_\Delta(a(V))$ (transformations cannot increase dimension) $= nr < n\aleph = \aleph$, a contradiction. Thus all nonzero elements a in a diameter 1 algebra must be "within striking distance of invertibility". In particular, algebras of diameter 1 containing matrices of finite rank must already be division algebras.

Algebras of diameter 1 have been called "purely infinite" and studied intensively in the setting of C^*-algebras.[1] In particular, J. Cuntz [2] introduced an algebra $\mathcal{O}_\infty$ which is the C^*-closure of the algebra of deep matrices with complex coordinates over a countable index set, and established the basic diameter 1 property making heavy use of the complete norm topology. We will develop a purely algebraic theory of deep matrices over arbitrary coordinate rings. We will work thoughout with *unital associative algebras* over an irrelevant (unital, associative, commutative) ring of scalars Φ. Andy Warhol used to say that each algebra (he meant, of course, only associative algebras) deserves to be famous for 10 minutes. We want to give the algebra of deep matrices a few pages in the limelight, in the hope that it may find useful employment in the algebraic community.

2. Heads and Bodies

We want to create an algebra of square matrices $A = \sum_{\mathbf{h},\mathbf{k}} a_{\mathbf{h},\mathbf{k}} E_{\mathbf{h}}^{\mathbf{k}}$ whose entries $a_{\mathbf{h},\mathbf{k}}$ come from some unital associative Φ-algebra $\mathcal{A}$, and whose deep matrix units $E_{\mathbf{h}}^{\mathbf{k}}$ have "deep" row- and column-indices $\mathbf{h}, \mathbf{k}$ from a set $\mathbf{H}(X)$ of "heads" based on some underlying nonempty index set X. The set of all "deep X-indices" or "X-heads"

$$\mathbf{H}(X) = \bigcup_{n=0}^{\infty} X^n$$

consists of all *finite* strings (n-tuples) $\mathbf{h} = (x_1, \ldots, x_n)$ of arbitrary *depth* $|\mathbf{h}| = n \geq 0$ whose individual indices x_i come from X. The number of heads is infinite. Notice that we include one important head, the *empty head* $\emptyset$ of depth 0. The reader may for concreteness think of X as the natural numbers $\mathbb{N} = \{1, 2, \ldots\}$, though neither countability nor ordering of the indices is relevant. Also, we are primarily interested in the case when the coordinate algebra $\mathcal{A}$ is a division algebra, or at least simple.

Our matrix units act in a gruesome way on a free *right* $\mathcal{A}$-module

$$V(X, \mathcal{A}) := \bigoplus_{\mathbf{b} \in \mathbf{B}} \mathbf{b}\mathcal{A}$$

with basis vectors $\mathbf{b}$ from the set of all "X-bodies"

$$\mathbf{B}(X) = \prod_{1}^{\infty} X$$

consisting of all *infinite* strings (sequences) $\mathbf{b} = (y_1, y_2, \ldots)$ of indices from X. The number of bodies is uncountable if $|X| \geq 2$. When $\mathcal{A}$ is commutative we can ignore the distinction between right and left modules.

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[1] I would like to thank Prof. Goodearl for directing me to the C^* literature. See [3] for several equivalent versions of the purely-infinite condition.

We cannot sew bodies together, but we can sew heads onto bodies: we can concatenate finite tuples with infinite sequences,

$$\mathbf{hb} := (x_1, \ldots, x_n, y_1, y_2, \ldots).$$

In addition to sewing heads on, we can also cut them off. The N^{th} **head and tail operations** $\eta_N : \mathbf{B}(X) \to \mathbf{H}(X),\ \tau_N : \mathbf{B}(X) \to \mathbf{B}(X)$ for finite $N = 0, 1, \ldots$ are defined by

$$\eta_N(\mathbf{b}) := (y_1, \ldots, y_N), \quad \tau_N(\mathbf{b}) = (y_{N+1}, y_{N+2}, \ldots) \quad (\mathbf{b} = (y_1, y_2, \ldots)).$$

Thus the head operation decapitates the N^{th} *head* (the first N indices) $\eta_N(\mathbf{b})$ from the body and carries it away, leaving behind the N^{th} *tail* $\tau_N(\mathbf{b})$ (all but the first N indices). We agree that $\eta_0(\mathbf{b}) = \emptyset$ is the empty head (no decapitation), and $\tau_0(\mathbf{b}) = \mathbf{b}$ is the identity map. The humpty-dumpty concatenation restores the original body by sewing its N^{th} head back on to its N^{th} tail:

$$\mathbf{b} = \eta_N(\mathbf{b})\tau_N(\mathbf{b}).$$

If we are careful we can even cut heads off heads, forming $\eta_N(\mathbf{h})$ as long as $N \leq |\mathbf{h}|$. We say that a finite or infinite string $\mathbf{d}$ *has head* or *begins with* $\mathbf{h}$, or that $\mathbf{h}$ *heads* $\mathbf{d}$ (written $\mathbf{h} \ll \mathbf{d}$) if $\mathbf{h} = \eta_N(\mathbf{d})$ is an initial segment of $\mathbf{d}$ for some N, i.e. $\mathbf{d}$ results from concatenation with $\mathbf{h}$. We say $\mathbf{h}$ is a *proper head* or *properly heads* or *properly begins* $\mathbf{d}$ (written $\mathbf{h} < \mathbf{d}$) if it is a proper initial segment:

$$\begin{gathered}\mathbf{h} \ll \mathbf{d} \in \mathbf{H} \text{ (resp. } \mathbf{B}) \text{ iff } \mathbf{d} = \mathbf{hd}' \text{ for some } \mathbf{d}' \in \mathbf{H} \text{ (resp. } \mathbf{B}),\\ \mathbf{h} < \mathbf{d} \in \mathbf{H} \text{ iff } \mathbf{h} \ll \mathbf{d} \neq \mathbf{h} \quad \text{(i.e. } \mathbf{d} = \mathbf{hd}' \text{ for } \mathbf{d}' \neq \emptyset)\end{gathered}$$

Note that always $\emptyset \ll \mathbf{h}$.

The relation of heading is a partial ordering of heads: it is reflexive, $\mathbf{h} \ll \mathbf{h}$, transitive, $\mathbf{j} \ll \mathbf{h} \ll \mathbf{k} \Longrightarrow \mathbf{j} \ll \mathbf{k}$, and is antisymmetric, $\mathbf{j} \ll \mathbf{h} \ll \mathbf{j} \Longrightarrow \mathbf{j} = \mathbf{h}$. Two heads **h,k** are **related** under this partial order (written $\mathbf{h} \sim \mathbf{k}$) if one is a head of the other, $\mathbf{h} \ll \mathbf{k}$ or $\mathbf{k} \ll \mathbf{h}$, otherwise they are **unrelated** (written $\mathbf{h} \not\sim \mathbf{k}$). The direction of a relation is determined by depth:

$$\begin{array}{l}\text{if } |\mathbf{h}| = |\mathbf{k}| \text{ then } \mathbf{h} \sim \mathbf{k} \Longleftrightarrow \mathbf{h} = \mathbf{k},\\ \text{if } |\mathbf{h}| < |\mathbf{k}| \text{ then } \mathbf{h} \sim \mathbf{k} \Longleftrightarrow \mathbf{h} < \mathbf{k},\\ \text{if } |\mathbf{h}| > |\mathbf{k}| \text{ then } \mathbf{h} \sim \mathbf{k} \Longleftrightarrow \mathbf{k} < \mathbf{h}.\end{array}$$

Note that each of our creatures is polycephalic, having lots of different heads (including an empty head), though fortunately all its heads are related.

The key anatomical result is

Theorem 1 (Heads)**.** (i) [Relatedness] *Let* $\mathbf{h}, \mathbf{k}, \mathbf{h}' \in \mathbf{H}(X)$ *be heads,* $\mathbf{d}, \mathbf{d}' \in \mathbf{H}(X) \cup \mathbf{B}(X)$ *be heads or bodies. Then* $\mathbf{h}$ *heads* $\mathbf{kd}$ *only if* **h**,**k** *are related; more precisely,* $\mathbf{h}$ *heads* $\mathbf{kd}$ *iff either* $\mathbf{h}$ *heads* $\mathbf{k}$*, or* $\mathbf{k}$ *properly heads* $\mathbf{h}$ *and the remainder of* $\mathbf{h}$ *heads* $\mathbf{d}$*:*

$$\mathbf{h} \ll \mathbf{kd} \Longleftrightarrow \begin{cases} (i) & \mathbf{h} \ll \mathbf{k} \text{ or}\\ (ii) & \mathbf{k} < \mathbf{h} = \mathbf{kh}' \text{ and } \emptyset \neq \mathbf{h}' \ll \mathbf{d} = \mathbf{h}'\mathbf{d}'.\end{cases}$$

If **h**,**kd** *are related then so are* **h**,**k***:*

$$\mathbf{h} \sim \mathbf{kd} \Longrightarrow \mathbf{h} \sim \mathbf{k}.$$

(ii) [Unrelatedness] *If* $\mathbf{h} \neq \mathbf{k}$ *are distinct heads in* $\mathbf{H}(X)$ *and* $y, z \in X$ *are indices not appearing in either head* ($y = z$ *allowed*)*, then* $\mathbf{h}y, \mathbf{k}z$ *are unrelated:*

$$\mathbf{h}y \not\sim \mathbf{k}z \qquad (y, z \notin \mathbf{h}, \mathbf{k}).$$

(*iii*) [Head Separation] *For any finite collection* $\mathbf{b}_1, \ldots, \mathbf{b}_n$ *of distinct bodies in* $\mathbf{B}(X)$, *there exists a head* $\mathbf{k}$ *such that*

$$\mathbf{k} \ll \mathbf{b}_1, \quad \textit{but} \quad \mathbf{k} \not\ll \mathbf{b}_i \quad \textit{for} \quad i = 2, \ldots, n.$$

Indeed, there is a natural number N *so that all the heads* $\eta_N(\mathbf{b}_i)$ *of depth* N *are already distinct.*

Proof. (1) Suppose $\mathbf{h} = (x_1, \ldots, x_r)$ heads $\mathbf{kd} = (y_1, \ldots, y_s, z_1, z_2, \ldots)$ for $\mathbf{k} = (y_1, \ldots, y_s)$ (x_i, y_j, z_k indices in X). When $r \le s$ (so $\mathbf{k}$ lasts as long as $\mathbf{h}$), we need $x_1 = y_1, x_2 = y_2, ..., x_r = y_r$, i.e. that $\mathbf{h} \ll \mathbf{k}$, as in (*i*). When $r > s$, so $\mathbf{k}$ stops before $\mathbf{h}$ does, we must have $y_1 = x_1, \ldots, y_s = x_s$ (i.e. $\mathbf{k} < \mathbf{h} = \mathbf{kh}'$ for $\mathbf{h}' = (w_1, \ldots, w_{r-s})$ of length $r - s > 0$) and $w_1 = x_{s+1} = z_1, w_2 = x_{s+2} = z_2, \ldots, w_{r-s} = x_r = z_{r-s}$. i.e. $\mathbf{h}' \ll \mathbf{d}$, as in (*ii*).

(2) follows since $\mathbf{kd} \ll \mathbf{h} \Longrightarrow \mathbf{k} \ll \mathbf{kd} \ll \mathbf{h}$.

(3) If $\mathbf{h}y \ll \mathbf{k}z$ then $\mathbf{h} \ll \mathbf{h}y \ll \mathbf{k}z \Longrightarrow \mathbf{h} \ll \mathbf{k}$ (since $\mathbf{h}$ does not involve z) $\Longrightarrow \mathbf{k} = \mathbf{hh}'$ for $\mathbf{h}' \neq \emptyset$ (since $\mathbf{k} \neq \mathbf{h}$). But then $\mathbf{h}y \ll \mathbf{k}z = \mathbf{hh}'z \Longrightarrow y \ll \mathbf{h}'z$ (cancelling $\mathbf{h}$), whereas y does not appear in the nonempty part $\mathbf{h}'$ of $\mathbf{k}$. Analogously $\mathbf{k}z \not\ll \mathbf{h}y$.

(4) Since the bodies are all distinct, for any two labels $i \neq j$ the bodies $\mathbf{b}_i, \mathbf{b}_j$ are distinct, and if N_{ij} is the first place they differ then their heads $\eta_N(\mathbf{b}_i) \neq \eta_N(\mathbf{b}_j)$ of length N already differ for any $N \geq N_{ij}$. If we take $N = \max_{i \neq j} N_{ij}$ to be the largest of these "differentiating places", any two bodies will already be different by their N^{th} place: $\eta_N(\mathbf{b}_i) \neq \eta_N(\mathbf{b}_j)$ if $i \neq j$. In particular, if we take $\mathbf{k} := \eta_N(\mathbf{b}_1)$ we have $\mathbf{k} \ll \mathbf{b}_1$ but $\mathbf{k} \not\ll \mathbf{b}_i$ for all other i (since their initial segment of depth N is $\eta_N(\mathbf{b}_i) \neq \eta_N(\mathbf{b}_1) = \mathbf{k}$). □

3. The Deep Matrix Algebra

Here we put our heads together to construct an algebra of "matrices" spanned by formal "matrix units" $E_{\mathbf{h}}^{\mathbf{k}}$ labelled by "deep" row and column indices $\mathbf{h},\mathbf{k}$.

Theorem 2 (Deep Matrix Algebra Construction)**.** *The* **deep matrix algebra** $\mathcal{E}(X, \mathcal{A})$ *based on* X *over* $\mathcal{A}$ *consists of the free left* $\mathcal{A}$*-module with the the basis of all* **deep matrix units** $E_{\mathbf{h}}^{\mathbf{k}}$ *for finite strings* $\mathbf{h}, \mathbf{k} \in \mathbf{H}(X)$, *together with the* **Deep Multiplication Rules** *for the products* $aE_{\mathbf{h}}^{\mathbf{i}} \cdot bE_{\mathbf{j}}^{\mathbf{k}} \quad (a, b \in \mathcal{A})$:

$$\text{(DMI)} \quad (aE_{\mathbf{h}}^{\mathbf{i}})(bE_{\mathbf{j}}^{\mathbf{k}}) = (aE_{\mathbf{h}}^{\mathbf{i}})(bE_{\mathbf{ij}'}^{\mathbf{k}}) = abE_{\mathbf{hj}'}^{\mathbf{k}}, \;\; \textit{if} \;\; \mathbf{i} \ll \mathbf{j} = \mathbf{ij}',$$

$$\text{(DMII)} \quad (aE_{\mathbf{h}}^{\mathbf{i}})(bE_{\mathbf{j}}^{\mathbf{k}}) = (aE_{\mathbf{h}}^{\mathbf{ji}'})(bE_{\mathbf{j}}^{\mathbf{k}}) = abE_{\mathbf{h}}^{\mathbf{ki}'} \;\; \textit{if} \;\; \mathbf{j} \ll \mathbf{i} = \mathbf{ji}',$$

$$\text{(DMIII)} \quad (aE_{\mathbf{h}}^{\mathbf{i}})(bE_{\mathbf{j}}^{\mathbf{k}}) = 0 \;\; \textit{if} \;\; \mathbf{i} \not\sim \mathbf{j} \; \textit{are unrelated} \quad (\mathbf{i} \not\ll \mathbf{j} \; \textit{and} \, \mathbf{j} \not\ll \mathbf{i}).$$

This is an associative algebra with unit $1_{deep} = E_{\emptyset}^{\emptyset}$. *The construction is an increasing function of both variables, and the construction for general* $\mathcal{A}$ *is just the usual scalar extension by* $\mathcal{A}$ *of the construction for the ground ring* Φ*:*

$$\mathcal{E}(X, \mathcal{A}) \subseteq \mathcal{E}(X, \mathcal{B}), \qquad \mathcal{E}(X, \mathcal{A}) \subseteq \mathcal{E}(Y, \mathcal{A}),$$
$$\mathcal{E}(X, \mathcal{A}) \cong \mathcal{A} \otimes_{\Phi} \mathcal{E}(X, \Phi)$$

under the natural inclusions for unital subalgebras $\mathcal{A} \subseteq \mathcal{B}$ *and subsets* $X \subseteq Y$, *and the natural isomorphism* $a \otimes E_{\mathbf{h}}^{\mathbf{k}} \to aE_{\mathbf{h}}^{\mathbf{k}}$. *In particular,* $\mathcal{E}(\cdot, X)$ *is a functor on unital associative algebras.*

We also have a functor $\mathcal{E}(X,\cdot)$ on unital associative $*$-algebras: if $\mathcal{A}$ carries an involution $a \to \bar{a}$ (e.g. if $\mathcal{A}$ is commutative, $\bar{a} = a$), then $\mathcal{E}(X,\mathcal{A})$ carries a natural conjugate transpose involution uniquely determined by

$$\big(aE^{\mathbf{k}}_{\mathbf{h}}\big)^* := \bar{a}E^{\mathbf{h}}_{\mathbf{k}}.$$

In particular, we always have a transpose involution on the subalgebra $\mathcal{E}(X,\Phi)$.

The deep matrix algebra is generated by the "forward and backward shifts" determined by elements of X :

$$E^{\mathbf{k}}_{\mathbf{h}} = E^{\emptyset}_{\mathbf{h}}E^{\mathbf{k}}_{\emptyset}, \quad \text{where for heads} \quad \mathbf{h} = (x_1,\dots,x_n),\ \mathbf{k} = (y_1,\dots,y_m)$$
$$E^{\emptyset}_{(x_1,\dots,x_n)} = E^{\emptyset}_{x_1}\cdots E^{\emptyset}_{x_n}, \qquad E^{(y_1,\dots,y_m)}_{\emptyset} = E^{y_m}_{\emptyset}\cdots E^{y_1}_{\emptyset}.$$

It can be characterized as the free algebra generated over $\mathcal{A}$ by "orthogonal shifts" S_x, S_x^* satisfying the defining relations

$$S_x^* S_y = \delta_{x,y}1$$

under the correspondence $S_{x_1}\cdots S_{x_n}S^*_{y_m}\cdots S^*_{y_1} = hk^* \to E^{\mathbf{k}}_{\mathbf{h}}$.

Proof. The Deep Multiplication Rules (1) for products of basis elements uniquely determine an algebra structure $\mathcal{E}(X,\mathcal{A})$; it is associative by a tedious direct calculation (superseded by the Deep Frankenstein Isomorphism 20.7(vii)). $E^{\emptyset}_{\emptyset}$ acts as unit from the left on the basis elements $E^{\mathbf{k}}_{\mathbf{j}}$ by the Deep Multiplication Rule (DMI) with $\mathbf{i} = \mathbf{h} = \emptyset$, and from the right on $E^{\mathbf{i}}_{\mathbf{h}}$ by (DMII) with $\mathbf{j} = \mathbf{k} = \emptyset$ (note that always $\emptyset \ll \mathbf{j},\mathbf{k}$ with trivial concatenations $\emptyset\mathbf{m} = \mathbf{m} = \mathbf{m}\emptyset$).

The natural inclusions (2) follow immediately from the Deep Multiplication Rules. Since $\mathcal{E}(X,\Phi)$ is free as Φ-module with basis $E^{\mathbf{k}}_{\mathbf{h}}$, the tensor product $\mathcal{A}\otimes_\Phi \mathcal{E}(X,\Phi)$ as well as $\mathcal{E}(X,\mathcal{A})$ are free as left $\mathcal{A}$-modules with bases $1\otimes E^{\mathbf{k}}_{\mathbf{h}}$ and $E^{\mathbf{k}}_{\mathbf{h}}$, and in view of the Deep Multiplication Rules the natural Φ-linear isomorphism $a\otimes E^{\mathbf{k}}_{\mathbf{h}} \to aE^{\mathbf{k}}_{\mathbf{h}}$ is an algebra isomorphism. Tensoring $\mathcal{A} \to \mathcal{A}\otimes_\Phi \mathcal{E}(X,\Phi)$ is always a functor (or, directly, note $\mathcal{A} \xrightarrow{\phi} \mathcal{A}'$ extends to $\mathcal{E}(X,\mathcal{A}) \xrightarrow{\mathcal{E}(\phi)} \mathcal{E}(X,\mathcal{A}')$ via $\mathcal{E}(\phi)(aE^{\mathbf{k}}_{\mathbf{h}}) = \phi(a)E^{\mathbf{k}}_{\mathbf{h}}$).

(3) The conjugate transpose involution $*$ certainly defines a linear transformation on $\mathcal{E}$ of period 2, $A^{**} = A$, which is an algebra anti-homomorphism $(AB)^* = B^*A^*$ on the basis elements A, B by straight-forward verification: when the middle deep indices $\mathbf{i}$,$\mathbf{j}$ are unrelated we have by (DMIII) $\big((aE^{\mathbf{i}}_{\mathbf{h}})(bE^{\mathbf{k}}_{\mathbf{j}})\big)^* = 0^* = 0 = (\bar{b}E^{\mathbf{j}}_{\mathbf{k}})(\bar{a}E^{\mathbf{h}}_{\mathbf{i}}) = \big(bE^{\mathbf{k}}_{\mathbf{j}}\big)^*\big(aE^{\mathbf{i}}_{\mathbf{h}}\big)^*$, while when $\mathbf{i} \ll \mathbf{j} = \mathbf{i}\mathbf{j}'$ we have by (DMI) $\big((aE^{\mathbf{i}}_{\mathbf{h}})(bE^{\mathbf{k}}_{\mathbf{j}})\big)^* = \big(abE^{\mathbf{k}}_{\mathbf{h}\mathbf{j}'}\big)^* = \bar{b}\,\bar{a}E^{\mathbf{h}\mathbf{j}'}_{\mathbf{k}} = (\bar{b}E^{\mathbf{j}}_{\mathbf{k}})(\bar{a}E^{\mathbf{h}}_{\mathbf{i}}) = \big(bE^{\mathbf{k}}_{\mathbf{j}}\big)^*\big(aE^{\mathbf{i}}_{\mathbf{h}}\big)^*$, and finally when $\mathbf{j} \ll \mathbf{i} = \mathbf{j}\mathbf{i}'$ we have by (DMII) $\big((aE^{\mathbf{i}}_{\mathbf{h}})(bE^{\mathbf{k}}_{\mathbf{j}})\big)^* = \big(abE^{\mathbf{k}\mathbf{i}'}_{\mathbf{h}}\big)^* = \bar{b}\,\bar{a}E^{\mathbf{h}}_{\mathbf{k}\mathbf{i}'} = (\bar{b}E^{\mathbf{j}}_{\mathbf{k}})(\bar{a}E^{\mathbf{h}}_{\mathbf{i}}) = \big(bE^{\mathbf{k}}_{\mathbf{j}}\big)^*\big(aE^{\mathbf{i}}_{\mathbf{h}}\big)^*$. Thus $*$ is an algebra involution.

(4) These generation formulas follow immediately from the Deep Multiplication Rules.

(5) Since the shallow matrix units satisfy $E^{x}_{\emptyset}E^{\emptyset}_{y} = \delta_{x,y}E^{\emptyset}_{\emptyset}$, the map $S_y \to E^{\emptyset}_{y}$, $S^*_x \to E^{x}_{\emptyset}$ induces an epimorphism $\mathcal{S} \to \mathcal{E}$ of the free algebra. The defining relations (5) (shortening any product with an S^*_x to the left of an S_y) show that $\mathcal{S}$ is spanned over $\mathcal{A}$ by elements hk^*, and these spanning elements are sent to the basis elements $E^{\mathbf{k}}_{\mathbf{h}} \in \mathcal{E}$ by (4). But then the hk^* must be $\mathcal{A}$-independent too, and the map is an isomorphism sending the natural $\mathcal{A}$-basis of $\mathcal{S}$ to that of $\mathcal{E}$. This establishes the description (5). □

Let us also comment on the "deepness" of these matrix units. Inside $\mathcal{E}_\emptyset := E^\emptyset_\emptyset \mathcal{E} E^\emptyset_\emptyset = \mathcal{E}$ of depth 0 we have "shallow" matrix units E^y_x $(x, y \in X)$, which by the Deep Multiplication Rules form an infinite family of ordinary matrix units $E^y_x E^w_z = \delta_{yz} E^w_x$ $(\delta_{yz} = 0 \,\text{if}\, y \neq z,\, \delta_{yz} = 1 \,\text{if}\, y = z)$, and thus span an infinite matrix subalgebra $\mathcal{M}_\infty \cong span\{E^y_x \mid x, y \in X\}$. But inside *any* diagonal subalgebra $\mathcal{E}_{\mathbf{m}} := E^{\mathbf{m}}_{\mathbf{m}} \mathcal{E} E^{\mathbf{m}}_{\mathbf{m}}$ (the span of all $E^{\mathbf{mk}}_{\mathbf{mh}}$) of arbitrary depth $|\mathbf{m}|$ we have another family of matrix units $E^{\mathbf{m}y}_{\mathbf{m}x}$ $(x, y \in X)$ with $E^{\mathbf{m}y}_{\mathbf{m}x} E^{\mathbf{m}w}_{\mathbf{m}z} = \delta_{yz} E^{\mathbf{m}w}_{\mathbf{m}x}$, which again spans an infinite matrix subalgebra $\cong \mathcal{M}_\infty$. Thus no matter how deeply we descend, we still find copies of $\mathcal{M}_\infty$ stretching below us. Indeed, $\mathcal{E}$ has a "fractal" nature: by the Deep Multiplication Rules every diagonal subalgebra $\mathcal{E}_{\mathbf{m}}$ is a clone (isomorphic copy) of the entire algebra $\mathcal{E}$ under the map $E^{\mathbf{h}}_{\mathbf{k}} \longrightarrow E^{\mathbf{mh}}_{\mathbf{mk}}$, in view of the rules (i) $E^{\mathbf{mi}}_{\mathbf{mh}} E^{\mathbf{mk}}_{\mathbf{mij}'} = E^{\mathbf{mk}}_{\mathbf{mhj}'}$ if $\mathbf{i} \ll \mathbf{j} = \mathbf{ij}'$, (ii) $E^{\mathbf{mji}'}_{\mathbf{mh}} E^{\mathbf{mk}}_{\mathbf{mj}} = E^{\mathbf{mki}'}_{\mathbf{mh}}$ if $\mathbf{j} \ll \mathbf{i} = \mathbf{ji}'$, (iii) $E^{\mathbf{mi}}_{\mathbf{mh}} E^{\mathbf{mk}}_{\mathbf{mj}} = 0$ if $\mathbf{i}, \mathbf{j}$ are not related $(\mathbf{i} \not\ll \mathbf{j} \,\text{and}\, \mathbf{j} \not\ll \mathbf{i})$.

4. The Scalar Multiple Theorem

THROUGHOUT THE REST OF THIS PAPER WE WILL ASSUME THAT X IS AN INFINITE INDEX SET. J. Cuntz showed that the complex C^*-algebra $\mathcal{O}_\infty$ of operators on a separable Hilbert space generated by a countable family of orthogonal isometries S_i, S_i^* satisfied the Deep Multiplication Rules [2, 1.2 p.175] and, making heavy use of the norm topology, had diameter 1 [2, 3.4 p.184].[2] We now turn to a direct computational proof that, for an arbitrary *infinite* index set X and arbitrary coordinate algebra $\mathcal{A}$, every nonzero deep matrix has a two-sided multiple which is a "scalar"; over a division algebra $\mathcal{A}$ this implies $\mathcal{E}(X, \mathcal{A})$ has diameter 1.

Theorem 3 (Scalar Multiple)**.** *Every nonzero element of the deep matrix algebra $\mathcal{E}(X, \mathcal{A})$ for an infinite set X has a "scalar multiple": if $0 \neq A \in \mathcal{E}(X, \mathcal{A})$ there exist a nonzero coordinate $0 \neq a \in \mathcal{A}$ and deep matrix units E, F (backward and forward shifts) with*

$$EAF = a1_{deep} \quad \text{and} \quad EF = \delta 1_{deep} \; (\delta = 1 \; or \, 0).$$

More specifically, let $A = \sum_{\mathbf{h},\mathbf{k}} a_{\mathbf{h},\mathbf{k}} E^{\mathbf{k}}_{\mathbf{h}} \neq 0$ be a finite sum with coefficients $a_{\mathbf{h},\mathbf{k}} \in \mathcal{A}$, and let $a_{\mathbf{h}_0,\mathbf{k}_0}$ be a nonzero coefficient which is minimal *in the sense that $a_{\mathbf{h},\mathbf{k}} = 0$ when $\mathbf{h} < \mathbf{h}_0$ is a proper initial segment, and also when $\mathbf{h} = \mathbf{h}_0$ but $\mathbf{k} < \mathbf{k}_0$ is a proper initial segment. Then*

$$EAF = a_{\mathbf{h}_0,\mathbf{k}_0} 1_{deep} \quad \text{and} \quad EF = \delta_{\mathbf{h}_0,\mathbf{k}_0} 1_{deep}.$$

for the backward shift $E = E^{\mathbf{h}_0 y}_\emptyset$ and forward shift $F = E^\emptyset_{\mathbf{k}_0 y}$ for any index $y \in X$ which does not appear in any of the finite number of deep indices $\mathbf{h},\mathbf{k}$ with $a_{\mathbf{h},\mathbf{k}} \neq 0$.

Proof. It suffices to find shifts $E = E^{\mathbf{h}_0 y}_\emptyset$, $F = E^\emptyset_{\mathbf{k}_0 y}$ which *isolate* the minimal matrix unit in the sense that $E E^{\mathbf{k}_0}_{\mathbf{h}_0} F = E^\emptyset_\emptyset$ but $E E^{\mathbf{k}}_{\mathbf{h}} F = 0$ for all other pairs of deep indices appearing in A. Then multiplication by E, F will pick out exactly the given miminal term of A and turn it into $a_{\mathbf{h}_0,\mathbf{k}_0} 1$, and automatically $EF = E^{\mathbf{h}_0 y}_\emptyset E^\emptyset_{\mathbf{k}_0 y} = \delta_{\mathbf{h}_0,\mathbf{k}_0} E^\emptyset_\emptyset$ by Heads Unrelatedness 20.1(iii).

[2]In [2, 1.13 p.179] Cuntz established similar results for C^*-algebra $\mathcal{O}_n$ of operators on a separable Hilbert space generated by a finite family of n orthogonal isometries S_i, S_i^* subject to the additional condition $\sum_{i=1}^n S_i S_i^* = 1$. In general, for finite $|X| = n < \infty$, the "correct" deep matrices require this extra condition $\sum_{i=1}^n x_i x_i^* = 1$, and require a slightly different treatment.

Certainly we have $E_{\emptyset}^{\mathbf{h}_0 y} E_{\mathbf{h}_0}^{\mathbf{k}_0} E_{\mathbf{k}_0 y}^{\emptyset} = E_{\emptyset}^{\mathbf{k}_0 y} E_{\mathbf{k}_0 y}^{\emptyset} = E_{\emptyset}^{\emptyset} = 1$ from the Deep Multiplication Rules for *any* index $y \in X$.

We claim that $E\,E_{\mathbf{h}}^{\mathbf{k}}\,F = 0$ for all other terms *as long as we choose y distinct from all indices x which appear in* $\mathbf{h}$,$\mathbf{k}$ (and we can do this for all the nonzero terms in a simultaneously since there are only finitely many of them and there is by hypothesis an *infinite* set X of indices y to choose from).

So assume $a_{\mathbf{h},\mathbf{k}}$ is some other *nonzero* coefficient. By the Deep Multiplication Rule (DMIII) we already have $E_{\emptyset}^{\mathbf{h}_0 y} E_{\mathbf{h}}^{\mathbf{k}} = 0$ unless $\mathbf{h}, \mathbf{h}_0 y$ are related. Since y was chosen not to appear in $\mathbf{h}$, $\mathbf{h}_0 y$ cannot be part of $\mathbf{h}$, so we must have $\mathbf{h} \ll \mathbf{h}_0 y$, and again since $\mathbf{h}$ does not contain y we must have $\mathbf{h} \ll \mathbf{h}_0$. By $a_{\mathbf{h},\mathbf{k}} \neq 0$ and *minimality* of $\mathbf{h}_0$ we cannot have $\mathbf{h} < \mathbf{h}_0$ a *proper* initial segment, so we must have $\mathbf{h} = \mathbf{h}_0$. Since we are working with different coefficients, we have

$$\mathbf{h} = \mathbf{h}_0, \qquad \mathbf{k} \neq \mathbf{k}_0.$$

But then by (DMII) $E\,E_{\mathbf{h}}^{\mathbf{k}}\,F = E_{\emptyset}^{\mathbf{h}_0 y} E_{\mathbf{h}_0}^{\mathbf{k}} E_{\mathbf{k}_0 y}^{\emptyset} = E_{\emptyset}^{\mathbf{k} y} E_{\mathbf{k}_0 y}^{\emptyset} = 0$ vanishes by the Deep Multiplication Rule (DMIII) since $\mathbf{k}y, \mathbf{k}_0 y$ are not related by Heads Unrelatedness 20.1(iii). □

Note that is crucial to isolate a *minimal* matrix unit, and it is crucial for isolation that X be an infinite set. Also, we must use *deep* matrix units; for ordinary $n \times n$ matrix units there is no trouble isolating a single matrix unit by multiplication since *all coefficients are automatically minimal*, but because $\emptyset$ is not allowed as an index we must create the identity matrix as a finite *sum* $\sum_{j=1}^{n} E_j^j$ of n matrix units rather than as a single matrix unit $E_{\emptyset}^{\emptyset}$.

The Scalar Multiple Theorem has important consequences for ideals and centers. First, it guarantees that ideals of $\mathcal{E}$ correspond to ideals of $\mathcal{A}$, just as for finite matrix algebras.

Theorem 4 (Ideal Lattice)**.** *The lattice of ideals $\mathcal{K}$ of $\mathcal{E}(X, \mathcal{A})$ is isomorphic to the lattice of ideals $\mathcal{I}$ of $\mathcal{A}$, since the ideals of $\mathcal{E}$ are precisely all*

$$\mathcal{E}(X, \mathcal{I}) := \sum_{\mathbf{h},\mathbf{k} \in \mathbf{B}(X)} \mathcal{I} E_{\mathbf{h}}^{\mathbf{k}} \qquad \text{for } \mathcal{I} \text{ defined by } \quad \mathcal{I}1_{deep} := \mathcal{K} \cap \mathcal{A}1_{deep}.$$

If $\mathcal{A}$ has an involution then the lattice of $*$-ideals of $\mathcal{E}(X, \mathcal{A})$ under the congjugate transpose involution is isomorphic to the lattice of $*$-ideals of $\mathcal{A}$.

Proof. Let $\mathcal{K}$ be an ideal of $\mathcal{E}$, and define $\mathcal{I}$ as above. Clearly $\mathcal{I}$ is an ideal of $\mathcal{A}$, and by definition $\mathcal{K} \supseteq (\mathcal{I}1_{deep})\mathcal{E} \supseteq \sum \mathcal{I} E_{\mathbf{h}}^{\mathbf{k}} = \mathcal{E}(X, \mathcal{I})$. We must establish the reverse inclusion, and it suffices by surgery to prove $\overline{\mathcal{K}} = \bar{0}$ in the quotient algebra $\mathcal{E}(X, \mathcal{A})/\mathcal{E}(X, \mathcal{I}) \cong \mathcal{E}(X, \overline{\mathcal{A}})$ for $\overline{\mathcal{A}} := \mathcal{A}/\mathcal{I}$ the quotient coordinate algebra. But by the Scalar Multiple Theorem 20.3 (applied to $\overline{\mathcal{A}}$), as soon as $\bar{0} \neq \bar{A} \in \overline{\mathcal{K}}$ there is a nonzero scalar $\bar{0} \neq \bar{a}1_{deep} = \bar{E}\,\bar{A}\,\bar{F} \in \overline{\mathcal{K}} \cap \overline{\mathcal{A}}1_{deep}$. Since the kernel $\mathcal{E}(X, \mathcal{I})$ is contained in $\mathcal{K}$, taking preimages gives $a1_{deep} \in \mathcal{K} \cap \mathcal{A}1_{deep}$ so by definition $a \in \mathcal{I}$ and $\bar{a} = \bar{0}$, a contradiction. Thus $\overline{\mathcal{K}}$ must be $\bar{0}$, as claimed.

In particular, *all* ideals are invariant under the transpose map, and $\mathcal{K}$ is invariant under the *conjugate transpose involution* iff $\mathcal{I}$ is invariant under the conjugation of $\mathcal{A}$. □

Theorem 5 (Simplicity)**.** *The deep matrix algebra $\mathcal{E}(X, \mathcal{A})$ is simple iff the coordinate algebra $\mathcal{A}$ is simple, and is $*$-simple iff the coordinate algebra is $*$-simple.* □

Secondly, the center of the deep matrix algebra consists of the scalar matrices coming from the center of the coordinate algebra, just as with finite matrix algebras.

Theorem 6 (Center). *The centralizer in the deep matrix algebra $\mathcal{E}(X, \mathcal{A})$ of the deep matrix units (even just the shallow backward or forward shifts) consists of the scalar multiples of the identity, and the center of $\mathcal{E}(X, \mathcal{A})$ corresponds to the central multiples of the identity:*

$$Centralizer_{\mathcal{E}}(E_{\emptyset}^{X}) = Centralizer_{\mathcal{E}}(E_{X}^{\emptyset}) = \mathcal{A}1_{deep},$$
$$Center(\mathcal{E}) = Center(\mathcal{A})1_{deep}.$$

Proof. It suffices to show a centralizer is a scalar. By the Scalar Multiple Theorem 20.3, if $C \neq 0$ we have $0 \neq a1_{deep} = ECF$ for matrix units $E = E_{\emptyset}^{\mathbf{k}}$, $F = E_{\mathbf{h}}^{\emptyset}$ with $EF = \delta 1_{deep}$. If C commutes with the shallow backward shifts $E_{\emptyset}^{x}$ it commutes with all backward shifts $E_{\emptyset}^{\mathbf{k}} = E_{\emptyset}^{(y_1,\ldots,y_n)} = E_{\emptyset}^{y_n} \cdots E_{\emptyset}^{y_1}$, so $0 \neq a1_{deep} = ECF = CEF = \delta C$. Then $\delta \neq 0$ forces $\delta = 1$ and $C = a1_{deep}$ is a scalar. Similarly, if C commutes with all backward shifts then $a1_{deep} = ECF = EFC = \delta C = C$. □

5. Frankenstein Actions

We can realize the abstract algebra of deep matrices as operators on the space spanned by all "bodies." Because we are dealing with an infinite index set X, the set $\mathbf{B}(X)$ of bodies is always uncountable. The standard matrix units E_i^j $(i, j \in \mathbb{N})$ have a natural representation as $\mathcal{A}$-linear transformations on a free right $\mathcal{A}$-module with basis $\{v_j\}$ via $E_i^j(v_k) = v_i \delta_{jk}$, so E_i^j replaces v_j by v_i and kills all other v_k. In a similar way, the deep matrix units $E_{\mathbf{h}}^{\mathbf{k}}$ have a natural representation as $\mathcal{A}$-linear operators $F_{\mathbf{h}}^{\mathbf{k}}$ on the **Frankenstein module**, the free right $\mathcal{A}$-module with basis of all bodies $\mathbf{b}$,

$$V(X, \mathcal{A}) = \bigoplus\nolimits_{\mathbf{b} \in \mathbf{B}} \mathbf{b}\mathcal{A},$$

where $F_{\mathbf{h}}^{\mathbf{k}}$ transforms basic bodies beginning with $\mathbf{k}$ into ones beginning with $\mathbf{h}$ according to the **basic Frankenstein Action Rules**

$$\text{(FAR)} \quad F_{\mathbf{h}}^{\mathbf{k}}(\mathbf{b}a) = 0 \text{ if } \mathbf{k} \not\ll \mathbf{b}, \quad F_{\mathbf{h}}^{\mathbf{k}}(\mathbf{k}\mathbf{b}'a) = \mathbf{h}\mathbf{b}'a \text{ if } \mathbf{k} \ll \mathbf{b} = \mathbf{k}\mathbf{b}' \quad (a \in \mathcal{A}).$$

Thus for heads $\mathbf{h}$,$\mathbf{k}$ the $\mathbf{h}^{th}$ "insertion" or "forward shift" or "sewing operator" (sewer, but watch the pronunciation!) $F_{\mathbf{h}}^{\emptyset}$ sews a new head onto the body (in front of its old one), the $\mathbf{k}^{th}$ "deletion" or "backward shift" or "chopping operator" (chopper) $F_{\emptyset}^{\mathbf{k}}$ removes the head $\mathbf{k}$ (so the operation is not a success, killing the patient, if it has a different $|k|$-th head), and the $\mathbf{hk}^{th}$ "chop-and-sewer" or general **Frankenstein operator** $F_{\mathbf{h}}^{\mathbf{k}}$ removes the head $\mathbf{k}$ and sews on the head $\mathbf{h}$ in its place. The **Frankenstein projection** $F_{\mathbf{k}}^{\mathbf{k}}$ kills all bodies not having $\mathbf{k}$ as head, but leaves bodies with head $\mathbf{k}$ alone (actually, it removes the head and then quickly sews it back on). In particular, $F_{\emptyset}^{\emptyset}$ is the identity operator.

We can take linear combinations of Frankenstein operators to form an algebra.

Theorem 7 (Frankenstein Algebra). *As $\mathcal{A}$-linear transformations on $V(X, \mathcal{A})_{\mathcal{A}}$, the Frankenstein operators have (for $\mathbf{b} \in \mathbf{B}(X), a \in \mathcal{A}$) the actions* (*i*) $F^{\emptyset}_{\emptyset}$ *is the identity operator* $F^{\emptyset}_{\emptyset}(\mathbf{b}a) = \mathbf{b}a$,

(*ii*) $F^{\emptyset}_{\mathbf{h}}$ *is the* $\mathbf{h}^{th}$ *"insertion" or "forward shift" or "sewer"*

$$F^{\emptyset}_{\mathbf{h}}(\mathbf{b}a) = \mathbf{hb}a,$$

(*iii*) $F^{\mathbf{k}}_{\emptyset}$ *is the* $\mathbf{k}^{th}$ *"deletion" or "backward shift" or "chopper"*

$F^{\mathbf{k}}_{\emptyset}(\mathbf{b}a) = 0$ *if* $\mathbf{k} \not\ll \mathbf{b}$, $F^{\mathbf{k}}_{\emptyset}(\mathbf{kd}a) = \mathbf{d}a$ *if* $\mathbf{k} \ll \mathbf{b} = \mathbf{kd}$

(*iv*) $F^{\mathbf{k}}_{\mathbf{h}} = F^{\emptyset}_{\mathbf{h}} F^{\mathbf{k}}_{\emptyset}$ *is the* $\mathbf{hk}^{th}$ *"chop-and-sewer"*

$F^{\mathbf{k}}_{\mathbf{h}}(\mathbf{b}a) = 0$ *if* $\mathbf{k} \not\ll \mathbf{b}$, $F^{\mathbf{k}}_{\mathbf{h}}(\mathbf{kb}a) = \mathbf{hd}a$ *if* $\mathbf{k} \ll \mathbf{b} = \mathbf{kd}$,

(*v*) *the Frankenstein projection* $F^{\mathbf{k}}_{\mathbf{k}}$ *is the projection onto the subspace of* V *spanned by all* $\mathbf{b}$ *beginning with* $\mathbf{k}$

$F^{\mathbf{k}}_{\mathbf{k}}(\mathbf{b}a) = 0$ *if* $\mathbf{k} \not\ll \mathbf{b}$, $F^{\mathbf{k}}_{\mathbf{k}}(\mathbf{kd}a) = \mathbf{kd}a$ *if* $\mathbf{k} \ll \mathbf{b} = \mathbf{kd}$.

The Frankenstein operators have the following multiplication table as linear transformations on the Frankenstein module $V(X, \mathcal{A})_{\mathcal{A}}$:

$$\text{(FrI)} \quad F^{\mathbf{i}}_{\mathbf{h}} F^{\mathbf{k}}_{\mathbf{j}} = F^{\mathbf{i}}_{\mathbf{h}} F^{\mathbf{k}}_{\mathbf{ij}'} = F^{\mathbf{k}}_{\mathbf{hj}'} \qquad (\mathbf{i} \ll \mathbf{j} = \mathbf{ij}')$$

$$\text{(FrII)} \quad F^{\mathbf{i}}_{\mathbf{h}} F^{\mathbf{k}}_{\mathbf{j}} = F^{\mathbf{ji}'}_{\mathbf{h}} F^{\mathbf{k}}_{\mathbf{j}} = F^{\mathbf{ki}'}_{\mathbf{h}} \qquad (\mathbf{j} \ll \mathbf{i} = \mathbf{ji}')$$

$$\text{(FrIII)} \quad F^{\mathbf{i}}_{\mathbf{h}} F^{\mathbf{k}}_{\mathbf{j}} = 0 \ \ \textit{if} \ \ \mathbf{i} \not\sim \mathbf{j} \ \textit{are unrelated} \ \ (\mathbf{i} \not\ll \mathbf{j} \ \textit{and} \ \mathbf{j} \not\ll \mathbf{i}).$$

(*vi*) *The distinguished basis of* $\mathbf{b}$*'s turns the right Frankenstein* $\mathcal{A}$*-module* $V(X, \mathcal{A})_{\mathcal{A}}$ *into an* $\mathcal{A}$*-bimodule via* $L_b\mathbf{b}a := b\mathbf{b}a$*, and the Frankenstein operators commute with this bimodule action. Thus the Frankenstein operators and left* $\mathcal{A}$*-multiplications generate a unital associative algebra, the* **Frankenstein algebra**

$$\mathcal{F}(X, \mathcal{A}) := L_{\mathcal{A}} F^{\mathbf{H}(X)}_{\mathbf{H}(X)} = \textstyle\sum_{\mathbf{h},\mathbf{k} \in \mathbf{H}(X)} \mathcal{A} F^{\mathbf{k}}_{\mathbf{h}} \subseteq End(V(X, \mathcal{A})_{\mathcal{A}}),$$

consisting of all Frankenstein transformations, *the finite* $\mathcal{A}$*-linear combinations* $\sum_{\mathbf{h},\mathbf{k}} a_{\mathbf{h},\mathbf{k}} F^{\mathbf{k}}_{\mathbf{h}}$ *of Frankenstein operators. The Frankenstein algebra is a free left* $\mathcal{A}$*-module with the Frankenstein operators as basis, and the Frankenstein module* $V(X, \mathcal{A})_{\mathcal{A}}$ *is naturally a left* $\mathcal{F}(X, \mathcal{A})$*-module.*

(*vii*) *There is a natural* **Deep Frankenstein Isomorphism**

$$\textstyle\sum_{\mathbf{h},\mathbf{k}} a_{\mathbf{h},\mathbf{k}} E^{\mathbf{k}}_{\mathbf{h}} \longrightarrow \sum_{\mathbf{h},\mathbf{k}} a_{\mathbf{h},\mathbf{k}} F^{\mathbf{k}}_{\mathbf{h}}$$

of the deep matrix algebra $\mathcal{E}(X, \mathcal{A})$ *with the Frankenstein algebra* $\mathcal{F}(X, \mathcal{A})$*, hence a faithful action of* $\mathcal{E}(X, \mathcal{A})$ *on* $V(X, \mathcal{A})$.

Proof. (1) These are all special cases of the Frankenstein Action Rules (FAR). Note for (*i*), (*ii*) that all bodies have $\mathbf{k} = \emptyset$ as one of their heads. For (*iv*), note that the general Frankenstein operator may, without changing the result, pause in mid-operation: chopping off head $\mathbf{k}$, pausing (temporarily sewing on an empty head), then resuming (removing the empty head) and sewing on the correct head $\mathbf{h}$.

(2) First note that the Frankenstein operators act only on the bodies $\mathbf{b}$, and hence commute with left and right multiplications by $\mathcal{A}$, which act only on the coefficients a. This allows us to forget about the coefficient a and prove the relations (FrI-III) only on bodies $\mathbf{b}$. In (FrI), $F^{\mathbf{i}}_{\mathbf{h}} F^{\mathbf{k}}_{\mathbf{ij}'}(\mathbf{b})$ vanishes unless $\mathbf{b} = \mathbf{kb}'$ begins with $\mathbf{k}$, in which case it produces $F^{\mathbf{i}}_{\mathbf{h}}(\mathbf{ij}'\mathbf{b}') = \mathbf{hj}'\mathbf{b}'$, which coincides with the action of $F^{\mathbf{k}}_{\mathbf{hj}'}$. In (FrII), $F^{\mathbf{ji}'}_{\mathbf{h}} F^{\mathbf{k}}_{\mathbf{j}}(\mathbf{b})$ vanishes again unless $\mathbf{b} = \mathbf{kb}'$ begins with $\mathbf{k}$, in which case it produces $F^{\mathbf{ji}'}_{\mathbf{h}}(\mathbf{jb}')$, which vanishes unless $\mathbf{ji}' \ll \mathbf{jb}'$, i.e. $\mathbf{i}' \ll \mathbf{b}' = \mathbf{i}'\mathbf{b}''$, so the whole operator vanishes unless $\mathbf{b} = \mathbf{ki}'\mathbf{b}''$ in which case it produces $F^{\mathbf{ji}'}_{\mathbf{h}}(\mathbf{ji}'\mathbf{b}'') = \mathbf{hb}''$, which is precisely the action of $F^{\mathbf{ki}'}_{\mathbf{h}}$. In (FrIII), as usual $F^{\mathbf{i}}_{\mathbf{h}} F^{\mathbf{k}}_{\mathbf{j}}(\mathbf{b})$ vanishes unless $\mathbf{b} = \mathbf{kb}'$, in which case it produces $F^{\mathbf{i}}_{\mathbf{h}}(\mathbf{jb}')$, which vanishes by Heads Relatedness

20.1(i) since we cannot have $\mathbf{i}$ beginning $\mathbf{jb}'$ if $\mathbf{i}$,$\mathbf{j}$ are not related, so the operator kills all basic bodies $\mathbf{b}$ and is the zero transformation.

(3) Since the Frankenstein operators, together with zero, form a semigroup by (2) commuting with left multiplications by $\mathcal{A}$, their finite $\mathcal{A}$-linear combinations form an algebra $\mathcal{F}$ of linear transformations. To see that $\mathcal{F}$ is free as a left $\mathcal{A}$-module, suppose some finite $\mathcal{A}$-linear combination of *distinct* Frankenstein operators with nonzero coefficients $a_{\mathbf{h},\mathbf{k}} \neq 0$ is the zero transformation, $\sum a_{\mathbf{h},\mathbf{k}} F^{\mathbf{k}}_{\mathbf{h}} = 0$. Following our usual procedure, choose a *minimal* head $\mathbf{k}_0$ among the $\mathbf{k}$'s this time (not among the $\mathbf{h}$'s!), so $\mathbf{k} \ll \mathbf{k}_0 \implies \mathbf{k} = \mathbf{k}_0$. *Since X is infinite* and only a finite number of $x_j \in X$ appear in the finite number of finite strings $\mathbf{k}$, there is at least one $y \in X$ which does not appear in any $\mathbf{k}$. Consider the body $\mathbf{b} := \mathbf{k}_0\mathbf{y}$ terminating in all y's (where $\mathbf{y} := (y, y, y, \ldots)$ denotes the constant sequence) . Then $F^{\mathbf{k}}_{\mathbf{h}}(\mathbf{b}) = 0$ unless $\mathbf{k} \ll \mathbf{k}_0\mathbf{y}$, which implies $\mathbf{k} \ll \mathbf{k}_0$ since $\mathbf{k}$ contains no y's, which in turn implies $\mathbf{k} = \mathbf{k}_0$ by minimality of $\mathbf{k}_0$. Then $0 = \sum a_{\mathbf{h},\mathbf{k}} F^{\mathbf{k}}_{\mathbf{h}}(\mathbf{b}) = \sum_{\mathbf{k}=\mathbf{k}_0} a_{\mathbf{h},\mathbf{k}_0} F^{\mathbf{k}_0}_{\mathbf{h}}(\mathbf{k}_0\mathbf{y}) = \sum_{\mathbf{k}=\mathbf{k}_0} a_{\mathbf{h},\mathbf{k}_0}\mathbf{hy}$; but the basic bodies $\mathbf{hy}$ are all distinct since these $\mathbf{h}$ are all *distinct* (the pairs ($\mathbf{h}$,$\mathbf{k}$) all have the same $\mathbf{k} = \mathbf{k}_0$ yet are distinct, so the $\mathbf{h}$ must be distinct), and none of them involve y, so by $\mathcal{A}$-freedom of the $\mathbf{b}$'s this would force the coefficient $a_{\mathbf{h},\mathbf{k}_0}$ of $\mathbf{hy}$ to be zero, a contradiction.

(4) The rule $\varphi(\sum_{\mathbf{h},\mathbf{k}} a_{\mathbf{h},\mathbf{k}} E^{\mathbf{k}}_{\mathbf{h}}) := \sum_{\mathbf{h},\mathbf{k}} a_{\mathbf{h},\mathbf{k}} F^{\mathbf{k}}_{\mathbf{h}}$ is a well-defined $\mathcal{A}$-linear *bijection* of free left $\mathcal{A}$-modules. This map is a homomorphism of algebras since $\varphi(AB) = \varphi(A)\varphi(B)$ on the basis matrix units (both deep and Frankenstein matrix units have the same multiplication rules (DMI-III), (FI-III), so it is an isomorphism of $\mathcal{E}$ on $\mathcal{F}$. □

6. Irreducible Actions

We will identify the irreducible submodules of the Frankenstein action, and thereby hangs a tail.

Theorem 8 (Tails). (*i*) *We say that two bodies* $\mathbf{b}, \mathbf{b}' \in \mathbf{B}(X)$ *have* **the same tail**, *or are* **tail-equivalent** $\mathbf{b} \sim \mathbf{b}'$, *if they become the same once you chop off a big enough head:* $\tau_N(\mathbf{b}) = \tau_{N'}(\mathbf{b}')$. *Algebraically this means that*

$$\mathbf{b} \sim \mathbf{b}' \iff \mathbf{b} = \mathbf{hd}, \mathbf{b}' = \mathbf{h}'\mathbf{d}$$

are obtained from the same tail $\mathbf{d}$ *by sewing on different heads* $\mathbf{h}, \mathbf{h}'$. *Note that we do not demand* $N = N'$, *i.e. that the heads be of the same depth. This gives an equivalence relation on sequences, and the equivalence classes are called the* **tail classes**.

We can get from any one body in a tail class to any other by means of Frankenstein operators,

$$\mathbf{b}' \sim \mathbf{b} \iff \mathbf{b}' = F^{\mathbf{h}}_{\mathbf{h}'}(\mathbf{b}) \in \mathcal{F}(\mathbf{b}) \quad (\textit{for some } \mathbf{h}, \mathbf{h}' \in \mathbf{H}(X)),$$

since by definition $\mathbf{b}' \sim \mathbf{b} \iff \mathbf{b}' = \mathbf{h}'\mathbf{d}, \mathbf{b} = \mathbf{hd}$ *for some tail* $\mathbf{d} \iff \mathbf{b}' = F^{\mathbf{h}}_{\mathbf{h}'}(\mathbf{b})$ *by the Frankenstein Action Rule* (FAR).

(*ii*) *For each tail-class* τ *we define the* **tail-submodule** *of the Frankenstein module* $V(X, \mathcal{A})_{\mathcal{A}}$ *by*

$$V_\tau(X, \mathcal{A}) := \textstyle\sum_{\mathbf{b}\in\tau} \mathbf{b}\mathcal{A}.$$

Because the Frankenstein operators only affect a finite number of indices in a body, they do not change tails ($F^{\mathbf{k}}_{\mathbf{h}}(\mathbf{b})$ *is* 0 *or some* $\mathbf{b}' \sim \mathbf{b}$), *so the Frankenstein transformations of the Frankenstein algebra* $\mathcal{F}(X, \mathcal{A})$ *don't either, and the* *tail-submodules are invariant under the Frankenstein action. Moreover, by* (*i*) *each* $V_\tau(X,\mathcal{A}) = \mathcal{F}(X,\mathcal{A})\mathbf{b}_\tau$ *is a cyclic right* $\mathcal{A}$*-module generated by any body* $\mathbf{b}_\tau$ *in the tail-class* τ*.*

(*iii*) *We have a direct decomposition*

$$V(X,\mathcal{A}) = \bigoplus_\tau V_\tau(X,\mathcal{A})$$

of the Frankenstein module V *into an uncountable number of invariant submodules* $V_\tau(X,\mathcal{A})$ *for the distinct tail-classes* τ*. More generally, for any two-sided ideal* $\mathcal{I}\lhd\mathcal{A}$ *we obtain an invariant* $\mathcal{F}$*-submodule*

$$V_\tau(X,\mathcal{I}) := \sum_{\mathbf{b}\in\tau} \mathbf{b}\mathcal{I} = \mathcal{I}\, V_\tau(X,\mathcal{A})$$

(*equality holding since* $\mathcal{I}$ *is a right ideal*)*, which is a left* $\mathcal{A}$*-module since* $\mathcal{I}$ *is a left ideal, and is Frankenstein-invariant since the tail-class is invariant under the Frankenstein operators* $F^{\mathbf{k}}_{\mathbf{h}}$*.*

It will be important in analyzing irreducible actions that *the Frankenstein operators act selectively.*

Theorem 9 (Body Separation)**.** *The Frankenstein projections separate bodies: for any finite collection* $\mathbf{b}_1,\ldots,\mathbf{b}_n$ *of distinct bodies, there exists a Frankenstein projection* $F^{\mathbf{k}}_{\mathbf{k}}$ *such that*

$$F^{\mathbf{k}}_{\mathbf{k}}(\mathbf{b}_1) = \mathbf{b}_1, \quad \textit{but} \quad F^{\mathbf{k}}_{\mathbf{k}}(\mathbf{b}_2) = \cdots = F^{\mathbf{k}}_{\mathbf{k}}(\mathbf{b}_n) = 0.$$

Proof. By Head Separation 20.1(iii) there is a head $\mathbf{k}$ such that $\mathbf{k} \ll \mathbf{b}_1,\ \mathbf{k} \not\ll \mathbf{b}_i$ for $i \neq 1$, and the result follows from Frankenstein Algebra 20.7(v). □

We can now describe all the invariant submodules of the Frankenstein representation

Theorem 10 (Frankenstein Submodule)**.** *The* $\mathcal{F}(X,\mathcal{A})$*-invariant submodules of the Frankenstein right* $\mathcal{A}$*-module are precisely the direct sums*

$$W = \bigoplus_\tau V_\tau(X,\mathcal{I}_\tau) \qquad (\mathcal{I}_\tau(W) := \{a\in\mathcal{A} \mid a(V_\tau)\subseteq W\} \lhd \mathcal{A}).$$

In particular, the irreducible invariant submodules are precisely all $V_\tau(X,\mathcal{I})$ *for minimal ideals* $\mathcal{I}\lhd\mathcal{A}$*. If* $\mathcal{A}$ *is a simple algebra, the irreducible invariant* $\mathcal{A}$*-submodules of the Frankenstein module are precisely the tail-submodules* $V_\tau(X,\mathcal{A})$*.*

Proof. For any $\mathcal{F}$-invariant right $\mathcal{A}$-submodule W the $\mathcal{I}_\tau(W)$ as defined above are two-sided ideals of $\mathcal{A}$: $\mathcal{A}\mathcal{I}_\tau\mathcal{A} \subseteq \mathcal{I}_\tau$ since $(\mathcal{A}\mathcal{I}_\tau\mathcal{A})(V_\tau) = \mathcal{A}(\mathcal{I}_\tau(\mathcal{A}(V_\tau))) \subseteq \mathcal{A}(\mathcal{I}_\tau(V_\tau))$ (since V_τ is $\mathcal{F}$-invariant) $\subseteq \mathcal{A}(W)$ (by definition of $\mathcal{I}_\tau$) $\subseteq W$ (since W is $\mathcal{F}$-invariant). Clearly $W \supseteq \bigoplus_\tau \mathcal{I}_\tau V_\tau$ by definition of $\mathcal{I}_\tau$; the trick is the reverse inclusion. For any $w = \sum_i \mathbf{b}_i a_i \in W$ we can by the Body Separation Theorem 20.9 pick out each individual $\mathbf{b}$-term using a suitable Frankenstein projection: $\mathbf{b}_i a_i = F^{\mathbf{h}}_{\mathbf{h}}(w) \in F^{\mathbf{h}}_{\mathbf{h}}(W) \subseteq W$. Then for any other $\mathbf{b}'_i = F^{\mathbf{h}}_{\mathbf{h}'}(\mathbf{b}_i)$ in the tail-class τ_i of $\mathbf{b}_i$) (using Tails 20.8(i)), and for any $a' \in \mathcal{A}$, we have $a_i(\mathbf{b}'_i a') = \mathbf{b}'_i a_i a' = F^{\mathbf{h}}_{\mathbf{h}'}(\mathbf{b}_i a_i)a' \in \mathcal{F}(W)a' \subseteq Wa' \subseteq W$ (using the fact that W is a right $\mathcal{A}$-module). This shows that $a_i(V_{\tau_i}) \subseteq W$, so a_i belongs to $\mathcal{I}_{\tau_i}$. Thus $w = \sum_i \mathbf{b}_i a_i = \sum_i a_i(\mathbf{b}_i) \in \sum_i \mathcal{I}_{\tau_i} V_{\tau_i} \subseteq \sum \mathcal{I}_\tau V_\tau$, giving the reverse inclusion, and $W = \bigoplus_\tau \mathcal{I}_\tau V_\tau = \bigoplus_\tau V(X,\mathcal{I}_\tau)$ as claimed.

If $\mathcal{A}$ is simple the only minimal ideal is $\mathcal{I} = \mathcal{A}$. Alternately, its only ideals are itself and 0, $\mathcal{I}_\tau = \mathcal{A}$ or 0 with $V_\tau(X,\mathcal{I}) = V_\tau(X,\mathcal{A})$ or $V_\tau(X,0) = 0$, so the only invariant submodules are the sums of certain V_τ's, the V_τ's are the unique minimal submodules, hence are irreducible. □

We also have a complete characterization of the $\mathcal{F}$-endomorphisms of any Frankenstein module.

Theorem 11 (Endomorphism)**.** *The only $\mathcal{F}(X,\mathcal{A})$-endomorphisms of the Frankenstein right $\mathcal{A}$-module $V(X,\mathcal{A})$ over an arbitrary coordinate ring $\mathcal{A}$ are the central coordinate multiplications on the individual tail-submodules $V_\tau := V_\tau(X,\mathcal{A})$,*

$$End_{\mathcal{F}}(V) = \bigoplus_\tau Center(\mathcal{A})1_{V_\tau},$$

particular the distinct tail-classes provide inequivalent representations of deep matrices: there are no nonzero homomorphism of V_τ into a different V_σ,

$$Hom_{\mathcal{F}}(V_\tau, V_\tau) = Center(\mathcal{A})1_{V_\tau}, \quad Hom_{\mathcal{F}}(V_\tau, V_\sigma) = 0 \quad (\sigma \neq \tau).$$

Proof. (1) The crux is that an $\mathcal{F}$-endomorphism φ must act diagonally: it can only scale up each body, $\varphi(\mathbf{b}) = \mathbf{b}a$, from the fact that we can single out $\mathbf{b}$ by the actions of the Frankenstein projections determined by its many heads, $\bigcap_{N=0}^{\infty} F_{\eta_N(\mathbf{b})}^{\eta_N(\mathbf{b})}(V) = \mathbf{b}\mathcal{A}$ since $F_{\eta_N(\mathbf{b})}^{\eta_N(\mathbf{b})}(\mathbf{b}') = 0$ as soon as $\mathbf{b}'$ has a different N^{th} head $\eta_N(\mathbf{b}') \neq \eta_N(\mathbf{b})$ than $\mathbf{b}$. (Alternately: write $\varphi(\mathbf{b}) = \mathbf{b}a + \sum_i \mathbf{b}_i a_i$ as a sum over distinct bodies, and apply $F_{\mathbf{k}}^{\mathbf{k}}$ of the Body Separation Theorem 20.9 fixing $\mathbf{b}$ and killing the $\mathbf{b}_i$, to get $\varphi(\mathbf{b}) = \varphi(F_{\mathbf{k}}^{\mathbf{k}}(\mathbf{b})) = F_{\mathbf{k}}^{\mathbf{k}}(\varphi(\mathbf{b})) = F_{\mathbf{k}}^{\mathbf{k}}(\mathbf{b}a + \sum_i \mathbf{b}_i a_i) = \mathbf{b}a$.) The multiplier a must be the same for all equivalent basis bodies because the Frankenstein operators act transitively on them by Tails 20.8(i): $\varphi(\mathbf{b}') = \varphi(F(\mathbf{b})) = F(\varphi(\mathbf{b})) = F(\mathbf{b}a) = F(\mathbf{b})a = \mathbf{b}'a$. Thus $\varphi\ |_{V_\tau} = a_\tau 1_{V_\tau}$ is a left multiplication on each tail-submodule. Since these multiplications by a must commute with left multiplications $\mathcal{A}E_\emptyset^\emptyset \subseteq \mathcal{F}$, the multipliers must lie in the center of $\mathcal{A}$ (and clearly all such central multiplications are $\mathcal{F}$-linear). This establishes (1).

(2) follows immediately from this and the direct sum decomposition Tails 20.8(iii) of V into V_τ's. □

This chapter is dedicated to J. Marshall Osborn on the occasion of his 70th birthday.

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Department of Mathematics, University of Virginia, Charlottesville, VA 22903 U.S.A.