FINE GRADINGS ON THE LIE ALGEBRA $\mathfrak{d}_4$

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ABSTRACT. We describe all the fine group gradings, up to equivalence, on the Lie algebra $\mathfrak{d}_4$. This problem is equivalent to finding the maximal abelian diagonalizable subgroups of the automorphism group of $\mathfrak{d}_4$. We prove that there are fourteen by using two different viewpoints. The first approach is computational: we get a full description of the gradings by using a particular implementation of the automorphism group of the Dynkin diagram of $\mathfrak{d}_4$ and some algebraic groups stuff. The second approach, more qualitative, emphasizes some algebraic aspects, as triality, and it is mostly devoted to gradings involving the outer automorphisms of order three.

1. INTRODUCTION

The increasing mathematical activity around gradings on Lie and other kind of algebras is a phenomenon which is in the background of the choices of bases and of maximal sets of quantum observables with additive quantum numbers, as pointed out by Patera et al. in a series of papers on the subject [27, 24, 25, 26]. The Cartan decomposition of a semisimple Lie algebra is one of the most interesting fine gradings (in fact the unique toral one), with a heavy influence in the structure of Lie algebras and in representation theory. But this influence invades also the nearby fields of particle physics via the usual identification of observables with generators in a Cartan subalgebra, and particles living comfortably in the root spaces of a suitable Lie algebra. So, for instance, it is possible to model the strong interaction of nature by means of the $\mathfrak{g}_2$ exceptional Lie algebra. The possibility of using this algebra for describing hypercharge and isospin third component, for a series of 14 elementary particles (quarks u, d, s and mesons π^+, K^+, K_0, π^0 together with their antiparticles), was highlighted by Gunaydin and Gursey in the early seventies (see [11].) In this nice description, two generators of the Cartan subalgebra are identified with the observables of hypercharge and isospin third component. These act simultaneously ad-diagonally on the root spaces (so that the roots give the quantum numbers.) And root space generators are the elementary particle mentioned above. So this is a physical picture of the fine toral grading of $\mathfrak{g}_2$. In a more recent development, this scheme is repeated when describing the strong interaction force by means also of $\mathfrak{g}_2$ (see for instance [10, p. 5].) Thus, strong interaction in nature may be described as the fine toral grading of $\mathfrak{g}_2$ with six gluons as long roots ($g^{r\bar{g}}$, $g^{\bar{r}g}$, $g^{r\bar{b}}$, $g^{\bar{r}b}$, $g^{\bar{g}b}$ and $g^{g\bar{b}}$), and color quarks q^r, $\bar{q}^r$, q^g, $\bar{q}^g$, q^b and $\bar{q}^b$ as short roots, so that each antiquark is the opossite root of the given quark. Along the lines used to describe strong interactions, other Lie algebras are

being used to tentatively describe the rest of the forces in nature. In many of these attempts, fine gradings appear as a common feature.

Other independent motivations toward gradings come from the theory of contraction of Lie algebras. As claimed in [9], contractions are important in physics because they explain in terms of Lie algebras why some theories arise as a limit of more exact theories. For example, the passage from the Poincaré algebra to the Galilei algebra (as $c \to \infty$.) Contraction consists in multiplying the generators of the symmetry by contraction parameters, such that when these parameters reach some singularity point one obtains a non-isomorphic Lie algebra with the same dimension. Graded contractions are a key ingredient when studying contractions that keep "undeformed" certain subalgebra (e.g. see [21].)

Gradings on Lie algebras are also of interest in the setting of Jordan algebras: given a group grading on $L = \sum_{g \in G} L_g$ and a nontrivial element $g \in G$, any element $k \in L_{g^{-1}}$ gives rise to a Jordan algebra structure in L_g by defining $x \circ y = (xk)y$. This also provides a bridge between group gradings and physics, by the classical work of Jordan [18].

The Lie algebra of type $\mathfrak{d}_4$, realized as the algebra of skew-symmetric matrices $\mathfrak{o}(8, \mathbb{K})$, is considered as exceptional by many authors. In spite of being a member of the family of classical Lie algebras $\mathfrak{d}_n$, it is the only one enjoying the benefits of a triality automorphism coming from the special triangular symmetry of its Dinkyn diagram. It can be said that this is the most symmetric algebra in the family. But more symmetries imply more gradings. This may be the reason why the study of gradings on $\mathfrak{d}_4$ does not match the general scheme of gradings in the rest of the algebras $\mathfrak{d}_n$. For instance, the gradings on the algebras $\mathfrak{o}(n, \mathbb{K})$ are computed in [1] and [12], but only for $n \neq 8$.

The main notions about Lie gradings are given in [27] by Patera and Zassenhaus, with a certain rectification in [7] about the existence on a grading group, which must be assumed. Continuation of that work are the papers [12], which deals with the gradings on the classical Lie algebras of types $\mathfrak{sl}(n, \mathbb{K})$ and $\mathfrak{o}(n, \mathbb{K})$, and [13], where the real case is considered. An alternative line of working is followed by Shestakov and Bahturin in [1], but again by using tools of associative algebras. The first work containing a treatment of the gradings on a exceptional Lie algebra is [3], which describes the gradings on $\mathfrak{g}_2$ by taking the octonions as a starting point (see also [2].) But the techniques used in that case are not enough to obtain the gradings on $\mathfrak{f}_4$, so that the same authors develop some computational techniques in [4] to obtain a complete description of the nontoral gradings on $\mathfrak{f}_4$. Such tools are applied successfully in this paper to obtain the fine gradings on $\mathfrak{d}_4$, but we want to remark the existence of alternative methods which could be useful possibly in algebras of bigger rank.

The paper is organized as follows. In Section 2 we give a quick review of the main notions relative to Lie gradings and their translation in terms of groups. Section 3 deals with the problem of finding fine gradings of $\mathfrak{d}_4$ from a computational approach. Since every MAD-group of aut $\mathfrak{d}_4$ lives in the normalizer of a maximal torus, the first task is to fix a maximal torus. Now, to compute the normalizer we need one automorphism extending each element in the isometry group of the root system. Next, in Theorem 1, we give an explicit expression of all the MAD's in terms of these elements, which, in particular, gives us their grading groups and the types of the gradings. Notice that the MAD's of aut $\mathfrak{d}_4$ that map into a 2-sylow of the group

of components of aut $\mathfrak{d}_4$, denoted by Out $\mathfrak{d}_4 \cong S_3$, are essentially described in [12] (at least, how to get them, although without specifying the properties, description or grading groups.) That is why, in Theorem 1, we prove that there are exactly three MAD's which map onto the 3-sylow of Out $\mathfrak{d}_4$, and we add a description of the remaining MAD's for its possible applications (a complete matrix description of all the fine group gradings on $\mathfrak{d}_4$ is given in [5].)

In Section 4 we obtain the same results from a completely different approach, that does not require any computer calculation. This allows to have a quick idea of which are these three MAD's containing outer 3-automorphisms. In fact, we can understand them deeply (with the concrete homogeneous components of the induced grading) only by knowing the gradings on the simpler Lie algebras $\mathfrak{g}_2$ and $\mathfrak{a}_2$ (or easier still, the gradings on the octonion algebra and on $M_{3\times3}(\mathbb{C})$.) This suggests an inductive process based in the knowledge the gradings on Lie algebras of less rank to compute the unknown gradings on the Lie algebras $\mathfrak{e}_6$, $\mathfrak{e}_7$ and $\mathfrak{e}_8$ (applied to $\mathfrak{e}_6$ with success in [6].)

2. Preliminary notions

If V is a finite-dimensional Lie algebra and G is an abelian group, we shall say that the decomposition $V = \oplus_{g \in G} V_g$ is a G-*grading* whenever for all $g, h \in G$, $V_g V_h \subset V_{gh}$ and G is generated by the set $\mathrm{Supp}(G) := \{g \in G \colon V_g \neq 0\}$, called the *support* of the grading. We say that two gradings $V = \oplus_{g \in G} X_g = \oplus_{g' \in G'} Y_{g'}$ are *equivalent* if the sets of homogeneous subspaces are the same up to isomorphism, that is, there are an automorphism $f \in \mathrm{aut}(V)$ and a bijection between the supports $\alpha \colon \mathrm{Supp}(G) \to \mathrm{Supp}(G')$ such that $f(X_g) = Y_{\alpha(g)}$ for any $g \in \mathrm{Supp}(G)$. A convenient invariant for equivalence is that of *type*. Suppose we have a grading on a finite dimensional algebra, then for each positive integer i we will denote (following [15]) by h_i the number of homogeneous components of dimension i. In this case we shall say that the grading is of type $(h_1, h_2, \ldots, h_l)$, for l the greatest index such that $h_l \neq 0$. Of course, the number $\sum_i i h_i$ agrees with the dimension of the algebra. We shall say that the G-grading is a *refinement* of the G'-grading if and only if each homogeneous component $Y_{g'}$ with $g' \in G'$ is a direct sum of some homogeneous components X_g. A grading is *fine* if its unique refinement is the given grading. Our objective is to classify fine gradings up to equivalence.

The ground field $\mathbb{K}$ will be supposed to be algebraically closed and of characteristic zero throughout this work. Notice that the group of automorphisms of the algebra V is an algebraic linear group. There is a deep relationship between gradings on V and quasitori of the group of automorphisms $\mathrm{aut}(V)$, according to [23, §3, p. 104]. If $V = \oplus_{g \in G} V_g$ is a G-grading, the map $\psi \colon \mathfrak{X}(G) = \hom(G, \mathbb{K}^\times) \to \mathrm{aut}(V)$ mapping each $\alpha \in \mathfrak{X}(G)$ to the automorphism $\psi_\alpha \colon V \to V$ given by $V_g \ni x \mapsto \psi_\alpha(x) := \alpha(g)x$ is a group homomorphism. Since G is finitely generated, then $\psi(\mathfrak{X}(G))$ is a quasitorus. And conversely, if Q is a quasitorus and $\psi \colon Q \to \mathrm{aut}(V)$ is a homomorphism, $\psi(Q)$ is formed by semisimple automorphisms and we have a $\mathfrak{X}(Q)$-grading $V = \oplus_{g \in \mathfrak{X}(Q)} V_g$ given by $V_g = \{x \in V \colon \psi(q)(x) = g(q)x \ \forall q \in Q\}$, with $\mathfrak{X}(Q)$ a finitely generated abelian group. If $V = \oplus_{g \in G} V_g$ is a G-grading, the set of automorphisms of V such that every V_g is contained in some eigenspace is an abelian group formed by semisimple automorphisms, which contains to $\psi(\mathfrak{X}(G))$. The grading is fine if and only if such set is a maximal abelian subgroup of semisimple elements,

usually called a *MAD* ("maximal abelian diagonalizable")-group. It is convenient to observe that the number of conjugacy classes of MAD-groups of $\mathrm{aut}(V)$ agrees with the number of equivalence classes of fine gradings on V.

3. Computational approach

First of all we must invoke a version of the Borel-Serre theorem ([28, Theorem 3.15, p. 92]) asserting that a supersolvable subgroup of semisimple elements in an algebraic group is contained in the normalizer of some maximal torus. In particular, this can be applied to finitely generated abelian groups. As we are able to implement a concrete maximal torus and its normalizer in a computer, we will use computational methods to find the maximal quasitori, which, up to conjugation, live in that normalizer.

Fix the symmetric matrix $C = \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix}$ and consider the Lie algebra $\mathcal{L} := \{x \in \mathrm{Mat}_{8\times 8}(\mathbb{K}) \colon x^t C = -Cx\}$, of type $\mathfrak{d}_4$. Take the Cartan subalgebra $\mathfrak{h}$ formed by the diagonal matrices of $\mathcal{L}$. Let $\mathcal{L} = \sum_{\alpha \in \mathfrak{h}^*} L_\alpha$ be the root decomposition relative to $\mathfrak{h}$, that is, $L_\alpha = \{x \in \mathcal{L} \colon [h, x] = \alpha(h)x \quad \forall h \in \mathfrak{h}\}$, and $\Phi = \{\alpha \in \mathfrak{h}^* - \{0\} \colon L_\alpha \neq 0\}$ is the root system. Let $e_{i,j}$ denote the elementary matrix whose all entries are trivial but the (i,j)-entry, which is 1. If we write $h_i = e_{i,i} - e_{i+4,i+4}$, then $\{h_i \colon i = 1, \ldots, 4\}$ is a basis of $\mathfrak{h}$.

If $h = \sum_{i=1}^4 w_i h_i$ is an arbitrary element in $\mathfrak{h}$, and we define $\alpha_i \colon \mathfrak{h} \to \mathbb{K}$ by $\alpha_i(h) = w_i - w_{i+1}$, for $i = 1, 2, 3$, and $\alpha_4(h) = w_3 + w_4$, then $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a basis of Φ. Indeed, if $b_{i,j} := e_{j,i} - e_{i+4,j+4}$, $c_{i,j} := e_{j,i+4} - e_{i,j+4}$ and $d_{i,j} := e_{i+4,j} - e_{j+4,i}$, we have $[h, b_{i,j}] = (w_j - w_i)b_{i,j}$, $[h, c_{i,j}] = (w_j + w_i)c_{i,j}$ and $[h, d_{i,j}] = (-w_i - w_j)d_{i,j}$. Thus we can choose $\mathcal{B} = \{h_i \colon i = 1, \ldots, 4\} \cup \{b_{i,j} \colon i \neq j, \, i, j = 1, \ldots, 4\} \cup \{c_{i,j}, d_{i,j} \colon i < j, \, i, j = 1, \ldots, 4\}$ a basis formed by root vectors, with b's, c's and d's ordered following the rows in the next array

(1)
$$
\begin{array}{lll}
b_{2,1} \in L_{\alpha_1} & b_{3,2} \in L_{\alpha_2} & b_{4,3} \in L_{\alpha_3} \\
c_{3,4} \in L_{\alpha_4} & b_{3,1} \in L_{\alpha_1+\alpha_2} & b_{4,1} \in L_{\alpha_1+\alpha_2+\alpha_3} \\
b_{4,2} \in L_{\alpha_2+\alpha_3} & c_{4,1} \in L_{\alpha_1+\alpha_2+\alpha_4} & c_{4,2} \in L_{\alpha_2+\alpha_4} \\
c_{2,1} \in L_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4} & c_{3,1} \in L_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} & c_{2,3} \in L_{\alpha_2+\alpha_3+\alpha_4}
\end{array}
$$

followed by the opposite roots (take $b_{j,i} \in L_{-\alpha}$ when $b_{i,j} \in L_\alpha$, and $d_{i,j} \in L_{-\alpha}$ when $c_{i,j} \in L_\alpha$) in the same order.

Take $\mathfrak{T} := \{t \in \mathrm{aut}\,\mathcal{L} \colon t|_{\mathfrak{h}} = \mathrm{id}\}$. This is a maximal torus of $\mathrm{aut}\,\mathcal{L}$ such that each element acts diagonally on the root spaces. More precisely, if the automorphism acts with eigenvalues α, β, γ and δ in L_{α_i} respectively for $i = 1, \ldots, 4$, then its matrix relative to the basis $\mathcal{B}$ is

$$
\mathrm{diag}\{1, 1, 1, 1, \alpha, \beta, \gamma, \delta, \alpha\beta, \alpha\beta\gamma, \beta\gamma, \alpha\beta\delta, \beta\delta, \alpha\beta^2\gamma\delta, \alpha\beta\gamma\delta, \beta\gamma\delta,
$$
$$
\tfrac{1}{\alpha}, \tfrac{1}{\beta}, \tfrac{1}{\gamma}, \tfrac{1}{\delta}, \tfrac{1}{\alpha\beta}, \tfrac{1}{\alpha\beta\gamma}, \tfrac{1}{\beta\gamma}, \tfrac{1}{\alpha\beta\delta}, \tfrac{1}{\beta\delta}, \tfrac{1}{\alpha\beta^2\gamma\delta}, \tfrac{1}{\alpha\beta\gamma\delta}, \tfrac{1}{\beta\gamma\delta}\}
$$

and the automorphism will be denoted by $t_{\alpha,\beta,\gamma,\delta}$.

In order to get the normalizer of $\mathfrak{T}$, we need to describe the abstract Weyl group of $\mathfrak{d}_4$. The Cartan matrix of $\mathfrak{d}_4$ is

$$
\begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & -1 \\
0 & -1 & 2 & 0 \\
0 & -1 & 0 & 2
\end{pmatrix}.
$$

So, for $E = \sum_{i=1}^{4} \mathbb{R}\alpha_i$ the euclidean space with the inner product $\langle \ , \ \rangle$, the Weyl group of $\mathfrak{d}_4$ is the subgroup $\mathcal{W}$ of $\mathrm{GL}(E)$ generated by the (simple) reflections s_i with $i = 1, 2, 3, 4$, given by $s_i(x) := x - \langle x, \alpha_i \rangle \alpha_i$, where the Cartan integers $\langle \cdot, \cdot \rangle$ are extracted as usual from the Cartan matrix above. Identifying $\mathrm{GL}(E)$ to $\mathrm{GL}(4, \mathbb{R})$ by means of the matrices relative to the $\mathbb{R}$-basis Δ, the reflections s_i are represented by

$$s_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

$$s_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad s_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Now the group of isometries of Φ is the semidirect product of the Weyl group with the group of automorphisms of the Dynkin diagram. These automorphisms come from permutations σ of $\{1, \ldots, 4\}$ such that $\langle \alpha_i, \alpha_j \rangle = \langle \alpha_{\sigma(i)}, \alpha_{\sigma(j)} \rangle$. Hence, denoting by

$$s_5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad s_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

respectively related to the pictures

the group of automorphisms of the Dynkin diagram of $\mathfrak{d}_4$ is $\langle s_5, s_6 \rangle \cong S_3$ and the group $\mathrm{aut}\,\Phi = \langle s_i : i = 1, \ldots 6 \rangle \equiv \mathcal{V} \cong \mathcal{W} \rtimes S_3$.

We shall consider $\mathcal{V} \subset \mathrm{GL}(4, \mathbb{R})$ ordered lexicographically, that is, first for any two different couples $(i, j), (k, l)$ such that $i, j, k, l \in \{1, 2, 3, 4\}$ we define $(i, j) < (k, l)$ if and only if either $i < k$ or $i = k$ and $j < l$, and second, for any two different matrices $\sigma = (\sigma_{ij})$, $\sigma' = (\sigma'_{ij})$ in $\mathcal{V}$, $\sigma < \sigma'$ if and only if $\sigma_{ij} < \sigma'_{ij}$ where (i, j) is the least element (with the previous order in the couples) such that $\sigma_{ij} \neq \sigma'_{ij}$. One possible way to compute this group with this particular enumeration is provided by the following code implemented with *Mathematica*:

```
V=Table[s_i,{i,6}];
a[L_,x_]:=Union[L, Table[L[[i]].x,{i,Length[L]}],
                    Table[x.L[[i]],{i,Length[L]}]]
Do[V=a[V,s_i],{i,6}]      (3 times repeated)
```

We get a list of $1152 = 2^7 3^2$ elements in the table V which is nothing but the group $\mathrm{aut}\,\Phi$. We are denoting by σ_i the i-th element of $\mathcal{V}$ lexicographically ordered. Now, for each $\sigma \in \mathrm{aut}\,\Phi$, it is possible to choose an automorphism $\tilde{\sigma} \in \mathrm{aut}\,\mathcal{L}$ mapping L_α into $L_{\sigma(\alpha)}$ such that $\tilde{\sigma}|_{\mathfrak{h}}$ agrees with $\sigma \in \mathrm{End}(\mathfrak{h}^*)$ by means of the identification between $\mathfrak{h}$ and $\mathfrak{h}^*$ given by the Killing form ($h \in \mathfrak{h} \mapsto K(h, -) \in \mathfrak{h}^*$.) Concretely, if v_α denotes the root vector specified in (1), we take as $\tilde{\sigma}$ the only automorphism

such that $\tilde{\sigma}(v_\alpha) = v_{\sigma(\alpha)}$ according to the isomorphism theorems in [16] (any other choice would have been in the form $\tilde{\sigma}t$ for some $t \in \mathfrak{T}$.) Thus we have a precise description of the normalizer of the maximal torus as

$$N_{\mathrm{aut}\,\mathfrak{d}_4}(\mathfrak{T}) = \{\tilde{\sigma}_i t_{x,y,z,u} \colon i = 1, \ldots, 1152, \ x, y, z, u \in \mathbb{K}^\times\} =: \mathfrak{N}.$$

The point is that all the MAD-groups of aut $\mathfrak{d}_4$ live in $\mathfrak{N}$ (up to conjugation), so we will be able to find a concrete description of them in terms of $\tilde{\sigma}_i$ and $t_{x,y,z,u}$. This approach allows us to know in detail the homogenous components.

Theorem 1. *There are fourteen maximal quasitori in* aut $\mathfrak{d}_4$. *They are isomorphic to*

$$\begin{array}{ll} \mathbb{Z}_2^n & (n = 5, 6, 7) \\ \mathbb{K}^\times \times \mathbb{Z}_2^n & (n = 3, 4, 5) \\ (\mathbb{K}^\times)^2 \times \mathbb{Z}_2^n & (n = 2, 3) \\ (\mathbb{K}^\times)^3 \times \mathbb{Z}_2 & \\ (\mathbb{K}^\times)^4 & \\ \mathbb{Z}_4 \times \mathbb{Z}_2^3 & \\ \mathbb{Z}_3^3, \ \mathbb{Z}_3 \times \mathbb{Z}_2^3, \ \mathbb{Z}_3 \times (\mathbb{K}^\times)^2 & \end{array}$$

and their precise descriptions, jointly with the types of the gradings induced by them, are summarized in the following table.

	Grading group	Automorphisms generating the group	Type	dim L_e
Q_1	$\mathbb{Z}_2^7$	$t_{-1,1,1,1}, t_{1,1,-1,-1}, t_{1,-1,1,1}$ $\tilde{\sigma}_1, \tilde{\sigma}_3, \tilde{\sigma}_{19}, \tilde{\sigma}_{259}$	(28)	0
Q_2	$\mathbb{Z}_2^5 \times \mathbb{Z}$	$t_{-1,1,1,1}, t_{1,1,-1,-1}, t_{1/u,u,1,1}$ $\tilde{\sigma}_1, \tilde{\sigma}_3, \tilde{\sigma}_{19}$	(28)	1
Q_3	$\mathbb{Z}_2^3 \times \mathbb{Z}^2$	$t_{-1,1,1,1}, t_{1/u,u,1,1}, t_{1/v,1,v,v}$ $\tilde{\sigma}_1, \tilde{\sigma}_3$	(26, 1)	2
Q_4	$\mathbb{Z}_2 \times \mathbb{Z}^3$	$t_{1/u,u,1,1}, t_{1/v,1,v,v}, t_{w,1,1,1/w^2}, \tilde{\sigma}_3$	(25, 0, 1)	3
Q_5	$\mathbb{Z}_2^3 \times \mathbb{Z}$	$t_{-1,1,1,1}, t_{1/v,1,v,v}, t_{1,-1,1,-1}, \tilde{\sigma}_{49}t_{1,-1,1,1}$	(25, 0, 1)	1
Q_6	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$	$t_{1,-1,1,1}, t_{-1,-1,1,1}, \tilde{\sigma}_{259}t_{-1,-1,-1,-1}, \tilde{\sigma}_7$	(24, 2)	0
Q_7	$\mathbb{Z}_2^4 \times \mathbb{Z}$	$t_{-1,1,-1,1}, t_{1,-1,1,1}, t_{1,1,1,u}, \tilde{\sigma}_{280}, \tilde{\sigma}_{634}$	(28)	1
Q_8	$\mathbb{Z}_2^6$	$t_{-1,1,1,1}, t_{1,-1,1,1}, t_{1,1,-1,-1}$ $\tilde{\sigma}_1 t_{i,1,i,-i}, \tilde{\sigma}_{259}t_{1,1,1,-1}, \tilde{\sigma}_{243}t_{-1,-i,1,1}$	(28)	0
Q_9	$\mathbb{Z}_2^5$	$t_{-1,1,1,1}, t_{1,-1,1,1}, t_{1,1,-1,1}, t_{1,1,1,-1}, \tilde{\sigma}_{259}$	(24, 0, 0, 1)	0
Q_{10}	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	$t_{-1,1,1,1}, t_{1/u,u,1,1}, t_{1/v,1,v,v}, \tilde{\sigma}_1$	(20, 4)	2
Q_{11}	$\mathbb{Z}^4$	$t_{u,1,1,1}, t_{1,v,1,1}, t_{1,1,w,1}, t_{1,1,1,z}$	(24, 0, 0, 1)	4
Q_{12}	$\mathbb{Z}_2^3 \times \mathbb{Z}_3$	$t_{1,-1,1,1}, t_{-1,-1,-1,-1}, \tilde{\sigma}_{20}$	(14, 7)	0
Q_{13}	$\mathbb{Z}_3 \times \mathbb{Z}^2$	$t_{1,y,1/y^2,1}, t_{x,1,1/x^3,x}, \tilde{\sigma}_4$	(26, 1)	2
Q_{14}	$\mathbb{Z}_3^3$	$t_{1,\omega,\omega,\omega}, t_{\omega,1,\omega,1}, \tilde{\sigma}_{59}$	(24, 2)	0

TABLE 1

The zero-component L_e in a fine grading of a Lie algebra L is an abelian subalgebra whose dimension is the dimension of the quasitorus producing such grading, as showed in [4, Prop. 10].

Observe that the type is not enough to determine the isomorphy classes of fine gradings on $\mathfrak{d}_4$, in contrast to the gradings on $\mathfrak{f}_4$. For instance there are 4 fine gradings with every homogeneous component one-dimensional.

Remark 1. The correspondence with the MAD-groups which could be obtained following the lines in [12] is given by: $T_{0,8}^{(0)} \cong Q_1$ (conjugated groups), $T_{2,6}^{(0)} \cong Q_2$, $T_{4,4}^{(0)} \cong Q_3$, $T_{6,2}^{(0)} \cong Q_4$, $T_{2,2}^{(1)} \cong Q_5$, $T_{0,2}^{(2)} \cong Q_6$, $T_{2,0}^{(2)} \cong Q_7$, $T_{0,1}^{(3)} \cong Q_8$, $T_{0,4}^{(1)} \cong Q_9$, $T_{4,0}^{(1)} \cong Q_{10}$, $T_{8,0}^{(0)} \cong Q_{11}$. Although that paper does not consider the outer automorphisms of order 3, it clearly shows how to find explicit expressions for the generators of the remaining MAD's, since $\operatorname{Int}\mathfrak{o}_K(8,\mathbb{C}) \cdot \mathbb{Z}_2 = \operatorname{Ad}\operatorname{O}_K(8,\mathbb{C})$ (for $\operatorname{Ad}P(x) = PxP^{-1}$.) Each of the $T_{2k,8/m-2k}^m$ is expressed in terms of a convenient symmetric matrix K.

Before proving Theorem 1, we are going to introduce some tools. Denote by $\pi\colon \mathfrak{N} \to \mathcal{V}$ the group epimorphism given by $\pi(\tilde{\sigma}_i t_{x,y,z,u}) = \sigma_i$. Notice that the action $\mathcal{V} \times \mathfrak{T} \to \mathfrak{T}$ given by $\sigma \cdot t = \tilde{\sigma}t\tilde{\sigma}^{-1} \in \mathfrak{T}$ does not depend on our choice of the concrete extension $\tilde{\sigma}$, but $\sigma \cdot t_{x,y,z,u} = t_{x',y',z',u'}$ for

$$
\begin{aligned}
x' &= x^{b_{11}}y^{b_{12}}z^{b_{13}}u^{b_{14}}, \\
y' &= x^{b_{21}}y^{b_{22}}z^{b_{23}}u^{b_{24}}, \\
z' &= x^{b_{31}}y^{b_{32}}z^{b_{33}}u^{b_{34}}, \\
u' &= x^{b_{41}}y^{b_{42}}z^{b_{43}}u^{b_{44}},
\end{aligned}
$$

if $\sigma = (b_{ij})_{i,j=1,\dots,4} \in \mathcal{V}$. Now consider the quasitori

$$
\begin{aligned}
\mathfrak{T}^{\langle j \rangle} &:= \{t \in \mathfrak{T}\colon \tilde{\sigma}_j \cdot t = t\}, \\
Q(j,t_0) &:= \langle \tilde{\sigma}_j t_0 \cup \mathfrak{T}^{\langle j \rangle}\rangle,
\end{aligned}
$$

if $j \in \{1,\dots 1152\}$ and $t_0 \in \mathfrak{T}$. Notice that the maximal quasitori not contained in $\operatorname{Int}(\mathfrak{d}_4)\cdot\mathbb{Z}_2$, that is, the ones related to the triality automorphism, will be proved to be precisely $Q_{12} = Q(20,\operatorname{id})$, $Q_{13} = Q(4,\operatorname{id})$ and $Q_{14} = Q(59,\operatorname{id})$. This explains the relevance of the considered quasitori. In some groups, like $\operatorname{aut}\mathfrak{f}_4$, every quasitorus is a subgroup of some $Q(j,\operatorname{id})$ (see [4].) Obviously, this is not our case (some of the Q_i's have more than 5 generators), but anyway, any Q minimal nontoral quasitorus (that is, Q is non toral but it does not contain properly any nontoral quasitorus of $\operatorname{aut}\mathfrak{d}_4$) is always conjugated to a subgroup of some $Q(j,t)$. As $Q(j,t)$ is isomorphic to some $Q(i,t')$ if σ_j is conjugated to σ_i, we only have to consider the indices of some representatives of the orbits up to conjugation in $\mathcal{V}$. There are 139 order 2 elements distributed in 7 orbits, 80 order 3 elements distributed in 3 orbits, 228 order 4 elements in 5 orbits, 464 order 6 elements in 7 orbits, 144 order 8 elements in only one orbit, and 96 order 12 elements in the same orbit. One choice of representatives of these orbits, jointly with the relevant quasitori related to them, is the following:

order	representative of the orbit	$\mathfrak{T}^{\langle j \rangle}$	$Q(j, \mathrm{id})$ is toral?
1	894	$\mathfrak{T} \cong \mathbb{K}^{\times 4}$	yes
2	1	$\{t_{\frac{x}{yz}, y, z, z} : x^2 = 1\} \cong \mathbb{K}^{\times 2} \times \mathbb{Z}_2$	no
2	3	$\{t_{x, y, z, \frac{1}{x^2 y^2 z}} : x, y, z \in \mathbb{K}^\times\} \cong \mathbb{K}^{\times 3}$	no
2	9	$\{t_{x, y, \frac{1}{x^2 y^2}, 1} : x, y \in \mathbb{K}^\times\} \cong \mathbb{K}^{\times 2}$	no
2	19	$\{t_{x, \frac{y}{x}, u, u} : y^2 = u^2 = 1\} \cong \mathbb{K}^\times \times \mathbb{Z}_2^2$	no
2	49	$\{t_{\frac{x}{z}, y, z, yz} : x^2 = y^2 = 1\} \cong \mathbb{K}^\times \times \mathbb{Z}_2^2$	no
2	259	$\{t_{x, y, z, u} : x^2 = y^2 = z^2 = u^2 = 1\} \cong \mathbb{Z}_2^4$	no
2	270	$\{t_{1, y, z, u} : y, z, u \in \mathbb{K}^\times\} \cong \mathbb{K}^{\times 3}$	yes
3	4	$\{t_{x, y, x^{-3} y^{-2}, x} : x, y \in \mathbb{K}^\times\} \cong \mathbb{K}^{\times 2}$	no
3	59	$\{t_{x, y, xy, y} : x^3 = y^3 = 1\} \cong \mathbb{Z}_3^2$	no
3	96	$\{t_{1, x, y, \frac{1}{xy}} : x, y \in \mathbb{K}^\times\} \cong \mathbb{K}^{\times 2}$	yes
4	2	$\{t_{x, \frac{y}{x^2}, x, x} : y^2 = 1\} \cong \mathbb{K}^\times \times \mathbb{Z}_2$	no
4	7	$\{t_{x, \frac{y}{x}, 1, 1} : y^2 = 1\} \cong \mathbb{K}^\times \times \mathbb{Z}_2$	no
4	30	$\{t_{1, y, z, \frac{1}{y^2 z}} : y, z \in \mathbb{K}^\times\} \cong \mathbb{K}^{\times 2}$	no
4	34	$\{t_{x, y, x, x} : x^2 = y^2 = 1\} \cong \mathbb{Z}_2^2$	no
4	46	$\{t_{y^2, y, y^2 u, u} : u^2 = 1 = y^4\} \cong \mathbb{Z}_2 \times \mathbb{Z}_4$	no
6	10	$\{t_{1, x, \frac{1}{x^2}, 1} : x \in \mathbb{K}^\times\} \cong \mathbb{K}^\times$	no
6	11	$\{t_{x^{-\frac{2}{3}}, x, x^{-\frac{2}{3}}, 1} : x \in \mathbb{K}^\times\} \cong \mathbb{K}^\times$	no
6	20	$\{t_{x, y, x, x} : x^2 = y^2 = 1\} \cong \mathbb{Z}_2^2$	no
6	55	$\{t_{x, y, 1, y} : x^2 = y^2 = 1\} \cong \mathbb{Z}_2^2$	no
6	56	$\{t_{1, x, \frac{1}{x^2}, x} : x \in \mathbb{K}^\times\} \cong \mathbb{K}^\times$	no
6	78	$\{t_{1, 1, x, \frac{1}{x}} : x \in \mathbb{K}^\times\} \cong \mathbb{K}^\times$	no
6	318	$\{t_{1, 1, 1, 1}\}$	no
8	8	$\{t_{1, y, 1, 1} : y^2 = 1\} \cong \mathbb{Z}_2$	no
12	58	$\{t_{1, 1, 1, 1}\}$	no

TABLE 2

Some comments on the torality of the above quasitori follow. The torality of $Q(96, \mathrm{id})$ and $Q(270, \mathrm{id})$ can be obtained by applying Lemma 2 below, since $\sigma_{96}, \sigma_{270} \in \mathcal{W}$. The remaining cases are nontoral: $Q(318, \mathrm{id})$ because $\sigma_{318} \notin \mathcal{W}$, and for any other index j because the identity component L_e in the grading induced by $Q(j, \mathrm{id})$ verifies that $\dim L_e < 4$ (as in [3, § 2.4] the torality of a grading can be characterized by $\mathrm{rank}\, L_e = \mathrm{rank}\, L$, taking into account that L_e is a reductive Lie algebra.)

The problem here is the proliferation of nontoral quasitori. Observe that $Q(j, \mathrm{id})$ is nontoral for 22 of the 25 chosen indices. This is not surprising, because the group $\mathrm{aut}\, \mathfrak{d}_4$ is smaller than $\mathrm{aut}\, \mathfrak{f}_4$ but of the same rank (in fact, $\mathcal{V}$ is isomorphic to the Weyl group of $\mathfrak{f}_4$), thus there are not so many elements to conjugate. That is why our aim will be finding only the maximal quasitori.

Notice that the Lie algebra fixed by $\tilde{\sigma}_3$ is of type $\mathfrak{b}_3$, so $\tilde{\sigma}_3$ is one of the order 2 automorphisms providing the symmetric pair $(\mathfrak{d}_4, \mathfrak{b}_3)$. Concretely s_6 is in the orbit of σ_3. On the other hand, s_5 is in the orbit of σ_4 so that $\tilde{\sigma}_4$ is an order 3 automorphism fixing a Lie algebra of type $\mathfrak{g}_2$. Thus $\mathrm{aut}\, \mathfrak{d}_4 = \mathrm{Int}\, \mathfrak{d}_4 \cdot \langle 1, \tilde{\sigma}_3 \rangle \cdot \langle 1, \tilde{\sigma}_4, \tilde{\sigma}_4^2 \rangle$. In fact, $Q(3, \mathrm{id}) = Q_4 \cong \mathbb{Z}_2 \times (\mathbb{K}^\times)^3$ and $Q(4, \mathrm{id}) = Q_{13} \cong \mathbb{Z}_3 \times (\mathbb{K}^\times)^2$ are maximal quasitori, since the automorphisms commuting with $\tilde{\sigma}_3$ (respectively,

$\tilde{\sigma}_4$) are just the extensions of automorphisms of $\mathfrak{b}_3$ (respectively of $\mathfrak{g}_2$), so that if we consider the maximal torus of $\mathfrak{b}_3$ (resp. $\mathfrak{g}_2$) we obtain Q_4 (resp. Q_{13}.) This will be explained with more detail in the next section.

In the proof of Theorem 1 and along the paper, we use extensively the following technical results.

Lemma 1. [4, Prop. 7, p. 27] *Let* $F = \{f_0, f_1, \ldots, f_n\} \subset \mathrm{aut}\,\mathfrak{d}_4$ *be a nontoral commutative family of semisimple elements such that* $\{f_1, \ldots, f_n\} \subset \mathrm{Int}\,\mathfrak{d}_4$ *is toral. Then, the subgroup generated by* F *is conjugated to some subgroup of the form* $\langle f, t_1, \ldots, t_n \rangle$ *where* $t_i \in \mathfrak{T}$ *and* $f \in \mathfrak{N}$ *is conjugated to* f_0. *Moreover, this can be done in such a way that* $F \cap \mathfrak{T} \subset \langle f, t_1, \ldots, t_n \rangle$.

Lemma 2. *If* L *is a simple Lie algebra,* T *is a torus of* $\mathrm{aut}\,L$ *and* H *is a toral subgroup of* $\mathrm{aut}\,L$ *commuting with* T, *then* HT *is toral.*

Proof. Let Z be the centralizer of H in $\mathrm{aut}\,L$. As H is toral, there is T' a maximal torus of $\mathrm{aut}\,L$ such that $H \subset T'$. Hence $T' \subset Z$ and it is also a maximal torus of Z. But $T \subset Z$ so that there is $p \in Z$ such that $pTp^{-1} \subset T'$. Consequently $p(HT)p^{-1} = HpTp^{-1} \subset HT' \subset (T')^2 \subset T'$ and HT is contained in the torus $p^{-1}T'p$. $\square$

Lemma 3. *Fix* $t \in \mathfrak{T}$ *and* $j \in \{1, \ldots, 1152\}$.

- *If* $\mathfrak{T}^{\langle j \rangle}$ *is finite, there is* $s \in \mathfrak{T}$ *such that* $\mathrm{Ad}\,s(\tilde{\sigma}_j t) = \tilde{\sigma}_j$.
- *If* $\mathfrak{T}^{\langle j \rangle}$ *is not finite, there are* $s \in \mathfrak{T}$ *and* $t' \in \mathfrak{T}^{\langle j \rangle}$ *such that* $\mathrm{Ad}\,s(\tilde{\sigma}_j t) = \tilde{\sigma}_j t'$.

Therefore $Q(j, t)$ *is conjugated to* $Q(j, \mathrm{id})$ *in all the cases. In particular* $Q_{12} \cong Q(20, t)$, $Q_{13} \cong Q(4, t)$ *and* $Q_{14} \cong Q(59, t)$ *for any* $t \in \mathfrak{T}$.

Proof. According to the proof and notations of [4, Prop. 6, p. 26], it is enough to check that $\mathfrak{T}^{\langle j \rangle} \cap \mathfrak{S}^{\langle j \rangle}$ is finite for the indices j corresponding to the representatives of the orbits. This is a straightforward computation. For instance, in our remarked cases $\mathfrak{T}^{\langle 59 \rangle} \cong \mathbb{Z}_3^2$, $\mathfrak{T}^{\langle 20 \rangle} \cong \mathbb{Z}_2^2$, and, although $\mathfrak{T}^{\langle 4 \rangle}$ is not finite, $\mathfrak{T}^{\langle 4 \rangle} \cap \mathfrak{S}^{\langle 4 \rangle} = \{t_{x,1,1,x} : x^3 = 1\} \cong \mathbb{Z}_3$ is so. $\square$

Notice for further use that the conjugation automorphism has been taken $\mathrm{Ad}\,s$ for certain $s \in \mathfrak{T}$, so that it does not "move" the normalizer $\mathfrak{N}$ nor any element in $\mathfrak{T}$.

Remark 2. One of the most useful tools in the computational approach to the gradings on $\mathfrak{f}_4$ was [4, Lemma 2], according to which every quasitorus of $\mathrm{aut}\,\mathfrak{f}_4$ such that $\mathfrak{X}(Q)$ has two generators is toral. Of course we cannot apply this result to $\mathrm{aut}\,\mathfrak{d}_4$, because one single outer automorphism generates a nontoral quasitorus. But even if we consider a subquasitorus Q of the connected component $\mathrm{Int}\,\mathfrak{d}_4$, it could happen that Q were nontoral with two generators. Indeed, take $Q = \langle \tilde{\sigma}_1, t_{-1,1,1,1} \rangle \cong \mathbb{Z}_2^2$, which is nontoral but $\sigma_1 \in \mathcal{W}$ (equivalently, $\tilde{\sigma}_1$ is an inner automorphism.) The nontorality of Q is consequence of the nontorality of $Q(1, \mathrm{id}) \cong Q \times (\mathbb{K}^\times)^2$, by applying Lemma 2. Moreover, it is possible to prove that if Q is a nontoral subquasitorus of $\mathrm{Int}\,\mathfrak{d}_4$ such that $\mathfrak{X}(Q)$ has two generators, Q can be subconjugated inside $Q(1, \mathrm{id})$. The source of these differences between the behavior of $\mathrm{aut}\,\mathfrak{f}_4$ and $\mathrm{Int}\,\mathfrak{d}_4$ is that $\mathrm{Int}\,\mathfrak{d}_4$ is not simply connected. Anyway, what we will use is a much weaker result: Note that if Q is the quasitorus generated by $\{\tilde{\sigma}_i t_1, t_2\}$ with $\sigma_i \in \mathcal{W}$ of order 3 and $t_j \in \mathfrak{T}$, then Q is obviously toral by Lemma 2 (σ_i would be conjugated

to σ_{96}), and also if $\sigma_i \in \mathcal{W}$ is of any other order but t_2 is contained in a subtorus of $\mathfrak{T}^{\langle i \rangle}$.

Proof of Theorem 1. First of all notice that these quasitori Q_i are not conjugated, since the grading groups $\mathfrak{X}(Q_i)$ are not isomorphic. Besides, we have remarked that $Q_1, \ldots, Q_{11}$ are conjugated to the MAD-groups obtained from [12], which cover all the MAD's contained in $\operatorname{Int}\mathfrak{d}_4 \cdot \langle 1, \tilde{\sigma}_3 \rangle$. Thus, what we are going to prove is that Q_{12}, Q_{13} and Q_{14} are maximal quasitori, and that if Q is a maximal quasitorus not conjugated to any subgroup of $\operatorname{Int}\mathfrak{d}_4 \cdot \langle 1, \tilde{\sigma}_3 \rangle$, then it is conjugated to one of these three quasitori.

We now check that Q_{14} is a maximal quasitorus. Let us prove that if $f \in \operatorname{aut}\mathfrak{d}_4$ commutes with Q_{14}, then f belongs to Q_{14}. With that purpose consider $Z = \mathfrak{C}_{\operatorname{aut}\mathfrak{d}_4}(\mathfrak{T}^{\langle 59 \rangle})$. The group $\langle Q_{14} \cup \{f\} \rangle$ is an abelian subgroup of Z, as well as its closure in the Zarisky topology. But this is again a quasitorus, whence it is contained in the normalizer of some maximal torus T of Z. In particular $\langle Q_{14} \cup \{f\} \rangle \subset N_Z(T)$. By construction also $\mathfrak{T} \subset Z$ so that there is some $p \in Z$ such that $pTp^{-1} = \mathfrak{T}$. Consequently $p\langle Q_{14} \cup \{f\} \rangle p^{-1} \subset N_{\operatorname{aut}\mathfrak{d}_4}(\mathfrak{T}) = \mathfrak{N}$ and $pfp^{-1}, p\tilde{\sigma}_{59}p^{-1} \in \mathfrak{N} \cap \mathfrak{C}_{\operatorname{aut}\mathfrak{d}_4}(\mathfrak{T}^{\langle 59 \rangle})$ with $ptp^{-1} = t$ for any $t \in \mathfrak{T}^{\langle 59 \rangle}$. Take $j_1, j_2 \in 1, \ldots, 1152$ and $t_1, t_2 \in \mathfrak{T}$ such that $pfp^{-1} = \tilde{\sigma}_{j_1}t_1, p\tilde{\sigma}_{59}p^{-1} = \tilde{\sigma}_{j_2}t_2$. A straightforward computation in the computer tell us that $\{i \in 1, \ldots, 1152 \colon \sigma_i \cdot t_{1,\omega,\omega,\omega} = t_{1,\omega,\omega,\omega}, \sigma_i \cdot t_{\omega,1,\omega,1} = t_{\omega,1,\omega,1}\} = \{59, 835, 894\}$. In particular j_i must be one of those indices, and $\sigma_{j_i} = \sigma_{59}^{n_i}$ for $n_i \in \{0, 1, 2\}$. Moreover, $n_2 \neq 0$, otherwise the grading produced by $pQ_{14}p^{-1}$ would be toral. We can take $t_2 = \operatorname{id}$ by replacing p by sp, being s the element in $\mathfrak{T}$ such that $s\tilde{\sigma}_{59}^{n_2}t_2s^{-1} = \tilde{\sigma}_{59}^{n_2}$ as in Lemma 3. As $\tilde{\sigma}_{59}^{n_1}t_1$ and $\tilde{\sigma}_{59}^{n_2}$ commute, as well as $\tilde{\sigma}_{59}^{n_1}$ and $\tilde{\sigma}_{59}^{n_2}$, then t_1 commutes with $\tilde{\sigma}_{59}$ so that $t_1 \in \mathfrak{T}^{\langle 59 \rangle}$. So $pfp^{-1} = \tilde{\sigma}_{59}^{n_1}t_1 \in Q(59, \operatorname{id}) = pQ_{14}p^{-1}$ and $f \in Q_{14}$.

Now let $f \in C_{\operatorname{aut}\mathfrak{d}_4}(Q_{13})$ and as before find an automorphism $p \in \operatorname{aut}\mathfrak{d}_4$ such that $ptp^{-1} = t$ for all $t \in \mathfrak{T}^{\langle 4 \rangle}$ and $pfp^{-1} = \tilde{\sigma}_{j_1}t_1, p\tilde{\sigma}_4p^{-1} = \tilde{\sigma}_{j_2}t_2 \in C_{\mathfrak{N}}(\mathfrak{T}^{\langle 4 \rangle})$ for $t_1, t_2 \in \mathfrak{T}$. Thus $\sigma_{j_i} \in \{\sigma_k \in \mathcal{V} \colon \mathfrak{T}^{\langle 4 \rangle} \subset \mathfrak{T}^{\langle k \rangle}\} = \{\sigma_4^n \colon n = 0, 1, 2\} \cdot \{1, \sigma_3\}$ (isomorphic to the group of permutations S_3.) Besides $p\tilde{\sigma}_4p^{-1}$ has order 3, so there is $l \in \{1, 2\}$ such that $\sigma_{j_2} = \sigma_4^l$. Now we can assume that $t_2 \in \mathfrak{T}^{\langle 4 \rangle}$, by replacing p by sp, being s the element as in Lemma 3 verifying $s\tilde{\sigma}_4^lt_2s^{-1} \in Q(4, \operatorname{id})$. Thus $pQ_{13}p^{-1} = Q(4, \operatorname{id})$. Note that σ_4 commutes neither with σ_3 nor $\sigma_3\sigma_4$ nor $\sigma_3\sigma_4^2$, since $\sigma_3\sigma_4 = \sigma_4^2\sigma_3$ with σ_3 of order 2. But pfp^{-1} commutes with $p\tilde{\sigma}_4p^{-1}$, so their projections by π commute and $\sigma_{j_1} \in \{\sigma_4^n \colon n = 0, 1, 2\}$. In particular $\tilde{\sigma}_{j_1}$ commutes with $\tilde{\sigma}_4^lt_2$ and hence t_1 does so. Hence $t_1 \in \mathfrak{T}^{\langle 4 \rangle}$, $pfp^{-1} = \tilde{\sigma}_{j_1}t_1 \in Q(4, \operatorname{id}) = pQ_{13}p^{-1}$ and $f \in Q_{13}$.

Finally, for $f \in C_{\operatorname{aut}\mathfrak{d}_4}(Q_{12})$ we again find an automorphism $p \in \operatorname{aut}\mathfrak{d}_4$ such that $ptp^{-1} = t$ for all $t \in \mathfrak{T}^{\langle 20 \rangle}$ and $pfp^{-1}, p\tilde{\sigma}_{20}p^{-1} \in C_{\mathfrak{N}}(\mathfrak{T}^{\langle 20 \rangle})$. Denote $J = \{\sigma_k \in \mathcal{V} \colon \mathfrak{T}^{\langle 20 \rangle} \subset \mathfrak{T}^{\langle k \rangle}\}$. As $p\tilde{\sigma}_{20}p^{-1} = \tilde{\sigma}_jt$ for some $\sigma_j \in J$, in particular $\sigma_j^2 \in J$ has order exactly 3 (not 1 because the grading induced by $\langle \tilde{\sigma}_{20}^2, \mathfrak{T}^{\langle 20 \rangle} \rangle$ is also nontoral.) But there are 32 order 3 elements in J, all of them in the orbit of σ_4. Hence $(\tilde{\sigma}_jt)^2$ is conjugated to $\tilde{\sigma}_4t_1$ without moving $\mathfrak{N}$, but also without moving $\mathfrak{T}^{\langle 20 \rangle}$, since the corresponding 32 conjugating elements in $\mathcal{V}$ can be taken inside J. Selecting now the order 2 elements $\sigma_i \in J$ commuting with σ_4 (looking for σ_j^3), there is only one, $\sigma_{259} = -\operatorname{id}$ (here id denotes the 4×4 identity matrix.) Replacing p, we have found that $pQ_{12}p^{-1} = \langle \tilde{\sigma}_4t_1, \tilde{\sigma}_{259}t_2, \mathfrak{T}^{\langle 20 \rangle} \rangle =: Q$ with $pfp^{-1} \in$

$\mathfrak{N}$ commuting with Q. Notice now that $\tilde{\sigma}_{40}Q\tilde{\sigma}_{40}^{-1} = Q(20, t_3)$, so there is $s \in \mathfrak{T}$ such that $(s\tilde{\sigma}_{40})Q(s\tilde{\sigma}_{40})^{-1} = Q_{12}$. Replace again p by $s\tilde{\sigma}_{40}p$ to get $pfp^{-1} \in \mathfrak{N}$ commuting with $pQ_{12}p^{-1} = Q_{12}$. As there are only 6 elements $\sigma_i \in J$ with $\sigma_i\sigma_{20} = \sigma_{20}\sigma_i$, they must be just $\{\sigma_{20}^n : n = 0, \ldots 5\}$. Thus there is n such that $pfp^{-1} = \tilde{\sigma}_{20}^n t_4$, which commutes with $\tilde{\sigma}_{20} \in pQ_{12}p^{-1}$, so that $t_4 \in \mathfrak{T}^{\langle 20 \rangle}$ and $pfp^{-1} \in Q_{12} = pQ_{12}p^{-1}$. We have already seen that Q_{12}, Q_{13} and Q_{14} are maximal abelian diagonalizable groups.

Suppose now that we have another Q maximal quasitorus contained in $\mathfrak{N}$ such that nor Q neither any of its conjugated quasitori are contained in $\text{Int}\,\mathfrak{d}_4 \cdot \{1, \tilde{\sigma}_3\}$. Recall that there are three orbits (under conjugation in $\mathcal{V}$) of order 3 elements in $\mathcal{V}$, given by the representatives σ_4, σ_{59} and σ_{96}. We are going to prove that the abelian group $\pi(Q)$ intersects the orbit of either σ_{59} or σ_4. Thus, we will be able to assume that either $\tilde{\sigma}_{59}t$ or $\tilde{\sigma}_4 t$ belongs to Q (for some $t \in \mathfrak{T}$), by conjugating by an element in $\mathfrak{N}$. Indeed, consider $\mathcal{V}_0 = \mathcal{W}$, $\mathcal{V}_1 = \mathcal{W}\sigma_3$, $\mathcal{V}_2 = \mathcal{W}\sigma_3\sigma_4$, $\mathcal{V}_3 = \mathcal{W}\sigma_3\sigma_4^2$, $\mathcal{V}_4 = \mathcal{W}\sigma_4$ and $\mathcal{V}_5 = \mathcal{W}\sigma_4^2$ (obviously $\mathcal{V} = \cup_{i=0}^6 \mathcal{V}_i$.) By hypothesis, $\pi(Q)$ is not contained in $\mathcal{V}_0 \cup \mathcal{V}_1$. It is neither contained in $\mathcal{V}_0 \cup \mathcal{V}_i$, for $i = 2, 3$, since they are conjugated to $\mathcal{V}_0 \cup \mathcal{V}_1$ by means of σ_4 and σ_4^2 respectively, since $\sigma_3\sigma_4 = \sigma_4^2\sigma_3$. Moreover, $\pi(Q)$ is not contained in $\mathcal{V}_0 \cup \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$, since $\pi(Q)$ is abelian, and none of the elements in $\mathcal{V}_i$ commutes with none of the elements in $\mathcal{V}_j$ nor $\mathcal{V}_k$, for $\{i, j, k\}$ distinct indices in $\{1, 2, 3\}$. Hence $\pi(Q) \cap (\mathcal{V}_4 \cup \mathcal{V}_5) \neq \emptyset$, but every element in $\mathcal{V}_4 \cup \mathcal{V}_5$ has order multiple of 3 $(3, 6, 12)$ and one of its powers is an order 3 element in the orbit of σ_{59} or σ_4 ($\sigma_{96} \in \mathcal{W}$), which also belongs to $\pi(Q)$.

First, suppose that $\sigma_4 \in \pi(Q)$. Note that $\pi(Q) = (\pi(Q) \cap (\mathcal{V}_0 \cup \mathcal{V}_1)) \cdot \langle \sigma_4 \rangle$, but, since it is abelian, $\pi(Q) = \pi(Q) \cap \mathcal{W} \cdot \langle \sigma_4 \rangle$ (take into account that σ_4 does not commute with any element in $\mathcal{V}_{1,2,3}$.) Suppose that there is an order 3 element in $\pi(Q) \cap \mathcal{W}$. As $\{\sigma \in \mathcal{W} : \sigma\sigma_4 = \sigma_4\sigma, \sigma^3 = \text{id}\} = \langle \sigma_{952} \rangle \cong \mathbb{Z}_3$, we have that $\sigma_{952} \in \pi(Q)$, but $\{\sigma \in \mathcal{W} : \sigma\sigma_4 = \sigma_4\sigma, \sigma\sigma_{952} = \sigma_{952}\sigma\} = \langle \sigma_{952}, \sigma_{259} = -\text{id} \rangle \cong \mathbb{Z}_6$, so that either $\pi(Q) = \langle \sigma_{952}, \sigma_4 \rangle$ or $\pi(Q) = \langle \sigma_{952}, -\sigma_4 \rangle$. In the first case there are $t_1, t_2 \in \mathfrak{T}$ with $Q = \langle \tilde{\sigma}_4 t_1, \tilde{\sigma}_{952}t_2, t_3 \rangle$, for $\langle t_3 \rangle = \mathfrak{T}^{\langle 4 \rangle} \cap \mathfrak{T}^{\langle 952 \rangle} = \{t_{x,1,1,x} : x^3 = 1\} \cong \mathbb{Z}_3$. As $\langle \tilde{\sigma}_{952}t_2, t_3 \rangle$ is necessarily toral by Remark 2, apply Lemma 1 to get that Q is conjugated to $\langle t_3, t_4, \tilde{\sigma}_i t_5 \rangle$ with $\tilde{\sigma}_i t_5$ conjugated to $\tilde{\sigma}_4 t_1$, and, in particular, $\sigma_i \notin \mathcal{W}$ (σ_i could be in other orbit, and could have order 3, 6 and 12, but still belongs to $\mathcal{V}_4 \cup \mathcal{V}_5$.) By maximality, Q is conjugated to $Q(i, t_5) \cong Q(i, \text{id})$ and $\mathfrak{T}^{\langle i \rangle} = \langle t_3, t_4 \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_r$. The only possibility, according to Table 2, is that σ_i is in the orbit of σ_{59} and Q is conjugated to $Q(59, \text{id}) = Q_{14}$. In the second case, there are $t_1, t_2 \in \mathfrak{T}$ with $Q = \langle \tilde{\sigma}_{1149}t_1, \tilde{\sigma}_{952}t_2 \rangle$ ($-\sigma_4 = \sigma_{1149}$), since $\mathfrak{T}^{\langle 1149 \rangle} \cap \mathfrak{T}^{\langle 952 \rangle} = \{t_{1,1,1,1}\}$. Obviously the grading induced by $\langle \tilde{\sigma}_{952}t_2 \rangle$ is toral and, by Lemma 1 there are $t_4, t_5 \in \mathfrak{T}$ such that Q is conjugated to $\langle t_4, \tilde{\sigma}_i t_5 \rangle$ for t_4 of order $3l$. By maximality of Q, this set coincides with $Q(i, t_5)$ and hence $\mathfrak{T}^{\langle i \rangle} \cong \mathbb{Z}_{3l}$, what it is a contradiction. Now we suppose that $\pi(Q) \cap \mathcal{W}$ does not contain order 3 elements. Hence $\pi(Q) \cap \mathcal{W} \subset \{\pm\text{id}, \pm\sigma_{111}, \pm\sigma_{211}, \pm\sigma_{249}\}$ (the only elements in $\mathcal{W}$ commuting with σ_4 of order coprime with 3.) Consequently the possibilities for $\pi(Q)$ are: $\langle \sigma_4 \rangle$, $\langle -\sigma_4 \rangle$, $\langle \sigma_4, \sigma_j \rangle$ and $\langle -\sigma_4, \sigma_j \rangle$, for some $j \in \{111, 211, 249\}$. If $\pi(Q) = \langle \sigma_4 \rangle$, then $Q = Q(4, t) \cong Q(4, \text{id}) = Q_{13}$. As $-\sigma_4$ is in the orbit of σ_{20}, if $\pi(Q) = \langle -\sigma_4 \rangle$, then $Q \cong Q(20, t) \cong Q(20, \text{id}) = Q_{12}$. If $\pi(Q) = \langle \sigma_4, \sigma_j \rangle$, there are $t_1, t_2 \in \mathfrak{T}$ such that $Q = \langle \tilde{\sigma}_4 t_1, \tilde{\sigma}_j t_2 \rangle \cdot \mathfrak{T}^{\langle 4 \rangle} \cap \mathfrak{T}^{\langle j \rangle}$. But $\mathfrak{T}^{\langle 4 \rangle} \cap \mathfrak{T}^{\langle j \rangle} \cong \mathbb{K}^\times$ for any of the three indices, so the grading induced by $\{\tilde{\sigma}_j t_2\} \cup (\mathfrak{T}^{\langle 4 \rangle} \cap \mathfrak{T}^{\langle j \rangle})$ is toral by Remark 2, and by

Lemma 1 there are $t_3, t_4 \in \mathfrak{T}$ and an index i such that $Q = \langle \tilde{\sigma}_i t_3, t_4 \rangle \cdot \mathfrak{T}^{\langle 4 \rangle} \cap \mathfrak{T}^{\langle j \rangle} = Q(i, t_3)$. In particular $\mathfrak{T}^{\langle i \rangle} = \langle t_4 \rangle \cdot \mathfrak{T}^{\langle 4 \rangle} \cap \mathfrak{T}^{\langle j \rangle} \cong \mathbb{Z}_{2s} \times \mathbb{K}^{\times}$, a contradiction (the only possibilities with $\mathfrak{T}^{\langle i \rangle} \cong \mathbb{Z}_{2s} \times \mathbb{K}^{\times}$ would be $i = 2, 7$, up to conjugation, but $\sigma_7 \in \mathcal{V}_0$ and $\sigma_2 \in \mathcal{V}_2$.) Finally, if $\pi(Q) = \langle -\sigma_4, \sigma_j \rangle$, there are $t_1, t_2 \in \mathfrak{T}$ such that $Q = \langle \tilde{\sigma}_{1149} t_1, \tilde{\sigma}_j t_2 \rangle \cdot \mathfrak{T}^{\langle 1149 \rangle} \cap \mathfrak{T}^{\langle j \rangle}$. Although $\mathfrak{T}^{\langle 1149 \rangle} \cap \mathfrak{T}^{\langle j \rangle} \cong \mathbb{Z}_2$, we can apply Remark 2 to get that the grading induced by $\{\tilde{\sigma}_j t_2\} \cup (\mathfrak{T}^{\langle 1149 \rangle} \cap \mathfrak{T}^{\langle j \rangle})$ is toral, because $\mathfrak{T}^{\langle 1149 \rangle} \cap \mathfrak{T}^{\langle j \rangle} \subset \mathfrak{T}^{\langle 4 \rangle} \cap \mathfrak{T}^{\langle j \rangle} \cong \mathbb{K}^{\times}$. Thus, by Lemma 1, the quasitorus Q is conjugated to $Q(i, t)$ for certain index i verifying $\mathfrak{T}^{\langle i \rangle} \cong \mathbb{Z}_{2l} \times \mathbb{Z}_2$. The only indices of representatives in these conditions are 34, 46, 20 and 55, but $\sigma_{55}, \sigma_{34} \in \mathcal{V}_0$ and $\sigma_{46} \in \mathcal{V}_1$ so that σ_i is in the orbit of σ_{20}, and $Q \cong Q(20, \mathrm{id}) = Q_{12}$.

Second, suppose that $\sigma_{59} \in \pi(Q)$. This time we can suppose that $\tilde{\sigma}_{59} \in Q$, by Lemma 3. As before, $\pi(Q) = (\pi(Q) \cap \mathcal{W}) \cdot \langle \sigma_{59} \rangle$ and by maximality $Q \cap \mathfrak{T} = \cap \{\mathfrak{T}^{\langle i \rangle} \cap \mathfrak{T}^{\langle 59 \rangle} : \sigma_i \in \pi(Q) \cap \mathcal{W}\}$. In particular $Q \cap \mathfrak{T} \subset \mathfrak{T}^{\langle 59 \rangle} \cong \mathbb{Z}_3^2$, so that $Q \cap \mathfrak{T} \cong \mathbb{Z}_3^{0,1,2}$. If $Q \cap \mathfrak{T} = \mathfrak{T}^{\langle 59 \rangle}$, then $Q_{14} = Q(59, \mathrm{id}) \subset Q$ and, by maximality of Q_{14} it follows $Q = Q_{14}$. If $Q \cap \mathfrak{T} = \langle t_1 \rangle \cong \mathbb{Z}_3$, then $\{\mathrm{id}\} \neq \pi(Q) \cap \mathcal{W} \subset \{\sigma_i \in \mathcal{W} : \sigma_i \sigma_{59} = \sigma_{59} \sigma_i\}$, which is a subgroup of $\mathcal{V}$ with only one order 2 element ($-\mathrm{id} = \sigma_{259}$.) If there were some element in $\pi(Q)$ of order not divisor of 3, then $\sigma_{259} \in \pi(Q)$ and $\mathbb{Z}_3 \cong Q \cap \mathfrak{T} \subset \mathfrak{T}^{\langle 259 \rangle} \cong \mathbb{Z}_2^4$, what is absurd. Hence $\pi(Q) \subset S := \{\sigma \in \mathcal{W} : \sigma^3 = \mathrm{id}, \sigma \sigma_{59} = \sigma_{59} \sigma\}$, which is a non abelian set with 9 elements. There must be $\sigma_i \in S$ such that $\pi(Q) \cap \mathcal{W} = \langle \sigma_i \rangle$ (otherwise another $\sigma_j \in S \setminus \langle \sigma_i \rangle$ would satisfy $\langle \sigma_i, \sigma_j \rangle \subset \pi(Q) \cap \mathcal{W}$ but then $\pi(Q) \cap \mathcal{W}$ would have at least 9 elements belonging to S, the whole S, but S is not abelian.) Thus, there is some $t_2 \in \mathfrak{T}$ such that $\langle \tilde{\sigma}_{59}, \tilde{\sigma}_i t_2, t_1 \rangle = Q$. As $\{\tilde{\sigma}_i t_2, t_1\}$ induces a toral grading by Remark 2 (σ_i has order 3), we can conjugate Q to $\langle t_1, t_3, f = \tilde{\sigma}_k t_4 \rangle$ with f a conjugate of $\tilde{\sigma}_{59}$ (hence of order just 3.) We can take $\sigma_k = \sigma_{59}$ or σ_4 by conjugating now inside $\mathfrak{N}$. In the first case $Q = Q(59, t_4) \cong Q_{14}$, and in the second case $Q \subsetneqq Q(4, t_4)$, a contradiction with the maximality of Q. Finally suppose that $Q \cap \mathfrak{T} = \{\mathrm{id}\}$. If $\pi(Q) \cap \mathcal{W} \subset S$, $\pi(Q) \cap \mathcal{W}$ would have only one order 3 generator and Q would be contained strictly in one of the quasitori in the paragraph above. Thus $-\mathrm{id} = \sigma_{259} \in \pi(Q)$. Notice also that $\langle \tilde{\sigma}_{59}, \tilde{\sigma}_{259} t \rangle$ is not a MAD (the second automorphism is inner, so the set is conjugated by Lemma 1 to $\langle \tilde{\sigma}_k t_1, t_2 \rangle$, where $\tilde{\sigma}_k t_1$ has order 3, and so σ_k is in the orbit of σ_{59} or σ_4, hence, it is strictly contained in $Q(4, t_1)$ or $Q(59, t_1)$), so that there is $\sigma_i \in \pi(Q) \cap \mathcal{W} \setminus \langle \sigma_{259} \rangle$. Let us check that σ_i cannot have order 4. In such case, $\sigma_i \sigma_{59}$ would have order 12 and it would be conjugated to σ_{58}, so there would be $t \in \mathfrak{T}$ such that $A = \langle (\tilde{\sigma}_{58} t)^n : n = 1 \dots 12 \rangle \subset Q \subset \mathfrak{N}$ (up to conjugation), but the centralizer $C_{\mathfrak{N}}(A) = A$ (since $\mathfrak{T}^{\langle 58 \rangle} = \{\mathrm{id}\}$ and $\{\sigma \in \mathcal{V} : \sigma \sigma_{58} = \sigma_{58} \sigma\}$ has 12 elements, obviously the powers of σ_{58}), so that $Q = A$. This is an absurd since A is not maximal in $\mathrm{aut}\, \mathfrak{d}_4$ (it is only maximal in $\mathfrak{N}$.) Because $\sigma_{58}^3 \in \mathcal{W}$, so $A \cong \langle (\tilde{\sigma}_{58} t)^4, (\tilde{\sigma}_{58} t)^3 \rangle$ is conjugated to $\langle \tilde{\sigma}_k t_1, t_2 \rangle$, strictly contained in $Q(4, t_1)$ or $Q(59, t_1)$, as above. Therefore σ_i has order either 6 or 3. It can be taken of order 3 (if $\sigma_i^3 = -\mathrm{id}$ then $-\sigma_i \in \pi(Q)$ has order 3.) Then $\pi(Q) = \langle \sigma_{59}, \sigma_{259} \sigma_i \rangle$ (any element in S different than $(\sigma_i)^{0,1,2}$ does not commute with σ_i), with $\sigma_i \sigma_{259} \in \mathcal{W}$ of order 6, so applying Lemma 1 we obtain that Q is conjugated to $\langle \tilde{\sigma}_k t_1, t_2 \rangle$ with $k = 4$ ($\sigma_k \notin \mathcal{W}$, $\sigma_k^3 = \mathrm{id}$ and $\mathbb{Z}_6 \subset \mathfrak{T}^{\langle k \rangle}$), strictly contained in $Q(4, t_1)$, a contradiction. $\square$

4. Algebraic approach

We are going to revisit the MAD-groups of aut $\mathfrak{d}_4$ which intersect some connected component with order 3 outer automorphisms, looking at them under the light of triality and related stuff. While Section 3 is of a computational nature, this is more conceptual and qualitative.

According to [19], there are two order 3 outer automorphisms of $\mathfrak{d}_4$, up to conjugation, related to the affine diagrams

obtained from [19, TABLE Aff 3, p. 55], which fix subalgebras of type $\mathfrak{g}_2$ and $\mathfrak{a}_2$, respectively. The first one is known by its relationship with the triality phenomenon, but the second one is also related to it and both cases can be treated in a uniform way.

4.1. **Automorphisms of $\mathfrak{d}_4$.** Denote by (C, q) a Cayley algebra with standard quadratic map q. As usual its canonical involution will be denoted by $x \mapsto \bar{x}$. Under our assumptions on the ground field, such a C is unique up to isomorphism. Now an algebra of type $\mathfrak{d}_4$ is $\mathfrak{o}(C, q) = \{g \in \mathfrak{gl}(C) : b(g(x), y) + b(x, g(y)) = 0 \quad \forall x, y \in C\}$, where b denotes the polar form of q. In this subsection we would like to get a description of $\mathcal{G} := \operatorname{aut}(\mathfrak{d}_4)$ which could be useful in our further study of quasitori and gradings on $\mathfrak{d}_4$. Denote by $\operatorname{GO}(C, q)$ the algebraic group of all bijective linear maps $f : C \to C$ such that there is $\lambda \in \mathbb{K}^\times$ such that $q(f(x)) = \lambda q(x)$ for any $x \in C$. The scalar λ is called the *multiplier* of f and it is usually denoted by $\mu(f)$ (see [20, §12] for more information.) It is well known (for instance [17, Exercise 15, p. 287]) that the identity component $\mathcal{G}_0$ of the algebraic group $\mathcal{G}$ is the group of all automorphisms of the form $g \mapsto f g f^{-1}$ where $f \in \operatorname{GO}(C, q)^+$ (notation as in [20, p. 154].) But two elements $f_1, f_2 \in \operatorname{GO}(C, q)^+$ induce the same automorphism $g \mapsto f_i g f_i^{-1}$ ($i = 1, 2$) if and only if $f_1 = \lambda f_2$ for some $\lambda \in \mathbb{K}^\times$. Thus we can identify $\mathcal{G}_0$ with $\operatorname{PGO}(C, q)^+ = \operatorname{GO}(C, q)^+ / \mathbb{K}^\times I$.

It is also well known that $\mathcal{G}$ is an extension $\mathcal{G} = \mathcal{G}_0 \cdot S_3$ where S_3 denotes the symmetric group of order 3. The aim of this subsection is to clarify the nature of this extension, which will enable us to make some explicit computations (for instance on centralizers.) To describe the extension $\mathcal{G}_0 \cdot S_3$ we need to explicit the action of the generators of S_3 on $\mathcal{G}_0$. If we take an element $f \in \operatorname{GO}(C, q)^+$, we shall denote by $\bar{f}$ the new element in $\operatorname{GO}(C, q)^+$ such that $\bar{f}(x) := \overline{f(\bar{x})}$ for any $x \in C$. The map $f \mapsto \bar{f}$ induces an order 2 automorphism $\sigma : \operatorname{PGO}(C, q)^+ \to \operatorname{PGO}(C, q)^+$. This will be identified with an order 2 permutation in S_3. Moreover, to describe the action on $\mathcal{G}_0$ of the cyclic order 3 permutation in S_3 we need to take into account the two (unique up to isomorphism) possible 8-dimensional symmetric compositions which can be constructed from (C, q) ([20, § 34].) The first one is the para-Hurwitz algebra $\bar{C}$ with multiplication $x \cdot y = \bar{x}\, \bar{y}$, while the second one is the Okubo algebra $C_{\varphi, \varphi^{-1}}$ whose multiplication is $x \cdot y = \varphi(\bar{x}) \varphi^{-1}(\bar{y})$, where $\varphi \in \operatorname{aut}(C)$ is an order three automorphism fixing a four-dimensional algebra (such an automorphism is unique up to conjugation.) It is easy to check that $\operatorname{aut}(\bar{C}) = \operatorname{aut}(C) = G_2$ while $\operatorname{aut}(C_{\varphi, \varphi^{-1}})$ is the identity connected component of $A_2 = \operatorname{aut}(\mathfrak{sl}(3, \mathbb{K}))$. Though this is a well known fact ([8]), one can check it by taking into account that the

Okubo algebra is isomorphic to $(\mathbb{P}, *) = (\text{Mat}_3(\mathbb{K})_0, *)$ the pseudo-octonion algebra constructed on the vector space of zero trace 3×3 matrices with entries in $\mathbb{K}$ ([22, 8].) Its product is given by $x * y := \mu xy + (1 - \mu)yx - \frac{1}{3}\text{tr}(xy)$ for $\mu = \frac{1-\omega}{3}$, with ω primitive cube root of unity and $\text{tr}(\cdot)$ denotes the matrix trace. If $p \in \text{GL}(3, \mathbb{K})$, then $x \mapsto pxp^{-1}$ is an automorphism of $\mathbb{P}$. That is, $\text{Int}(\mathfrak{sl}(3, \mathbb{K})) \subset \text{aut}(\mathbb{P})$. On the other hand, the product $[x, y] := xy - yx$ is $[x, y] = (2\omega - 1)^{-1}(x * y - y * x)$. Thus any element in $\text{aut}(\mathbb{P})$ is an automorphism of the Lie algebra $\mathfrak{sl}(3, \mathbb{K})$. Therefore it is an automorphism of $\text{Mat}_3(\mathbb{K})$ or the opposite of an antiautomorphism of the same algebra. But this last possibility does not provide an automorphism of $\mathbb{P}$ so $\text{aut}(\mathbb{P}) \cong \text{Int}(\mathfrak{sl}(3, \mathbb{K})) = \text{PGL}(3, \mathbb{K})(= (A_2)_0)$.

If we denote by S to any of the symmetric compositions introduced in the previous paragraph, and by $*$ its product, then it is well known ([20, Proposition 35.4]) that for any $t \in \text{GO}(C, q)^+$ with multiplier $\mu(t)$ there are elements $t^+, t^- \in \text{GO}(C, q)^+$ such that (t, t^-, t^+) is an $admissible\ triple$, that is, $\mu(t)^{-1}t(x * y) = t^-(x) * t^+(y)$ for any $x, y \in C$. Moreover t^+ and t^- are unique up to scalar multiplication by some nonzero λ and λ^{-1} respectively. It is also a standard fact that (t, t^-, t^+) is admissible if and only if (t^-, t^+, t) is admissible. Now the map $t \mapsto t^-$ induces an order 3 automorphism $\theta_S : \text{PGO}(C, q)^+ \to \text{PGO}(C, q)^+$. We shall write $\theta := \theta_{\bar{C}}$ and $\theta' := \theta_{C_{\varphi, \varphi^{-1}}}$. Denoting by $[t]$ the equivalence class in $\text{PGO}(C, q)^+$ of $t \in \text{GO}(C, q)^+$, we have $\theta_S([t]) = [t^-]$ so that $\theta_S^3 = 1$ and it can be easily proved that $\theta_S \sigma = \sigma \theta_S^2$, which together with $\sigma^2 = 1$ provide a group monomorphism $S_3 \to \text{aut}(\text{PGO}(C, q)^+)$. Thus we get a description of $\text{aut}(\mathfrak{d}_4)$ as a semidirect product

$$\mathcal{G} \cong \text{PGO}(C, q)^+ \rtimes S_3$$

where the product of elements is given by

$$(2) \qquad \sigma \cdot [t] = [\bar{t}]\sigma, \quad \theta_S \cdot [t] = [t^-]\theta_S, \quad \theta_S^2 \cdot [t] = [t^+]\theta_S^2$$

while products of θ_S's and σ's are governed by the corresponding relations in S_3.

Observe now that we can consider $\text{aut}\,S \subset \mathcal{G}_0$ independently of the symmetric composition algebra considered $(S, *)$. In the para-Hurwitz case, $\text{aut}\,\bar{C} = \text{aut}\,C \subset \text{O}(C, q)^+ \subset \text{GO}(C, q)^+$, and, by composing with the canonical projection $\text{GO}(C, q)^+ \to \text{PGO}(C, q)^+$, we still have a monomorphism $\text{aut}(C) \hookrightarrow \text{PGO}(C, q)^+$. In the Okubo case, straightforward computations show that $(x * y) * x = q(x)y$, so that any $f \in \text{aut}(S, *)$ verifies $q(f(x)) = q(x)$, that is, $\text{aut}\,S \subset \text{O}(C, q)$. Moreover, $\text{aut}\,S \subset \text{O}(C, q)^+$ because $\text{aut}\,S$ is connected $(= (A_2)_0.)$ As before, the composition of $\text{aut}\,S \hookrightarrow \text{GO}(C, q)^+$ with the projection $\text{GO}(C, q)^+ \to \text{PGO}(C, q)^+$ is still an embedding (the only multiple of $f \in \text{aut}\,S$ which is also an automorphism is f itself.)

Now we can compute certain centralizers easily.

Proposition 1. *For $S = \bar{C}$ or $C_{\varphi, \varphi^{-1}}$, the centralizer in $\mathcal{G}$ of θ_S is $C_{\mathcal{G}}(\theta_S) = \text{aut}(S) \cup \text{aut}(S)\theta_S \cup \text{aut}(S)\theta_S^2$.*

Proof. Recalling that $\mathcal{G}_0 = \text{PGO}(C, q)^+$ we have

$$\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_0\theta_S \cup \mathcal{G}_0\theta_S^2 \cup \mathcal{G}_0\sigma \cup \mathcal{G}_0\sigma\theta_S \cup \mathcal{G}_0\sigma\theta_S^2.$$

Taking an element $[f] \in \mathcal{G}_0 = \text{PGO}(C, q)^+$ and imposing the condition that it centralizes θ_S, we get $\theta_S[f] = [f^-]\theta_S = [f]\theta_S$ which implies that there is an admissible triple (f, f, f^+) and consequently an admissible triple (f, f^+, f). Since f^- and f^+ are uniquely determined by f (up to nonzero scalar multiples), we

have $[f] = [f^+] = [f^-]$. This implies that a certain nonzero multiple of f is an automorphism of S. So $[f] \in \operatorname{aut}(S)$. If we take now $n = 1$ or 2 and an element $[f]\theta_S^n \in \mathcal{G}_0\theta_S^n \cap C_{\mathcal{G}}(\theta_S)$, by imposing the commutativity condition we get also $[f] = [f^+] = [f^-]$ as before. The rest of the possibilities ($[f] \in \mathcal{G}_0\sigma\theta_S^{0,1,2}$) are not compatible with the commutativity condition. $\square$

Corollary 1. *Let S be the para-Hurwitz or the Okubo algebra. For any maximal quasitorus Q of $\operatorname{aut}(S)$ the group $\hat{Q} := \langle Q \cup \{\theta_S\}\rangle$ is a maximal quasitorus of $\mathcal{G} = \operatorname{aut}\mathfrak{d}_4$.*

Proof. Suppose that $r \in \mathcal{G}$ is a semisimple element commuting with $\hat{Q}$. Then $r \in C_{\mathcal{G}}(\theta_S) = H \cup H\theta_S \cup H\theta_S{}^2$, for $H := \operatorname{aut}(S)$. If $r \in H$ then $r \in C_H(Q)$ and by the maximality of Q we have $r \in Q \subset \hat{Q}$. In case that $r \in H\theta_S$ then $r\theta_S^2 \in H$ so that $r\theta_S^2 \in C_H(Q)$ and therefore $r\theta_S^2 \in Q$ implying $r \in \hat{Q}$ (the other possibility $r \in H\theta_S^2$ is similar.) $\square$

We know that the order 3 outer automorphisms of $\mathfrak{d}_4$ fall into two categories: those whose fix a subalgebra of type $\mathfrak{a}_2$ and the ones whose fix a subalgebra of type $\mathfrak{g}_2$, corresponding respectively to those whose centralizer intersected with $\mathcal{G}_0$ is isomorphic to $\operatorname{Int}\mathfrak{a}_2$ and $\operatorname{aut}\mathfrak{g}_2$, that is, conjugated to θ' and θ respectively. Next we study which is the category of $\varphi^2\theta$, which has order 3 since $\varphi \in \operatorname{aut}C \subset \mathcal{G}_0$ commutes with θ.

Corollary 2. *The centralizer of $\varphi^2\theta$ in $\mathcal{G}_0$ is isomorphic to $\operatorname{Int}(\mathfrak{sl}(3,\mathbb{K}))$. In particular $\varphi^2\theta$ is conjugated to θ'.*

Proof. Take $[t] \in \operatorname{PGO}(C,q)^+ = \mathcal{G}_0$ such that $\varphi^2\theta[t] = [t]\varphi^2\theta$. Then $\varphi^2[t^-] = [t]\varphi^2$ or $[t^-] = [\varphi t \varphi^{-1}]$. A little more computation reveals that $[t^+] = [\varphi^{-1}t\varphi]$. Thus there is an admissible triple $(t, \varphi t\varphi^{-1}, \varphi^{-1}t\varphi)$ for $\bar{C}$. By definition, this means that $\mu(t)^{-1}t(\bar{x}\bar{y}) = \overline{\varphi t\varphi^{-1}(x)}\ \overline{\varphi^{-1}t\varphi(y)} = t(\varphi^{-1}(x)) * t(\varphi(y))$ for any $x, y \in C$, if $*$ denotes now the product of $C_{\varphi,\varphi^{-1}}$. Making $a = \varphi^{-1}(x)$, $b = \varphi(y)$, we get $\mu(t)^{-1}t(\varphi(\bar{a})\varphi^{-1}(\bar{b})) = t(a) * t(b)$ or equivalently $\mu(t)^{-1}t(a * b) = t(a) * t(b)$ so that some nonzero multiple of t is an automorphism of $\mathbb{P}$. This implies that $C_{\mathcal{G}_0}(\varphi^2\theta) \subset \operatorname{aut}(\mathbb{P}) \cong \operatorname{Int}(\mathfrak{sl}(3,\mathbb{K}))$, so that $C_{\mathcal{G}_0}(\varphi^2\theta)$ can not be isomorphic to G_2. $\square$

Therefore the centralizer $C_{\mathcal{G}}(\theta)$ contains a conjugated of θ'.

4.2. **On certain quasitori of $\operatorname{aut}(\mathfrak{d}_4)$ and their induced gradings.** For any of the previously considered symmetric composition algebras S (para-Hurwitz or Okubo algebras) we had $\operatorname{aut}(S,*)$ embedded in $\mathcal{G} = \operatorname{aut}(\mathfrak{d}_4)$. If we denote by $f \mapsto f^\diamond$ the mentioned embedding, we can use it to construct maximal quasitori in $\mathcal{G}$ by *mixing* maximal quasitori in $\operatorname{aut}(S,*)$ together with θ_S, according to Corollary 1.

First, consider the case $S = \bar{C}$ the para-Hurwitz algebra. Since $\operatorname{aut}(C) = \operatorname{aut}(\bar{C}) = G_2$, there is a lot of available information on this group. Let

$$B = \{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$$

be the *standard basis* of the Cayley algebra C, defined by

$$\begin{aligned}
e_1u_j &= u_j = u_je_2, & u_iu_j &= v_k = -u_ju_i, & u_iv_i &= e_1, \\
e_2v_j &= v_j = v_je_1, & -v_iv_j &= u_k = v_jv_i, & v_iu_i &= e_2,
\end{aligned}$$

where e_1 and e_2 are orthogonal idempotents, (i, j, k) is any cyclic permutation of $(1, 2, 3)$, and the remaining relations are null. Denote by $t_{\alpha,\beta}$ the automorphism of

C whose matrix in the standard basis is the diagonal matrix

$$\operatorname{diag}(1, 1, \alpha, \beta, (\alpha\beta)^{-1}, \alpha^{-1}, \beta^{-1}, \alpha\beta),$$

where $\alpha, \beta \in \mathbb{K}^\times$. The set of these automorphisms is a maximal torus of $\operatorname{aut}(C)$. In particular,

$$P_1 := \langle \theta, t_{\alpha,\beta}^\diamond \colon \alpha, \beta \in \mathbb{K}^\times \rangle \cong (\mathbb{K}^\times)^2 \times \mathbb{Z}_3$$

is a maximal quasitorus of $\mathcal{G}$. Consider the $\mathbb{Z}_3$-grading induced by θ on $\mathcal{L} := \mathfrak{d}_4$. This is $\mathcal{L} = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}} \oplus \mathcal{L}_{\bar{2}}$ where the $\mathcal{L}_{\bar{0}}$-modules $\mathcal{L}_{\bar{1}}$ and $\mathcal{L}_{\bar{2}}$ in the grading are dual for the Killing form, hence seven-dimensional (the fixed subalgebra of $\mathfrak{d}_4$ by θ is $\operatorname{Der} C = \mathfrak{g}_2 = \mathcal{L}_{\bar{0}}$.) Besides they are irreducible (see [19, Prop 8.6, Ch 8, p. 138]), so that they are isomorphic to the natural $\mathfrak{g}_2$-module C_0. Since all the components of the $\mathbb{Z}^2$-grading on C_0 are one-dimensional (with generators $e_1 - e_2, u_1, u_2, u_3, v_1, v_2$ and v_3), and the same happens for the non-zero homogeneous components on $\operatorname{Der}(C)$ (the root spaces), the $\mathbb{Z}^2 \times \mathbb{Z}_3$-grading on $\mathcal{L}$ induced by P_1 is of type $(12, 1) + 2(7, 0) = (26, 1)$.

Now, consider the maximal quasitorus of $\operatorname{aut}(C)$ generated by $\{t_{1,-1}, t_{-1,1}, f\}$, where f is the automorphism given by the following matrix relative to the standard basis

$$f = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

Again all the components of the grading induced on C_0 are one-dimensional, with generators $e_1 - e_2, u_1 \pm v_1, u_2 \pm v_2$ and $u_3 \pm v_3$, but all the non-zero homogeneous components of $\mathcal{L}_{\bar{0}}$ are Cartan subalgebras of $\mathfrak{g}_2$, hence two-dimensional. Thus, the maximal quasitorus of $\mathcal{G}$ given by

$$P_2 := \langle \theta, t_{1,-1}^\diamond, t_{-1,1}^\diamond, f^\diamond \rangle \cong \mathbb{Z}_2^3 \times \mathbb{Z}_3 \cong \mathbb{Z}_2^2 \times \mathbb{Z}_6$$

induces a $\mathbb{Z}_2^2 \times \mathbb{Z}_6$-grading on $\mathcal{L}$ of type $(0, 7) + 2(7, 0) = (14, 7)$.

Second, consider the Okubo algebra which is isomorphic to the pseudo-octonion algebra $\mathbb{P}$ as previously mentioned. If $p \in \operatorname{GL}(3, \mathbb{K})$, the map

$$\operatorname{In}(p) \colon \mathbb{P} \to \mathbb{P}, \quad \operatorname{In}(p)(x) = pxp^{-1}$$

is an automorphism of $\mathbb{P}$. Take

$$p_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad p_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},$$

order-three invertible matrices verifying $p_1 p_2 = \omega p_2 p_1$. Note that

$$\operatorname{In}(p_1) \operatorname{In}(p_2)(x) = p_1 p_2 x (p_1 p_2)^{-1} = \omega p_2 p_1 x \omega^{-1} (p_2 p_1)^{-1} = \operatorname{In}(p_2) \operatorname{In}(p_1)(x),$$

that is, $\langle \operatorname{In}(p_1), \operatorname{In}(p_2) \rangle \le \operatorname{aut}(\mathbb{P})$ is an abelian subgroup of automorphisms isomorphic to $\mathbb{Z}_3^2$. Hence

$$P_3 := \langle \theta', \operatorname{In}(p_1)^\diamond, \operatorname{In}(p_2)^\diamond \rangle \cong \mathbb{Z}_3^3$$

is another maximal quasitorus of $\mathcal{G}$. Consider again the $\mathbb{Z}_3$-grading $\mathcal{L} = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}} \oplus \mathcal{L}_{\bar{2}}$ induced in $\mathfrak{d}_4$ by θ'. Then $\mathcal{L}_{\bar{0}}$ is 8-dimensional (isomorphic to $\mathfrak{sl}(3, \mathbb{K})$), and $\mathcal{L}_{\bar{1}}$ and $\mathcal{L}_{\bar{2}}$ are $\mathcal{L}_{\bar{0}}$-dual irreducible modules (again [19, Prop. 8.6, Ch. 8, p. 138]), hence of dimension 10. More precisely, if V is a tridimensional vector space, the Lie algebra $\mathcal{L}_{\bar{0}}$ is isomorphic to $\mathfrak{sl}(V)$ and the other components are isomorphic to $\mathfrak{sl}(V)$-modules of type $S^3(V)$ and $S^3(V^*)$. When considering the $\mathbb{Z}_3$-grading on V given by an arbitrary basis $\{v_0, v_1, v_2\}$, the map $f_1 \colon V \to V$ given by $f_1(v_i) = v_{i+1}$ is extended to $\mathcal{L}$ as $\mathrm{In}(p_1)^\diamond$, and splits $S^3(V)$ into "pieces" of size $4, 3$ and 3 respectively. The same happens to $\mathrm{In}(p_2)^\diamond$, which is the extension to $\mathcal{L}$ of $f_2 \colon V \to V$ given by $f_2(v_i) = \omega^i v_i$. Both automorphisms together split $S^3(V)$ in one subspace of dimension 2 and the remaining ones of dimension 1. Thus, the $\mathbb{Z}_3^3$-grading induced on $\mathfrak{d}_4$ is of type $(8, 0) + 2(8, 1) = (24, 2)$.

To summarize, as $\langle t_{1,-1}, t_{-1,1}, f \rangle \cong \mathbb{Z}_3^3$ and $\langle t_{\alpha, \beta} \colon \alpha, \beta \in \mathbb{K}^\times \rangle \cong (\mathbb{K}^\times)^2$ are MAD-groups of $\mathrm{aut}\, \mathfrak{g}_2$ (the only MAD's, according to [3]), and $\langle \mathrm{In}(p_1), \mathrm{In}(p_2) \rangle \cong \mathbb{Z}_3^2$ is a MAD of the group of automorphisms of the pseudo-octonions algebra, by Corollary 1 we have that the quasitori P_i are maximal (and because of their types, $P_1 \cong Q_{13}$, $P_2 \cong Q_{12}$ and $P_3 \cong Q_{14}$.)

In order to prove that the P_i's are all the maximal quasitori (which intersect the θ-component), we prove first that any other MAD would have some order 3 outer automorphism.

Consider the maximal torus T of $\mathcal{G}_0$ induced by the elements $t_{\lambda, \alpha, \beta, \gamma} \in \mathrm{GO}(C, q)$ whose matrix in the standard basis of C is

$$(3) \qquad \mathrm{diag}(\lambda, \lambda^{-1}, \alpha, \beta, \gamma, \alpha^{-1}, \beta^{-1}, \gamma^{-1}.)$$

Straightforward computations reveal that for $t \in T$ one also has $\bar{t}, t^+, t^- \in T$ and if $t = t_{\lambda, \alpha, \beta, \gamma}$, then $\bar{t} = t_{\bar{\lambda}^{-1}, \bar{\alpha}, \bar{\beta}, \bar{\gamma}}$ and $t^\pm = t_{\lambda^\pm, \alpha^\pm, \beta^\pm, \gamma^\pm}$ where

$$\lambda^- = \frac{1}{\sqrt{\alpha\beta\gamma\lambda}}, \quad \alpha^- = \sqrt{\frac{\lambda\alpha}{\beta\gamma}}, \quad \beta^- = \sqrt{\frac{\lambda\beta}{\alpha\gamma}}, \quad \gamma^- = \sqrt{\frac{\lambda\gamma}{\alpha\beta}},$$

$$(4) \qquad \lambda^+ = \sqrt{\frac{\alpha\beta\gamma}{\lambda}}, \quad \alpha^+ = \sqrt{\frac{\alpha}{\beta\gamma\lambda}}, \quad \beta^+ = \sqrt{\frac{\beta}{\alpha\gamma\lambda}}, \quad \gamma^+ = \sqrt{\frac{\gamma}{\alpha\beta\lambda}}.$$

Notice that σ, θ and θ' belong to $N(T)$, according to their actions on $T \subset \mathcal{G}_0$ given in (2).

Lemma 4. *For any $t \in T$ there is some $s \in T$ such that $ts(s^-)^{-1} \in T^{(\theta)} := T \cap C_{\mathcal{G}}(\theta)$.*

Proof. We must take into account that $t_{\lambda, \alpha, \beta, \gamma} \in T^{(\theta)}$ if and only if $\lambda = 1$ and $\alpha\beta\gamma = 1$. Thus, making $t = t_{\lambda, \alpha, \beta, \gamma}$ and $s = t_{x,y,u,v}$ the fact that $ts(s^-)^{-1} \in T^{(\theta)}$ is equivalent to proving that the equations

$$\lambda x = x^-, \quad \alpha\beta\gamma yuv = u^- v^- y^-,$$

have some solution in x, y, u and v. But writing x^-, y^-, u^- and v^- as functions of x, y, u and v according to the relations in (4), the resulting equations are $\alpha\beta\gamma(uvy)^{3/2} = x\sqrt{x}$, $\lambda x\sqrt{x} = (uvy)^{-1/2}$ which can be solved in x, y, u, v for any given $\lambda, \alpha, \beta, \gamma$. $\square$

Proposition 2. *Let Q be a MAD-group of $\mathcal{G}$ such that $Q \cap \mathcal{G}_0 \theta \neq \emptyset$. Then there is an order 3 outer automorphism in Q.*

Proof. Recall that $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_0\sigma \cup \mathcal{G}_0\sigma\theta \cup \mathcal{G}_0\sigma\theta^2 \cup \mathcal{G}_0\theta \cup \mathcal{G}_0\theta^2$. If $f \in \mathcal{G}_0\theta$ has order $3^k \cdot m$ with $\gcd(m, 3) = 1$, then f^m has order 3^k and belongs to $\mathcal{G}_0\theta$ or to $\mathcal{G}_0\theta^2$, which is conjugated to $\mathcal{G}_0\theta$. Thus we can consider from the beginning an element $f \in \mathcal{G}_0\theta$ with minimum order 3^k. If $k = 1$ we are done, so suppose $k > 1$. We must note that there is a maximal toral subgroup B of Q and a maximal torus T of $\mathcal{G}$ such that $B \subset T$ and $Q \subset N(T)$ (see [4, Sect. 5].) Then $B = T \cap Q$ and for each $f \in Q \setminus T$ the subquasitorus generated by $\{f\} \cup (Q \cap T)$ is nontoral. As T is conjugated to the torus given by (3), replace θ, θ' and σ by their corresponding conjugated automorphisms inside $N(T)$. Denote by $\pi\colon N(T) \to N(T)/T \cong \operatorname{aut}\Phi = \mathcal{W} \rtimes S_3$ the canonical projection (S_3 is generated by $\pi(\theta)$ and $\pi(\sigma)$.) Consider also $T^{\langle\sigma\rangle} := \{t \in T \mid \sigma \cdot t = t\}$, for $\cdot$ the action of the Weyl group $\mathcal{W}$ on the torus. Notice that $Q \cap T = \cap_{\sigma\in\pi(Q)} T^{\langle\sigma\rangle}$, by the maximality of Q. Now, $\pi(f) \neq 1$ since $f \notin T$, hence $\pi(f)$ is an order 3 element (since the Weyl group of $\mathfrak{d}_4$ has no elements of order 3^n for $n > 1$.) In particular $\pi(f)$ is conjugated to $\pi(\theta)$ or $\pi(\theta')$. More concretely, it is conjugated to $\pi(\theta)$, because every element in $N(T)$ projecting in $\pi(\theta')$ has order just 3, following [4, Lemma 1, p. 26], so in such case f would also have order 3. Thus, by conjugating by an element in $\mathcal{W}$ if necessary (so we do not change T nor $N(T)$) we may suppose that $\pi(f) = \pi(\theta)$, and $f = t\theta \in Q$ for some $t \in T$. Now conjugating by a suitable element $s \in T$ we have $st\theta s^{-1} = st(s^-)^{-1}\theta$ and by Lemma 4 the toral element s can be chosen so that $st(s^-)^{-1} \in T^{(\theta)}$. Thus from here on, we suppose $f = t\theta$ with $t \in T^{(\theta)}$.

The quasitorus Q being an abelian group, has no element of the connected components $\mathcal{G}_0\sigma$, $\mathcal{G}_0\sigma\theta$ or $\mathcal{G}_0\sigma\theta^2$. Therefore $\pi(Q) = (\pi(Q)\cap\mathcal{W})\langle\pi(\theta)\rangle$. Note that there is not $g \in Q$ with $\pi(g) \in \mathcal{W}$ of order 3. Because $B \cup \{g\}$ would be non toral, but it is contained in $T^{(g)} \cup \{g\}$, toral according to Table 2 (and immediate by Lemma 2.) Therefore any element in $\pi(Q) \cap \mathcal{W}$ has order a power of 2. If $\pi(Q) \cap \mathcal{W} = \{1\}$ we have $\pi(Q) = \langle\pi(\theta)\rangle$ and $Q \cap T = T^{\langle\pi(\theta)\rangle} = T^{(\theta)} \ni t$ so that $\theta \in Q$ would be an order 3 element. Otherwise, there is $g \in Q$ with $\pi(g) \in \mathcal{W}$ of order 2. Thus $B \subset T^{\langle\pi(gf)\rangle}$, but for the order 6 elements $\pi(gf) = \pi(g\theta) \in \operatorname{aut}\Phi$ we know by Table 2 that $T^{\langle\pi(gf)\rangle}$ is either a one-dimensional torus, or isomorphic to $\mathbb{Z}_2^2$, or $\{\operatorname{id}\}$. The last two options are not possible because $t^3 = f^3 \in B$ has order 3^{k-1}. Now the quasitorus generated by $T^{\langle\pi(gf)\rangle} \cup \{g\}$ is toral by Lemma 2, so that the quasitorus generated by $B \cup \{g\}$ is also toral, a contradiction with the election of B. $\square$

Up to this point we have "mixed" MAD-groups of $\operatorname{aut}(S)$ (where $S = \bar{C}$ or $\mathbb{P}$) with θ_S so as to obtain MAD-groups of $\mathcal{G} = \operatorname{aut}(\mathfrak{d}_4)$. Notice that in $\operatorname{Int}(\mathfrak{a}_2)$ there is another MAD-group, the maximal torus, but when mixing with θ' the obtained quasitori will turn out to be conjugated to P_1, as a consequence of Corollary 2. Thus we arrive at the main theorem.

Theorem 2. P_1, P_2 and P_3 are, up to conjugation, the only maximal quasitori of $\mathcal{G}$ not contained in $\mathcal{G}_0 \cup \mathcal{G}_0\sigma$.

Proof. Let $S = \bar{C}$ or $\mathbb{P}$ and consider a MAD-group Q of $\mathcal{G}$ containing some element in the connected component of the element θ. If Q contains some element conjugated to $\theta' = \theta_{\mathbb{P}}$, we can suppose that θ' belongs to Q. Hence $Q \subset C_{\mathcal{G}}(\theta') = (A_2)_0 \cup (A_2)_0\theta' \cup (A_2)_0\theta'^2$ applying Proposition 1. Thus, there is a quasitorus Q' of $(A_2)_0 = \operatorname{aut}(\mathbb{P})$ such that $Q = \langle Q' \cup \{1, \theta', \theta'^2\}\rangle$. Moreover, the maximality of Q implies that of Q' in $(A_2)_0$. There are only two MAD-groups of $(A_2)_0$ (there are four

MAD-groups of A_2, according to [14], but two of them have outer automorphisms), and, when we mix them with θ' we get two maximal quasitori: P_3 and

$$P_4 := \langle \theta', \mathrm{In}(\alpha E_{1,1} + \alpha^{-1} E_{2,2} + E_{3,3})^{\diamond}, \mathrm{In}(E_{1,1} + \beta^{-1} E_{2,2} + \beta E_{3,3})^{\diamond} : \alpha, \beta \in \mathbb{K}^{\times} \rangle$$

$$\cong \mathbb{Z}_3 \times (\mathbb{K}^{\times})^2,$$

obtained by joining the maximal torus of A_2 with θ'.

On the other hand, if Q does not contain any element conjugated to θ', by Proposition 2 there is an element conjugated to $\theta = \theta_{\bar{C}}$ in Q. Again Q is a copy of the direct product of a MAD-group of aut $\mathfrak{g}_2$ with $\{1, \theta, \theta^2\}$. Notice that there are 2 MAD's of the group G_2 (see [3]), which provide just P_1 and P_2 when mixing with θ. But Q cannot be conjugated to P_2 because in P_2 there is an automorphism conjugated to θ', by Corollary 2.

Therefore we have proved that there are only three MAD's in the stated conditions, P_1, P_3 and P_4 (in particular, P_2 and P_4 are conjugated.) $\square$

References

[1] Y. A. Bahturin, I. P. Shestakov and M. V. Zaicev. *Gradings on simple Jordan and Lie algebras*. Journal of Algebra **283** (2005), 849–868.

[2] Y. A. Bahturin and M. Tvalavadze. *Group gradings on G_2*. ArXiv: math.RA/0703468.

[3] C. Draper and C. Martín. *Gradings on $\mathfrak{g}_2$*. Linear Algebra and its Applications **418** (2006), 85–111.

[4] C. Draper and C. Martín. *Gradings on the Albert algebra and on $\mathfrak{f}_4$*. Preprint Jordan Theory Preprint Archives 232, March 2007.

[5] C. Draper and A. Viruel. *Group gradings on $o(8, \mathbb{C})$*. Reports on Math. Physics. **61** (2008), 263–278.

[6] C. Draper, C. Martín and A. Viruel. *Fine gradings on $\mathfrak{e}_6$*. In preparation.

[7] A. Elduque. *A Lie grading which is not a semigroup grading*. Linear Algebra and its Applications **418** (2006), 312–314.

[8] A. Elduque. *Composition algebras with large derivation algebras*. Journal of Algebra **190** (1997), 372–404.

[9] A. Fialowski and M. De Montigny. *On Deformations and Contractions of Lie Algebras*. Symmetry, Integrability and Geometry: Methods and Applications Vol. 2 (2006), Paper 048, 10 pages.

[10] A. Garret Lisi. *An Exceptionally Simple Theory of Everithing*. Arxiv: 0711.0770v1.

[11] M. Gunaydin and F. Gursey. *Quark structure and octonions*. J. Math. Phys **14**, no. 11 (1973), 1651–1667.

[12] M. Halvíček, J. Patera and E. Pelantová. *On Lie gradings II*. Linear Algebra and its Applications **277** (1998), 97–125.

[13] M. Halvíček, J. Patera and E. Pelantová. *On Lie gradings III. Gradings of the real forms of classical Lie algebras (dedicated to the memory of H. Zassenhaus)*. Linear Algebra and its Applications **314** (2000), 1–47.

[14] M. Halvíček, J. Patera and E. Pelantová. *Fine gradings of the real forms of sl$(3, \mathbb{C})$*. Phys. Atomic Nuclei 61, **12** (1998), 2183–2186.

[15] W. H. Hesselink. *Special and Pure Gradings of Lie Algebras*. Math. Z. 179 (1982), 135–149.

[16] J. E. Humphreys. *Introduction to Lie algebras and Representation Theory*. Springer-Verlag, 1972.

[17] N. Jacobson. *Lie algebras*. Dover Publications, Inc., New York, 1979.

[18] P. Jordan. *Über Verallgemeinerungsmöglichkeiten des Formalismus der Quantenmechanik*. Nachr. Ges. Wiss. Göttingen, (1933), 209–214.

[19] V. G. Kac. *Infinite dimensional Lie algebras*. Cambridge University Press, 1990.

[20] M-A. Knus, A. Merkurjev, M. Rost and J-P. Tignol. *The Book of involutions*. American Mathematical Society Colloquium Publications, Vol 44, 1998.

[21] X.-D. Leng and J. Patera: *J. Phys. A. Math. Gen.* **27**, 1233–1250 (1994.)

[22] S. Okubo. *Introduction to Octonion and Other Non-Associative Algebras in Physics*. Cambridge University Press 1995.

[23] A. L. Onishchnik, E. B. Vinberg (Editors.) *Lie Groups and Lie Algebras III, Encyclopaedia of Mathematical Sciences, Vol. 41*. Springer-Verlag. Berlin, 1991.

[24] J. Patera, E. Pelantov and M. Svobodov: *Fine gradings of $o(5, \mathbb{C})$, $sp(4, \mathbb{C})$ and of their real forms*. J. Math. Phys. **42** (2001), 3839-3853.

[25] J. Patera, E. Pelantová and M. Svobodova. *The eight fine gradings of $sl(4, \mathbb{C})$ and $o(6, \mathbb{C})$*. J. math. Phys. **43** (2002), 6353–6378.

[26] J. Patera, E. Pelantová and M. Svobodova. *Fine gradings of $o(4, \mathbb{C})$*. J. Phys. 45 (2004), 2188–2198.

[27] J. Patera and H. Zassenhaus. *On Lie gradings. I.* Linear Algebra and its Applications **112** (1989), 87–159.

[28] V. P. Platonov. *The theory of algebraic linear groups and periodic groups*. American Society Translations. Serie 2 (69) (1968), 61–110.

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