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Cubic and symmetric compositions over rings

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Abstract. We consider generalized symmetric compositions over a ring k on the one hand, and unital algebras with multiplicative cubic forms on the other. Given a primitive sixth root of unity in k, we construct functors between these categories which are equivalences if 3 is a unit in k. This extends to arbitrary base rings, and with new proofs, results of Elduque and Myung on non-degenerate symmetric compositions and separable alternative algebras of degree 3 over fields. It also answers a problem posed in "The Book of Involutions" [boi, 34.26].

Introduction

Consider a finite-dimensional separable alternative algebra A of degree 3 over a field k of characteristic not 3, containing a primitive third root of unity. Okubo [ok] and later Faulkner [foa] found on the space A^0 of trace zero elements a remarkable multiplication $\star$ which has the composition and associativity property

$$q(x \star y) = q(x)q(y), \qquad b(x \star y, z) = b(x, y \star z), \tag{1}$$

where q is $-1/3$ times the quadratic trace of A and b is the polar form of q. In [boi, §34], the terminology "symmetric compositions" was coined for non-associative algebras over a field with a non-degenerate quadratic form satisfying (1). There is also a construction in the opposite direction: Given a symmetric composition on a k-vector space M of dimension $\geqslant 2$, there is a naturally defined multiplication $\bullet$ and a cubic form N on $A = k \oplus M$ making A a separable alternative algebra of degree 3 with generic norm N. Thus one obtains an equivalence between the categories of separable alternative algebras of degree 3, and symmetric compositions of dimension $\geqslant 2$, always under the restrictions on k mentioned earlier. This result is due to Elduque-Myung [elmy], see also [boi, Theorem 34.23]. Results of Schafer [schafer:n] show that separable alternative algebras of degree 3 are the same as unital algebras of dimension $\geqslant 3$ with non-degenerate multiplicative cubic forms. By this detour, the symmetric composition property (1) is equivalent to the multiplicativity of N with respect to $\bullet$. As remarked in [boi, 34.26], it would be nice to have a direct proof of this fact. Such a proof was attempted by Tschupp [tsch]. While his construction of a symmetric composition from A and N works well, the opposite direction — from symmetric compositions to algebras with multiplicative cubic forms — contains errors.

The object of this paper is to give a direct proof of this correspondence without making any non-degeneracy assumptions, and at the same time to extend the theory as far as possible to arbitrary base rings.

Specifically, we work in the following setting. Let k be an arbitrary commutative ring. On the one hand, we consider *unital cubic compositions*, that is, pairs (A, N) consisting of a unital k-algebra A and a multiplicative cubic form N on A. Apart from assuming that the unit element of A be a unimodular vector, there are no restrictions on the k-module structure of A. Hence, the cubic form N must be interpreted as a polynomial law in the sense of Roby [roby]. With an obvious definition of morphism, unital cubic compositions over k form a category $\mathbf{uccomp}_k$.

On the other hand, there is the category $\mathbf{scomp}_k$ of (generalized) *symmetric compositions* $(M, q, \star)$, where M is a k-module with a quadratic form q and a multiplication $\star$ satisfying (1). We make no assumptions on the structure of M as a k-module or on non-degeneracy of q. Let $\boldsymbol{\pi}_6(k) = \{\alpha \in k : \alpha^2 - \alpha + 1 = 0\}$ denote the set of primitive sixth roots of unity in k (sixth and third primitive roots of unity are in bijection via $\alpha \mapsto -\alpha$; formulas tend to become simpler using sixth roots of unity). Given $\alpha \in \boldsymbol{\pi}_6(k)$, we define functors

$$\mathbf{C}_\alpha : \mathbf{uccomp}_k \to \mathbf{scomp}_k, \qquad \mathbf{A}_\alpha : \mathbf{scomp}_k \to \mathbf{uccomp}_k$$

and show that there are natural transformations

$$\mathbf{A}_{\alpha^{-1}} \circ \mathbf{C}_\alpha \to \mathrm{Id}_{\mathbf{uccomp}_k}, \qquad \mathbf{C}_\alpha \circ \mathbf{A}_{\alpha^{-1}} \to \mathrm{Id}_{\mathbf{scomp}_k}, \tag{2}$$

which are isomorphisms if 3 is a unit in k. This yields the desired correspondence.

Here is a more detailed description of the contents. In §1, we establish notation and collect some facts on polynomial laws. In particular, we introduce the kernel of a polynomial law, a notion only implicitly contained in [roby].

It is useful to develop the above-mentioned correspondence first on a level not involving any multiplications and, hence, not requiring the existence of $\alpha \in \boldsymbol{\pi}_6(k)$. Accordingly, we consider the category $\mathbf{ucform}_k$ of unital cubic forms (instead of unital cubic compositions) on the one hand, and the category $\mathbf{qcform}_k$ of modules with a quadratic and a cubic form (instead of symmetric compositions) on the other. In §2 we show that there are functors $\mathbf{C}$: $\mathbf{ucform}_k \to \mathbf{qcform}_k$ and $\mathbf{A}$: $\mathbf{qcform}_k \to \mathbf{ucform}_k$ which are equivalences provided $3 \in k^\times$.

§3 contains, after some auxiliary results on unital compositions, the construction of the functor $\mathbf{C}_\alpha$. Let (A, N) be a unital cubic composition. Since we do not assume 3 invertible in k, there is no direct sum decomposition of A into $k \cdot 1_A$ and the space of trace zero elements. But there is a naturally induced quadratic form q on the quotient $\dot{A} = A/k \cdot 1_A$, as well as a family of multiplications depending on a scalar parameter α. Theorem 3.6 shows that $\dot{A}$ becomes a symmetric composition if either $\alpha \in \boldsymbol{\pi}_6(k)$ or A is commutative.

In §4, we construct the algebra $\mathbf{A}_\alpha(M, q, \star)$ of a symmetric composition (Theorem 4.1) and prove the existence of the natural transformations (2) (Proposition 4.10). On $k \oplus M$ we consider the cubic form $N(\lambda \oplus x) = \lambda^3 - \lambda q(x) + b(x, x \star x)$ and a family of multiplications, depending on a parameter $\alpha \in k$. There is an explicit formula for the lack of multiplicativity of N (Lemma 4.8) which shows that $\alpha \in \boldsymbol{\pi}_6(k)$ or commutativity of M is sufficient for N to be multiplicative. The proof is purely computational but not at all straightforward.

The previous constructions hinge on the existence of a primitive sixth root of unity in k. Since this is in general not the case, we introduce in §5 the quadratic k-algebra $K = k[\mathbf{t}]/(\mathbf{t}^2 - \mathbf{t} + 1)$ (the affine algebra of the scheme $\boldsymbol{\pi}_6$) and define the category $\mathbf{uccomp}^{(2)}_{K/k}$ of unital cubic compositions of the second kind. The algebra K is étale if and only if $3 \in k^\times$ but it always has surjective trace form. This allows us to develop a sufficient part of Galois descent theory and to show that there are functors $\tilde{\mathbf{C}}$ and $\tilde{\mathbf{A}}$ between $\mathbf{uccomp}^{(2)}_{K/k}$ and $\mathbf{scomp}_k$ which are isomorphisms if $3 \in k^\times$ (Proposition 5.10).

The final §6 discusses the transfer of regularity conditions, such as non-degeneracy, separability or strictness, by the functors $\mathbf{C}_\alpha$ and $\mathbf{A}_\alpha$.

Throughout, k denotes an arbitrary unital commutative ring and k-**alg** the category of commutative associative unital k-algebras. Unsubscripted tensor products are understood over k. For a k-module X and $R \in k\text{-}\mathbf{alg}$, the base change $X \otimes R$ is often abbreviated to X_R. Similarly, the base extension of a linear map $f\colon X \to Y$ is denoted f_R. The symbol $\mathbb{N}$ denotes the natural numbers including zero.

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1. Generalities

1.1. Polynomial laws [roby]**.** For any k-module X, define the functor $X_{\mathbf{a}}$ on $k\text{-}\mathbf{alg}$ with values in sets by $X_{\mathbf{a}}(R) = X \otimes R$ for all $R \in k\text{-}\mathbf{alg}$, and by $X_{\mathbf{a}}(\varphi) = \mathrm{Id}_X \otimes \varphi\colon X_{\mathbf{a}}(R) \to X_{\mathbf{a}}(S)$, for all k-algebra homomorphisms $\varphi\colon R \to S$. A *polynomial law* f on X with values in a k-module V is a natural transformation $f\colon X_{\mathbf{a}} \to V_{\mathbf{a}}$ of functors. Thus for every $R \in k\text{-}\mathbf{alg}$, $f_R\colon X_R = X \otimes R \to V_R$ is a map, and for every k-algebra homomorphism $\varphi\colon R \to S$, the diagram

$$\begin{array}{ccc} X_R & \xrightarrow{\;f_R\;} & V_R \\ {\scriptstyle \mathrm{Id}_X \otimes \varphi}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}_V \otimes \varphi} \\ X_S & \xrightarrow[\;f_S\;]{} & V_S \end{array}$$

is commutative. As long as this does not cause confusion, we often write simply $f(x)$ instead of $f_R(x)$, for an element $x \in X_R$. Denote by $\mathcal{P}(X, V)$ the set (actually, a k-module) of V-valued polynomial laws on X.

A polynomial law $f \in \mathcal{P}(X, V)$ is said to be *homogeneous of degree* d if $f(rx) = r^d f(x)$, for all $r \in R$, $x \in X_R$, $R \in k\text{-}\mathbf{alg}$. Traditionally, homogeneous polynomial laws of degree $\geqslant 1$ with values in $V = k$ are called *forms*.

For example, polynomial laws of degree zero are in bijection with elements of V, and those of degree 1 are naturally identified with linear maps from X to V [roby, Corollary of Prop. I.6]. A polynomial law of degree 2 is the same as a quadratic map $q\colon X \to V$ in the usual sense [roby, Prop. II.1].

Polynomial laws can be composed in the obvious way: If $f \in \mathcal{P}(X, Y)$ and $g \in \mathcal{P}(Y, V)$ then $g \circ f \in \mathcal{P}(X, V)$ is given by $(g \circ f)_R = g_R \circ f_R$, for all $R \in k\text{-}\mathbf{alg}$. If f and g are homogeneous of degree m and n then $g \circ f$ is homogeneous of degree mn. In particular, composition with linear maps does not change the degree.

1.2. Extension and restriction of scalars. Let $k' \in k\text{-}\mathbf{alg}$ and denote by $\varphi\colon k \to k'$ the ring homomorphism making k' a k-algebra. For a k'-module X' let ${}_kX'$ be the k-module whose underlying abelian group is that of X', but with scalar operation of k given by $\lambda \cdot x := \varphi(\lambda)x$, for all $\lambda \in k$, $x \in X'$. In particular, if $S \in k'\text{-}\mathbf{alg}$ then ${}_kS \in k\text{-}\mathbf{alg}$. Now let X and V be k-modules and $f \in \mathcal{P}(X, V)$. The *base change of f from k to k'* is the polynomial law $f \otimes k' \in \mathcal{P}(X \otimes k', V \otimes k')$ given as follows. There is a canonical isomorphism $(X \otimes_k k') \otimes_{k'} S \cong X \otimes_k ({}_kS)$. Treating this as in identification, we put

$$(f \otimes k')_S(x) = f_{({}_kS)}(x), \tag{1.2.1}$$

for all $x \in (X \otimes_k k') \otimes_{k'} S$, $S \in k'$-**alg**.

There is also *restriction of scalars* for polynomial laws. Let X' and V' be k'-modules and let $f' \in \mathcal{P}(X', V')$ be a polynomial law. For any $R \in k$-**alg** there is a canonical isomorphism ${}_kX' \otimes_k R \cong X' \otimes_{k'} (k' \otimes_k R)$. Treating this as an identification, ${}_kf' \in \mathcal{P}({}_kX', {}_kV')$ is given by the formula

$$({}_kf')_R(x) = f'_{k'\otimes_k R}(x) \tag{1.2.2}$$

for all $x \in {}_kX' \otimes_k R$, $R \in k$-**alg**.

1.3. Homogeneous components. An arbitrary $f \in \mathcal{P}(X, V)$ determines a family $(f_d)_{d\in\mathbb{N}}$ of homogeneous polynomial laws such that

$$f(x) = \sum_{d\geqslant 0} f_d(x), \tag{1.3.1}$$

for all $x \in X_R$ and $R \in k$-**alg** (for every x, only finitely many $f_d(x)$ are different from zero, but there may be infinitely many non-zero f_d).

More generally, for all $(i_1, \dots, i_n) \in \mathbb{N}^n$ there are unique polynomial laws $f_{i_1\dots i_n}$ on X^n, multi-homogeneous of multi-degree $(i_1, \dots, i_n)$, such that

$$f(x_1 + \cdots + x_n) = \sum_{(i_1,\dots,i_n)\in\mathbb{N}^n} f_{i_1\dots i_n}(x_1, \dots, x_n), \tag{1.3.2}$$

for all $x_j \in X_R$ and $R \in k$-**alg**. The $f_{i_1\dots i_n}$ are called *polarizations* of f. If f is homogeneous of degree d then $f_{i_1\dots i_n} = 0$ unless $i_1 + \cdots + i_n = d$. If $i_j = 0$ then $f_{i_1\dots i_n}$ does not depend on the j-th variable, which then may be omitted in the notation. The polarizations satisfy the symmetry property

$$f_{i_1\dots i_n}(x_1, \dots, x_n) = f_{i_{\sigma(1)}\dots i_{\sigma(n)}}(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

for σ in the symmetric group. For example, if $f = f_3$ is a cubic form, we have the expansions

$$\begin{aligned} f(x+y) &= f(x) + f_{21}(x,y) + f_{12}(x,y) + f(y), \\ f(x+y+z) &= f(x) + f(y) + f(z) + f_{21}(x,y) + f_{21}(x,z) + f_{21}(y,z) \\ &\quad + f_{12}(x,y) + f_{12}(x,z) + f_{12}(y,z) + f_{111}(x,y,z), \end{aligned}$$

where $f_{21}(x,y) = f_{12}(y,x)$ is quadratic in x and linear in y, and $f_{111}(x,y,z)$ is trilinear and symmetric.

Let $z \in X$ and $i \in \mathbb{N}$. There is a unique polynomial law $\partial_z^{[i]} f \in \mathcal{P}(X, V)$, called the *$i$-th divided directional derivative of f in direction z*, satisfying

$$f(x + z \otimes r) = \sum_{i\geqslant 0} (\partial_z^{[i]} f)(x) r^i, \tag{1.3.3}$$

for all $r \in R$, $x \in X_R$, $R \in k$-**alg**. For $i = 1$, this is the usual directional derivative $\partial_z f$ of f in direction z. In general, $(\partial_z)^i = i!\partial_z^{[i]}$ which explains the terminology "divided derivative". In terms of the polarizations f_{ij}, we have

$$\big(\partial_z^{[i]} f\big)(x) = \sum_{j\geqslant 0} f_{ij}(z \otimes 1_R, x),$$

for all $x \in X_R$, $R \in k$-**alg**. If f is homogeneous of degree d then $f_{ij} = 0$ for $i+j \neq d$, and hence

$$\big(\partial_z^{[i]} f\big)(x) = f_{i,d-i}(z \otimes 1_R, x)$$

is homogeneous of degree $d - i$ (in x) if $i \leqslant d$, and zero if $i > d$.

1.4. Definition. Let $f \in \mathcal{P}(X, V)$. The *kernel* of f, denoted $\mathrm{Ker}(f)$, is the set of all $z \in X$ such that

$$f(x + z \otimes r) = f(x), \tag{1.4.1}$$

for all $x \in X_R$, $R \in k$-**alg**. We say f is *non-degenerate* if $\mathrm{Ker}(f) = \{0\}$. Condition (1.4.1) can be reformulated as follows:

$$z \in \mathrm{Ker}(f) \quad \Longleftrightarrow \quad \partial_z^{[i]} f = 0 \quad \text{for all } i \geqslant 1. \tag{1.4.2}$$

Indeed, $\Longleftarrow$ is clear from (1.3.3), and $\Longrightarrow$ follows also from (1.3.3) by replacing R with $R[\mathbf{t}]$, the polynomial ring in one variable, putting $r = \mathbf{t}$ and comparing coefficients at powers of $\mathbf{t}$. If f_d denotes as before the homogeneous component of degree d of f, then

$$\mathrm{Ker}(f) = \bigcap_{d \geqslant 0} \mathrm{Ker}(f_d). \tag{1.4.3}$$

Indeed, the homogeneous component of degree d of $\partial_z^{[i]} f$ is $\partial_z^{[i]} f_{d+i}$, and a polynomial law is zero if and only if all its homogeneous components vanish (by uniqueness of homogeneous components).

Examples. (a) The kernel of a polynomial law of degree 1, i.e., a linear map, is the usual kernel of the linear map.

(b) Let q be a polynomial law of degree 2, i.e., a quadratic map. Denoting by b the polar form of q, we have

$$z \in \mathrm{Ker}(q) \quad \Longleftrightarrow \quad q(z) = b(x, z) = 0, \tag{1.4.4}$$

for all $x \in X$.

(c) Let $f \in \mathcal{P}(X, V)$ be cubic, i.e., of degree 3. Then $\big(\partial_z^{[1]} f\big)(x) = f_{21}(x, z)$ and $\big(\partial_z^{[2]} f\big)(x) = f_{21}(z, x)$. Since these polynomial laws are of degree 2 and 1 (in x), they vanish if and only if they vanish on X. Thus

$$z \in \mathrm{Ker}(f) \quad \Longleftrightarrow \quad f(z) = f_{21}(z, x) = f_{21}(x, z) = 0, \tag{1.4.5}$$

for all $x \in X$.

1.5. Lemma. *Let $f \in \mathcal{P}(X, V)$. Then:*

(a) $\mathrm{Ker}(f)$ *is a submodule of* X.

(b) *For every $R \in k$-**alg**, define*

$$\mathbf{Ker}(f)(R) := \mathrm{Ker}(f \otimes R) \subset X \otimes R,$$

where $f \otimes R$ is the base change of f from k to R as in 1.2. Then $\mathbf{Ker}(f)$ is a sub-functor of $X_{\mathbf{a}}$; i.e., for every k-algebra homomorphism $\varphi\colon R \to S$, we have $X_{\mathbf{a}}(\varphi)\big(\mathbf{Ker}(f)(R)\big) \subset \mathbf{Ker}(f)(S)$.

(c) *Let $j\colon \mathrm{Ker}(f) \to X$ be the inclusion map and, for $R \in k$-**alg**, let $j_R\colon \mathrm{Ker}(f) \otimes R \to X \otimes R$ be its R-linear extension. Then*

$$j_R\big(\mathrm{Ker}(f) \otimes R\big) \subset \mathrm{Ker}(f \otimes R) \tag{1.5.1}$$

*for all $R \in k$-**alg**.*

(d) *If $R \in k$-**alg** is faithfully flat and $f \otimes R$ is non-degenerate then f is non-degenerate.*

Remark. In general, the map j_R of (c) is not injective nor is its image equal to $\mathrm{Ker}(f \otimes R)$. Also, non-degeneracy is in general not preserved under base change, even in case of fields.

Proof. (a) This follows easily from the definition (1.4.1).

(b) Let $z \in \mathbf{Ker}(f)(R) = \mathrm{Ker}(f \otimes R)$. Then $X_{\mathbf{a}}(\varphi)(z) = z \otimes 1_S$, where we canonically identify $X_R \otimes_R S \cong X \otimes_k S = X_{\mathbf{a}}(S)$. Now let T be an S-algebra, $t \in T$ and $x \in X_{\mathbf{a}}(T)$. We must show that

$$f\big(x + (z \otimes 1_S) \otimes t\big) = f(x).$$

Under the canonical identification $(X_R \otimes_R S) \otimes_S T \cong X_R \otimes_R T$, we have $(z \otimes 1_S) \otimes t = z \otimes t$, where we consider T as an R-algebra by means of $R \to S \to T$. Since $z \in \mathrm{Ker}(f \otimes R)$, it follows that $f(x + z \otimes t) = f(x)$, as desired.

(c) By (b), the canonical homomorphism $\varphi\colon k \to R$ making R a k-algebra induces a map $\mathrm{Ker}(f) \to \mathrm{Ker}(f \otimes R)$ given by $z \mapsto j(z) \otimes 1_R$. Since $\mathrm{Ker}(f \otimes R)$ is an R-submodule of $X \otimes R$ by (a), it follows that $j_R(\sum z_i \otimes r_i) = \sum \big(j(z_i) \otimes 1_R\big) r_i \in \mathrm{Ker}(f \otimes R)$ for all $z_i \in \mathrm{Ker}(f)$ and $r_i \in R$, which proves (1.5.1).

(d) Since R is flat, the map $j_R\colon \mathrm{Ker}(f) \otimes R \to \mathrm{Ker}(f \otimes R) \subset X \otimes R$ is injective. Thus non-degeneracy of $f \otimes R$ implies $\mathrm{Ker}(f) \otimes R = \{0\}$ and therefore also $\mathrm{Ker}(f) = \{0\}$, since R is faithfully flat.

1.6. Proposition. *Let X and V be k-modules, let $Z \subset X$ be a submodule and denote by $\pi\colon X \to X/Z$ the canonical map. For $g \in \mathcal{P}(X/Z, V)$, let $\pi^*(g) := g \circ \pi$ be the pullback of g to a polynomial law on X as in* 1.1. *Then the map $g \mapsto \pi^*(g)$ is a bijection*

$$\pi^* : \mathcal{P}(X/Z, V) \xrightarrow{\cong} \{f \in \mathcal{P}(X, V) : Z \subset \mathrm{Ker}(f)\}.$$

Denoting by π_* its inverse, we have

$$\mathrm{Ker}\,\big(\pi^*(g)\big) = \pi^{-1}\big(\mathrm{Ker}(g)\big), \tag{1.6.1}$$
$$\mathrm{Ker}\,\big(\pi_*(f)\big) = \pi\big(\mathrm{Ker}(f)\big). \tag{1.6.2}$$

Proof. Let $i\colon Z \to X$ be the inclusion map and define $\alpha_1 = \mathrm{pr}_1\colon X \times Z \to X$ and $\alpha_2\colon X \times Z \to X$, $(x, z) \mapsto x + i(z)$. Then the sequence of sets

$$X \times Z \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} X \xrightarrow{\ \pi\ } X/Z$$

is exact in the sense of [`roby`, IV, No. 8]. Hence by [`roby`, Theorem IV.4], the map π^* is a bijection between $\mathcal{P}(X/Z, V)$ and the set of all $f \in \mathcal{P}(X, V)$ for which $f \circ \alpha_1 = f \circ \alpha_2$. This condition means that

$$f(x) = f\big(x + i_R(y)\big) \tag{1.6.3}$$

all $R \in k$-**alg**, $x \in X_R$, $y \in Z_R$. Thus we must show that (1.6.3) holds if and only if $Z \subset \mathrm{Ker}(f)$.

(a) Suppose (1.6.3) is true and let $z \in Z$. Also let $R \in k$-**alg**, $r \in R$ and $x \in X_R$. Then $z \otimes r \in Z_R$, so $f(x + i(z) \otimes r) = f(x + i_R(z \otimes r)) = f(x)$, showing that $i(z) = z \in \mathrm{Ker}(f)$.

(b) Conversely, let $Z \subset \operatorname{Ker}(f)$. This means that the inclusion map $i\colon Z \to X$ factors via $\operatorname{Ker}(f)$, i.e., $i = j \circ i'$ where $i'\colon Z \to \operatorname{Ker}(f)$ and $j\colon \operatorname{Ker}(f) \to X$ are the inclusions. Tensoring with R yields $i_R = j_R \circ i'_R$, and by Lemma 1.5(c), j_R takes values in $\operatorname{Ker}(f \otimes R)$. Hence $i_R(y) \in \operatorname{Ker}(f \otimes R)$, which in particular implies that (1.6.3) holds.

To complete the proof, let $f = \pi^*(g)$. Then we have $f(x + y \otimes r) = g\big((\pi_R(x) + \pi(y) \otimes r\big)$ for all $y \in X$, $x \in X_R$, $R \in k$-**alg**. Using the fact that $\pi_R\colon X_R \to (X/Z)_R$ is surjective, it is now easy to verify that (1.6.1) and (1.6.2) hold. The details are left to the reader.

1.7. Corollary. *Any $f \in \mathcal{P}(X, V)$ induces a non-degenerate polynomial law $\pi_*(f)$ on $X/\operatorname{Ker}(f)$.*

Proof. Put $Z = \operatorname{Ker}(f)$. Then (1.6.2) shows $\operatorname{Ker}\big(\pi_*(f)\big) = \pi\big(\operatorname{Ker}(f)\big) = \pi(Z) = \{0\}$.

2. Unital forms

2.1. Definition. A *unital form of degree d* over k or a *unital d-form* is a triple $\mathfrak{X} = (X, N, 1_X)$ consisting of a k-module X, a form (i.e., a k-valued homogeneous polynomial law) N of degree d, and a unimodular vector $1_X \in X$ satisfying $N(1_X) = 1$. Here an element $u \in X$ is called unimodular if $\alpha(u) = 1$ for some linear form α on X. In case $d = 2$ these are the unital quadratic forms studied in [`uqf`], and for $d = 3$ they are called *unital cubic forms*. As in loc. cit., one shows that a vector $1_X \in X$ with $N(1_X) = 1$ is automatically unimodular if either X is finitely generated and projective or $d \in k^\times$.

A morphism $\varphi\colon \mathfrak{X} \to \mathfrak{X}'$ of unital d-forms is a linear map $\varphi\colon X \to X'$ satisfying $\varphi(1_X) = 1_{X'}$ and $N' \circ \varphi = N$. Unless this leads to confusion, we will often write 1 instead of 1_X. We put $\dot{X} := X/k \cdot 1_X$ and denote the canonical map $\kappa\colon X \to \dot{X}$ by $x \mapsto \dot{x}$. Then the fact that 1_X is unimodular is equivalent to the sequence $0 \to k \to X \to \dot{X} \to 0$ being split-exact. Note that the property of being a unital d-form is stable under base change and descends from faithfully flat base extensions.

2.2. The trace forms. Let $\mathfrak{X}$ be a unital d-form. The *trace forms* of $\mathfrak{X}$ are the forms T_i of degree i defined by

$$T_i = \partial_{1_X}^{[d-i]} N, \tag{2.2.1}$$

cf. (1.3.3). Since N has degree d, we have the expansion

$$N(1_X \otimes r + x) = \sum_{i=0}^{d} T_i(x) r^{d-i}, \tag{2.2.2}$$

for all $r \in R$, $x \in X_R$, $R \in k$-**alg**. Clearly $T_0 = 1_k$, $T_d = N$ and $T_i = 0$ for $i > d$. Morphisms $\varphi\colon \mathfrak{X} \to \mathfrak{X}'$ of unital d-forms are compatible with the trace forms in the sense that $T'_i \circ \varphi = T_i$. The *linear and quadratic trace forms* are the linear and quadratic forms

$$T := T_1, \quad Q := T_2.$$

For indices $(i_1, \dots, i_p) \in \mathbb{N}^p$ with $i_1 + \cdots + i_p = i$, let $T_{i_1 \dots i_p}$ denotes the corresponding polarization of T_i, cf. 1.3. In particular,

$$B := T_{11} \quad \text{and} \quad \varPhi := T_{111}$$

denote the polar form of the quadratic trace form Q and the total linearization of the cubic form T_3, respectively.

From (2.2.2) we see that $\sum_{i=0}^{d} T_i(1)\mathbf{t}^i = N(1_X \otimes \mathbf{t} + 1_X) = N((1+\mathbf{t})1_X) = (1+\mathbf{t})^d$, and hence

$$T_i(1_X) = \binom{d}{i}. \tag{2.2.3}$$

Similarly, expand $N\big((\mathbf{s}+\mathbf{t})1_X + x\big) = N\big(\mathbf{t}1 + (x + \mathbf{s}1)\big)$ and compare coefficients of $\mathbf{s}^m\mathbf{t}^n$ to see that

$$T_{ij}(x, 1_X) = \binom{d-i}{j} T_i(x). \tag{2.2.4}$$

2.3. The characteristic polynomial and the discriminant. Let $\mathfrak{X} = (X, N, 1)$ be a unital form of degree d and let $\mathcal{P}(X) := \mathcal{P}(X, k)$ be the algebra of polynomial laws on X with values in k [roby, V.4]. Then $T_i \in \mathcal{P}(X)$, and we introduce the monic polynomial

$$\chi(\mathbf{t}) = \sum_{i=0}^{d} (-1)^i T_i\, \mathbf{t}^{d-i} \in \mathcal{P}(X)[\mathbf{t}],$$

called the *characteristic polynomial* of $\mathfrak{X}$. Evaluation of a polynomial law at $x \in X_R$ ($R \in k$-**alg**) yields a homomorphism $\mathcal{P}(X) \to R$. Writing $\chi(\mathbf{t}; x) \in R[\mathbf{t}]$ for the polynomial obtained by evaluating the coefficients of χ at x, we have

$$\chi(\mathbf{t}; x) = N(\mathbf{t}1_X - x).$$

The discriminant $\Delta \in \mathcal{P}(X)$ of $\chi(\mathbf{t})$ will also be called the *discriminant of* $\mathfrak{X}$. For $d = 2$, this is the quadratic form $\Delta = T^2 - 4Q$ while for $d = 3$, it is the sextic form

$$\Delta = -4T^3N + T^2Q^2 + 18TQN - 4Q^3 - 27N^2.$$

To shorten notation, we will often write T_x instead of $T(x) = T_1(x)$, as long as this is not in conflict with the notation T_i for the trace form of degree i.

2.4. Lemma. *Let $\mathfrak{X} = (X, N, 1)$ be a unital cubic form. Define quadratic and cubic forms H_2 and H_3 on X by*

$$H_2(x) = T(x)^2 - 3Q(x), \tag{2.4.1}$$
$$\begin{aligned} H_3(x) &= 2T(x)^3 - 9T(x)Q(x) + 27N(x) \qquad (2.4.2)\\ &= -T(x)^3 + 3T(x)H_2(x) + 27N(x). \end{aligned}$$

(a) *Then $1_X \in \mathrm{Ker}(H_2) \cap \mathrm{Ker}(H_3)$. Hence by Prop. 1.6, H_2 and H_3 induce quadratic and cubic forms q and h on $\dot{X} = X/k \cdot 1$, given by*

$$q(\dot{x}) = H_2(x), \quad h(\dot{x}) = H_3(x),$$

*for all $x \in X_R$, $R \in k$-**alg**. The polar forms H_{11} and H_{21} of H_2 and H_3 are*

$$H_{11}(x, y) = 2T(x)T(y) - 3B(x, y), \tag{2.4.3}$$
$$\begin{aligned} H_{21}(x, y) &= 6T_x^2T_y - 9T_yQ(x) - 9T_xB(x, y) + 27N_{21}(x, y) \qquad (2.4.4)\\ &= -3T_x^2T_y + 3T_yH_2(x) + 3T_xH_{11}(x, y) + 27N_{21}(x, y). \end{aligned}$$

(b) 27 *times the discriminant of $\mathfrak{X}$ is expressible by H_2 and H_3:*

$$-27\Delta = H_3^2 - 4H_2^3. \tag{2.4.5}$$

Proof. (a) By (2.2.3) and (2.2.4) we have $T(1) = 3 = Q(1)$ and $B(x,1) = T_{11}(x,1) = \binom{3-1}{1}T_1(x) = 2T(x)$. Hence for all $r \in R$, $x \in X_R$,

$$\begin{aligned} H_2(x + 1_X \otimes r) &= \{T(x) + rT(1)\}^2 - 3\{Q(x) + B(x,1)r + Q(1)r^2\} \\ &= T_x^2 + 6rT_x + 9r^2 - 3Q(x) - 6rT_x - 9r^2 = H_2(x). \end{aligned}$$

Similarly, using (2.2.2),

$$\begin{aligned} H_3(x + 1 \otimes r) &= -\{T_x + 3r\}^3 + 3(T_x + 3r)H_2(x + 1 \otimes r) + 27N(x + 1 \otimes r) \\ &= -\{T_x^3 + 9rT_x^2 + 27r^2T_x + 27r^3\} + 3T_xH_2(x) + 9rH_2(x) \\ &\quad + 27\{r^3 + r^2T_x + rQ(x) + N(x)\} = H_3(x). \end{aligned}$$

Formulas (2.4.3) and (2.4.4) follow easily from (2.4.1) and (2.4.2).

(b) This follows by a lengthy but straightforward computation.

Remark. It can be shown that H_2 and H_3 are the essentially unique (up to a scalar factor) quadratic (resp., cubic) forms on $\dot{X}$ which are linear combinations of T^2 and Q (resp., of T^3, TQ and N) and for which 1_X lies in the kernel.

2.5. Definition. Let $\mathbf{ucform}_k$ denote the category of unital cubic forms over k, with morphisms defined in 2.1. As suggested by Lemma 2.4, we consider also the following category $\mathbf{qcform}_k$: Its objects are triples $\mathfrak{M} = (M, f_2, f_3)$ where M is a k-module and f_2 and f_3 is a quadratic and a cubic form on M, respectively. Morphisms are linear maps preserving these forms. From Lemma 2.4(a), it follows that there is a functor

$$\mathbf{C} : \mathbf{ucform}_k \to \mathbf{qcform}_k,$$

given by $\mathbf{C}(\mathfrak{X}) = (\dot{X}, q, h)$ on objects, while for a morphism $\varphi\colon \mathfrak{X} \to \mathfrak{X}'$ of unital cubic forms, $\mathbf{C}(\varphi)$ is the induced map $\dot{\varphi}\colon \dot{X} \to \dot{X}'$.

There is also a functor $\mathbf{A}\colon \mathbf{qcform}_k \to \mathbf{ucform}_k$ in the opposite direction: For $\mathfrak{M} = (M, f_2, f_3) \in \mathbf{qcform}_k$, let $\mathbf{A}(\mathfrak{M}) = (k \oplus M, N_{\mathfrak{M}}, 1_k \oplus 0)$, where $N_{\mathfrak{M}}$ is the cubic form on $k \oplus M$ given by

$$N_{\mathfrak{M}}(\lambda \oplus x) = \lambda^3 - 3\lambda f_2(x) + f_3(x), \tag{2.5.1}$$

for all $\lambda \in R$, $x \in M_R$, $R \in k$-**alg**. For a morphism $\psi\colon \mathfrak{M} \to \mathfrak{M}'$, define $\mathbf{A}(\psi)$ by

$$\mathbf{A}(\psi)(\lambda \oplus x) = \lambda \oplus \psi(x) \quad (\lambda \in k,\ x \in M).$$

Note that the linear and quadratic trace of $N_{\mathfrak{M}}$ are

$$T_{\mathfrak{M}}(\lambda \oplus x) = 3\lambda, \tag{2.5.2}$$
$$Q_{\mathfrak{M}}(\lambda \oplus x) = 3(\lambda^2 - f_2(x)), \tag{2.5.3}$$

as follows easily from the definition.

2.6. Proposition. *Let* $\mathbf{C}$ *and* $\mathbf{A}$ *be the functors defined above.*

(a) *There is a natural transformation*

$$\zeta : \mathbf{A} \circ \mathbf{C} \to \mathrm{Id}_{\mathbf{ucform}_k}$$

as follows: Given $\mathfrak{X} = (X, N, 1) \in \mathbf{ucform}_k$*, define*

$$\zeta_{\mathfrak{X}} : (\mathbf{A} \circ \mathbf{C})(X) = k \oplus \dot{X} \to X, \qquad \zeta_{\mathfrak{X}}(\lambda \oplus \dot{x}) = \big(\lambda - T(x)\big)1_X + 3x, \tag{2.6.1}$$

where T *is the linear trace form of* $\mathfrak{X}$*, and* $\dot{X} = X/k \cdot 1$ *and* $x \mapsto \dot{x}$ *are as in* 2.1.

(b) *If* $3 \in k^\times$ *then* $\zeta_{\mathfrak{X}}$ *is an isomorphism with inverse*

$$\zeta_{\mathfrak{X}}^{-1}(x) = \frac{1}{3}\big(T(x) \oplus \dot{x}\big).$$

In general, denoting by ${}_3\dot{X}$ *the* 3*-torsion elements of* $\dot{X}$*, there is an exact sequence*

$$0 \longrightarrow {}_3\dot{X} \xrightarrow{\ i\ } k \oplus \dot{X} \xrightarrow{\ \zeta_{\mathfrak{X}}\ } X \xrightarrow{\ p\ } \dot{X}/3\dot{X} \longrightarrow 0,$$

where $p(x) = \dot{x} + 3\dot{X}$ *and* i *is given as follows: Choose a linear form* α *on* X *with* $\alpha(1_X) = 1$*. Then* $i(\dot{x}) = \big(T(x) - \alpha(3x)\big) \oplus \dot{x}$*.*

(c) *There is a natural transformation*

$$\vartheta : \mathbf{C} \circ \mathbf{A} \to \mathrm{Id}_{\mathbf{qcform}_k}$$

given as follows: For $\mathfrak{M} = (M, f_2, f_3) \in \mathbf{qcform}_k$*, the* k*-module underlying* $(\mathbf{C} \circ \mathbf{A})(\mathfrak{M})$ *is* $(k \oplus M)/k \cdot (1 \oplus 0)$ *which is canonically identified with* M*. Then define*

$$\vartheta_{\mathfrak{M}} \colon (\mathbf{C} \circ \mathbf{A})(M) = M \to M, \qquad \vartheta_{\mathfrak{M}}(x) := 3x. \tag{2.6.2}$$

If $3 \in k^\times$ *then* $\vartheta_{\mathfrak{M}}$ *is an isomorphism, while in general we have the exact sequence*

$$0 \longrightarrow {}_3M \xrightarrow{\ \mathrm{inc}\ } M \xrightarrow{\ \vartheta_{\mathfrak{M}}\ } M \xrightarrow{\ \mathrm{can}\ } M/3M \longrightarrow 0.$$

Proof. (a) From $T(1_X) = 3$ one sees easily that $\zeta_{\mathfrak{X}}$ is a well-defined map. Clearly $\zeta_{\mathfrak{X}}(1 \oplus 0) = 1_X$, so it remains to check that $\zeta_{\mathfrak{X}}$ preserves cubic forms. Using (2.2.2) in case $d = 3$, we compute, in any base ring extension,

$$\begin{aligned} N\big(\zeta_{\mathfrak{X}}(\lambda \oplus \dot{x})\big) &= N\big((\lambda - T_x)1_X + 3x\big) \\ &= (\lambda - T_x)^3 + (\lambda - T_x)^2 T(3x) + (\lambda - T_x)Q(3x) + N(3x) \\ &= \lambda^3 - 3\lambda\big(T_x^2 - 3Q_x\big) + \big\{2T_x^3 - 9T_xQ_x + 27N(x)\big\} \\ &= \lambda^3 - 3\lambda H_2(x) + H_3(x) \\ &= \lambda^3 - 3\lambda q(\dot{x}) + h(\dot{x}) = N_{\mathfrak{M}}(\lambda \oplus \dot{x}). \end{aligned}$$

Finally, naturality of ζ means that, for a morphism $\varphi \colon \mathfrak{X} \to \mathfrak{X}'$ of unital cubic forms, the diagram

$$\begin{array}{ccc} (\mathbf{A} \circ \mathbf{C})(\mathfrak{X}) & \xrightarrow{\ \zeta_{\mathfrak{X}}\ } & \mathfrak{X} \\ \downarrow {\scriptstyle (\mathbf{A}\circ\mathbf{C})(\varphi)} & & \downarrow {\scriptstyle \varphi} \\ (\mathbf{A} \circ \mathbf{C})(\mathfrak{X}') & \xrightarrow[\ \zeta_{\mathfrak{X}'}\ ]{} & \mathfrak{X}' \end{array}$$

commutes, which is straightforward to check.

(b) It is easily verified that $\zeta_{\mathfrak{X}}^{-1}$ is given by the indicated formula if $3 \in k^\times$. To prove the remaining statements, we show first that i is well-defined. Indeed,

$T - 3\alpha$ vanishes on 1_X because $T(1) = 3$ and $\alpha(1) = 1$, and thus induces a linear form α' on $\dot{X}$. Moreover, if also $\beta(1) = 1$ and $\dot{x} \in {}_3\dot{X}$, then $(\beta' - \alpha')(\dot{x}) = \big(T(x) - 3\beta(x)\big) - \big(T(x) - 3\alpha(x)\big) = (\alpha - \beta)(3x) = 0$ because $3\dot{x} = 0$ implies $3x \in k \cdot 1_X$.

Next, injectivity of i is clear (project onto the second factor), and $\mathrm{Im}(i) \subset \mathrm{Ker}(\zeta_{\mathfrak{X}})$ follows from $(\zeta_{\mathfrak{X}} \circ i)(\dot{x}) = \zeta_{\mathfrak{X}}\big((T(x) - \alpha(3x)) \oplus \dot{x}\big) = 3\big(x - \alpha(x)1\big) = 0$, because $3\dot{x} = 0$ implies $3x = \mu 1$ where $\mu = \alpha(3x)$. Conversely, let $\lambda \oplus \dot{x} \in \mathrm{Ker}(\zeta_{\mathfrak{X}})$. Then $\big(\lambda - T(x)\big)1_X + 3x = 0$, i.e., $3x = \big(T(x) - \lambda\big)1_X$. Hence $\alpha(3x) = T(x) - \lambda$ and therefore $\lambda \oplus \dot{x} = \big(T(x) - \alpha(3x)\big) \oplus \dot{x} = i(\dot{x})$.

(c) We check that $\vartheta_{\mathfrak{M}}$ preserves quadratic and cubic forms. Note that $T_{\mathfrak{M}}(0 \oplus x) = 0$, $Q_{\mathfrak{M}}(0 \oplus x) = -3f_2(x)$ and $N_{\mathfrak{M}}(0 \oplus x) = f_3(x)$, for $x \in M_R$, $R \in k$-**alg**, by (2.5.2), (2.5.3) and (2.5.1). Since we identify $x \in M$ with $\mathrm{can}(0 \oplus x) \in (k \oplus M)/k \cdot (1 \oplus 0)$, it follows from Lemma 2.4 that $q(x) = H_2(0 \oplus x) = (-3)\big(-3f_2(x)\big) = 9f_2(x) = f_2\big(\vartheta_{\mathfrak{M}}(x)\big)$, and $h(x) = H_3(0 \oplus x) = 27f_3(x) = f_3\big(\vartheta_{\mathfrak{M}}(x)\big)$, as required. It is easily checked that $\vartheta_{\mathfrak{M}}$ depends functorially on $\mathfrak{M}$, so ϑ is indeed a natural transformation of functors. The final statement is evident.

2.7. Corollary. *If $3 \in k^\times$, then the categories* **ucform**$_k$ *and* **qcform**$_k$ *are equivalent.*

3. From unital cubic to symmetric compositions

3.1. Definition. Let k be a commutative ring. A *unital composition of degree d* is a quadruple $\mathfrak{A} = (X, N, 1, \cdot)$ such that $\mathfrak{X} = (X, N, 1)$ is a unital form of degree d as in 2.1, $A = (X, \cdot, 1)$ is a unital k-algebra, and both structures are related by the requirement that N be multiplicative:

$$N(xy) = N(x)N(y) \quad \text{for all } x, y \in X \otimes R,\ R \in k\text{-}\mathbf{alg}. \tag{3.1.1}$$

Depending on context, we will write $\mathfrak{A} = (\mathfrak{X}, \cdot)$ or $\mathfrak{A} = (A, N)$. This definition is the special case where the values of the form N lie in k and A is unital, of the more general definition of Roby [roby2]. Note that we assume neither associativity conditions on A nor non-degeneracy conditions on N. In case $d = 2$ or $d = 3$, we speak of a unital quadratic or cubic composition.

It is tempting to call $\mathfrak{A}$ a "composition algebra of degree d", and indeed this terminology is used in [bb] (except for a different definition of form). However, this may be in conflict with the notion of degree of an algebra in cases where such a degree is well-defined, e.g., when A is associative or alternative and finite-dimensional over a field. In these cases, the degree of A will in general be different from the degree d of N. For example, $A = k$ (as a k-algebra) has degree 1, but for any d, the form $N(x) = x^d$ makes the pair (k, N) a unital composition of degree d. Also, $A = k \times k$ with component-wise operations is an algebra of degree 2, but the form $N(x_1, x_2) = x_1$ makes $(k \times k, N)$ a unital composition of degree 1.

By a morphism $\varphi\colon \mathfrak{A} \to \mathfrak{A}'$ of unital compositions of degree d we mean a linear map of the underlying modules preserving forms, units and multiplications. Unital compositions of degree d form a category, and there is an obvious forgetful functor Ω from unital compositions to unital forms omitting the product. Note that (A^{op}, N) is a unital composition of degree d along with (A, N).

3.2. Lemma. *Let (A, N) be a unital composition of degree d. With the notations introduced in* 2.2*, the following formulas hold for all $x, y, z \in A$:*

$$T(xy) = T(x)T(y) - B(x, y) = T(yx), \tag{3.2.1}$$

$$
\begin{aligned}
T\big(x(yz)\big) &= T(x)T(y)T(z) - B(x,yz) - B(y,xz) - B(z,xy) - \Phi(x,y,z) && (3.2.2)\\
&= T\big((xy)z\big), && (3.2.3)\\
B(x,1) &= (d-1)T(x), && (3.2.4)\\
\Phi(x,y,1) &= (d-2)B(x,y). && (3.2.5)
\end{aligned}
$$

Proof. Consider the algebra $R = k(\varepsilon,\delta,\eta)$ with relations $\varepsilon^2 = \delta^2 = \eta^2 = 0$. Then, since N is multiplicative and R is associative,

$$
\begin{aligned}
N\big((1+\varepsilon x)[(1+\delta y)(1+\eta z)]\big) &= N(1+\varepsilon x)N(1+\delta y)N(1+\eta z)\\
&= N\big([(1+\varepsilon x)(1+\delta y)](1+\eta z)\big). \qquad (3.2.6)
\end{aligned}
$$

Using (2.2.2) and the relations in R, we have

$$
\begin{aligned}
N(1+\varepsilon x)N(1+\delta y)N(1+\eta z) &= 1 + \varepsilon T(x) + \delta T(y) + \eta T(z)\\
&\quad + \varepsilon\delta T(x)T(y) + \varepsilon\eta T(x)T(z) + \delta\eta T(y)T(z) + \varepsilon\delta\eta T(x)T(y)T(z).
\end{aligned}
$$

For easier notation, let us put

$$
\begin{aligned}
&x_1 = x, \quad x_2 = y, \quad x_3 = z,\\
&x_4 = x_1x_2, \quad x_5 = x_1x_3, \quad x_6 = x_2x_3, \quad x_7 = x_1(x_2x_3),\\
&\varepsilon_1 = \varepsilon, \quad \varepsilon_2 = \delta, \quad \varepsilon_3 = \eta,\\
&\varepsilon_4 = \varepsilon_1\varepsilon_2, \quad \varepsilon_5 = \varepsilon_1\varepsilon_3, \quad \varepsilon_6 = \varepsilon_2\varepsilon_3, \quad \varepsilon_7 = \varepsilon_1\varepsilon_2\varepsilon_3.
\end{aligned}
$$

Then

$$
(1+\varepsilon x)[(1+\delta y)(1+\eta z)] = 1 + \sum_{i=1}^{7} \varepsilon_i x_i = 1 + u,
$$

where $u := \sum_{i=1}^{7} \varepsilon_i x_i$. By the general expansion formula (2.2.2), $N(1+u) = \sum_{i=0}^{d} T_i(u)$. We show first that $T_p(u) = 0$ for $p \geqslant 4$. Indeed,

$$
\begin{aligned}
T_p(u) &= \sum_{i=1}^{7} T_p(x_i)\varepsilon_i^p\\
&\quad + \sum_{1\leqslant i<j\leqslant 7} \; \sum_{l+m=p} T_{lm}(x_i,x_j)\varepsilon_i^l\varepsilon_j^m\\
&\quad + \sum_{1\leqslant i<j<k\leqslant 7} \; \sum_{l+m+n=p} T_{lmn}(x_i,x_j,x_k)\varepsilon_i^l\varepsilon_j^m\varepsilon_k^n\\
&\quad + \sum_{1\leqslant i_1<i_2<i_3<i_4\leqslant 7} \; \sum_{l_1+\cdots+l_4=p} T_{l_1\ldots l_4}(x_{i_1},\ldots,x_{i_4})\varepsilon_{i_1}^{l_1}\cdots\varepsilon_{i_4}^{l_4}\\
&\quad + \quad \cdots \quad + \sum_{l_1+\cdots+l_7=p} T_{l_1\ldots l_7}(x_1,\ldots,x_7)\varepsilon_1^{l_1}\cdots\varepsilon_7^{l_7}.
\end{aligned}
$$

(The summation runs over all partitions of p in at most 7 parts). Now consider the products of powers of the ε_i. Since $\varepsilon_i^2 = 0$, the first sum vanishes as soon as $p \geqslant 2$. Likewise, the second sum vanishes for $p \geqslant 3$, because then either $l \geqslant 2$ or $m \geqslant 2$. In the third sum, the only product of type $\varepsilon_i^l\varepsilon_j^m\varepsilon_k^n$ which is non-zero is $\varepsilon_1\varepsilon_2\varepsilon_3$, as follows from the definition of the $\varepsilon_4,\ldots,\varepsilon_7$ above. Hence the third sum vanishes for $p \geqslant 4$, and yields only the term $\varepsilon_1\varepsilon_2\varepsilon_3\Phi(x,y,z)$ for $p = 3$. Furthermore, again by definition of the ε_i, any product of more than three different ε_i vanishes, so the remaining sums vanish as well. This yields

$$
\begin{aligned}
T_1(u) &= \varepsilon T(x) + \delta T(y) + \eta T(z) + \varepsilon\delta T(xy) + \varepsilon\eta T(xz) + \delta\eta T(yz) \\
&\quad + \varepsilon\delta\eta T(x(yz)), \\
T_2(u) &= \varepsilon\delta B(x,y) + \varepsilon\eta B(x,z) + \delta\eta B(y,z), \\
&\quad + \varepsilon\delta\eta\big\{B(x,yz) + B(y,xz) + B(z,xy)\big\}, \\
T_3(u) &= \varepsilon\delta\eta \Phi(x,y,z).
\end{aligned}
$$

Adding everything up and comparing the coefficients of $\varepsilon\delta$ shows (3.2.1). Comparing the coefficients of $\varepsilon\delta\eta$ yields (3.2.2), and (3.2.3) follows by passing to the opposite algebra. Formula (3.2.4) is a special case of (2.2.4). Finally, put $z = 1$ in (3.2.2) and use (3.2.1) and (3.2.4):

$$
\begin{aligned}
\Phi(x,y,1) &= T(x)T(y)T(1) - B(x,y) - B(y,x) - B(xy,1) - T(xy) \\
&= dT(x)T(y) - 2B(x,y) - (d-1)T(xy) - T(xy) \\
&= d\big\{T(x)T(y) - T(xy)\big\} - 2B(x,y) = (d-2)B(x,y).
\end{aligned}
$$

Remark. The formulas $T(xy) = T(yx)$ and $T\big((xy)z\big) = T\big(x(yz)\big)$ are expressed by saying that T is commutative and associative. To save parentheses, we will often write T_x instead of $T(x)$ and $xy \cdot z$ instead of $(xy)z$. Because of commutativity and associativity of T we have $T_{xy\cdot x} = T_{x\cdot yx} = T_{yx\cdot x}$ for which we simply write T_{xyx}. In particular, $T(x^3) := T(x \cdot x^2) = T(x^2 \cdot x)$ is well-defined although $x \cdot x^2$ may be different from $x^2 \cdot x$.

3.3. Corollary. *Let (A, N) be a unital composition of degree d. Then Φ is given by the following formula involving only the linear trace form T and the multiplication in A:*

$$
\Phi(x,y,z) = T_{xyz} + T_{zyx} - T_xT_{yz} - T_yT_{zx} - T_zT_{xy} + T_xT_yT_z. \tag{3.3.1}
$$

Proof. Substitute (3.2.1) into (3.2.2) and use commutativity and associativity of T.

The first five formulas of the following lemma are also found in [bb, Lemma 1].

3.4. Lemma. *Let (A, N) be a unital cubic composition and use the notations of 2.2 and 3.2. Then the following formulas hold for all u, x, y, z in all base extensions of A:*

$$
\begin{aligned}
T(u)N(x) &= N_{21}(x, ux) = N_{21}(x, xu), && (3.4.1)\\
T(u)N_{21}(x,y) &= \Phi(x,y,ux) + N_{21}(x,uy) && (3.4.2)\\
&= \Phi(x,y,xu) + N_{21}(x,yu), && (3.4.3)\\
T(u)\Phi(x,y,z) &= \Phi(ux,y,z) + \Phi(x,uy,z) + \Phi(x,y,uz) && (3.4.4)\\
&= \Phi(xu,y,z) + \Phi(x,yu,z) + \Phi(x,y,zu), && (3.4.5)\\
Q(x)T(y) &= B(x,xy) + N_{21}(x,y), && (3.4.6)\\
Q(xy) &= Q(yx) = Q(x)Q(y) - \Phi(x,y,xy), && (3.4.7)\\
B(xy,x) &= B(yx,x) = T_{xy}T_x - T_{xyx}, && (3.4.8)\\
Q(x)Q(y) + Q(xy) &= T(xy \cdot yx) - T_xT_{yxy} - T_yT_{xyx} + T_xT_yT_{xy}, && (3.4.9)\\
3N(x) - T(x)Q(x) &= T(x^3) - T(x)T(x^2). && (3.4.10)
\end{aligned}
$$

Proof. Let $u, x \in A \otimes R$, $R \in k$-**alg**. Then we have in $A \otimes R(\varepsilon)$, where $R(\varepsilon)$ is the algebra of dual numbers:

$$N(x+\varepsilon ux) = N\big((1+\varepsilon u)x\big) = N(1+\varepsilon u)N(x) = N(x)N(1+\varepsilon u) = N(x+\varepsilon xu),$$

since $R(\varepsilon)$ is commutative. Expanding both sides and comparing terms at ε yields (3.4.1), since $N(1+\varepsilon u) = 1+\varepsilon T(u)$. Linearizing (3.4.1) with respect to x in direction y and then in direction z shows that (3.4.2) – (3.4.5) hold.

Next, consider $R(\varepsilon, \delta)$ where $\varepsilon^3 = \delta^3 = 0$. Then it follows by expanding

$$N\big((1+\varepsilon x)(1+\delta y)\big) = N(1+\varepsilon x)N(1+\delta y)$$

and comparing terms at $\varepsilon^2\delta$ and $\varepsilon^2\delta^2$ that (3.4.6) holds, as well as

$$Q(x)Q(y) = Q(xy) + \Phi(x, y, xy). \tag{3.4.11}$$

Putting $u = y$ in (3.4.2) and (3.4.3) yields

$$\Phi(x, y, xy) = \Phi(x, y, yx) = T(y)N_{21}(x, y) - N_{21}(x, y^2).$$

Now (3.4.11) shows that (3.4.7) holds. Formula (3.4.8) is a consequence of (3.2.1) and commutativity and associativity of T (Lemma 3.2) while (3.4.9) follows by adding $Q(xy)$ to both sides of (3.4.11), using (3.3.1) to evaluate $\Phi(x, y, xy)$ and recalling that $2Q(xy) = B(xy, xy) = T_{xy}^2 - T(xy \cdot xy)$ by (3.2.1).

Finally, we prove (3.4.10). Put $x = y$ in (3.4.6). This shows $N_{21}(x, x) = T(x)Q(x) - B(x, x^2) = T(x)Q(x) - T(x)T(x^2) + T(x \cdot x^2)$ (by (3.2.1)). On the other hand, $3N(x) = N_{21}(x, x)$ since N is homogeneous of degree 3, whence (3.4.10).

3.5. Definition. A *symmetric composition over k* is a triple $\mathfrak{C} = (M, q, \star)$ consisting of a k-module M, a bilinear multiplication $\star : M \times M \to M$, and a quadratic form q on M such that q is multiplicative and the polar form b of q is associative with respect to $\star$, i.e.,

$$q(x \star y) = q(x)q(y), \tag{3.5.1}$$

$$b(x \star y, z) = b(x, y \star z), \tag{3.5.2}$$

for all $x, y, z \in M$. This generalizes the usual definition [boi, §34] inasmuch as non-degeneracy assumptions on q are not imposed and k is an arbitrary ring instead of a field. Hence, $\mathfrak{C}$ should perhaps be called a generalized symmetric composition, but we will drop the epithet “generalized” for brevity. In any case, non-degeneracy of q can be forced by dividing out the kernel of q, see Lemma 4.4. Note that passing to the opposite multiplication yields again a symmetric composition. — Given a symmetric composition, we introduce the cubic form

$$h(x) = b(x, x \star x). \tag{3.5.3}$$

We can now formulate the main result of this section.

3.6. Theorem. *Let (A, N) be a unital cubic composition, and let $\alpha, \beta \in k$ satisfy $\alpha + \beta = 1$.*

(a) *There is a well-defined multiplication $\star$ on $\dot{A} = A/k \cdot 1$ such that*

$$\kappa(x) \star \kappa(y) = \kappa\big((1+\alpha)xy + (1+\beta)yx - T_x y - T_y x\big) \tag{3.6.1}$$

for all $x, y \in A$, where κ: $A \to \dot{A}$ is the canonical map.

(b) *Denote by $q = \kappa_*(H_2)$ the quadratic form on $\dot{A}$ as in 2.4 and assume that.*

$$3(\alpha\beta - 1)[A, A] = 0, \tag{3.6.2}$$

where $[x, y] = xy - yx$ denotes the commutator in A. Then the triple $(\dot{A}, q, \star)$ is a symmetric composition, and its associated cubic form is $h = \kappa_*(H_3)$.

3.7. Remarks and special cases. (a) Switching α and β in (3.6.1) or replacing A by the opposite algebra A^{op} amounts to replacing $\star$ by the opposite multiplication.

(b) Condition (3.6.2) holds if either $3 = 0$ in k or A is commutative or $\alpha\beta = 1$. We discuss these cases in turn.

(i) Assume $3 = 0$ in k. Then we obtain a one-parameter family (because $\beta = 1 - \alpha$) of symmetric compositions on $\dot{A}$, all with the same quadratic form q. Note, however, that q is the square of a linear form, because $T(1_A) = 3$, so T induces a linear form t on $\dot{A}$, and then $q = t^2$ by (2.4.1).

(ii) If A is commutative the symmetric composition $(\dot{A}, q, \star)$ is commutative and independent of the choice of α.

(iii) Since $\beta = 1 - \alpha$, the condition $\alpha\beta = 1$ is equivalent to $\alpha^2 - \alpha + 1 = 0$. Let $\Phi_n(\mathbf{t})$ be the n-th cyclomotic polynomial and define the *functor of primitive n-th roots of unity* $\boldsymbol{\pi}_n$ from k-**alg** to sets by

$$\boldsymbol{\pi}_n(R) = \{r \in R : \Phi_n(r) = 0\} \qquad (R \in k\text{-}\mathbf{alg}).$$

In particular, $\Phi_3(\mathbf{t}) = \mathbf{t}^2 + \mathbf{t} + 1$ and $\Phi_6(\mathbf{t}) = \mathbf{t}^2 - \mathbf{t} + 1$, so $\boldsymbol{\pi}_6 \cong \boldsymbol{\pi}_3$ under $r \mapsto -r$. We have preferred to use primitive sixth rather than third roots of unity because the formulas become more natural and involve fewer minus signs. Formulas closer to [`elmy`, `boi`] are obtained by writing $\alpha = -\omega$ where $\omega \in \boldsymbol{\pi}_3(k)$. For easy reference, we formulate the following consequence of Theorem 3.6:

3.8. Corollary. *Let (A, N) be a unital cubic composition and let $\alpha \in \boldsymbol{\pi}_6(k)$ be a primitive sixth root of unity. Then $\dot{A}$ becomes a symmetric composition with quadratic form $q = \kappa_*(H_3)$, product*

$$\kappa(x) \star \kappa(y) = \kappa\big((1+\alpha)xy + (1+\alpha^{-1})yx - T_x y - T_y x\big) \tag{3.8.1}$$

and associated cubic form $h = \kappa_*(H_3)$.

The proof of Theorem 3.6 rests on the following lemmas. We use the notations introduced in Lemma 2.4; in particular, H_{11} is the polar form of H_2.

3.9. Lemma. *Let $\alpha, \beta \in k$ and consider the multiplication*

$$x * y := (1+\alpha)xy + (1+\beta)yx - T_x y - T_y x \tag{3.9.1}$$

on A. Then

$$x * 1 = 1 * x = (\alpha + \beta - 1)x - T_x \cdot 1, \tag{3.9.2}$$

$$H_{11}(x * y, z) - H_{11}(x, y * z) = (\alpha + \beta - 1)(T_x T_{yz} - T_z T_{xy}), \tag{3.9.3}$$

for all $x, y, z \in A$.

Proof. (3.9.2) is immediate from the definition and $T(1) = 3$. By (2.4.3), $H_{11}(x, y) = 2T_x T_y - 3B(x, y)$. Now a simple computation using (3.2.1) and associativity of T shows that

$$\begin{aligned} H_{11}(x * y, z) = 2T_x T_y T_z - 3T_x T_{yz} - 3T_y T_{zx} - (2 + \alpha + \beta)T_z T_{xy} \\ + 3(1+\alpha)T_{xyz} + 3(1+\beta)T_{zyx}. \end{aligned} \tag{3.9.4}$$

Subtracting from (3.9.4) the formula obtained by cyclically permuting x, y, z yields (3.9.3).

3.10. Lemma. *Let $\alpha, \beta \in k$ satisfy $\alpha + \beta = 1$ and define $x * y$ as in* (3.9.1). *Then*

$$H_2(x)H_2(y) - H_2(x * y) = 3(\alpha\beta - 1)T\bigl(xy \cdot [x, y]\bigr), \tag{3.10.1}$$

$$H_2(x)^2 = H_2(x * x). \tag{3.10.2}$$

Proof. Put $\lambda = \alpha\beta - 1$. Then

$$(1 + \alpha)^2 + (1 + \beta)^2 = 3 - 2\lambda \quad \text{and} \quad (1 + \alpha)(1 + \beta) = 3 + \lambda.$$

Hence, by the symmetry of $Q(xy)$ in x and y ((3.4.7)), the symmetry $B(xy, x) = B(yx, x)$ ((3.4.8)) and the standard formula $B(u, v) = T(u)T(v) - T(uv)$ ((3.2.1)) as well as $2Q(u) = B(u, u)$,

$$\begin{aligned}
Q(x * y) &= Q\bigl((1 + \alpha)xy + (1 + \beta)yx - (T_x y + T_y x)\bigr) \\
&= (1 + \alpha)^2 + (1 + \beta)^2)Q(xy) + (1 + \alpha)(1 + \beta)B(xy, yx) + Q\bigl(T_x y + T_y x\bigr) \\
&\quad - B\bigl((1 + \alpha)xy + (1 + \beta)yx, T_x y + T_y x\bigr) \\
&= (3 - 2\lambda)Q(xy) + (\lambda + 3)B(xy, yx) \\
&\quad + T_x^2 Q(y) + T_y^2 Q(x) + T_x T_y B(x, y) \\
&\quad - 3B\bigl(xy, T_x y + T_y x\bigr) \\
&= \lambda\bigl\{B(xy, yx) - B(xy, xy)\bigr\} + 3Q(xy) + 3B(xy, yx) \\
&\quad + T_x^2 Q(y) + T_y^2 Q(x) + T_x T_y B(x, y) \\
&\quad - 3B\bigl(xy, T_x y + T_y x\bigr).
\end{aligned}$$

Here the coefficient of λ is, by commutativity of T and (3.2.1),

$$B(xy, yx) - B(xy, xy) = T_{xy}T_{yx} - T(xy \cdot yx) - T_{xy}^2 + T(xy \cdot xy) = T\bigl(xy \cdot [x, y]\bigr).$$

Replacing systematically $B(u, v)$ by $T_u T_v - T_{uv}$, we obtain

$$\begin{aligned}
Q(x * y) = \lambda T\bigl(xy \cdot [x, y]\bigr) &+ T_x^2 T_y^2 + 3\bigl\{T_x T_{yxy} + T_y T_{xyx} + T_{xy}^2 - T(xy \cdot yx)\bigr\} \\
&- 7T_x T_y T_{xy} + 3Q(xy) + T_x^2 Q(y) + T_y^2 Q(x).
\end{aligned} \tag{3.10.3}$$

Again by commutativity of T and $\alpha + \beta = 1$, we have $T(x * y) = 3T_{xy} - 2T_x T_y$. Hence,

$$\begin{aligned}
H_2(x)H_2(y) - H_2(x * y) = &\bigl\{T_x^2 - 3Q(x)\bigr\}\bigl\{T_y^2 - 3Q(y)\bigr\} \\
&- \bigl\{3T_{xy} - 2T_x T_y\bigr\}^2 + 3Q(x * y).
\end{aligned}$$

Let us put $\delta = H_2(x)H_2(y) - H_2(x * y) - 3\lambda T\bigl(xy \cdot [x, y]\bigr)$. Then by (3.10.3),

$$\begin{aligned}
\delta &= T_x^2 T_y^2 - 3Q(x)T_y^2 - 3Q(y)T_x^2 + 9Q(x)Q(y) \\
&\quad - 9T_{xy}^2 + 12T_x T_y T_{xy} - 4T_x^2 T_y^2 \\
&\quad + 3T_x^2 T_y^2 + 9\bigl\{T_x T_{yxy} + T_y T_{xyx} + T_{xy}^2 - T(xy \cdot yx)\bigr\} \\
&\quad - 21T_x T_y T_{xy} + 9Q(xy) + 3T_x^2 Q(y) + 3T_y^2 Q(x) \\
&= 9\bigl\{Q(x)Q(y) + Q(xy) + T_x T_{yxy} + T_y T_{xyx} - T_x T_y T_{xy} - T(xy \cdot yx)\bigr\}
\end{aligned}$$

and the expression in braces vanishes by (3.4.9). This proves (3.10.1), and (3.10.2) is an immediate consequence.

3.11. Proof of Theorem 3.6**.** It only remains to show that the cubic form h associated with $(\dot{A}, q, \star)$ is indeed $\kappa_*(H_3)$. By (3.9.4), (3.4.10), and the definition of H_3 in 2.4,

$$\begin{aligned} h\big(\kappa(x)\big) = b\big(\kappa(x), \kappa(x) \star \kappa(x)\big) &= H_{11}(x, x * x) = 2T_x^3 - 9T_x T(x^2) + 9T(x^3) \\ &= 2T_x^3 + 9\big(3N(x) - T_x Q(x)\big) = H_3(x), \end{aligned}$$

as desired.

3.12. Definition. Let $\mathbf{scomp}_k$ denote the category of symmetric compositions over k, where morphisms are k-linear maps preserving products and quadratic forms, and recall the category $\mathbf{qcform}_k$ of 2.5. Then there is a functor

$$\Upsilon\colon \mathbf{scomp}_k \to \mathbf{qcform}_k, \qquad (M, q, \star) \mapsto (M, q, h). \tag{3.12.1}$$

Let $\mathbf{uccomp}_k$ denote the category of unital cubic compositions over k, and let $\alpha \in \boldsymbol{\pi}_6(k)$. We define a functor

$$\mathbf{C}_\alpha\colon \mathbf{uccomp}_k \to \mathbf{scomp}_k$$

as follows. For a unital cubic composition $\mathfrak{A} = (A, N)$ and a primitive sixth root of unity α, let $\mathbf{C}_\alpha(\mathfrak{A}) = (\dot{A}, q, \star)$ be the symmetric composition defined in Cor. 3.8. For a morphism $\varphi\colon \mathfrak{A} \to \mathfrak{A}'$ let $\mathbf{C}_\alpha(\varphi)\colon \mathbf{C}_\alpha(\mathfrak{A}) \to \mathbf{C}_\alpha(\mathfrak{A}')$ be the induced map $\dot{\varphi}\colon \dot{A} \to \dot{A}'$. Then $\mathbf{C}_\alpha\colon \mathbf{uccomp}_k \to \mathbf{scomp}_k$ is compatible with the functor $\mathbf{C}$ of 2.5 in the sense that the diagram

$$\begin{array}{ccc} \mathbf{uccomp}_k & \xrightarrow{\ \mathbf{C}_\alpha\ } & \mathbf{scomp}_k \\ {\scriptstyle \Omega}\big\downarrow & & \big\downarrow{\scriptstyle \Upsilon} \\ \mathbf{ucform}_k & \xrightarrow[\ \mathbf{C}\ ]{} & \mathbf{qcform}_k \end{array}$$

is commutative, where Ω is the functor forgetting the multiplication, cf. 3.1.

3.13. The connection with [tsch]**.** Let (A, N) be a unital cubic composition and suppose $3 \in k^\times$. Then $A = k \cdot 1_A \oplus A^0$ where $A^0 = \operatorname{Ker}(T)$. Let $\omega \in \boldsymbol{\pi}_3(k)$ be a primitive third root of unity, and define a multiplication $\circledast$ on A^0 by

$$x \circledast y = \frac{1-\omega}{3} xy + \frac{1-\omega^{-1}}{3} yx - \frac{1}{3} T(xy) 1_A, \tag{3.13.1}$$

see [tsch]. The last term serves to make $x \circledast y$ lie in A^0. Also, define a quadratic form n on A^0 by

$$n(x) = -\frac{1}{3} Q(x). \tag{3.13.2}$$

It is proved in [tsch, 3.1], for k a field of characteristic not 2 or 3, that n is multiplicative and the polar form b_n of n is associative with respect to $\circledast$. This result, without assuming characteristic $\neq 2$, can now be recovered in our setting as follows. Let $\alpha = -\omega \in \boldsymbol{\pi}_6(k)$ and let $(\dot{A}, q, \star) = \mathbf{C}_\alpha(A, N)$. Then the map

$$\varphi = \kappa \circ \mathrm{inc} : A^0 \to A \to \dot{A}, \quad x \mapsto \frac{1}{3} \dot{x},$$

is an isomorphism $\varphi\colon (A^0, n, \circledast) \xrightarrow{\cong} (\dot{A}, q, \star)$. Indeed, φ is clearly an isomorphism of k-modules. From (3.13.1) and (3.8.1) we see that φ preserves products, and from (3.13.2) and Lemma 2.4(a) it follows that φ preserves quadratic forms. Since $(\dot{A}, q, \star)$ is a symmetric composition, so is $(A^0, n, \circledast)$.

4. From symmetric to unital cubic compositions

4.1. Theorem. *Let $(M, q, \star)$ be a symmetric composition over k and let $\alpha, \beta \in k$ with $\alpha + \beta = 1$. On the k-module $A := k \oplus M$ consider the unital cubic form*

$$N(\lambda \oplus x) = \lambda^3 - 3\lambda q(x) + b(x, x \star x), \tag{4.1.1}$$

and define a multiplication $\bullet$ with unit element $1_A := 1 \oplus 0$ and

$$(0 \oplus x) \bullet (0 \oplus y) = b(x, y) \oplus \big(\alpha x \star y + \beta y \star x\big) \tag{4.1.2}$$

for $x, y \in M$. Assume that

$$(\alpha\beta - 1)[M, M] = 0 \tag{4.1.3}$$

where $[x, y] = x \star y - y \star x$ denotes the commutator. Then $(A, N, 1_A, \bullet)$ is a unital cubic composition.

4.2. Remarks. (a) The cubic form N is of course the cubic form $N_{\mathfrak{M}}$ defined in (2.5.1), for $\mathfrak{M} = \varUpsilon(\mathfrak{C}) = (M, q, h)$ as in (3.12.1) and $h(x) = b(x, x \star x)$ as in (3.5.3).

(b) Interchanging α and β in (4.1.2) or replacing $\mathfrak{C}$ by $\mathfrak{C}^{\mathrm{op}}$ amounts to replacing the multiplication $\bullet$ by its opposite.

(c) If $\mathfrak{C}$ is commutative then (4.1.3) is satisfied for any choice of α, and the product $\bullet$ is again commutative and independent of α.

(d) As in 3.7(b)(iii), the conditions $\alpha + \beta = \alpha\beta = 1$ are equivalent to $\alpha \in \boldsymbol{\pi}_6(k)$. Hence we have the following corollary.

4.3. Corollary. *Let $(M, q, \star)$ be a symmetric composition and let $\alpha \in \boldsymbol{\pi}_6(k)$ be a primitive sixth root of unity. Then $k \oplus M$ becomes a unital cubic composition with unit element $1 \oplus 0$, cubic form* (4.1.1) *and product*

$$(0 \oplus x) \bullet (0 \oplus y) = b(x, y) \oplus \big(\alpha x \star y + \alpha^{-1} y \star x\big). \tag{4.3.1}$$

Remark. For the case where k is a field of characteristic different from 2 and 3 and q is non-degenerate, Tschupp [`tsch`, 3.2] attempted to prove this result, but his proof contains errors. Specifically, it is claimed in [`tsch`, Lemma 3.10] that, in our notation, $h(x \star y) = b\big((x \star y) \star (x \star y), y \star x\big)$, and in [`tsch`, Lemma 3.11] that $b\big(y \star (y \star x), (y \star x) \star x\big) = q(x)q(y)b(x, y)$. The following example, due to H. P. Petersson, disproves these formulas: Let $M = \mathrm{Mat}_2(k)$ with $q(x) = \det(x)$ and $x \star y = \bar{x}\bar{y}$, where $\bar{x} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ for $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then it suffices to put $x = e_{12}$ and $y = e_{21}$ (matrix units).

Theorem 4.1 will be a consequence of the following Lemmas 4.6 – 4.8. We note that P. Alberca and C. Martín [`ident`] have recently proved and extended these lemmas with the aid of a computer.

In the interest of readability, we denote in this section the multiplication in a symmetric composition simply by juxtaposition; thus $xy = x \star y$, and in particular, $x^2 = x \star x$. Also, to save parentheses, we will often write $x \cdot yz$ instead of $x(yz)$ etc.

4.4. Lemma. *With the above notations, the polar forms h_{21} and h_{111} of h are given by*

$$h_{21}(x,y) = 3b(x^2,y), \qquad h_{111}(x,y,z) = 3b(xy+yx,z). \tag{4.4.1}$$

Writing $Z := \mathrm{Ker}(q)$, we have

$$Z \subset \mathrm{Ker}(h), \qquad Z \star M + M \star Z \subset Z. \tag{4.4.2}$$

Hence there is a symmetric composition $\mathfrak{C}/\mathrm{Ker}(q) := (M/Z, \bar{q}, \bar{\star})$ whose quadratic form $\bar{q}$ (induced from q) is non-degenerate.

Proof. The formulas (4.4.1) follow easily from associativity of b ((3.5.2)). If $z \in Z$ and $x \in M$ then $h(z) = b(z,z^2) = 0$ and $h_{21}(z,x) = 3b(z^2,x) = 3b(z,zx) = 0$ as well as $h_{21}(x,z) = 3b(z,x^2) = 0$, which shows $Z \subset \mathrm{Ker}(h)$. Finally, $q(zx) = q(z)q(x) = 0$ and $b(zx,y) = b(z,xy) = 0$ so $ZM \subset Z$, and similarly $MZ \subset Z$. The last statement follows from Cor. 1.7.

4.5. Formulas in symmetric compositions. There is a number of formulas which are known for non-degenerate symmetric compositions [boi, §34] but which hold only modulo $\mathrm{Ker}(q)$ in general. Therefore, we introduce the notation

$$x \equiv y \quad :\Longleftrightarrow \quad x - y \in \mathrm{Ker}(q).$$

First of all, linearization of $q(xy) = q(x)q(y)$ ((3.5.1)) yields

$$b(xy,xz) = b(yx,zx) = q(x)b(y,z), \tag{4.5.1}$$

$$b(x_1,x_2)b(x_3,x_4) = b(x_1x_3,x_2x_4) + b(x_1x_4,x_2x_3). \tag{4.5.2}$$

Associativity of b is equivalent to the fact that $b(xy,z)$ is invariant under cyclic permutation of x,y,z. As before, let $[x,y] := xy - yx$ denote the commutator. Then:

$$\text{The trilinear form } (x,y,z) \mapsto b\big([x,y],z\big) \text{ is alternating.} \tag{4.5.3}$$

Next, we have flexibility modulo $\mathrm{Ker}(q)$:

$$xy \cdot x \equiv x \cdot yx \equiv q(x)y, \tag{4.5.4}$$

Indeed, for all $v \in M$, $b(xy \cdot x, v) = b(xy,xv) = q(x)b(y,v) = b\big(q(x)y,v\big)$ by associativity and (4.5.1), and

$$\begin{aligned} q\big(xy \cdot x - q(x)y\big) &= q(xy \cdot x) - q(x)b(xy \cdot x, y) + q\big(q(x)y\big) \\ &= q(x)^2q(y) - q(x)^2b(y,y) + q(x)^2q(y) = 0. \end{aligned}$$

This proves $xy \cdot x \equiv q(x)y$, and the second formula follows by passing to the opposite multiplication. An immediate consequence is

$$(xx^2)x \equiv (x^2x)x \equiv x(xx^2) \equiv x(x^2x) \equiv q(x)x^2. \tag{4.5.5}$$

Since $\mathrm{Ker}(q)$ is a submodule, (4.5.4) can be linearized:

$$xy \cdot z + zy \cdot x \equiv x \cdot yz + z \cdot yx \equiv b(x,z)y. \tag{4.5.6}$$

Specializing z to y in (4.5.6) and also interchanging x and y yields

$$xy \cdot y + y^2x \equiv xy^2 + y \cdot yx \equiv b(x,y)y, \tag{4.5.7}$$

$$x^2y + yx \cdot x \equiv yx^2 + x \cdot xy \equiv b(x,y)x. \tag{4.5.8}$$

These imply the commutator formulas

$$[x,y^2] \equiv xy \cdot y - y \cdot yx, \tag{4.5.9}$$

$$[x^2,y] \equiv x \cdot xy - yx \cdot x, \tag{4.5.10}$$

$$[x^2,y^2] \equiv (x^2y)y - y(yx^2) \equiv x(xy^2) - (y^2x)x. \tag{4.5.11}$$

4.6. Lemma. *Let $x, y \in M$. We introduce the abbreviations*

$$s := xy, \quad t := yx, \quad a := b(x, y^2)b(y, x^2), \quad c := b_{x,y} q_x q_y, \quad e := b_{x,y} b_{s,t}.$$

Then the following identities hold:

$$h(x)h(y) = b(s, x^2y^2) + b(xy^2, x^2y), \tag{4.6.1}$$
$$b(s^2, t) = b(t^2, s) = b(s, y^2x^2) = b(t, x^2y^2) = a - c, \tag{4.6.2}$$
$$h(s) - h(t) = h([x, y]), \tag{4.6.3}$$
$$h(s) = a - b(sy, xs), \tag{4.6.4}$$
$$h(t) = a - b(tx, yt), \tag{4.6.5}$$
$$b(xs, yt) = e - c, \tag{4.6.6}$$
$$b(x, y)^3 = b(xy^2, yx^2) + c + e. \tag{4.6.7}$$

Proof. Formula (4.6.1) follows by applying (4.5.2) to $h(x)h(y) = b(x, x^2)b(y, y^2)$. For (4.6.2), we use again (4.5.2), applied to $x_1 = x$, $x_2 = y^2$, $x_3 = y$, $x_4 = x^2$, and then (4.5.5) and the definition of c:

$$a = b(x, y^2)b(y, x^2) = b(xy, y^2x^2) + b(xx^2, y^2y) = b(xy, y^2x^2) + c.$$

Since $a - c$ is symmetric in x and y, we also have $b(yx, x^2y^2) = a - c$. Next, by associativity of b and again (4.5.2), as well as (4.5.4) and (4.5.1),

$$\begin{aligned} a &= b(xy, y)b(yx, x) = b(xy \cdot yx, yx) + b(xy \cdot x, y \cdot yx) \\ &= b(st, t) + q(x)b(y, y \cdot yx) = b(s, t^2) + q(x)b(yy, yx) = b(s, t^2) + c. \end{aligned}$$

Interchanging x and y switches s and t and leaves $a - c$ fixed. Hence this proves (4.6.2). Now expand and use (4.4.1) and (4.6.2) to obtain (4.6.3):

$$\begin{aligned} h([x, y]) &= h(s - t) = h(s) - h_{21}(s, t) + h_{21}(t, s) - h(t) \\ &= h_s - 3b(s^2, t) + 3b(t^2, s) - h_t = h_s - h_t. \end{aligned}$$

Next, we rewrite a in a third form, using associativity of b and again the linearized composition formula (4.5.2):

$$a = b(xy, x)b(xy, y) = b(s, x)b(s, y) = b(s^2, xy) + b(sy, xs) = h(s) + b(sy, xs).$$

This is (4.6.4), and (4.6.5) follows by interchanging x and y.

By (4.5.4), $xt \equiv q(x)y$ and $ys \equiv q(y)x$. Hence, again by (4.5.2),

$$e = b(x, y)b(s, t) = b(xs, yt) + b(xt, ys) = b(xs, yt) + c,$$

which proves (4.6.6).

For (4.6.7), observe first that $b_{x,y}^2 = b(x^2, y^2) + b_{s,t}$ follows from (4.5.2). Now multiply this by $b_{x,y}$ and obtain

$$\begin{aligned} b_{x,y}^3 &= b_{x,y} b(x^2, y^2) + e = b(xx^2, yy^2) + b(xy^2, yx^2) + e \quad \text{(by (4.5.2))} \\ &= b(xy^2, yx^2) + c + e \quad \text{(by (4.5.4))}. \end{aligned}$$

This completes the proof.

4.7. Lemma. *We use the abbreviations introduced in Lemma* 4.6. *Then for all* $x, y \in M$,

$$b\big(s, [x, y]\big) = 2q_x q_y - b_{s,t}, \tag{4.7.1}$$
$$b\big(sy, [x, y]\big) = q_y b(y, x^2) - b(sy, t), \tag{4.7.2}$$
$$b(sy, t) = b(x, y)b(x, y^2) - q_x h_y, \tag{4.7.3}$$
$$b\big(xs, [x, y^2]\big) = a + c - e - h_s, \tag{4.7.4}$$
$$h_x h_y = b^3_{x,y} + 3(a - e) - h_s - h_t. \tag{4.7.5}$$

Proof. Formula (4.7.1) is easy: $b\big(s, [x, y]\big) = b(xy, xy) - b(xy, yx) = 2q_x q_y - b_{s,t}$ by (4.5.1). Similarly, (4.7.2) follows from $b\big(sy, [x, y]\big) = b(sy, xy) - b(sy, yx) = q_y b(s, x) - b(sy, t)$. Next, associativity of b and (4.5.2) as well as (4.5.4) imply

$$\begin{aligned} b(x, y)b(x, y^2) &= b(x, y)b(t, y) = b(xt, y^2) + b(xy, yt) \\ &= b(x \cdot yx, y^2) + b(s, yt) = q(x)b(y, y^2) + b(sy, t) \end{aligned}$$

which is (4.7.3).

To prove (4.7.4), we have $a - h_s = b(xs, sy)$ by (4.6.4) and $e - c = -b(xs, yt)$ by (4.6.6). Adding these two formulas yields

$$a - c + e - h_s = b(xs, sy - yt) = b(xs, [x, y^2]),$$

because of (4.5.9).

For (4.7.5), we begin by rewriting the right hand side, substituting from (4.6.7), (4.6.4) and (4.6.5):

$$\begin{aligned} \text{rhs} &= \big\{b(xy^2, yx^2) + c + e\big\} + 3(a - e) + \big\{b(sy, xs) - a\big\} + \big\{b(tx, yt) - a\big\} \\ &= b(xy^2, yx^2) + (a - c) - 2(e - c) + b(sy, xs) + b(tx, yt). \end{aligned}$$

Replace here $a - c$ by $b(s, y^2x^2)$ (using (4.6.2)) and $e - c$ by $b(xs, yt)$ (using (4.6.6)). Then

$$\begin{aligned} \text{rhs} &= b(xy^2, yx^2) + b(xy, y^2x^2) - 2b(xs, yt) + b(sy, xs) + b(tx, yt) \\ &= b(xy^2, yx^2) + b(xy, y^2x^2) + b(xs, sy - yt) + b(yt, tx - xs). \end{aligned}$$

Now use (4.6.1) to expand the left hand side of (4.7.5) and form the difference δ of the left and right hand side:

$$\delta = b\big(s, [x^2, y^2]\big) + b\big(xy^2, [x^2, y]\big) + b(xs, yt - sy) + b(yt, xs - tx).$$

By (4.5.10), $xs - tx \equiv [x^2, y]$ so

$$\delta = b\big(s, [x^2, y^2]\big) + b\big(xy^2 + yt, [x^2, y]\big) + b(xs, yt - sy).$$

Furthermore, by (4.5.7), $xy^2 + yt \equiv b(x, y)y$, and therefore the middle term is

$$b\big(xy^2 + yt, [x^2, y]\big) = b(x, y)b(y, [x^2, y]) = 0,$$

because of (4.5.3). It follows from (4.5.11) and (4.5.9) and associativity of b that

$$\begin{aligned} \delta &= b(s, [x^2, y^2]) + b(xs, yt - sy) = b(s, x(xy^2) - (y^2x)x) + b(xs, y^2x - xy^2) \\ &= b(sx, xy^2) - b(xs, y^2x) + b(xs, y^2x - xy^2) = b(sx - xs, xy^2). \end{aligned}$$

Here $sx = (xy)x \equiv q_x y$ by (4.5.4), and $b(xs, xy^2) = q_x b(s, y^2)$ by (4.5.1). Thus

$$\begin{aligned} \delta &= q_x\big\{b(y, xy^2) - b(s, y^2)\big\} = q_x\big\{b(yx, y^2) - b(xy, y^2)\big\} \\ &= -q_x b([x, y], y^2) = q_x b([y^2, y], x) = 0, \end{aligned}$$

by (4.5.3) and (4.5.4). This completes the proof.

4.8. Lemma. *With the notations of Theorem* 4.1*, the formula*

$$N(v)\,N(w) - N(v \bullet w) = (1-\alpha\beta)\bigl\{3b\bigl(x\cdot xy,\ [x,y^2]\bigr) + (1+\beta)h\bigl([x,y]\bigr)\bigr\} \\ + 3(1-\alpha\beta)b\bigl([x,y],\ -\lambda xy\cdot y + \mu yx\cdot x + \lambda\mu xy\bigr) \tag{4.8.1}$$

holds for all $v = \lambda \oplus x$, $w = \mu \oplus y$ *in all base extensions of* $k \oplus M$.

Proof. Since all constructions are compatible with base change, it suffices to prove (4.8.1) for $\lambda, \mu \in k$ and $x, y \in M$. We use the abbreviations introduced in Lemma 4.6, put

$$L := N(\lambda \oplus x)N(\mu \oplus y), \qquad R := N\bigl((\lambda \oplus x) \bullet (\mu \oplus y)\bigr),$$

and expand L and R as polynomials in λ, μ:

$$L = \sum_{i,j} L_{ij}\lambda^i\mu^j, \qquad R = \sum_{i,j} R_{ij}\lambda^i\mu^j.$$

Then (4.8.1) will be a consequence of the following relations:

$$L_{00} - R_{00} = (1-\alpha\beta)\bigl\{(1+\beta)h([x,y]) + 3b\bigl(xs,[x,y^2]\bigr)\bigr\}, \tag{4.8.2}$$
$$L_{10} - R_{10} = -3(1-\alpha\beta)b\bigl(sy,[x,y]\bigr), \tag{4.8.3}$$
$$L_{01} - R_{01} = -3(1-\alpha\beta)b\bigl(tx,[y,x]\bigr), \tag{4.8.4}$$
$$L_{11} - R_{11} = 3(1-\alpha\beta)b\bigl(s,[x,y]\bigr), \tag{4.8.5}$$
$$L_{ij} - R_{ij} = 0 \quad \text{otherwise.} \tag{4.8.6}$$

Let us prove (4.8.2). Clearly, $L_{00} = N(0 \oplus x)N(0 \oplus y) = h_xh_y$. The expansion of $R_{00} = N\bigl(b_{x,y} \oplus (\alpha s + \beta t)\bigr)$ yields

$$\begin{aligned} R_{00} &= b_{x,y}^3 - 3b_{x,y}q(\alpha s + \beta t) + h(\alpha s + \beta t) \\ &= b_{x,y}^3 - 3b_{x,y}\bigl\{(\alpha^2+\beta^2)q_xq_y + \alpha\beta b_{s,t}\bigr\} \\ &\qquad + \alpha^3h_s + \alpha^2\beta h_{21}(s,t) + \alpha\beta^2h_{21}(t,s) + \beta^3h_t. \end{aligned}$$

By (4.4.1) and (4.6.2), $h_{21}(s,t) = 3b(s^2,t) = 3b(t^2,s) = h_{21}(t,s) = 3(a-c)$, and $\alpha^2\beta + \alpha\beta^2 = \alpha\beta(\alpha+\beta) = \alpha\beta$ as well as $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = 1 - 2\alpha\beta$. Hence,

$$\begin{aligned} R_{00} &= b_{x,y}^3 - 3(1-2\alpha\beta)c - 3\alpha\beta e + \alpha^3h_s + 3\alpha\beta(a-c) + \beta^3h_t \\ &= b_{x,y}^3 - 3c + 3\alpha\beta(a+c-e) + \alpha^3h_s + \beta^3h_t. \end{aligned} \tag{4.8.7}$$

Observe that $1+\alpha^3 = (1+\alpha)(1-\alpha+\alpha^2)$ and $1-\alpha+\alpha^2 = 1-\alpha(1-\alpha) = 1-\alpha\beta$. Hence $1+\alpha^3 = (1-\alpha\beta)(1+\alpha)$ and by symmetry, $1+\beta^3 = (1-\alpha\beta)(1+\beta)$. Now (4.8.7) and (4.7.5) imply

$$\begin{aligned} L_{00} - R_{00} &= 3(1-\alpha\beta)(a+c-e) - (1+\alpha^3)h_s - (1+\beta^3)h_t \\ &= (1-\alpha\beta)\bigl\{3(a+c-e) - (1+\alpha)h_s - (1+\beta)h_t\bigr\}. \end{aligned} \tag{4.8.8}$$

This formula is symmetric in (α, x) and (β, y) but contains no explicit commutators. To introduce them, we use (4.7.4) and (4.6.3):

$$\begin{aligned}L_{00}-R_{00} &= (1-\alpha\beta)\big\{3b\big(xs,[x,y^2]\big)+(2-\alpha)h_s-(1+\beta)h_t\big\}\\ &= (1-\alpha\beta)\big\{3b\big(xs,[x,y^2]\big)+(1+\beta)(h_s-h_t)\big\}\\ &= (1-\alpha\beta)\big\{3b\big(xs,[x,y^2]\big)+(1+\beta)h\big([x,y]\big)\big\}.\end{aligned}$$

This proves (4.8.2). — Next, we establish the relations involving $L_{i0}-R_{i0}$ for $i>0$. On the left, we have

$$N(\lambda\oplus x)N(y)=(\lambda^3-3\lambda q_x+h_x)h_y,$$

whence

$$L_{10}=-3q_xh_y,\quad L_{20}=0,\quad L_{30}=h_y.$$

On the right, expansion yields

$$\begin{aligned}N\big((\lambda\oplus x)\bullet(0\oplus y)\big) &= N\big(b_{x,y}\oplus(\alpha s+\beta t+\lambda y)\big)\\ &= R_{00}-3\lambda b_{x,y}\big(\alpha b_{s,y}+\beta b_{t,y}\big)-3\lambda^2 b_{x,y}q_y\\ &\quad+\lambda h_{21}(\alpha s+\beta t,y)+\lambda^2 h_{21}(y,\alpha s+\beta t)+\lambda^3 h_y.\end{aligned}\tag{4.8.9}$$

Here $b_{s,y}=b_{t,y}=b(x,y^2)$ because b is associative, and, by (4.4.1),

$$\begin{aligned}h_{21}(\alpha s+\beta t,y) &= 3b\big((\alpha s+\beta t)^2,y\big)\\ &= 3\big\{\alpha^2 b(s^2,y)+\beta^2 b(t^2,y)+\alpha\beta b(st,y)+\alpha\beta b(ts,y)\big\}.\end{aligned}$$

Now $b(y,s^2)=b(ys,s)=q_yb(x,s)=q_yb_{y,x^2}$ and $b(t^2,y)=b(t,ty)=q_yb(t,x)=q_yb(y,x^2)$ as well as $b(st,y)=b(s,ty)=q_yb(s,x)=q_yb(y,x^2)$, using (4.5.4). Hence R_{10}, the coefficient of λ in (4.8.9), is

$$R_{10}=3\big\{-b_{x,y}b(x,y^2)+(\alpha^2+\beta^2+\alpha\beta)q_yb(y,x^2)+\alpha\beta b(ts,y)\big\}.$$

Furthermore, $\alpha^2+\beta^2+\alpha\beta=(\alpha+\beta)^2-\alpha\beta=1-\alpha\beta$, so

$$R_{10}=3\big\{-b_{x,y}b(x,y^2)+(1-\alpha\beta)q_yb(y,x^2)+\alpha\beta b(ts,y)\big\}.$$

Hence, using (4.7.3) and (4.7.2),

$$\begin{aligned}L_{10}-R_{10} &= 3\big\{-q_xh_y+b_{x,y}b(x,y^2)-(1-\alpha\beta)q_yb(y,x^2)-\alpha\beta b(sy,t)\big\}\\ &= 3(1-\alpha\beta)\big\{b(sy,t)-q_yb(y,x^2)\big\}\\ &= -3(1-\alpha\beta)\,b\big(sy,[x,y]\big),\end{aligned}$$

which proves (4.8.3). Formula (4.8.4) follows by symmetry.

Next,

$$R_{20}=-3b_{x,y}q_y+3b(y^2,\alpha s+\beta t)=0,$$

because $b(y^2,s)=b(y^2,xy)=q_yb_{x,y}=b(y^2,t)$ by (4.5.1), and $\alpha+\beta=1$. Since $R_{30}=h_y$, we obtain $L_{i0}-R_{i0}=0$ for $i\geqslant 2$.

Let us consider (4.8.5). Clearly, $L_{11}=9q_xq_y$. On the other hand, a straightforward expansion and (4.4.1) shows that

$$\begin{aligned}R_{11} &= -3(\alpha^2+\beta^2)q_xq_y-3\alpha\beta b_{s,t}+\alpha h_{111}(x,y,s)+\beta h_{111}(x,y,t)\\ &= -3(\alpha^2+\beta^2)q_xq_y-3\alpha\beta b_{s,t}+3\big\{\alpha b(s,s)+\alpha b(t,s)+\beta b(s,t)+\beta b(t,t)\big\}\end{aligned}$$

Now $b(s,s)=b(xy,xy)=2q_xq_y$ as well as $b(t,t)=b(yx,yx)=2q_xq_y$. Hence

$$\begin{aligned}R_{11} &= -3(\alpha^2+\beta^2)q_xq_y-3\alpha\beta b_{s,t}+6(\alpha+\beta)q_xq_y+3b_{s,t}\\ &= 3\big\{(2-\alpha^2-\beta^2)q_xq_y+(1-\alpha\beta)b_{s,t}\big\}.\end{aligned}$$

From $\alpha+\beta=1$ it follows that $\alpha^2+\beta^2=1-2\alpha\beta$, so we finally obtain, using (4.7.1),

$$\begin{aligned}L_{11}-R_{11} &= 3\big\{(3-1-2\alpha\beta)q_xq_y-(1-\alpha\beta)b_{s,t}\big\}\\ &= 3(1-\alpha\beta)\big(2q_xq_y-b_{s,t}\big)=3(1-\alpha\beta)b\big(s,[x,y]\big).\end{aligned}$$

This proves (4.8.5). Finally, it is easily seen that $L_{ij}-R_{ij}=0$ for the remaining cases $(ij)=(33),(32),(23),(31),(13),(22),(21),(12)$. This completes the proof of Lemma 4.8 and also of Theorem 4.1.

4.9. Definition. Let $\alpha \in \boldsymbol{\pi}_6(k)$ be a primitive sixth root of unity. We define a functor

$$\mathbf{A}_\alpha\colon \mathbf{scomp}_k \to \mathbf{uccomp}_k$$

from symmetric to unital cubic compositions as follows. For a symmetric composition $\mathfrak{C} = (M, q, \star)$, let $\mathbf{A}_\alpha(\mathfrak{C}) = (k \oplus M, N, 1 \oplus 0, \bullet)$ be the unital cubic composition defined in Cor. 4.3. For a morphism $\psi\colon \mathfrak{C} \to \mathfrak{C}'$ of symmetric compositions let $\mathbf{A}_\alpha(\psi)\colon \mathbf{A}_\alpha(\mathfrak{C}) \to \mathbf{A}_\alpha(\mathfrak{C}')$ be defined by $\lambda \oplus x \mapsto \lambda \oplus \psi(x)$. Then $\mathbf{A}_\alpha$ is compatible with the functor $\mathbf{A}\colon \mathbf{qcform}_k \to \mathbf{uccomp}_k$ of 2.5, i.e., the diagram

$$\begin{array}{ccc} \mathbf{scomp}_k & \xrightarrow{\mathbf{A}_\alpha} & \mathbf{uccomp}_k \\ {\scriptstyle \Upsilon}\downarrow & & \downarrow{\scriptstyle \Omega} \\ \mathbf{qcform}_k & \xrightarrow[\mathbf{A}]{} & \mathbf{ucform}_k \end{array}$$

is commutative, cf. also the diagram in 3.12.

We now extend Prop. 2.6 to the functors $\mathbf{C}_\alpha$ and $\mathbf{A}_\alpha$:

4.10. Proposition. *Let $\alpha \in \boldsymbol{\pi}_6(k)$ be a primitive sixth root of unity and put $\beta = \alpha^{-1} = 1-\alpha$. Composing the functors $\Omega\colon \mathbf{uccomp}_k \to \mathbf{ucform}_k$ and $\Upsilon\colon \mathbf{scomp}_k \to \mathbf{qcform}_k$ with the natural transformations ζ and ϑ of 2.6 yields natural transformations*

$$\zeta'\colon \mathbf{A}_\beta \circ \mathbf{C}_\alpha \to \mathrm{Id}_{\mathbf{uccomp}_k}, \qquad \vartheta'\colon \mathbf{C}_\alpha \circ \mathbf{A}_\beta \to \mathrm{Id}_{\mathbf{scomp}_k}.$$

If $3 \in k^\times$ then ζ' and ϑ' are isomorphisms.

Proof. (a) In more detail, ζ' is given as follows. For a unital cubic composition $\mathfrak{A} = (A, N)$ let $\Omega(\mathfrak{A}) = \mathfrak{X}$ be the unital cubic form obtained by omitting the multiplication, cf. 3.1. Then $\zeta'_\mathfrak{A} := \zeta_\mathfrak{X}$ is defined as in (2.6.1). Thus by Prop. 2.6(a), it remains to show that $\zeta_\mathfrak{X}$ is a homomorphism of algebras. Consider the product $*$ on A as in (3.9.1), where now $\beta = 1 - \alpha = \alpha^{-1}$, so $\alpha\beta = \alpha + \beta = 1$ and $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = -1$. Then

$$\begin{aligned} \beta x * y + \alpha y * x &= \beta\big\{(1+\alpha)xy + (1+\beta)yx - T_x y - T_y x\big\} \\ &\quad + \alpha\big\{(1+\alpha)yx + (1+\beta)xy - T_y x - T_x y\big\} \\ &= (\beta + 1 + \alpha + 1)xy + (\beta + \beta^2 + \alpha + \alpha^2)yx - T_x y - T_y x \\ &= 3xy - T_x y - T_y x. \end{aligned} \tag{4.10.1}$$

Since $\zeta_\mathfrak{X}$ preserves unit elements, we may restrict attention to elements of the form $v = 0 \oplus \dot{x}$ and $w = 0 \oplus \dot{y}$ of $k \oplus \dot{X}$, where $x, y \in A$ and $\dot{x} = \kappa(x)$. Then

$$\zeta_\mathfrak{X}(0 \oplus x) \cdot \zeta_\mathfrak{X}(0 \oplus y) = (3x - T_x \cdot 1_A)(3y - T_y \cdot 1_A), \tag{4.10.2}$$

while by (4.3.1) (with α replaced by α^{-1}) and (4.10.1),

$$\begin{aligned} \zeta_\mathfrak{X}\big((0 \oplus \dot{x}) \bullet (0 \oplus \dot{y})\big) &= \zeta_\mathfrak{X}\Big(b(\dot{x}, \dot{y}) \oplus (\alpha^{-1}\dot{x} \star \dot{y} + \alpha\dot{y} \star \dot{x})\Big) \\ &= \zeta_\mathfrak{X}\Big(\big(2T_xT_y - 3B(x,y)\big) \oplus \kappa(\alpha^{-1}x * y + \alpha y * x)\Big) \\ &= \zeta_\mathfrak{X}\Big(\big(2T_xT_y - 3B(x,y)\big) \oplus \kappa(3xy - T_x y - T_y x)\Big) \\ &= \big(2T_xT_y - 3B(x,y) - T(3xy) + 2T_xT_y\big)1_A + 3(3xy - T_x y - T_y x). \end{aligned}$$

This agrees with (4.10.2) because $4T_xT_y - 3B(x,y) - 3T(xy) = 4T_xT_y - 3T_xT_y$ (by (3.2.1)) $= T_xT_y$.

(b) For a symmetric composition $\mathfrak{C} = (M, q, \star)$ let $\mathfrak{M} = \Upsilon(\mathfrak{C}) = (M, q, h) \in$ $\mathbf{qcform}_k$ as in (3.12.1). Then $\vartheta'_{\mathfrak{C}} := \vartheta_{\mathfrak{M}}$ as in (2.6.2). Since $\vartheta_{\mathfrak{M}}$ preserves quadratic forms by 2.6(b), it remains to show that it preserves products. For $v = \lambda \oplus x$ and $w = \mu \oplus y$ in $k \oplus M$ the traces are $T_{\mathfrak{M}}(v) = 3\lambda$ and $T_{\mathfrak{M}}(w) = 3\mu$ by (2.5.2). Now use the definition of the $*$-product in (3.9.1), where the algebra product is taken in the algebra $\mathbf{A}_\beta(\mathfrak{C})$ with product as in (4.3.1) but with α and β interchanged. Then a similar computation as in (4.10.1) shows

$$\begin{aligned} v * w &= (1+\alpha) v \bullet w + (1+\beta) w \bullet v - T_{\mathfrak{M}}(v) w - T_{\mathfrak{M}}(w) v \\ &= (1+\alpha)\big\{(\lambda\mu + b_{x,y}) \oplus (\beta x \star y + \alpha y \star x + \lambda y + \mu x)\big\} \\ &+ (1+\beta)\big\{(\lambda\mu + b_{x,y}) \oplus (\beta y \star x + \alpha x \star y + \mu x + \lambda y)\big\} - 3\lambda w - 3\mu v \\ &= \big\{(3\lambda\mu + 3b_{x,y}) \oplus 3(x \star y + \lambda y + \mu x)\big\} - 3\lambda w - 3\mu v \\ &= 3\big\{(b_{x,y} - \lambda\mu) \oplus x \star y\big\}. \end{aligned}$$

Hence the product $\dot{*}$ induced from $*$ on $(k \oplus M)/k \cdot 1$ is, after the identification with M, simply given by $x \,\dot{*}\, y = 3x \star y$. This shows that indeed $\vartheta_{\mathfrak{M}}(x \,\dot{*}\, y) = 3(3x \star y) = (3x) \star (3y) = \vartheta_{\mathfrak{M}}(x) \star \vartheta_{\mathfrak{M}}(y)$.

Finally, it is clear from Prop. 2.6 that ζ' and ϑ' are isomorphisms if $3 \in k^\times$. Hence:

4.11. Corollary. *If $3 \in k^\times$ and if k contains a primitive sixth root of unity then the categories* $\mathbf{uccomp}_k$ *and* $\mathbf{scomp}_k$ *are equivalent.*

We will see in the next section that the condition on the existence of the sixth root of unity can be omitted after replacing unital cubic compositions over k by unital cubic compositions of the second kind over a suitable quadratic extension of k.

5. Unital cubic compositions of the second kind

5.1. Quadratic algebras. Following [knus, III, §4], we mean by a quadratic algebra an algebra $K \in k$-**alg** which is finitely generated and projective of rank 2 as a k-module. Such an algebra has a canonical linear form t_K, the trace, and a quadratic form n_K, the norm, as well as an involution ι which satisfy

$$\begin{gathered} x^2 - t_K(x)x + n_K(x)1_K = 0, \\ x + \iota(x) = t_K(x)1_K, \qquad x\iota(x) = n_K(x)1_K, \end{gathered}$$

for all $x \in K$.

Being of rank 2, K is in particular faithful as a k-module. By [knus, I, (1.3.5)], the map $\lambda \mapsto \lambda 1_K$ is an isomorphism of k onto a direct summand of K. Hence $\dot{K} := K/k \cdot 1$ is finitely generated and projective of rank 1 over k.

5.2. Conjugations. Let K be a quadratic algebra and let Y be a K-module. A *conjugation* on Y is a k-linear map j from Y to itself which satisfies $j(j(y)) = y$ and $j(y\mu) = j(y)\iota(\mu)$, for all $y \in Y$ and $\mu \in K$. Note that this definition generalizes the definition of a Galois descent datum [knus, III, p. 115] insofar as K is not assumed to be étale over k.

Let $\mathbf{mod}_k$ be the category of k-modules and let $\mathbf{mod}_{K/k}$ be the category whose objects are pairs (Y, j) consisting of a K-module Y and a conjugation j, with morphisms K-linear maps commuting with the conjugations. By abuse of notation, the explicit reference to j will often be suppressed. For $X \in \mathbf{mod}_k$ let

$$\mathbf{E}(X) = (X \otimes_k K,\, \mathrm{Id}_X \otimes \iota),$$

be the base extension from k to K (with conjugation $\mathrm{Id}_X \otimes \iota$ induced from ι), and for $(Y, j) \in \mathbf{mod}_{K/k}$, let

$$\mathbf{D}(Y, j) = \mathrm{Ker}(\mathrm{Id}_Y - j) = \{x \in Y : j(x) = x\},$$

the fixed point set of j. Then it is easy to see that we have functors

$$\mathbf{E} : \mathbf{mod}_k \to \mathbf{mod}_{K/k} \text{ (“extension”)}, \qquad \mathbf{D} : \mathbf{mod}_{K/k} \to \mathbf{mod}_k \text{ (“descent”)},$$

and that $\mathbf{D}$ is right adjoint to $\mathbf{E}$, i.e., that there are natural bijections

$$\mathbf{mod}_{K/k}\big(\mathbf{E}(X), Y\big) \cong \mathbf{mod}_k\big(X, \mathbf{D}(Y)\big).$$

By general facts from category theory [maclane], there are natural transformations

$$\eta : \mathrm{Id}_{\mathbf{mod}_k} \to \mathbf{D} \circ \mathbf{E}, \qquad \varepsilon : \mathbf{E} \circ \mathbf{D} \to \mathrm{Id}_{\mathbf{mod}_{K/k}}, \tag{5.2.1}$$

which in the present situation are given as follows:

$$\begin{aligned} \eta_X(x) &= x \otimes 1_K \in \mathbf{D}(\mathbf{E}(X)) && (x \in X \in \mathbf{mod}_k), \\ \varepsilon_Y(y \otimes \mu) &= y\mu \in Y && (y = j(y) \in Y \in \mathbf{mod}_{K/k}). \end{aligned}$$

Standard results on Galois descent [knus, Chapter III] say that if K is étale, then η and ε are isomorphisms, and $\mathbf{D}$ and $\mathbf{E}$ are quasi-inverse equivalences of categories. More generally, there is still the following result.

5.3. Lemma. *Let K be a quadratic algebra and put $K^0 := \mathrm{Ker}(t_K)$ and $\dot{K} := K/k \cdot 1_K$. Then the following conditions are equivalent:*

- (i) *K has surjective trace,*
- (ii) *$\mathrm{Id} - \iota$ induces an isomorphism $\dot{K} \xrightarrow{\cong} K^0$,*
- (iii) *for all $X \in \mathbf{mod}_k$, the map $x \mapsto x \otimes 1$ is an isomorphism of X onto the fixed point set of $j := \mathrm{Id}_X \otimes \iota$ in $X \otimes K = \mathbf{E}(X)$,*
- (iv) *$\eta : \mathrm{Id}_{\mathbf{mod}_k} \to \mathbf{D} \circ \mathbf{E}$ is an isomorphism.*

Proof. (i) $\Longrightarrow$ (ii): Clearly $\varphi := \mathrm{Id} - \iota = 2\mathrm{Id} - 1_K \otimes t_K$ vanishes on $k \cdot 1_K$ and takes values in K^0, thus inducing a homomorphism $\dot{K} \to K^0$. Since t_K is surjective the sequence $0 \longrightarrow K^0 \longrightarrow K \xrightarrow{t_K} k \longrightarrow 0$ is exact, so K^0 is finitely generated and projective of rank 1. Therefore, it suffices to show that $\varphi \otimes \boldsymbol{\kappa}(\mathfrak{p}) \neq 0$ for all prime ideals $\mathfrak{p}$ of k, where $\boldsymbol{\kappa}(\mathfrak{p})$ denotes the quotient field of $k/\mathfrak{p}$. After changing base from k to $\boldsymbol{\kappa}(\mathfrak{p})$, we may assume k is a field and then have to show that $\varphi \neq 0$. If $\mathrm{char}(k) = 2$, $\varphi = 1_K \otimes t_K \neq 0$ is clear. If $\mathrm{char}(k) \neq 2$, $K = k \cdot 1_K \oplus K^0$, and $\varphi\big|K^0 = 2\mathrm{Id} \neq 0$.

(ii) $\Longrightarrow$ (iii): By (ii), we have an exact sequence

$$0 \longrightarrow k \xrightarrow{\ i\ } K \xrightarrow{\mathrm{Id}-\iota} K^0 \longrightarrow 0 \tag{5.3.1}$$

where $i(\lambda) = \lambda \cdot 1_K$, and it splits because K^0 is projective. Hence tensoring (5.3.1) with X yields the exact sequence $0 \longrightarrow X \longrightarrow X \otimes K \xrightarrow{\mathrm{Id}-j} X \otimes K^0 \longrightarrow 0$. This shows that the fixed point set of j in $X \otimes K$ is $X \otimes 1 \cong X$.

(iii) $\iff$ (iv): Evident.

(iii) $\Longrightarrow$ (i): Let $I \subset k$ be the image of t_K and put $X = k/I$. Then $IX = 0$ and $-x = x$ for all $x \in X$ because $2 = t_K(1_K) \in I$. Hence for all $x \in X$ and $\mu \in K$,

$$(\mathrm{Id}_X \otimes \iota)(x \otimes \mu) = x \otimes (t_K(\mu) \cdot 1_K - \mu) = t_K(\mu)x \otimes 1_K - x \otimes \mu = x \otimes \mu.$$

This shows $\mathrm{Id}_X \otimes \iota$ is the identity on $X \otimes K$, so by (iii), the map $x \mapsto x \otimes 1_K$ is an isomorphism $X \cong X \otimes K$ of k-modules. On the other hand, the sequence $0 \longrightarrow k \longrightarrow K \xrightarrow{\mathrm{can}} \dot{K} \longrightarrow 0$ is split-exact. Tensoring with X yields the (split-)exact sequence $0 \longrightarrow X \xrightarrow{\cong} X \otimes K \longrightarrow X \otimes \dot{K} \longrightarrow 0$ which implies $X \otimes \dot{K} = 0$. Since $\dot{K}$ is faithfully flat, it follows that $X = 0$.

5.4. Modules with additional structure. The foregoing considerations extend in a straightforward manner to the situation where $\mathbf{mod}_k$ is replaced by a category of modules with some additional algebraic structure, for example, a (not necessarily associative) multiplication. Then a conjugation is required to preserve the additional structure, and the analogously defined functors $\mathbf{D}$ and $\mathbf{E}$ have the same properties as above. We will need in particular the case of modules equipped with polynomial laws which is more involved. Let $\mathbf{pol}_k$ be the category whose objects are triples (X, V, f) consisting of k-modules X and V and a polynomial law f on X with values in V. A morphism from (X, V, f) to (X', V', f') is a pair of k-linear maps $\varphi\colon X \to X'$ and $\psi\colon V \to V'$ such that $\psi_R \circ f_R = f'_R \circ \varphi_R$ for all $R \in k$-$\mathbf{alg}$. Similarly, define $\mathbf{pol}_{K/k}$ to be the category whose objects are quintuples (Y, j_Y, W, j_W, g), where (Y, j_Y) and (W, j_W) are in $\mathbf{mod}_{K/k}$ and $g \in \mathcal{P}(Y, W)$ is a polynomial law, compatible with the conjugations in the sense that the diagram

$$\begin{array}{ccc} ({}_kY)_{\mathbf{a}} & \xrightarrow{\ {}_kg\ } & ({}_kW)_{\mathbf{a}} \\ {\scriptstyle {}_kj_Y}\downarrow & & \downarrow{\scriptstyle {}_kj_W} \\ ({}_kY)_{\mathbf{a}} & \xrightarrow[{}_kg]{} & ({}_kW)_{\mathbf{a}} \end{array} \tag{5.4.1}$$

is commutative. Here ${}_k(\ )$ is restriction of scalars to k, cf. (1.2.2). Morphisms in $\mathbf{pol}_{K/k}$ are defined similarly as in $\mathbf{pol}_k$. Then there is a base extension functor $\mathbf{E} : \mathbf{pol}_k \to \mathbf{pol}_{K/k}$ given by

$$\mathbf{E}(X, V, f) = \big(\mathbf{E}(X), \mathbf{E}(V), \mathbf{E}(f) = f \otimes K\big).$$

There is also a descent functor which requires more care:

5.5. Proposition. *Given $(Y, j_Y, W, j_W, g) \in \mathbf{pol}_{K/k}$, there is a "descended" polynomial law $f = \mathbf{D}(g) \in \mathcal{P}\big(\mathbf{D}(Y), \mathbf{D}(W)\big)$, uniquely determined by the condition that, for all $R \in k$-$\mathbf{alg}$, and putting $X = \mathbf{D}(Y)$ and $V := \mathbf{D}(W)$ for short, the following diagram is commutative:*

$$\begin{array}{ccc} X \otimes R & \xrightarrow{\ f_R\ } & V \otimes R \\ {\scriptstyle i_X \otimes \mathrm{Id}_R}\downarrow & & \downarrow{\scriptstyle i_V \otimes \mathrm{Id}_R} \\ ({}_kY) \otimes R & \xrightarrow[({}_kg)_R]{} & ({}_kW) \otimes R \end{array} \tag{5.5.1}$$

Here $i_X\colon X \to {}_kY$ and $i_V\colon V \to {}_kW$ denote the inclusion maps. This defines a functor $\mathbf{D} : \mathbf{pol}_{K/k} \to \mathbf{pol}_k$, compatible with taking homogeneous components and *right adjoint to* $\mathbf{E} : \mathbf{pol}_k \to \mathbf{pol}_{K/k}$*, so there are natural transformations* η *and* ε *as in* (5.2.1).

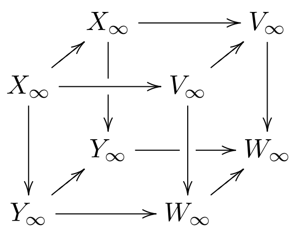
Proof. Let $k_\infty = k[\mathbf{t}_1, \mathbf{t}_2, \ldots]$ be the polynomial ring in countably many indeterminates, and let k-$\mathbf{alg}'$ be the full subcategory of k-$\mathbf{alg}$ having the single object k_∞. For a k-module M, let $M_{\mathbf{a}'}$ be the restriction of the functor $M_{\mathbf{a}}$ to k-$\mathbf{alg}'$. By [roby, Prop. IV.4], a polynomial law $f \in \mathcal{P}(X, V)$ is the same as a natural transformation $f'\colon X_{\mathbf{a}'} \to V_{\mathbf{a}'}$. More precisely, any $f \in \mathcal{P}(X, V)$ induces by restriction a natural transformation $f'\colon X_{\mathbf{a}'} \to V_{\mathbf{a}'}$, and this establishes a bijection between $\mathcal{P}(X, V)$ and the set of natural transformations from $X_{\mathbf{a}'}$ to $V_{\mathbf{a}'}$. Since k_∞ is free as a k-module, the exact sequence

$$0 \longrightarrow X \xrightarrow{\ i_X\ } {}_kY \xrightarrow{\ \mathrm{Id}-j_Y\ } {}_kY$$

remains exact upon tensoring with k_∞. Hence $i_X \otimes \mathrm{Id}_{k_\infty}$ maps $X \otimes k_\infty$ isomorphically onto the fixed point set of $j_Y \otimes \mathrm{Id}_{k_\infty}$, and the same is true for i_V. Since $({}_kg)_{k_\infty}$ commutes with the conjugations by (5.4.1), it induces a unique map $f_{k_\infty}\colon X \otimes k_\infty \to V \otimes k_\infty$. It remains to show that this defines a natural transformation $X_{\mathbf{a}'} \to V_{\mathbf{a}'}$; i.e., that for every k-algebra homomorphism $\varphi\colon k_\infty \to k_\infty$, the diagram

$$\begin{array}{ccc} X_\infty & \xrightarrow{\ f_\infty\ } & V_\infty \\ {\scriptstyle \mathrm{Id}\otimes\varphi}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}\otimes\varphi} \\ X_\infty & \xrightarrow[\ f_\infty\ ]{} & V_\infty \end{array} \tag{5.5.2}$$

is commutative, where we have abbreviated $f_\infty := f_{k_\infty}$ and $X_\infty := X \otimes k_\infty$ and similarly for V_∞. This follows by chasing the diagram below whose sides and bottom are commutative and where the vertical arrows are injective, being induced from i_X and i_V:



The proof of the remaining statements is left to the reader.

5.6. Corollary. *Let K be a quadratic algebra with surjective trace. Let $V \in \mathbf{mod}_k$ and $(Y, j) \in \mathbf{mod}_{K/k}$, and let $g \in \mathcal{P}\big(Y, \mathbf{E}(V)\big)$ be a polynomial law compatible with the conjugations j on Y and $\mathrm{Id}\otimes\iota$ on $\mathbf{E}(V) = V \otimes K$. Then the descended polynomial law $f = \mathbf{D}(g)$ takes values in V. In particular, if $V = k$ and g is a form of degree d then $\mathbf{D}(g)$ is a form of degree d on $X = \mathbf{D}(Y)$.*

Proof. By Prop. 5.5, $\mathbf{D}(g)$ takes values in $\mathbf{D}\big(\mathbf{E}(V)\big)$ which is canonically isomorphic to V by Lemma 5.3.

5.7. Definition. Let K be a quadratic k-algebra with surjective trace. An *involution of the second kind* on a K-algebra $\tilde{A}$ (not necessarily associative or unital) is a conjugation J as in 5.2 which is, in addition, an anti-automorphism of the k-algebra ${}_k\tilde{A}$. The category $\mathbf{uccomp}^{(2)}_{K/k}$ of *unital cubic compositions of the second kind* is defined as follows. Its objects are triples $(\tilde{A}, \tilde{N}, J)$ where $(\tilde{A}, \tilde{N}) \in \mathbf{uccomp}_K$ is a unital cubic composition over K and J is an involution of the second kind of $\tilde{A}$ which is compatible with $\tilde{N}$ in the sense of 5.4; i.e., such that $(\tilde{A}, J, K, \iota, \tilde{N}) \in \mathbf{pol}_{K/k}$. Morphisms are defined as expected. Here, the descended k-module $\mathcal{H} := \{x \in \tilde{A} : J(x) = x\}$ is the set of hermitian elements of $\tilde{A}$. In general, it does not inherit an algebra structure from $\tilde{A}$ because J is not an automorphism of $\tilde{A}$ but if $\tilde{A}$ is associative or alternative, it will be a Jordan algebra over k. Since K has surjective trace, $\tilde{N}$ induces by Cor. 5.6 a k-valued cubic form $N = \mathbf{D}(\tilde{N})$ on $\mathcal{H}$.

We also introduce the following category $\mathbf{scomp}_{K/k}$. Its objects are pairs $(\tilde{\mathfrak{C}}, j)$ where $\tilde{\mathfrak{C}} = (\tilde{M}, \tilde{q}, \tilde{\star}) \in \mathbf{scomp}_K$ is a symmetric composition over K and j is a conjugation on $\tilde{M}$ compatible with $\tilde{q}$ and $\tilde{\star}$ as explained in 5.4. In particular,

$$j(x \,\tilde{\star}\, y) = j(x) \,\tilde{\star}\, j(y) \tag{5.7.1}$$

(no reversal of factors). By Cor. 5.6, $\tilde{q}$ induces a quadratic form $q = \mathbf{D}(\tilde{q})$ with values in k on $M = \mathbf{D}(\tilde{M})$. Because of (5.7.1), $\tilde{\star}$ induces a multiplication $\star = \mathbf{D}(\tilde{\star})$ on M, making $\mathbf{D}(\tilde{\mathfrak{C}}, j) := (M, q, \star)$ a symmetric composition over k. We thus have a descent functor

$$\mathbf{D} : \mathbf{scomp}_{K/k} \to \mathbf{scomp}_k \,, \tag{5.7.2}$$

which is, as in 5.2, right adjoint to the base extension functor

$$\mathbf{E} : \mathbf{scomp}_k \to \mathbf{scomp}_{K/k} \,.$$

Since K has surjective trace, the unit η of this adjunction is, by Lemma 5.3, an isomorphism

$$\eta : \mathrm{Id}_{\mathbf{scomp}_k} \xrightarrow{\cong} \mathbf{D} \circ \mathbf{E} \,. \tag{5.7.3}$$

We now specialize the quadratic algebra K to

$$K := k[\mathbf{t}]/(\mathbf{t}^2 - \mathbf{t} + 1). \tag{5.7.4}$$

Thus K is a free quadratic algebra with basis 1 and $\rho := \mathrm{can}(\mathbf{t})$ and the relation $\rho^2 - \rho + 1 = 0$, and the functor $\boldsymbol{\pi}_6$ of 3.7(c) is (represented by) the affine scheme defined by K; i.e., $\boldsymbol{\pi}_6(R)$ is in natural bijection with $\mathrm{Hom}_{k\text{-}\mathbf{alg}}(K, R)$, by associating with $r \in \boldsymbol{\pi}_6(R)$ the homomorphism $K \to R$ sending $\rho \mapsto r$. In particular, $t_K(\rho) = n_K(\rho) = 1$ and $\rho \in \boldsymbol{\pi}_6(K)$. The involution ι of K is given on ρ by

$$\iota(\rho) = 1 - \rho = \rho^{-1}. \tag{5.7.5}$$

The discriminant of $\mathbf{t}^2 - \mathbf{t} + 1$ is -3, so K is étale if and only if $3 \in k^\times$.

5.8. The functor $\tilde{\mathbf{C}}$. Let $(\tilde{A}, \tilde{N}, J)$ be a unital cubic composition of the second kind over K/k, with K as in (5.7.4). Then $(\tilde{A}, \tilde{N})$ is a unital cubic composition over K and $\rho \in \boldsymbol{\pi}_6(K)$, so we may apply the functor $\mathbf{C}_\rho$ of 3.12 and obtain a symmetric composition $\mathbf{C}_\rho(\tilde{A}, \tilde{N}) = (\tilde{A}/K \cdot 1, \tilde{q}, \tilde{\star})$ over K. Moreover, J leaves $K \cdot 1$ stable and hence induces a conjugation j on the K-module $\tilde{A}/K \cdot 1$. We claim that j is compatible with $\tilde{q}$ and $\tilde{\star}$, so that

$$\tilde{\mathbf{C}}_\rho(\tilde{A}, \tilde{N}, J) := \big(\mathbf{C}_\rho(\tilde{A}, \tilde{N}), j\big) \in \mathbf{scomp}_{K/k}. \tag{5.8.1}$$

Indeed, let $\tilde{T}$ and $\tilde{Q}$ be the linear and quadratic trace forms of $\tilde{N}$, cf. 2.2. From compatibility of $\tilde{N}$ and J and since J fixes the unit element of $\tilde{A}$, it follows that $\tilde{T}$ and $\tilde{Q}$ are compatible with J as well. Hence, the same is true of the quadratic form $\tilde{H}_2 = \tilde{T}^2 - 3\tilde{Q}$, which implies that the quadratic form $\tilde{q}$ induced by $\tilde{H}_2$ on

$\tilde{M} := \tilde{A}/K\cdot 1$ is compatible with j. The multiplication $\tilde{\star}$ on $\tilde{M}$ is induced from the multiplication

$$x \,\tilde{*}\, y = (1+\rho)xy + (1+\rho^{-1})yx - \tilde{T}(x)y - \tilde{T}(y)x \tag{5.8.2}$$

on $\tilde{A}$, cf. (3.8.1). By applying J to this equation and using (5.7.5) as well as $\tilde{T}\big(J(x)\big) = \iota\big(\tilde{T}(x)\big)$, one sees that $J(x \,\tilde{*}\, y) = J(x) \,\tilde{*}\, J(y)$. Hence the multiplication $\tilde{\star}$ on $\tilde{M}$ is compatible with the conjugation j. This proves (5.8.1), and it is easily checked that we have in fact a functor $\tilde{\mathbf{C}}_\rho : \mathbf{uccomp}^{(2)}_{K/k} \to \mathbf{scomp}_{K/k}$. Combining this with the descent functor (5.7.2), we obtain a functor

$$\tilde{\mathbf{C}} := \mathbf{D}\circ\tilde{\mathbf{C}}_\rho : \mathbf{uccomp}^{(2)}_{K/k} \to \mathbf{scomp}_k. \tag{5.8.3}$$

Remark. The symmetric composition $(M,q,\star) = \tilde{\mathbf{C}}(\tilde{A},\tilde{N},J)$ can also be obtained as follows. Instead of first forming the quotient $\tilde{A}/K\cdot 1$ and then taking the fixed point set of j, one can take first the fixed point set $\mathcal{H} = \mathbf{D}(\tilde{A},J)$, which is a k-submodule of $\tilde{A}$ containing $k\cdot 1$, and then pass to the quotient $\dot{\mathcal{H}} = \mathcal{H}/k\cdot 1$. Indeed, since (by definition) the unit element of $\tilde{A}$ is a unimodular vector, we have the split-exact sequence $0\to K\to\tilde{A}\to\tilde{M}\to 0$ of K-modules. Applying the functor $\mathbf{D}$ yields the split-exact sequence $0\to k\to\mathcal{H}\to M\to 0$ (where $M := \mathbf{D}(\tilde{M})$) of k-modules, so $M\cong\dot{\mathcal{H}}$. Moreover, it is easily seen that $\mathcal{H}$ is closed under the multiplication (5.8.2). Since the quadratic form $\tilde{H}_2$ restricted to $\mathcal{H}$ takes values in k, it follows that q and $\star$ on M are also induced from the corresponding structures on $\mathcal{H}$.

5.9. The functor $\tilde{\mathbf{A}}$. There is a functor $\tilde{\mathbf{A}}$ in the opposite direction as follows. Let $(\tilde{\mathfrak{C}},j)\in\mathbf{scomp}_{K/k}$, cf. 5.7. We apply the functor $\mathbf{A}_{\rho^{-1}}$ of 4.9 to $\tilde{\mathfrak{C}}$ and obtain a unital cubic composition $\mathbf{A}_{\rho^{-1}}(\tilde{\mathfrak{C}}) = (K\oplus\tilde{M},\tilde{N})\in\mathbf{uccomp}_K$ over K. Now extend the conjugation j on $\tilde{M}$ to a conjugation J on $\tilde{A} := K\oplus\tilde{M}$ by

$$J(\mu\oplus x) := \iota(\mu)\oplus j(x) \qquad (\mu\in K,\ x\in\tilde{M}).$$

Then J is an involution of the second kind of $\tilde{A}$. Indeed, it is clear that J fixes the unit element $1_K\oplus 0$, so it suffices to prove the antiautomorphism property for elements of the form $0\oplus x$. Since the action of j on $\tilde{M}$ is compatible with $\tilde{\star}$ and $\tilde{q}$, we have $j(x\,\tilde{\star}\,y) = j(x)\,\tilde{\star}\,j(y)$ and $\iota\big(\tilde{b}(x,y)\big) = \tilde{b}(j(x),j(y))$. Hence it follows from (5.7.5) that

$$\begin{aligned} J\big((0\oplus x)\bullet(0\oplus y)\big) &= J\Big(\tilde{b}(x,y)\oplus(\rho^{-1}x\,\tilde{\star}\,y + \rho y\,\tilde{\star}\,x)\Big)\\ &= \tilde{b}(j(x),j(y))\oplus\big(\rho j(x)\,\tilde{\star}\,j(y) + \rho^{-1}j(y)\,\tilde{\star}\,j(x)\big)\\ &= J(0\oplus y)\bullet J(0\oplus x). \end{aligned}$$

The cubic form $\tilde{N}$ of $\tilde{A}$ is given by $\tilde{N}(\mu\oplus x) = \mu^3 - 3\mu\tilde{q}(x) + \tilde{b}(x, x\tilde{\star}x)$. This readily implies that $\tilde{N}$ is compatible with the conjugation J on $\tilde{A}$. Altogether, we see that $\tilde{\mathbf{A}}_{\rho^{-1}}(\tilde{\mathfrak{C}},j) := \big(\mathbf{A}_{\rho^{-1}}(\tilde{\mathfrak{C}}),J\big)$ is a unital cubic composition of the second kind and one checks easily that this defines a functor $\tilde{\mathbf{A}}_{\rho^{-1}} : \mathbf{scomp}_{K/k}\to\mathbf{uccomp}^{(2)}_{K/k}$. Now combine this with the extension functor $\mathbf{E}$ to obtain a functor

$$\tilde{\mathbf{A}} := \tilde{\mathbf{A}}_{\rho^{-1}}\circ\mathbf{E} : \mathbf{scomp}_k\to\mathbf{uccomp}^{(2)}_{K/k}. \tag{5.9.1}$$

5.10. Proposition. *With K, $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{C}}$ as above, there are natural transformations*

$$\tilde{\zeta} : \tilde{\mathbf{A}} \circ \tilde{\mathbf{C}} \to \mathrm{Id}_{\mathbf{uccomp}^{(2)}_{K/k}}, \tag{5.10.1}$$

$$\tilde{\vartheta} : \tilde{\mathbf{C}} \circ \tilde{\mathbf{A}} \to \mathrm{Id}_{\mathbf{scomp}_k}. \tag{5.10.2}$$

If $3 \in k^\times$ then $\tilde{\zeta}$ and $\tilde{\vartheta}$ are isomorphisms, and hence the category of symmetric compositions over k is equivalent to the category of unital cubic compositions of the second kind.

Proof. By Prop. 2.6(a) and Prop. 4.10(a), there is a natural transformation $\zeta' : \mathbf{A}_{\rho^{-1}} \circ \mathbf{C}_\rho \to \mathrm{Id}_{\mathbf{uccomp}_K}$, given as follows: If $\tilde{\mathfrak{A}} = (\tilde{A}, \tilde{N}) \in \mathbf{uccomp}_K$ then

$$\zeta'_{\tilde{\mathfrak{A}}}(\mu \oplus \dot{x}) = \big(\mu - \tilde{T}(x)\big) \cdot 1 + 3x, \tag{5.10.3}$$

for all $x \in \tilde{A}$, $\mu \in K$. Here $\tilde{T}$ denotes the linear trace form of the cubic form $\tilde{N}$. Now suppose that $(\tilde{\mathfrak{A}}, J) = (\tilde{A}, \tilde{N}, J)$ is a unital cubic composition of the second kind. Since the conjugation J is compatible with $\tilde{T}$, and since the conjugation on $\tilde{\mathbf{C}}_\rho(\tilde{\mathfrak{A}}, J)$ is induced from J by passing to the quotient modulo $K \cdot 1$, it is clear from (5.10.3) that $\zeta'_{\tilde{\mathfrak{A}}}$ commutes with the respective conjugations. Hence we obtain a natural transformation

$$\zeta'' : \tilde{\mathbf{A}}_{\rho^{-1}} \circ \tilde{\mathbf{C}}_\rho \to \mathrm{Id}_{\mathbf{uccomp}^{(2)}_{K/k}}$$

by defining $\zeta''_{(\tilde{\mathfrak{A}},J)} = \zeta'_{\tilde{\mathfrak{A}}}$. The desired natural transformation (5.10.1) is then obtained by combining ζ'' with the natural transformation $\varepsilon : \mathbf{E} \circ \mathbf{D} \to \mathrm{Id}$, cf. (5.2.1):

$$\tilde{\zeta} : \tilde{\mathbf{A}} \circ \tilde{\mathbf{C}} = \tilde{\mathbf{A}}_{\rho^{-1}} \circ \mathbf{E} \circ \mathbf{D} \circ \tilde{\mathbf{C}}_\rho \xrightarrow{\varepsilon} \tilde{\mathbf{A}}_{\rho^{-1}} \circ \tilde{\mathbf{C}}_\rho \xrightarrow{\zeta''} \mathrm{Id}_{\mathbf{uccomp}^{(2)}_{K/k}}.$$

Similarly, by Prop. 2.6(c) and Prop. 4.10(b), there is a natural transformation $\vartheta' : \mathbf{C}_\rho \circ \mathbf{A}_{\rho^{-1}} \to \mathrm{Id}_{\mathbf{scomp}_K}$ which is simply multiplication by 3. If $(\tilde{\mathfrak{C}}, j) \in \mathbf{scomp}_{K/k}$ then this map obviously commutes with the conjugations. Hence we obtain a natural transformation

$$\vartheta'' : \tilde{\mathbf{C}}_\rho \circ \tilde{\mathbf{A}}_{\rho^{-1}} \to \mathrm{Id}_{\mathbf{scomp}_{K/k}}$$

by defining $\vartheta''_{(\tilde{\mathfrak{C}},j)} = \vartheta'_{\tilde{\mathfrak{C}}}$. The asserted natural transformation $\tilde{\vartheta}$ is then obtained by combining ϑ'' with the inverse of η of (5.7.3):

$$\tilde{\vartheta} : \tilde{\mathbf{C}} \circ \tilde{\mathbf{A}} = \mathbf{D} \circ \tilde{\mathbf{C}}_\rho \circ \tilde{\mathbf{A}}_{\rho^{-1}} \circ \mathbf{E} \xrightarrow{\vartheta''} \mathbf{D} \circ \mathbf{E} \xrightarrow{\eta^{-1}} \mathrm{Id}_{\mathbf{scomp}_k}.$$

The last statement is clear from the corresponding statement in Prop. 4.10.

6. Transfer of regularity conditions

6.1. Separable polynomial laws. While non-degeneracy of a polynomial law descends from faithfully flat base extensions by Lemma 1.5(d), this property is not preserved under base extension even in the case of fields:

Example. Let k be an imperfect field of characteristic p, suppose $a \in k$ is not a p-th power, and consider the polynomial law f of degree p on $X = k \times k$ with values in k given by

$$f(x) = x_1^p - ax_2^p$$

for all $x = (x_1, x_2) \in X_R = R \times R$, $R \in k\text{-}\mathbf{alg}$. Then f is anisotropic ($f(x) \neq 0$ for all $0 \neq x \in X$), in particular, $\mathrm{Ker}(f) = 0$. Now put $R = k[\mathbf{t}]/(\mathbf{t}^p - a)$ and let $\tau = \mathrm{can}(\mathbf{t})$, so that $\tau^p = a$. Then $z = (\tau, 1) \in \mathrm{Ker}(f \otimes R)$ since f is additive.

Therefore, we introduce the following notion: A polynomial law $f \in \mathcal{P}(X, V)$ is said to be *separable* if X and V are finitely generated and projective k-modules and $f \otimes F$ is non-degenerate, for all fields $F \in k\text{-}\mathbf{alg}$. This generalizes the separable quadratic forms of [uqf, §3]. It can be shown that a linear map $f\colon X \to V$ between finitely generated and projective modules, considered as a polynomial law of degree 1, is separable if and only if it is an isomorphism of X onto a direct summand of V.

By Lemma 6.2 below, separability behaves well under base change. Note, however, that separability does not imply non-degeneracy: Let $p \in \mathbb{Z}$ be a prime. The form f of degree p on $\mathbb{Z}$ given by $f(x) = x^p$ (for all $x \in R$, R a $\mathbb{Z}$-algebra, i.e., an arbitrary commutative ring) is clearly separable, because $f \otimes F$ is anisotropic for any field F. But let for example $R = \mathbb{Z}/p^2\mathbb{Z}$. Then $z = p1_R \in R$ is not zero, satisfies $z^2 = pz = 0$, and hence for all $S \in R\text{-}\mathbf{alg}$ and all $x \in S$, we have

$$f(z_S + x) = (p1_S + x)^p = x^p,$$

because $p^p = p^2 p^{p-2} = 0$ and $\binom{p}{i} \equiv 0 \pmod{p}$ for $0 < i < p$. Hence $z \in \mathrm{Ker}(f \otimes R)$.

6.2. Lemma. *Let $f \in \mathcal{P}(X, V)$.*

(a) *If f is separable then so is $f \otimes R$, for all $R \in k$-**alg**.*

(b) *Conversely, let $R \in k$-**alg** be faithfully flat and $f \otimes R$ separable. Then f is separable.*

Proof. (a) It is well known that finitely generated and projective modules remain so under base extension. Any field $F \in R\text{-}\mathbf{alg}$ can be considered, by restriction of scalars, as a field in $k\text{-}\mathbf{alg}$. With the usual identifications, $(f \otimes_k R) \otimes_R F = f \otimes_k F$ whence $\mathrm{Ker}\big((f \otimes_k R) \otimes_R F\big) = 0$.

(b) If X_R and V_R are finitely generated and projective over R then so are X and V over k by [bac, I, §3.6, Prop. 12]. Let $F \in k\text{-}\mathbf{alg}$ be a field. After replacing k by F and correspondingly R by $R \otimes F$ (which is still faithfully flat over F, i.e., $\neq \{0\}$), we may assume that k is a field, and then must show that f is non-degenerate. Let $\mathfrak{m} \subset R$ be a maximal ideal and $L = R/\mathfrak{m}$. Then L is an extension field of k via $k \to R \to L$. Since $f \otimes R$ is separable, $(f \otimes_k R) \otimes_R L$ is non-degenerate. But $(f \otimes_k R) \otimes_R L = f \otimes_k L$, and L is obviously faithfully flat over k (being an extension field). Hence $f \otimes_k L$ non-degenerate implies f non-degenerate by Lemma 1.5(d).

6.3. Definition. A unital d-form $\mathfrak{X} = (X, N, 1)$ is called *separable* if N is separable in the sense of 6.1. Similarly, a quadratic-cubic form $\mathfrak{M} = (M, f_2, f_3) \in \mathbf{qcform}_k$ as in 2.5 is called separable if $f = f_2 + f_3$ is separable. We show next how the functors $\mathbf{C}$ and $\mathbf{A}$ of 2.5 interact with non-degeneracy and separability.

6.4. Proposition. *Let $\mathfrak{X} = (X, N, 1) \in \mathbf{ucform}_k$ and $\mathfrak{M} = \mathbf{C}(\mathfrak{X}) = (\dot{X}, q, h) \in \mathbf{qcform}_k$ as in 2.5. The canonical map $X \to \dot{X} = X/k \cdot 1$ is denoted by κ.*

(a) *Then $\kappa\big(\mathrm{Ker}(N)\big) \subset \mathrm{Ker}(q + h)$, and equality holds if $3 \in k^\times$.*

(b) *If* $\mathbf{C}(\mathfrak{X})$ *is separable then so is* $\mathfrak{X}$*, and the converse holds if* $3 \in k^\times$*.*

Proof. (a) By Lemma 2.4(a), $q+h = \kappa_*(H_2+H_3)$. Hence Prop. 1.6 shows that $\dot{z} \in \mathrm{Ker}(q+h)$ if and only if $z \in \mathrm{Ker}(H_2+H_3)$. By (1.4.3), we have $\mathrm{Ker}(H_2+H_3) = \mathrm{Ker}(H_2) \cap \mathrm{Ker}(H_3)$. One sees by a straightforward computation, using (2.4.1) – (2.4.4) as well as the description of the kernel of a quadratic and a cubic form in (1.4.4) and (1.4.5), that $z \in \mathrm{Ker}(H_2) \cap \mathrm{Ker}(H_3)$ if and only if

$$T(z)^2 = 3Q(z), \tag{6.4.1}$$
$$2T(z)T(x) = 3B(z,x) \quad \text{for all } x \in X, \tag{6.4.2}$$
$$3T(z)Q(z) = 27N(z), \tag{6.4.3}$$
$$9T(x)Q(z) = 27N_{21}(z,x) \quad \text{for all } x \in X, \tag{6.4.4}$$
$$9T(z)Q(x) = 27N_{21}(x,z) \quad \text{for all } x \in X. \tag{6.4.5}$$

We also have

$$z \in \mathrm{Ker}(N) \quad \Longrightarrow \quad z \in \mathrm{Ker}(T) \cap \mathrm{Ker}(Q). \tag{6.4.6}$$

Indeed, $z \in \mathrm{Ker}(N)$ implies $N_{21}(x,z) = N_{21}(z,x) = 0$ for all $x \in X$. Putting $x = 1$ yields $T(z) = N_{21}(1,z) = 0$ and $Q(z) = N_{21}(z,1) = 0$. Moreover, $B(x,z) = \Phi(x,z,1) = 0$ follows by linearizing the condition $N_{21}(x,z) = 0$ with respect to x in direction z.

Now (6.4.1) – (6.4.6) show that $\mathrm{Ker}(N) \subset \mathrm{Ker}(H_2+H_3)$, and hence $\kappa\big(\mathrm{Ker}(N)\big) \subset \kappa\big(\mathrm{Ker}(H_2+H_3)\big) = \mathrm{Ker}(q+h)$, by (1.6.2).

Suppose $3 \in k^\times$. Then $X = k \cdot 1 \oplus X^0$ where $X^0 = \mathrm{Ker}(T)$, so $\kappa\colon X^0 \to \dot{X}$ is an isomorphism of k-modules. For an element $w \in \mathrm{Ker}(q+h)$ let $z_0 \in X^0$ be its inverse image. Then $z_0 \in \mathrm{Ker}(H_2) \cap \mathrm{Ker}(H_3)$ by (1.6.1), so (6.4.1) and (6.4.3) – (6.4.5) show that $z_0 \in \mathrm{Ker}(N)$, and therefore $w = \kappa(z_0) \in \kappa\big(\mathrm{Ker}(N)\big)$, as desired.

(b) As 1_X is a unimodular vector, $X \cong k \oplus \dot{X}$ as a k-module. Hence X is finitely generated and projective if and only if $\dot{X}$ is so. Now let $\mathbf{C}(\mathfrak{X})$ be separable and let $F \in k$-**alg** be a field. After extending the base from k to F, we may assume that k itself is a field. Then $\kappa\big(\mathrm{Ker}(N)\big) \subset \mathrm{Ker}(q+h) = \{0\}$ implies $\mathrm{Ker}(N) \subset k \cdot 1$. But if $\lambda \cdot 1 \in \mathrm{Ker}(N)$ then $0 = N(\lambda \cdot 1) = \lambda^3$ implies $\lambda = 0$, because k is a field. Hence $\mathrm{Ker}(N) = \{0\}$.

Now assume $3 \in k^\times$ and $\mathbf{C}(\mathfrak{X})$ separable. As before, we may assume k to be a field. Then $\mathrm{Ker}(q+h) = \kappa\big(\mathrm{Ker}(N)\big) = \kappa(\{0\}) = \{0\}$.

6.5. Proposition. *Let* $\mathfrak{M} = (M, f_2, f_3) \in \mathbf{qcform}_k$ *and let* $N_{\mathfrak{M}}$ *be the cubic form of* $\mathbf{A}(\mathfrak{M})$ *as in* (2.5.1).

(a) *Then* $\nu \oplus z \in \mathrm{Ker}(N_{\mathfrak{M}})$ *if and only if*

$$z \in \mathrm{Ker}(3f_2), \tag{6.5.1}$$
$$3\nu = \nu^3 + f_3(z) = 0, \tag{6.5.2}$$
$$f_{21}(x,z) = f_{21}(z,x) = 0 \quad \textit{for all } x \in M. \tag{6.5.3}$$

In particular, $0 \oplus \mathrm{Ker}(f_2+f_3) \subset \mathrm{Ker}(N_{\mathfrak{M}})$*. If* k *has no* 3*-torsion, then*

$$\mathrm{Ker}(N_{\mathfrak{M}}) = 0 \oplus \mathrm{Ker}(f_2+f_3). \tag{6.5.4}$$

(b) *If* $\mathbf{A}(\mathfrak{M}) \in \mathbf{ucform}_k$ *is separable then so is* $\mathfrak{M}$*, and the converse holds provided* $3 \in k^\times$*.*

Proof. (a) Put $N = N_{\mathfrak{M}}$ for short. From (2.5.1) it follows easily that

$$N_{21}(\lambda \oplus x,\, \mu \oplus y) = 3\lambda^2\mu - 3\mu f_2(x) - 3\lambda f_{11}(x,y) + f_{21}(x,y).$$

Here f_{11} and f_{21} denote the polarizations of f_2 and f_3. By (1.4.5), $\nu \oplus z \in \mathrm{Ker}(N)$ if and only if $N(\nu \oplus z) = N_{21}(\nu \oplus z, \lambda \oplus x) = N_{21}(\lambda \oplus x, \nu \oplus z) = 0$ for all $\lambda \oplus x \in k \oplus M$. It is straightforward to show that these conditions are equivalent to (6.5.1) – (6.5.3), and are implied by $\nu = 0$ and $z \in \mathrm{Ker}(N)$. If k has no 3-torsion then (6.5.1) – (6.5.3) are equivalent to $\nu = 0$ and $z \in \mathrm{Ker}(f_2) \cap \mathrm{Ker}(f_3)$.

(b) This follows by observing that M is finitely generated and projective if and only if $k \oplus M$ is so, and by applying (a) to $\mathfrak{M} \otimes F$ where $F \in k$-**alg** a field.

6.6. Definition. Recall (3.1) the functor Ω from unital compositions to unital forms which omits the multiplication. For a unital degree d composition $\mathfrak{A} = (\mathfrak{X}, \cdot) = (A, N)$, we define separability by

$$\begin{aligned} \mathfrak{A} \text{ is separable} &\iff \Omega(\mathfrak{A}) = \mathfrak{X} \text{ is separable}, \\ &\iff N \text{ is separable}. \end{aligned} \tag{6.6.1}$$

Thus separability of $\mathfrak{A}$ is *not* defined in terms of the algebra A (this would be rather awkward, since we don't even know whether A is alternative), but solely in terms of the form N.

A symmetric composition $\mathfrak{C} = (M, q, \star)$ is called separable if the quadratic form q is separable. Recall from (3.12.1) the functor Υ: **scomp** $\to$ $\mathbf{qcform}_k$, $\mathfrak{C} \mapsto (M, q, h)$. Since Υ is compatible with base change, and $\mathrm{Ker}(q \otimes F + h \otimes K) = \mathrm{Ker}(q \otimes F) \cap \mathrm{Ker}(h \otimes F)$ (by (1.4.3)) $= \mathrm{Ker}(q \otimes F)$ (by (4.4.2)), we see that

$$\mathfrak{C} \text{ is separable} \iff \Upsilon(\mathfrak{C}) = (M, q, h) \text{ is separable}. \tag{6.6.2}$$

Note that the underlying module of a separable unital or symmetric composition is, by definition, finitely generated and projective.

Now combine Prop. 6.4(b) and 6.5(b) with (6.6.1) and (6.6.2) and the compatibility of the functors Ω and Υ with $\mathbf{C}_\alpha$ and $\mathbf{C}$ as well as $\mathbf{A}_{\alpha^{-1}}$ and $\mathbf{A}$. Then Prop. 4.10 implies:

6.7. Corollary. *Let $3 \in k^\times$ and $\alpha \in \boldsymbol{\pi}_6(k)$ a primitive sixth root of unity. Then $\mathbf{C}_\alpha$ and $\mathbf{A}_{\alpha^{-1}}$ induce quasi-inverse equivalences between the categories of separable unital cubic compositions and separable symmetric compositions.*

It is possible to formulate a twisted version of this result, using the functors $\tilde{\mathbf{C}}$ and $\tilde{\mathbf{A}}$ of §5. The details are omitted.

6.8. Strictness. We call a symmetric composition $\mathfrak{C} = (M, q, \star)$ *strict* if the congruences (4.5.4) are equalities, i.e., if

$$(x \star y) \star x = x \star (y \star x) = q(x)y, \tag{6.8.1}$$

for all $x, y \in M$. From 4.5 it is clear that

$$q \text{ non-degenerate} \implies \mathfrak{C} \text{ strict},$$

and it can be shown that separable symmetric compositions are strict as well.

In a similar vein, let us call a unital cubic composition $\mathfrak{A} = (A, N)$ *strict* if A is an alternative algebra and satisfies the characteristic polynomial χ of 2.3, i.e.,

$$\chi(x;x) = x^3 - T(x)x^2 + Q(x)x - N(x)1 = 0$$

for all $x \in A \otimes R$, $R \in k$-**alg**. — Here the relation between non-degeneracy and strictness is less clear than in case of symmetric compositions. It is claimed in [bb, Satz 3] that N non-degenerate implies A alternative. Similarly, it is claimed in [berg, p. 142, Satz 4] and [bb, Satz 4] that non-degeneracy of N implies that A satisfies χ. I have not been able to verify the arguments given by Baumgartner and Bergmann. On the other hand, Schafer [schafer:n, Theorem 2, Theorem 3] proved that this is true if k is a field of characteristic different from 2 or 3.

The following result is the analogue, in our setting, of [elmy, Prop. 4.1, Prop. 4.2]. We leave it to the reader to formulate and prove a twisted version like [elmy, Prop. 5.1, Prop. 5.2].

6.9. Proposition. *The functors $\mathbf{C}_\alpha$ and $\mathbf{A}_\alpha$ preserve strictness.*

Proof. Let $\mathfrak{A} = (A, N) \in \mathbf{uccomp}_k$ be strict. We compute the product $(x*y)*x$, defined as in (3.9.1), where now $\beta = \alpha^{-1} = 1 - \alpha$. A straightforward computation shows that

$$(x*y)*x = 3(x^2y+xyx+yx^2) - 3T_x(xy+yx) - 3T_yx^2 + 3\big(T_xT_y - T_{xy}\big)x + T_x^2y. \quad (6.9.1)$$

On the other hand, $x + y$ satisfies the polynomial $\chi(\mathbf{t}; x + y)$, whence

$$(x + y)^3 - T_{x+y}(x + y)^2 = N(x + y) \cdot 1 - Q(x + y)(x + y).$$

By expanding and comparing the terms quadratic in x and linear in y, we obtain

$$x^2y + xyx + yx^2 - T_x(xy + yx) - T_yx^2 = N_{21}(x, y)1 - Q_xy - B(x, y)x. \quad (6.9.2)$$

Combining (6.9.1) and (6.9.2) and using (3.2.1) yields

$$(x * y) * x = (T_x^2 - 3Q_x)y + 3N_{21}(x, y) \cdot 1. \quad (6.9.3)$$

Now pass to the quotient $\dot{A} = A/k \cdot 1$ and recall that the quadratic form q on $\mathbf{C}_\alpha(\mathfrak{A})$ is induced from $H_2(x) = T_x^2 - 3Q_x$. Then (6.9.3) implies $(\dot{x} \star \dot{y}) \star \dot{x} = q(\dot{x})\dot{y}$, and the second equation of (6.8.1) is proved similarly.

Let $\mathfrak{C} = (M, q, \star)$ be strict and $\mathbf{A}_\alpha(\mathfrak{C}) = \mathfrak{A} = (A, N)$, thus $A = k \oplus M$ with the multiplication $\bullet$ of (4.3.1) and the cubic form N of (4.1.1). We first show that A is alternative. It suffices to prove the alternative law for elements of the form $u = 0 \oplus x$ and $v = 0 \oplus y$. Then the assertion follows by a straightforward computation, using (6.8.1) and the identities (4.5.7) and (4.5.8) which follow from (6.8.1) by linearization and are now equalities (instead of congruences modulo $\mathrm{Ker}(q)$).

The fact that A satisfies the characteristic polynomial is likewise a straightforward computation, using the formulas (2.5.2) and (2.5.3) for the trace and quadratic trace of N.

References