CORRIGENDUM TO: "FACTORING SKEW POLYNOMIALS OVER HAMILTON'S QUATERNION ALGEBRA AND THE COMPLEX NUMBERS" [J. ALGEBRA 427 (2015), 20-29]

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Abstract. The proofs for [7, Theorem 2 (i), Corollary 3 (i), (iii), Theorem 5 (i), (iii), Corollary 6] and [8, Theorem 2 (i)], only hold if σ is an F-algebra automorphism and δ is F-linear.

1. Correction

1.1. Let F be a real closed field. We use the notation of [7, 8]:

In [8, Theorem 2 (i)] we claim that every non-constant polynomial in a skew-polynomial ring $D[t;\sigma,\delta]$, D the quaternion division algebra over F, decomposes into a product of linear factors. Regrettably, the proofs for [7, Theorem 2 (i)] and its improvement [8, Theorem 2 (i)] (and hence for [7, Corollary 3 (i), (iii)]) are not complete, since the nonassociative algebra S_f used in the argument is not always a finite-dimensional F-algebra, as claimed. S_f is an algebra over $F_0 = \{a \in D \,|\, ah = ha \text{ for all } h \in S_f\}$, a subfield of D [6]. However, when σ or δ are not F-linear, F_0 is a a proper subfield of F. Hence S_f is an infinite-dimensional algebra over F_0 and the argument in the proof breaks down, as it relies on the fact that a finite-dimensional algebra over F has dimension 1, 2, 4 or 8. It still works if we assume that σ and δ are F-linear maps, in that case the algebras S_f are indeed finite-dimensional algebras over F.

Now σ is an F-linear map if and only if is an inner automorphism of D, i.e. $\sigma = \sigma_a$, $\sigma_a(u) = aua^{-1}$ for some $a \in D^\times$, and δ is an F-linear map if and only if $\delta = \delta_d$ is an inner σ-derivation δ_d, i.e. $\delta_d(u) = ud - d\sigma(u)$ for some $d \in D$ [2, Proposition 2.1.4]. The linear change of variables $z = ta - da$ reduces σ to id and δ to 0 [2, p. 51].

Thus for F-linear maps σ and δ, $D[t;\sigma,\delta] = D_L[t]$ by a linear change of variables, and we have proved a result equivalent to the Fundamental Theorem of Algebra for $D_L[t]$ ([5] or [3, Theorem (16.5), p. 271]) using nonassociative algebra.

1.2. For the same reasons as in Section 1.1., the proofs of [7, Theorem 5, (i), (iii)] and [7, Corollary 6] only work if σ and δ are F-linear maps, since only then $F \subset F_0$ and S_f is a finite-dimensional algebra over F, i.e. $\sigma = id$ or the non-trivial automorphism of the quadratic field extension $F(\sqrt{-1})/F$ (there exist other automorphisms, cf. [9]). Let σ be

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the non-trivial automorphism of the quadratic field extension $F(\sqrt{-1})/F$. The corrected versions of [7, Theorem 5, (i)] and [7, Corollary 6] are:

Theorem 1. *Every polynomial $f(t) \in F(\sqrt{-1})[t;\sigma]$ of degree $m > 3$ decomposes into the product of linear, quadratic or quartic irreducible polynomials. No quartic irreducible polynomial is a two-sided element in $F(\sqrt{-1})[t;\sigma]$.*

Corollary 2. *Let $^-$ be complex conjugation. Then every polynomial $f(t) \in \mathbb{C}[t;^-]$ of degree ≥ 1 decomposes into the product of linear and quadratic irreducible polynomials.*

Note that by [1, Theorem 3.1] there are no quartic irreducible polynomials $f(t) \in F(\sqrt{-1})[t;\sigma]$, improving Theorem 1 and that Corollary 2 was recently also proved in [1, Corollary 3.2] and briefly mentioned in [4].

We now obtain the following stronger version of [7, Corollary 9]:

Corollary 3. *(i) If S_f is a finite-dimensional real division algebra, then $R = \mathbb{C}[t;^-]$, $f \in R$ is irreducible of degree 2, and if S_f is not associative then $\mathrm{Nuc}_l(S_f) = \mathrm{Nuc}_m(S_f) = \mathbb{C}$ and $\mathrm{Nuc}_r(S_f) \in \{\mathbb{R}, \mathbb{C}\}$.*
(ii) Every four-dimensional real division algebra A with multiplication $\star$ which is a two-dimensional vector space over $\mathbb{C} \subset \mathrm{Nuc}_l(A) \cap \mathrm{Nuc}_m(A)$ is isomorphic to S_f for a suitable irreducible $f \in \mathbb{C}[t;^-]$ of degree two.
(iii) Every four-dimensional real division algebra A which is a two-dimensional vector space over $\mathbb{C} \subset \mathrm{Nuc}_m(A) \cap \mathrm{Nuc}_r(A)$ is isomorphic to $(S_f)^{op}$ for a suitable irreducible $f \in \mathbb{C}[t;^-]$ of degree two.

Proof. (i) If S_f with $R = \mathbb{C}[t;\sigma,\delta]$ is a finite-dimensional real division algebra then $f \in R$ is irreducible of degree 2, and if S_f is not associative then $\mathrm{Nuc}_l(S_f) = \mathrm{Nuc}_m(S_f) = \mathbb{C}$ and $\mathrm{Nuc}_r(S_f) \in \{\mathbb{R}, \mathbb{C}\}$ [6, (1), p. 13-08]. Suppose σ or δ are not $\mathbb{R}$-linear. Then F_0 is properly contained in $\mathbb{R}$ and S_f is an infinite-dimensional algebra over F_0, contradicting our assumption. Thus both σ and δ are $\mathbb{R}$-linear and we can assume either $R = \mathbb{C}[t;^-]$ by a linear change of variables, or $R = \mathbb{C}[t]$ by [6, (11)]. There are no irreducible polynomials of degree 2 in $\mathbb{C}[t]$, so $R = \mathbb{C}[t;^-]$.
(ii) Every four-dimensional real division algebra A with multiplication $\star$ which is a two-dimensional vector space over $\mathbb{C} \subset \mathrm{Nuc}_l(A) \cap \mathrm{Nuc}_m(A)$ is isomorphic to S_f for a suitable irreducible $f \in \mathbb{C}[t;\sigma,\delta]$, $f = t^2 - d_1 t - d_0$, where σ and δ are defined via

$$t \star a = \sigma(a) \star t + \delta(a)$$

for all $a \in D$ [7, Corollary 9]. The assertion now follows from (i).
(iii) Just apply (ii) to $(S_f)^{op}$. □

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