# ON COMMUTING $U$-OPERATORS IN JORDAN ALGEBRAS

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Abstract. Recently J.A.Anquela, T.Cortés, and H.Petersson [2] proved that for elements $x, y$ in a non-degenerate Jordan algebra $J$, the relation $x \circ y = 0$ implies that the $U$-operators of $x$ and $y$ commute: $U_x U_y = U_y U_x$. We show that the result may be not true without the assumption on non-degeneracy of $J$. We give also a more simple proof of the mentioned result in the case of linear Jordan algebras, that is, when $char\, F \neq 2$.

Dedicated to Professor Amin Kaidi
on the occasion of his 65-th annyversary

## 1. An Introduction

In recent paper [2] J.A.Anquela, T.Cortés, and H.Petersson have studied the following question for Jordan algebras:

(1) does the relation $x \circ y = 0$ imply that the quadratic operators $U_x$ and $U_y$ commute?

They proved that the answer is positive for non-degenerate Jordan algebras, and left open the question in general case.

We show that the answer to question (1) is negative in general case. We give also a more simple proof of the result for linear non-degenerate Jordan algebras, that is, over a field $F$ of characteristic $\neq 2$.

## 2. A counter-example

Let us remind some facts on Jordan algebras. We use for references the books [1, 4], and the paper [3].

Consider the free special Jordan algebra $SJ[x, y, z]$ and the free associative algebra $F\langle x, y, z\rangle$ over a field $F$. Let $*$ be the involution of $F\langle x, y, z\rangle$ identical on the set $\{x, y, z\}$. Denote $\{u\} = u + u^*$ for $u \in F\langle x, y, z\rangle$, then $\{u\} \in SJ[x, y, z]$ [1]. Below $ab$ is the product in $F\langle x, y, z\rangle$ and $a \circ b = ab + ba$ and $aU_b = bab$ are linear and quadratic operations in $SJ[x, y, z]$.

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For an ideal $I$ of $SJ[x,y,z]$ denote by $\hat{I}$ the ideal of $F\langle x,y,z\rangle$ generated by $I$. By Cohn's Lemma [1, lemma 1.1], the quotient algebra $J = SJ[x,y,z]/I$ is special if and only if $I = \hat{I} \cap SJ[x,y,z]$.

**Lemma 1.** *The following equality holds*

$$z[U_x, U_y] = \{(x \circ y)zxy\} - zU_{x\circ y}.$$

*Proof.* We have in $F\langle x,y,z\rangle$

$$\begin{aligned} z[U_x, U_y] &= yxzxy - xyzyx = (y\circ x)zxy - xyzxy - xyzyx = \\ &= (y \circ x)zxy - xyz(x \circ y) = \{(x \circ y)zxy\} - (x \circ y)z(x \circ y). \end{aligned}$$

□

**Theorem 1.** *Let $I$ denote the ideal of $SJ[x,y,z]$ generated by $x \circ y = xy + yx$ and $J = SJ[x,y,z]/I$. Then for the images $\bar{x}, \bar{y}$ of the elements $x, y$ in $J$ we have $\bar{x} \circ \bar{y} = 0$ but $[U_{\bar{x}}, U_{\bar{y}}] \neq 0$.*

*Proof.* It suffices to show that $k = z[U_x, U_y] \notin I$. By lemma 1, $k = \{(x \circ y)zxy\}$ $(mod\ I)$. Now, the arguments from the proof of [1, theorem 1.2], show that $k \notin I$ when $F$ is a field of characteristic not 2 (see also [1, exercise 1, page 12]).

The result is also true in characteristic 2 for quadratic Jordan algebras. In this case, one needs certain modifications concerning the generation of ideals in quadratic case. The author is grateful to T. Cortés and J.A. Anquela who correct the first "naive" author's proof and suggest the proper modifications which we give below.

We have to prove that $\{(x \circ y)zxy\} \notin I$. By [6, (1.9)], the ideal $I$ is the outer hull of $F(x \circ y) + U_{x\circ y}\widehat{SJ[x,y,z]}$, where $\widehat{J}$ denotes the unital hull of $J$. Assume that there exists a Jordan polynomial $f(x,y,z,t) \in SJ[x,y,z,t]$ with all of its Jordan monomials containing the variable $t$, such that $\{(x \circ y)zxy\} = f(x,y,z,x\circ y)$. By degree considerations, $f = g + h$, where $g, h \in SJ[x,y,z,t]$, $g$ is multilinear, and $h(x,y,z,t)$ is a linear combination of $U_t z$ and $z \circ t^2$. On the other hand, arguing as in [1, Theorem 1.2], $g \in SJ[x,y,z,t] \subseteq H(F\langle x,y,z,t\rangle, *)$, and because of degree considerations and the fact that $z$ occupies inside position in the associative monomials of $\{(x \circ y)zxy\}$, $g$ is a linear combination of

$$\{xzyt\}, \{xzty\}, \{tzxy\}, \{tzyx\}, \{yztx\}, \{yzxt\},$$

and $h$ is a scalar multiple of $U_t z$. Hence $f$ has the form

$$\begin{aligned} f(x,y,z,t) &= \alpha_1\{xzyt\} + \alpha_2\{xzty\} + \alpha_3\{tzxy\} \\ &+ \alpha_4\{tzyx\} + \alpha_5\{yztx\} + \alpha_6\{yzxt\} \\ &+ \alpha_7 tzt, \end{aligned}$$

and therefore

$$\begin{aligned}\{(x\circ y)zxy\} &= \alpha_1\{xzy(x\circ y)\}+\alpha_2\{xz(x\circ y)y\}+\alpha_3\{(x\circ y)zxy\}\\ &+ \alpha_4\{(x\circ y)zyx\}+\alpha_5\{yz(x\circ y)x\}+\alpha_6\{yzx(x\circ y)\}\\ &+ \alpha_7(x\circ y)z(x\circ y),\end{aligned}$$

Comparing coefficients as in [1, Theorem 1.2], we get

$$\alpha_1=\alpha_2=\alpha_5=\alpha_6=0,$$
$$\alpha_3=\lambda+1, \alpha_4=\lambda, \alpha_7=-2\lambda,$$

for some $\lambda\in F$. Going back to $f$, we get

$$f=(\lambda+1)\{tzxy\}+\lambda\{tzyx\}-2\lambda tzt=\{tzxy\}+\lambda\{tz(x\circ y)\}-2\lambda U_t z,$$

so that $\{tzxy\}\in SJ[x,y,z,t]$, which is a contradiction.

In fact, the standard arguments with Grassmann algebra do not work in characteristic 2, to prove that $\{tzxy\}\notin SJ[x,y,z,t]$, but one can check directly (or with aid of computer) that the space of symmetric multilinear elements in $F\langle x,y,z,t\rangle$ has dimension 12 while the similar space of Jordan elements has dimension 11.

□

## 3. The non-degenerate case

Here we will give another proof of the main result from [2] that the answer to question (1) is positive for nondegenerate algebras, for the case of linear Jordan algebras (over a field $F$ of characteristic $\neq 2$).

Let J be a linear Jordan algebra, $a\in J$, $R_a: x\mapsto xa$ be the operator of right multiplication on $a$, and $U_a=2R_a^2-R_{a^2}$.

As in [2], due to the McCrimmon-Zelmanov theorem [5], it suffices to consider Albert algebras. We will need only the fact that an Albert algebra $A$ is *cubic*, that is, for every $a\in A$, holds the identity

$$a^3=t(a)a^2-s(a)a+n(a),$$

where $t(a), s(a), n(a)$ are correspondingly linear, quadratic, and cubic forms on $A$ [1]. Linearizing the above identity on $a$, we get the identity

$$\begin{aligned}2((ab)c+(ac)b+(bc)a) &= 2(t(a)bc+t(b)ac+t(c)ab)\\ &\quad -s(a,b)c-s(a,c)b-s(b,c)a+n(a,b,c),\end{aligned}$$

where $s(a,b)=s(a+b)-s(a)-s(b)$ and $n(a,b,c)=n(a+b+c)-n(a+b)-n(a+c)-n(b+c)+n(a)+n(b)+n(c)$ are bilinear and trilinear forms. In particular, we have

(1) $\quad a^2b + 2(ab)a = t(b)a^2 + 2t(a)ab - s(a,b)a - s(a)b + \frac{1}{2}n(a,a,b).$

**Lemma 2.** *Let $a, b \in J$ with $ab = 0$. Then $[U_a, U_b] = [R_{a^2}, R_{b^2}]$.*

*Proof.* Linearizing the Jordan identity $[R_x, R_{x^2}] = 0$, one obtains

$$[R_{a^2}, R_b] = -2[R_{ab}, R_a] = 0,$$

and similarly $[R_a, R_{b^2}] = 0$. Therefore,

$$[U_a, U_b] = [2R_a^2 - R_{a^2}, 2R_b^2 - R_{b^2}] = 4[R_a^2, R_b^2] + [R_{a^2}, R_{b^2}].$$

Furthermore, $[R_a^2, R_b^2] = [R_a, R_b^2 R_a + R_a R_b^2]$. By the operator Jordan identity [1, (1.$O_2$)],

$$R_b^2 R_a + R_a R_b^2 = -R_{(ba)b} + 2R_{ab}R_b + R_{b^2}R_a = R_{b^2}R_a,$$

therefore $[R_a^2, R_b^2] = [R_a, R_{b^2}R_a] = [R_a, R_{b^2}]R_a = 0$, which proves the lemma. □

**Theorem 2.** *Let $J$ be a cubic Jordan algebra and $a, b \in J$ with $ab = 0$. Then $[U_a, U_b] = 0$.*

*Proof.* For any $c \in J$ we have by Lemma 2 and by the linearization of the Jordan identity $(x, y, x^2) = 0$

$$c[U_a, U_b] = c[R_{a^2}, R_{b^2}] = (a^2, c, b^2) = -2(a^2b, c, b).$$

By (1), we have

$$\begin{aligned} (a^2b, c, b) &= t(b)(a^2, c, b) - s(a)(b, c, b) - s(a,b)(a, c, b) \\ &= -2t(b)(ab, c, a) - s(a,b)(a, c, b) = -s(a,b)(a, c, b). \end{aligned}$$

Substituting $c = a$, we get $(a^2b, a, b) = ((a^2b)a)b = (a^2(ba))b = 0$, which implies $0 = s(a,b)(a,a,b) = s(a,b)(a^2b)$. Therefore, $s(a,b) = 0$ or $a^2b = 0$. In both cases this implies $c[U_a, U_b] = 0$. □

**Corollary 1.** *In an Albert algebra $A$, the equality $ab = 0$ implies $[U_a, U_b] = 0$.*

In connection with the counter-example above, we would like to formulate an open question. Let $f, g \in SJ[x, y, z]$ such that $g \in \widehat{(f)}$ but $g \notin (f)$, where $(f)$ and $\widehat{(f)}$ are the ideals generated by $f$ in $SJ[x, y, z]$ and in $F\langle x, y, z\rangle$, respectively. Then the quotient algebra $SJ[x, y, z]/(f)$ is not special, due to Cohn's Lemma. It follows from the results of [7] that the quotient algebra $\widehat{(f)}/(f)$ is degenerated. The question we want to ask is the following:

*If $f = 0$ in a nondegenerate Jordan algebra $J$, should also be $g = 0$?*

Of course, there is a problem of writing $f$ and $g$ in arbitrary Jordan algebra, we know only what they are in $SJ[x, y, z]$, but in the free Jordan algebra

$J[x, y, z]$ they have many pre-images (up to $s$-identities), and one may choose pre-images for which the question has a negative answer. For example, the answer is probably negative for $f = x \circ y$ and $g = z[U_x, U_y] + G(x, y, z)$, where $G(x, y, z)$ is the Glennie $s$-identity [1].

But assume that $f$ and $g$ are of degree less then 2 on $z$, then by the Macdonald-Shirshov theorem they have unique pre-images in $J[x, y, z]$, and we may ask: if $f = 0$ in a non-degenerate Jordan algebra $J$, should also be $g = 0$?

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