THE COMPOSITION STRUCTURE OF ALTERNATIVE AND MALCEV ALGEBRAS.

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ABSTRACT. We have constructed the element of degree 5 from the associative kernel of the free alternative algebra. We show that this element has a minimal degree. Using this element we obtain decomposition of the varieties of alternative and Malcev algebras.

1. INTRODUCTION

All algebras in this work are considered over a field F of characteristic 0. We use standard definitions and notation from [1].

We will denote by Alt, Mal the variety of the alternative and Malcev algebras respectively. Let $Ass[x]$, $Alt[X]$, $Mal[X]$ be a free associative, a free alternative, a free Malcev algebras with set of generators $X=\{x_1,...,x_n,...\}$. The subalgebra $SMal[X]$ of $Alt[X]^{(-)}$ generated by the set X is called *free special Malcev* algebra. The elements of $SMal[X]$ are called *Malcev polynomials*. Let π denote the canonical homomorphism $\pi : Mal[X]\to SMal[X]$.

We will denote by $\bullet$ the multiplication in the algebra $Mal[X]$, by $[a,b]$ the multiplication in the algebra $SMal[X]$, where $[a,b]=ab-ba$ is the commutator of $a,b\in Alt[X]$.

We will use right-handed bracketing in nonassociative words. For example, $m=x_1\bullet(x_1\bullet x_2)\bullet(x_1\bullet x_2)\bullet x_1$ means $m=((x_1\bullet(x_1\bullet x_2))\bullet(x_1\bullet x_2))\bullet x_1$. We will denote by $T(a)$, $a\in Alt[X]$ the T-ideal in $Alt[X]$ generated by a.

Definition. The *nucleus* (the *associative center*) of the algebra $Alt[X]$ is the set

$$N(Alt[X])=\{a\in Alt[X] \mid (a,Alt[X],Alt[X])=0\},$$

where $(a,b,c)=(ab)c-a(bc)$ is the associator of $a,b,c\in Alt[X]$.

An ideal $I\lhd Alt[X]$ is called a *nuclear ideal* if $I\subseteq N(Alt[X])$, the maximal nuclear ideal is called the *associative kernel* and is denoted by $U(Alt[X])$. A nonzero element $a\in U(Alt[X])$ is called a *nuclear element.*

The *Lie center* of the algebra $Mal[X]$ is the set

$$Z(Mal[X])=\{a\in Mal[X] \mid J(a,Alt[X],Alt[X])=0,$$

where $J(a,b,c)=(a\bullet b)\bullet c+(b\bullet c)\bullet a+(c\bullet a)\bullet b$ is the Jacobian of $a,b,c\in Mal[X]$.

Let us shortly review the main results of this paper. We have constructed the element of the associative kernel of $Alt[X]$ of the minimal degree.

Theorem 1. The element $l=l(x,y)=[x,[x,y]^2]\in U$, $x,y\in X$, where $U=U(ALT[X])$ is the associative kernel of the free alternative algebra $ALT[X]$,and $l=l(x,y)$ has a minimal degree.

Definition. The variety $\mathfrak{M}$ is called *composition* of the varieties $\mathfrak{N}_1$, $\mathfrak{N}_2$ and denoted by $\mathfrak{M}=\mathfrak{N}_1\circ\mathfrak{N}_2$ if $\forall A\in\mathfrak{M}$ $\exists B\lhd A$, that $A/B\in\mathfrak{N}_1$, $B\in\mathfrak{N}_2$.

Let Alt, Ass, $L = Var(l)$ denote the varieties of alternative algebras, associative algebras and alternative algebras with the identity $l = l(x, y)$, respectively. Using $l = l(x, y)$ we obtain decomposition of the variety of the alternative algebras.

Theorem 2. $Alt = Ass \circ L = L \circ Ass$.

We have constructed the element $m = m(x, y) = x \bullet (x \bullet y) \bullet (x \bullet y) \bullet x \in Mal[X]$ which is an analog of $l = l(x, y)$ for Malcev algebras.

Theorem 3. The element $m = m(x, y) = x \bullet (x \bullet y) \bullet (x \bullet y) \bullet x \in Z(Mal[X])$ and $\pi(m) \in T(l)$.

Let Mal, Lie, $M = Var(m)$ denote the varieties of Malcev algebras, Lie algebras and Malcev algebras with the identity $m = m(x, y)$, respectively. Using $m = m(x, y)$ we obtain decomposition of the variety of Malcev algebras.

Theorem 4. $Mal = Lie \circ M = M \circ Lie$.

Note that the centers of the free alternative and Malcev algebras were investigated by M. Slater [2], G. Dorofeev [3], I. Shestakov [4], V. Filippov [5], I. Hentzel and L. Peresi [6]. First examples of decomposition of the varieties Alt and Mal were obtained by V. Filippov [7, 8].

2. THE NUCLEAR ELEMENT $l = l(x, y) = [x, [x, y]^2]$

We will use the next notations in $A = Alt[X]$:

$$a \circ b = \frac{1}{2}(ab + ba),$$
$$aD_{b,c} = a \circ b \circ c - a \circ c \circ b,$$
$$J(a,b,c) = [[a,b],c] + [[b,c],a] + [[c,a],b],$$

where $a, b, c \in A$.

The next identities are well-known in the alternative algebras (see [1]):

$$12aD_{b,c} = J(a,b,c) + 3[a,[b,c]], \tag{1}$$
$$J(a,b,c) = 6(a,b,c), \tag{2}$$
$$[a \circ b, c] = [a, c \circ b] + [b, c \circ a] = [a,c] \circ b + [b,c] \circ a, \tag{3}$$
$$(a^2, b, c) = 2(a,b,c) \circ a = 2(a, b \circ a, c), \tag{4}$$
$$(x \circ y)D_{a,b} = xD_{a,b} \circ y + yD_{a,b} \circ x, \tag{5}$$
$$(a,b,c) \circ [a,b] = (a,b,c \circ [a,b]) = 0, \tag{6}$$
$$([a,b]^2, b, c) = 0, \tag{7}$$
$$([a,b]^2, x, y) \circ [a,b] = 0, \tag{8}$$
$$([x,[x,y]^2], a, b) = 0, \tag{9}$$
$$2[J(x,y,z),t] = J([x,y],z,t) + J([y,z],x,t) + J([z,x],y,t), \tag{10}$$
$$J([a,b],x,y) = [J(a,x,y),b] + [J(b,x,y),a] - 2J(a,b,[x,y]), \tag{11}$$
$$([a,b],b,c) = (a,b,[c,b]) = [b,(a,b,c)]. \tag{12}$$

Definition. The element $n \in N(A)$ is called an J *-annihilator element* of the algebra A if

$$n(A,A,A) = (A,A,A)n = 0.$$

The J *-annihilator* of the algebra A is the set of all J -annihilator elements of A and is denoted by JA.

Lemma 1. JA is a nuclear ideal.

PROOF. It is sufficient to prove that $JA \lhd A$. Let $n \in JA$ and $x \in A$. We only need to show that $nx, xn \in JA$. We have

$$ab = \frac{1}{2}[a,b] + a \circ b, \tag{13}$$

Therefore, it remains to prove that $[n,x], n \circ x \in JA$.

But

$$(n \circ x, a, b) \underset{(4)}{=} (n,a,b) \circ x + (x,a,b) \circ n = 0,$$
$$([n,x],a,b) \underset{(11)}{=} [(n,a,b),x] + [(x,a,b),n] - 2(n,x,[a,b]).$$

Hence, $[n,x], x \circ n \in N(A)$.

Set $d = (a,b,c)$, then

$$d[n,x] \underset{(13)}{=} \frac{1}{2}[d,[n,x]] + d \circ [n,x],$$

and

$$[d,[n,x]] \underset{(2)}{=} 6(d,n,x) - [n,[x,d]] - [x,[d,n]] \underset{(10)}{=} 0,$$
$$d \circ [n,x] \underset{(3)}{=} [n \circ d, x] - [d,x] \circ n \underset{(10)}{=} 0.$$

This gives $d[n,x] = 0$. Similarly, $d[n,x] = 0$. Consequently, $[n,x] \in JA$.

Furthermore

$$d(n \circ x) \underset{(13)}{=} \frac{1}{2}[d, n \circ x] + d \circ (n \circ x) \underset{(3)}{=} \frac{1}{2}([d,n] \circ x + [d,x] \circ d) + nD_{x,d} + n \circ d \circ x \underset{(10),(1)}{=} 0.$$

Similarly, $(n \circ x) \circ d = 0$. Consequently, $n \circ x \in JA$.

The lemma is proved.

Our next goal is to prove that $l = l(x,y) = [x,[x,y]^2]$ is a nuclear element in $A = Alt[X]$, for $x, y \in X$. Note that $l \underset{(9)}{\in} N(A)$ and $l(a,b,c) \underset{(13),(10)}{=} l \circ (a,b,c) \underset{(13),(10)}{=} (a,b,c)l$. Therefore $l \in JA$ if and only if

$$[x,[x,y]^2] \circ (a,b,c) = 0,$$

for $a,b,c \in X$. We will prove it in a few steps.

In the following lemmas 2 and 3, we will construct the intermediate identities.

Lemma 2. In the variety Alt the following identities are valid:

$$[x,y]\circ(x^2,a,b)=-[x,a]\circ(x^2,y,b)\,, \tag{14}$$
$$[x,y]\circ(x^2,a,b)=[x^2,y]\circ(x,a,b)\,, \tag{15}$$
$$([x,a]\circ b,x^2,y)=([x^2,a]\circ b,x,y)\,, \tag{16}$$
$$([y,c],x,y)\circ(x^2,a,b)=([y,c],x^2,y)\circ(x,a,b)\,, \tag{17}$$
$$([x,a]D_{[y,c],x},x,y)=0\,, \tag{18}$$
$$((x,a,b)D_{[y,c],x},x,y)=0\,. \tag{19}$$

PROOF. The identity (14):

$$[x,y]\circ(x^2,a,b)+[x,a]\circ(x^2,y,b)\underset{(6)}{=}-[b,y]\circ(x^2,a,x)+[b,a]\circ(x^2,y,x)=0\,.$$

The identity (15):

$$[x,y]\circ(x^2,a,b)-[x^2,y]\circ(x,a,b)\underset{(3),(4)}{=}2([x,y]\circ(x,a,b\circ x)-[x,y\circ x]\circ(x,a,b))$$
$$\underset{(6)}{=}-2([x,a]\circ(x,y,b\circ x)-[x,a]\circ(x,y\circ x,b))\underset{(4)}{=}0\,.$$

The identity (16):

$$([x,a]\circ b,x^2,y)-([x^2,a]\circ b,x,y)\underset{(4)}{=}(([x,a],x^2,y)-([x^2,a],x,y))\circ b+(b,x^2,y)\circ[x,a]$$
$$-(b,x,y)\circ[x^2,a]\underset{(4),(15)}{=}0\,.$$

The identity (17):

$$([y,c],x,y)\circ(x^2,a,b)-([y,c],x^2,y)\circ(x,a,b)\underset{(12)}{=}(c,[y,x],y)\circ(x^2,a,b)-(c,[y,x^2],y)\circ(x,a,b)$$
$$\underset{(4)}{=}(c,[y,x]\circ(x^2,a,b),y)-(c,[y,x^2]\circ(x,a,b),y)-(c,(x^2,a,b),y)\circ[y,x]+(c,(x,a,b),y)\circ[y,x^2]$$
$$\underset{(15),(6)}{=}((x,(x^2,a,b),y)-(x^2,(x,a,b),y))\circ[y,c]\underset{(4)}{=}0\,.$$

The identity (18):

$$2([x,a]D_{[y,c],x},x,y)\underset{(3),(4)}{=}([x,a]\circ[y,c],x^2,y)-([x^2,a]\circ[y,c],x,y)\underset{(16)}{=}0\,.$$

The identity (19):

$$2((x,a,b)D_{[y,c],x},x,y)\underset{(4)}{=}((x,a,b)\circ[y,c],x^2,y)-((x^2,a,b)\circ[y,c],x,y)\underset{(4)}{=}(((x,a,b),x^2,y)-((x^2,a,b),x,y))\circ[y,c]$$
$$+([y,c],x^2,y)\circ(x,a,b)-([y,c],x,y)\circ(x^2,a,b)\underset{(4),(17)}{=}0\,.$$

The lemma is proved.

Let $f(a,b,c,...)\in Alt[X]$, where $a,b,c\in X$ and $d_a(f)=d_b(f)=d_c(f)$. Set

$$\Delta f=\sum_{\sigma\in S_3}(-1)^{\sigma}f(y_{\sigma(1)},y_{\sigma(2)},y_{\sigma(3)},...)\,,$$

where $y_1=a,y_2=b,y_3=c$.

For example, $\Delta(a,b,c)=6(a,b,c)\underset{(2)}{=}J(a,b,c)$.

Lemma 3. In the variety *Alt* the following identities are valid:

$$\Delta([x,a,b]D_{[y,c],x},x,y)=0, \tag{20}$$

$$\Delta(xD_{[x,a],[x,b]},y,c)=0, \tag{21}$$

PROOF. The identity (20):

$$2\Delta([x,a,b]D_{[y,c],x},x,y)=\Delta(([x,a,b]+[b,x,a])D_{[y,c],x},x,y)=\Delta((J(x,a,b)+[x,[a,b]])D_{[y,c],x},x,y)$$
$$=\Delta(J(x,a,b)D_{[y,c],x},x,y)+\Delta([x,[a,b]]D_{[y,c],x},x,y)\underset{(18),(19)}{=}0.$$

The identity (21):

$$\Delta(xD_{[x,a],[x,b]},y,c)\underset{(3)}{=}\Delta([x^2,a]\circ[x,b],y,c)\underset{(4)}{=}\Delta([x^2,a],y,c)\circ[x,b]-\Delta([x,a],y,c)\circ[x^2,b]$$
$$\underset{(6)}{=}-\Delta(([x^2,a],y,x)\circ[c,b]+([x^2,a],b,c)\circ[x,y]+([x^2,a],b,x)\circ[c,y]-\Delta([x,a],y,x^2)\circ[c,b]$$
$$-([x,a],b,x^2)\circ[c,y]-([x,a],b,c)\circ[x^2,y])\underset{(3),(4)}{=}-\Delta(([x^2,a],b,c)\circ[x,y]-([x,a],b,c)\circ[x^2,y]).$$

From (12) we have

$$\Delta([x^2,a]\circ[x,b],y,c)\underset{(4)}{=}\Delta([x^2,a],y,c)\circ[x,b]-\Delta([x,a],y,c)\circ[x^2,b]\underset{(12)}{=}-\Delta([x^2,y],a,c)\circ[x,b]$$
$$+(x^2,y,[c,a])\circ[x,b]+(x^2,a,[c,y])\circ[x,b]+([x,y],a,c)\circ[x^2,b]-(x,y,[c,a])\circ[x^2,b]$$
$$+(x,a,[c,y])\circ[x^2,b])\underset{(15)}{=}-\Delta(([x^2,y],a,c)\circ[x,b]-([x,y],a,c)\circ[x^2,b])=\Delta(([x^2,y],b,c)\circ[x,a]$$
$$-([x,y],b,c)\circ[x^2,a]).$$

Therefore

$$\Delta(([x^2,a],b,c)\circ[x,y]-([x,a],b,c)\circ[x^2,y])=-\Delta(([x^2,y],b,c)\circ[x,a]-([x,y],b,c)\circ[x^2,a])$$
$$=\Delta(([x^2,b],y,c)\circ[x,a]-([x,b],y,c)\circ[x^2,a]). \tag{22}$$

On the other hand

$$2\Delta(xD_{[x,y],[x,a]},b,c)\underset{(3)}{=}\Delta(([x^2,y]\circ[x,a],b,c)-([x^2,a]\circ[x,y],b,c))\underset{(4)}{=}\Delta(([x^2,y],b,c)\circ[x,a]$$
$$+([x,a],b,c)\circ[x^2,y]-([x^2,a],b,c)\circ[x,y]-([x,y],b,c)\circ[x^2,a])\underset{(6)}{=}\Delta(([x^2,y],b,c)\circ[x,a]$$
$$-([x,y],b,c)\circ[x^2,a])-([x,a],b,x^2)\circ[c,y]-([x,a],y,c)\circ[x^2,b]-([x,a],y,x^2)\circ[c,b]$$
$$+([x^2,a],b,x)\circ[c,y]+([x^2,a],y,c)\circ[x,b]+([x^2,a],y,x)\circ[c,b])\underset{(4)}{=}\Delta(([x^2,y],b,c)\circ[x,a]$$
$$-([x,y],b,c)\circ[x^2,a])-([x,a],y,c)\circ[x^2,b]+([x^2,a],y,c)\circ[x,b]))\underset{(22)}{=}2\Delta(([x^2,y],b,c)\circ[x,a]$$
$$-([x,y],b,c)\circ[x^2,a])).$$

Thus

$$\Delta(xD_{[x,a],[x,b]},y,c)=-\Delta(xD_{[x,a],[x,y]},b,c).$$

From this we conclude that the polynomials $\Delta(xD_{[x,a],[x,b]},y,c)$, $\Delta([x^2,a]\circ[x,b],y,c)$ are skew-symmetric with respect to a,b,c,y. We will denote this property by (s).

Finally,

$$\Delta([x^2,a]\circ[x,b],y,c)\underset{(4)}{=}\Delta(([x^2,a],y,c)\circ[x,b]-([x,a],y,c)\circ[x^2,b])\underset{(12)}{=}\Delta(-([y,a],x^2,c)\circ[x,b]$$
$$+[(a,y,c),x^2]\circ[x,b]+[(x^2,y,c),a]\circ[x,b]+([y,a],x,c)\circ[x^2,b]-[(a,y,c),x]\circ[x^2,b]-[(x,y,c),a]\circ[x^2,b]$$
$$\underset{(15),(10)}{=}\Delta([(x^2,y,c),a]\circ[x,b]-[(x,y,c),a]\circ[x^2,b])\underset{(3),(15)}{=}\Delta([x^2,b,a]\circ(x,y,c)-[x,b,a]\circ(x^2,y,c))$$
$$=\frac{1}{2}\Delta(([x^2,b,a]+[a,x^2,b])\circ(x,y,c)-([x,b,a]+[a,x,b])\circ(x^2,y,c))\underset{(15)}{=}\frac{1}{2}\Delta(J(x^2,b,a)\circ(x,y,c)$$
$$-J(x,b,a)\circ(x^2,y,c))\underset{(2)}{=}3\Delta((x^2,b,a)\circ(x,y,c)-(x,b,a)\circ(x^2,y,c))\underset{(s)}{=}0.$$

The lemma is proved.

Set $k=[x,y]$.

Lemma 4. The algebra $A=Alt[X]$ satisfies the identities

$$(y,a,b)\circ[k^2,x]=0. \tag{23}$$
$$(a,b,c)\circ[k^2,x]=([x,a,y]\circ[x^2,y],b,c)-([x^2,a,y]\circ[x,y],b,c). \tag{24}$$
$$(a,b,c)\circ[k^2,x]=([x,a,b]\circ[y,c],x^2,y)-([x^2,a,b]\circ[y,c],x,y) \tag{25}$$
$$+([x^2,b]\circ[y,c],[x,a],y)-([x,b]\circ[y,c],[x^2,a],y).$$

PROOF. The identity (23):

$$(y,a,b)\circ[k^2,x]\underset{(6),(9)}{=}-(y,x,b)\circ[k^2,a]\underset{(3)}{=}-2(y,x,b)\circ[k,a\circ k]\underset{(3),(6)}{=}2k\circ[(y,x,b),a\circ k]$$
$$\underset{(10)}{=}(([y,x],b,a\circ k)+([x,b],y,a\circ k)+([b,y],x,a\circ k))\circ k\underset{(4)}{=}-\frac{1}{2}(k^2,b,a\circ k)+(([x,b]\circ k,y,a)$$
$$+([x,b]\circ a,y,k)+([b,y]\circ k,x,a)+([b,y]\circ a,x,k))\circ k\underset{(4),(8),(6)}{=}-([x,b]\circ k,y,x)\circ[a,y]-([b,y]\circ k,x,y))\circ[x,a]$$
$$\underset{(6),(9)}{=}0.$$

The identity (24):

$$(a,b,c)\circ[k^2,x]\underset{(22)}{=}-2(y,b,c)\circ[[x,a]\circ[x,y],x]\underset{(4),(9)}{=}-2(y\circ[[x,a]\circ[x,y],x],b,c).$$

But

$$4y\circ[[x,a]\circ[x,y],x]\underset{(3)}{=}2([[x,a],[x^2,y]]+[[x,y],[x^2,a]])\circ y\underset{(3)}{=}[[x,a],[x^2,y^2]]-2[x,a,y]\circ[x^2,y]$$
$$+[[x,y^2],[x^2,a]]+2[x^2,a,y]\circ[x,y]\underset{(3)}{=}2[[x,a],[x,y^2]\circ x]+[[x,y^2],[x^2,a]]+2[x^2,a,y]\circ[x,y]$$
$$-2[x,a,y]\circ[x^2,y]\underset{(3)}{=}[[x^2,a],[x,y^2]]+2[[x,a]\circ[x,y^2],x]+[[x,y^2],[x^2,a]]+2[x^2,a,y]\circ[x,y]$$

$$-2[x,a,y]\circ[x^2,y]=2[[x,a]\circ[x,y^2],x]+2[x^2,a,y]\circ[x,y]-2[x,a,y]\circ[x^2,y]$$

$$\underset{(9)}{\equiv}2([x^2,a,y]\circ[x,y]-[x,a,y]\circ[x^2,y]) \bmod N(A),$$

Hence

$$(a,b,c)\circ[k^2,x]=([x,a,y]\circ[x^2,y],b,c)-([x^2,a,y]\circ[x,y],b,c).$$

The identity (25):

From (7) we have

$$([x,y]^2,b,c)\underset{(7)}{=}-2([x,b]\circ[x,y],y,c)\underset{(7)}{=}2(([c,b]\circ[x,y],y,x)+([x,b]\circ[c,y],y,x))$$
$$\underset{(7)}{=}2([x,b]\circ[c,y],y,x).$$

Therefore

$$([x,y]\circ[z,y],b,c)=([x,b]\circ[y,c],z,y)+([z,b]\circ[y,c],x,y) \quad . \tag{26}$$

Consequently,

$$(a,b,c)\circ[k^2,x]\underset{(24)}{=}([x,a,y]\circ[x^2,y],b,c)-([x^2,a,y]\circ[x,y],b,c)\underset{(26)}{=}([x,a,b]\circ[y,c],x^2,y)$$
$$+([x^2,b]\circ[y,c],[x,a],y)-([x^2,a,b]\circ[y,c],x,y)-([x,b]\circ[y,c],[x^2,a],y)$$

The lemma is proved.

Theorem 1. The element $l=l(x,y)=[x,[x,y]^2]\in U$, $x,y\in X$, where $U=U(ALT[X])$ is an associative kernel of the free alternative algebra $ALT[X]$,and $l=l(x,y)$ has a minimal degree.

PROOF. First of all, we note that there are no elements of degree 4 in $N(A)$ (see [6]), neither in $U\subseteq N$. It remains to prove that

$$(a,b,c)\circ[k^2,x]=0. \tag{27}$$

We have

$$6(a,b,c)\circ[k^2,x]=\Delta(a,b,c)\circ[[x,y]^2,x]\underset{(25)}{=}\Delta(([x,a,b]\circ[y,c],x^2,y)-([x^2,a,b]\circ[y,c],x,y)$$
$$+([x^2,b]\circ[y,c],[x,a],y)-([x,b]\circ[y,c],[x^2,a],y)).$$

But

$$2\Delta[x^2,a,b]=\Delta([x^2,a,b]-[x^2,b,a])=\Delta(J(x^2,a,b)+[x^2,[a,b]])\underset{(3),(4)}{=}2\Delta(J(x,a,b)$$
$$+[x,[a,b]])\circ x=4\Delta[x,a,b]\circ x.$$

Hence

$$\Delta(([x,a,b]\circ[y,c],x^2,y)-([x^2,a,b]\circ[y,c],x,y)\underset{(4)}{=}2\Delta(([x,a,b]\circ[y,c]\circ x,x,y)-([x,a,b]\circ x\circ[y,c],x,y)$$
$$=2\Delta([x,a,b]D_{[y,c],x},x,y)\underset{(20)}{=}0.$$

On the other hand

$$\Delta(([x^2,b]\circ[y,c],[x,a],y)-([x,b]\circ[y,c],[x^2,a],y)) \underset{(4)}{=} \Delta(([x^2,b]\circ[y,c],[x,a],y)$$
$$-([x,b],[x^2,a]\circ[y,c],y)-([y,c],[x^2,a]\circ[x,b],y)) \underset{(3)}{=} \Delta(([x^2,b]\circ[y,c],[x,a],y)$$
$$+([x,a],[x^2,b]\circ[y,c],y)-2([y,c],x\circ[x,a]\circ[x,b],y)) = \Delta(xD_{[x,a],[x,b]},[y,c],y) \underset{(21)}{=} 0 .$$

Consequently, $(a,b,c)\circ[k^2,x]=0$. The theorem is proved.

2. THE COMPOSITION STRUCTURE OF ALTERNETIVE ALGEBRAS.

Let $D(A)$ be the associator ideal of $A=Alt[X]$, $L(A)$ - the T-ideal of A generated by $l=l(x,y)=[x,[x,y]^2]$. It is clear that $D(A)$ is a T-ideal of A.
We will denote by $a\dfrac{d}{dx}$ the operator of the differential substitution $x\to a$. For example,

$$(a,b,c)\frac{d}{dy}[x,[x,y]^2]=2[x,[x,y]\circ[x,(a,b,c)],$$

$$(a,b,c)\frac{d}{dx}[x,[x,y]^2]=[(a,b,c),[x,y]^2]+2[x,[(a,b,c),y]\circ[x,y]].$$

Let $\mathfrak{M}$ be a variety of F -algebras, $\mathfrak{M}[X]$ is the free algebra in $\mathfrak{M}$. Let a set $V\subseteq\mathfrak{M}[X]$. We will denote by $I_{\mathfrak{M}[X]}(V)$ the ideal of $\mathfrak{M}[X]$ generated by V, by $(V)_F$ the F-module of $\mathfrak{M}[X]$ generated by V, by $T_{\mathfrak{M}[X]}(V)$ the T-ideal of $\mathfrak{M}[X]$ generated by V. We will denote by $\mathfrak{M}(V)$ the variety defined by the set of the identities V. Let an algebra $B\in\mathfrak{M}$. Let us denote by $\mathfrak{M}(B)$ the variety generated by the algebra B.

Let $f=f(x_1,...,x_n)$ be a homogenous polynomial in $\mathfrak{M}[X]$ and $b_1,...,b_n\in\mathfrak{M}[X]$. Set $f(x_1,...,x_n)\big|_{x_1=b_1,...,x_n=b_n}=f(b_1,...,b_n)$ for $b_1,...,b_n\in\mathfrak{M}[X]$.

Definition. A F-module $B\subseteq\mathfrak{M}[X]$ is called f*-bimodule* of $\mathfrak{M}[X]$ if $\forall a\in B,\ b_1,...,b_n\in\mathfrak{M}[X]$

$$\left.(a\frac{d}{dx_i}f(x_1,...,x_n))\right|_{x_1=b_1,...,x_n=b_n}=0, \tag{28}$$

for $i=1,...,n$.

It is evident that if an algebra $B\subseteq\mathfrak{M}[X]$ is a f-bimodule then $\forall b_1,...,b_n\in B$ $f(x_1,...,x_n)\big|_{x_1=b_1,...,x_n=b_n}\underset{(28)}{=}0$. Therefore $B\in\mathfrak{M}(f)$.

Proposition 1. Let B is a T-ideal and a f-bimodule of $\mathfrak{M}[X]$ then $\mathfrak{M}=\mathfrak{M}(B)\circ\mathfrak{M}(f)$.

PROOF. Let an algebra $C\in\mathfrak{M}$ and $\varphi:\mathfrak{M}[X]\to C$ is the canonical homomorphism. Then $C/\varphi(B)\in\mathfrak{M}(B)$ and $\varphi(B)\in\mathfrak{M}(f)$. The proposition is proved.

Our goal is to prove that $D(A)$ is a l-bimodule, $L(A)$ is an ass-bimodule, where $ass=ass(x,y,z)=(x,y,z)$. In this case, from Proposition 1it follows Theorem 1.

A set $V \subseteq A$ is called *Lie-closed* if $[V,a] \subseteq V$. It is easy to show that if V is Lie-closed than $I_A(V) = V + V \circ A$ (see [ ]).

Set $L_0(A) = ([x,[x,y]]^2 \mid x,y \in A)_F$, $D_0(A) = ((x,y,z) \mid x,y \in A)_F$.

Now, we will describe the structure of T-ideals $D(A)$ and $l(A)$.

Proposition 2. $D(A) = D_0(A) + D_0(A) \circ A, L(A) = L_0(A) + L_0(A) \circ A$.

PROOF. It suffices to prove that $D_0(A)$ and $L_0(A)$ are Lie-closed. The first equality follows from (2), (10). The second follows from the next identity:

$$[[x,[x,y]^2],z] \underset{(9),(3)}{=} [[x,z],[x,y]^2] + 2[x,[[x,y],z] \circ [x,y]] \underset{(6)}{=} [[x,z],[x,y]^2]$$

$$+2[x,[[x,z],y] \circ [x,y]] + 2[x,[[x,[y,z]] \circ [x,y]],$$

i.e.

$$[[x,[x,y]^2],z] = [[x,z],[x,y]^2] + 2[x,[[x,z],y] \circ [x,y]] + 2[x,[[x,[y,z]] \circ [x,y]]. \quad (29)$$

The proposition is proved.

Let $f(x,x^2,a,...,b),\ g(a,b,...,c) \in Alt[X]$, where $x,a,...,b,...,c \in X$ and $x,a,...,b,...,c$ are different. Set

$$\delta(f) = f(x,x^2,a,...,b) - f(x^2,x,a,...,b),$$
$$\Delta(a,b)(g) = g(a,b,...,c) - g(b,a,...,c),$$
$$(\delta \circ \Delta(a,b))(f) = \delta(\Delta(a,b)(f))$$

For example, we have from (15)

$$\delta([x,y] \circ (x^2,a,b)) = 0, \quad (30)$$

Lemma 4. The algebra $A = Alt[X]$ satisfies the identities:

$$\delta(([[x^2,y],a],b,x)) = 0, \quad (31)$$
$$\delta([(a,b,[x^2,y]),x]) = 0, \quad (32)$$
$$[(a,b,x) \circ [x,y],x] = 0, \quad (33)$$
$$(a,b,c)\frac{d}{dy}[x,[x,y]^2] = 0, \quad (34)$$
$$\delta([x^2,y] \circ (a,b,[x,y])) = 0, \quad (35)$$
$$\delta([x^2,y] \circ [d,[x,y]]) = 0, \quad (36)$$
$$2[(a,b,[x,y]),x] \circ y = [(a,b,[x,y^2]),x], \quad (37)$$
$$2[(a,b,y) \circ [x,y],x] = (a,b,[x,y]^2), \quad (38)$$
$$(a,b,c)\frac{d}{dx}[x,[x,y]^2] = 0, \quad (39)$$
$$\delta([[x^2,z],[x,d]]) = 0, \quad (40)$$

$$\delta([[x^2,z],[x,y]] \circ d) = 0, \quad (41)$$

$$(\delta\circ\Delta(y,z))([x^2,z,y]\circ[x,d]+[x^2,z,d]\circ[x,y])=0\,, \qquad (42)$$

$$(\delta\circ\Delta(y,z))([[x^2,z],[x,d\circ y]])=0\,, \qquad (43)$$

where $d=(a,b,c)$.

PROOF. The identity (31):

$$([[x^2,y],a],b,x)\underset{(3)}{=}2([[x,y]\circ x,a],b,x)\underset{(3)}{=}2([[x,y],a]\circ x,b,x)+2([x,y]\circ[x,a],b,x)$$

$$\underset{(4),(7)}{=}([[x,y],a],b,x^2)\,.$$

The identity (32):

$$2\delta([(a,b,[x^2,y]),x])\underset{(10)}{=}\delta(([a,b],[x^2,y],x)+([b,[x^2,y]],a,x)+([[x^2,y],a],b,x))\underset{(4),(31)}{=}0\,.$$

The identity (33): $\quad 2[(a,b,x)\circ[x,y],x]\underset{(4),(3)}{=}[(a,b,[x^2,y]),x]-[(a,b,[x,y]),x^2]\underset{(32)}{=}0\,.$

The identity (34):

$$(a,b,c)\frac{d}{dy}[x,[x,y]^2]\underset{(10)}{=}[x,[y,x]\circ(([a,b],c,x)+([b,c],a,x)+([c,a],b,x))]\underset{(33)}{=}0\,.$$

The identity (35):

$$\delta([x^2,y]\circ(a,b,[x,y]))\underset{(12)}{=}\delta([x^2,y]\circ(-(a,y,[x,b])+([a,y],b,x)+([a,b],y,x))$$
$$\underset{(30)}{=}-\delta([x^2,y]\circ(a,y,[x,b])\underset{(6)}{=}\delta([a,y]\circ(x^2,y,[x,b]))\underset{(5)}{=}0$$

The identity (36):

$$2\delta([x^2,y]\circ[d,[x,y]])\underset{(10)}{=}\delta([x^2,y]\circ(([a,b],c,[x,y])+([b,c],a,[x,y])+([c,a],b,[x,y]))\underset{(35)}{=}0$$

The identity (37):

$$4[(a,b,[x,y]),x]\circ y\underset{(10)}{=}2(([a,b],[x,y],x)+([b,[x,y]],a,x)+([[x,y],a],b,x)\circ y$$

$$\underset{(3),(4),(6)}{=}([a,b],[x,y^2],x)+2([b,[x,y]]\circ y,a,x)+2([[x,y],a]\circ y,b,x)-2(y,a,x)\circ[b,[x,y]]$$

$$-2(y,b,x)\circ[[x,y],a]\underset{(3),(4),(6)}{=}([a,b],[x,y^2],x)+([b,[x,y^2]],a,x)-2([b,y]\circ[x,y],a,x)$$

$$+([[x,y^2],a],b,x)-2([y,a]\circ[x,y],b,x)+2[x,y]\circ[b,(y,a,x)]+2[x,y]\circ[(y,b,x),a]$$

$$\underset{(10),(7),(11)}{=} 2[(a,b,[x,y^2]),x].$$

The identity (38):

$$2[(a,b,y)\circ[x,y],x] \underset{(3),(4)}{=} [(a,b,[x,y^2]),x]-2[(a,b,[x,y])\circ y,x] \underset{(3)}{=} [(a,b,[x,y^2]),x]-2[(a,b,[x,y]),x]\circ y$$

$$-2(a,b,[x,y])\circ[y,x] \underset{(4),(37)}{=} (a,b,[x,y]^2).$$

The identity (39):

$$(a,b,c)\frac{d}{dx}[x,[x,y]^2] \underset{(10)}{=} [(a,b,c),[x,y]^2]+[x,[(([a,b],c,y)+([b,c],a,y)+([c,a],b,y))\circ[x,y]]$$

$$\underset{(10),(38)}{=} [(a,b,c),[x,y]^2]-[(a,b,c),[x,y]^2]=0.$$

The identity (40):

$$[[x^2,z],[x,J]] \underset{(3)}{=} 2[[x,z]\circ x,[x,J]] \underset{(3)}{=} [[x,z],[x^2,J]]+2[x,[x,z]\circ[x,J]] \underset{(34)}{=} [[x,z],[x^2,J]].$$

The identity (41):

$$[[x^2,z],[x,y]]\circ J \underset{(3)}{=} 2[x,[x,z]\circ[x,y]]\circ J+[[x,z],[x^2,y]]\circ J \underset{(27)}{=} [[x,z],[x^2,y]]\circ J.$$

The identity (42):

$$(\delta\circ\Delta(y,z))([x^2,z,d]\circ[x,y]) = (\delta\circ\Delta(y,z))(\circ J(x^2,z,d)-[z,d,x^2]-[d,x^2,z])\circ[x,y]$$

$$\underset{(30),(10),(3)}{=} (\delta\circ\Delta(y,z))[x,y,z]\circ[d,x^2] \underset{(\delta),(\Delta)}{=} -(\delta\circ\Delta(y,z))[x^2,z,y]\circ[x,d].$$

The identity (43):

$$(\delta\circ\Delta(y,z))([[x^2,z],[x,d\circ y]]) \underset{(3)}{=} (\delta\circ\Delta(y,z))([[x^2,z],[x,d]]\circ y+[[x^2,z],[x,y]]\circ d+[[x^2,z],y]\circ[x,d]$$

$$+[[x^2,z],d]\circ[x,y]) \underset{(40),(41),(42)}{=} 0.$$

The lemma is proved.

Note that from (34), (39) it follows that $D_0(A)=((x,y,z)\mid x,y\in A)_F$ is a l-bimodule. Now we will prove that $D_0(A)\circ A$ is a l-bimodule too.

Write $p(x_1,x_2,x_3\mid y_1,y_2)=\sum_{\sigma\in S_3}[x_{\sigma(1)},[x_{\sigma(2)},y_1]\circ[x_{\sigma(3)},y_2]]$, for $x_1,x_2,x_3,y_1,y_2\in A$. Note that $p(x_1,x_2,x_3\mid y_1,y_2)$ is an identity in the Cayley–Dickson algebra. Let us list the elementary properties of the polynomial $p(x_1,x_2,x_3\mid y_1,y_2)$.

Proposition 3. The algebra $A = Alt[X]$ satisfies the identities:

$$\forall \sigma \in S_3 \; p(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)} \mid y_2, y_1) = p(x_1, x_2, x_3 \mid y_1, y_2),$$

$$p(x,x,x \mid y,y) = 6l(x,y) = 6[x,[x,y]^2],$$

$$p(x,x,x \mid x,y) = p(y,x,x \mid x,x) = 0, \quad (44)$$

$$p(z,x,y \mid x,x) + p(z,x,x \mid y,x) = 0, \quad (45)$$

$$p(z,x,x \mid y,y) + 4p(z,x,y \mid x,y) + p(z,y,y \mid x,x) = 0, \quad (46)$$

$$P(z^2,x,x \mid y,y) = p_1 + p_2 + p_3 - 2p_4 + 6p_5 + 2p_6 - 4p_7 + 2p_8 + 2p_9 - 4p_{10}, \quad (47)$$

where

$$p_1 = p(z,z,y \mid x^2,y),\; p_2 = p(z,z,y \mid x\circ y,x),\; p_3 = p(z,z,x \mid y\circ x,y),\; p_4 = p(z,x,y \mid y\circ z,x),$$

$$p_5 = p(z,x,y \mid x\circ z,y),\; p_6 = p(z,x^2,y \mid z,y),\; p_7 = p(z,x\circ y,x \mid z,y),\; p_8 = p(z,x\circ z,x \mid y,y),$$

$$p_9 = p(z,y^2,x \mid z,x),\; p_{10} = p(z,y\circ z,x \mid x,y).$$

PROOF. The identity (44) is trivial. The identities (45), (46) are the linearization of (44). The elegant identity (47) is a polynomial on three generators. It is easy to check that (47) is an identity in the Cayley–Dickson algebra; so is in the free associative algebra. By A. Iltyakov's results [9], it is an identity in $A = Alt[X]$. The proposition is proved.

Lemma 5. The F-module $D_0(A)\circ A$ is a l-bimodule and the next identities are valid in A.

$$p(x,x,x \mid d\circ y,y) = 0, \quad (48)$$
$$p(x,x,x \mid d\circ z,y) = 0, \quad (49)$$
$$p(d\circ x,x,x \mid y,y) = p(d\circ y,x,x \mid y,y) = 0, \quad (50)$$
$$p(d\circ z,x,x \mid y,y) = 0, \quad (51)$$

where $d = (a,b,c)$.

PROOF. It is clear that from (49), (51) it follows that $D_0(A)\circ A$ is a l-bimodule.
The identity (48):

$$2[x,[x,d\circ y]\circ[x,y]] \underset{(3)}{=} \delta([[x^2,d\circ y],[x,y]]) \underset{(3)}{=} \delta(([[x^2,d]\circ y,[x,y]]) + ([[x^2,y]\circ d,[x,y]]))$$

$$\underset{(3)}{=} \delta(\frac{1}{2}[[x^2,d],[x,y^2]] + [y,[x^2,d]\circ[x,y]] + [x^2,y]\circ[d,[x,y]] + [[x^2,y],[x,y]]\circ d) \underset{(40),(30),(36),(41)}{=} 0.$$

The identity (49):

$$2[x,[x,d\circ z]\circ[x,y]] \underset{(3)}{=} \delta([[x^2,y],[x,d\circ z]]) \underset{(48)}{=} \frac{1}{2}(\delta\circ\Delta(y,z))([[x^2,y],[x,d\circ z]]) \underset{(43)}{=} 0.$$

The identity (50):

$$[x\circ d,[x,y]^2]+2[x,[x\circ d,y]\circ[x,y]] \underset{(3),(49),(27)}{=} x\circ[d,[x,y]^2]-2[y,[x\circ d,x]\circ[x,y]]$$

$$\underset{(3)}{=} x\circ[d,[x,y]^2]-[y,[d,x^2]\circ[x,y]] \underset{(34),(3)}{=} x\circ[d,[x,y]^2]+2x\circ[x,[d,x]\circ[x,y]] \underset{(39)}{=} 0,$$

$$[y\circ d,[x,y]^2]+2[x,[y\circ d,y]\circ[x,y]] \underset{(3),(27)}{=} y\circ[d,[x,y]^2]+[x,[d,y^2]\circ[x,y]] \underset{(34)}{=} y\circ[d,[x,y]^2]$$

$$-[y^2,[d,x]\circ[x,y]] \underset{(3)}{=} y\circ[d,[x,y]^2]+2y\circ[y,[x,d]\circ[x,y]] \underset{(39)}{=} 0.$$

The identity (51):

$$p(z\circ d,x,x\,|\,y,y) \underset{(47),(49),(34),(39)}{=} p(z,d\circ x,x\,|\,y,y)-2p(z,d\circ y,x\,|\,x,y) \underset{(50)}{=} -\frac{1}{2}p(z\circ d,x,x\,|\,y,y)$$

$$+2p(y,d\circ z,x\,|\,x,y)+2p(z,d\circ x,y\,|\,x,y) \underset{(50)}{=} -\frac{1}{2}p(z\circ d,x,x\,|\,y,y)+2p(y,d\circ z,x\,|\,x,y)$$

$$-p(z,d\circ y,y\,|\,x,x) \underset{(50)}{=} -\frac{1}{2}p(z\circ d,x,x\,|\,y,y)+2p(y,d\circ z,x\,|\,x,y)+\frac{1}{2}p(z\circ d,y,y\,|\,x,x).$$

Hence

$$3p(z\circ d,x,x\,|\,y,y)=4p(d\circ z,x,y\,|\,x,y)+p(z\circ d,y,y\,|\,x,x).$$

But

$$p(z\circ d,x,x\,|\,y,y) \underset{(46)}{=} -4p(d\circ z,x,y\,|\,x,y)-p(z\circ d,y,y\,|\,x,x).$$

Consequently,

$$p(z\circ d,x,x\,|\,y,y)=0.$$

The lemma is proved.

Lemma 6. $D(A)$ is a l-bimodule, $L(A)$ is an ass-bimodule.

PROOF. From the lemmas 4, 5 it follows that $D(A)$ is a l-bimodule. From proposition 2, the identities (9), (4), (27) it follows that $L(A)$ is an ass-bimodule. The lemma is proved.

Theorem 2. $Alt = Ass\circ L = L\circ Ass$.

PROOF. From the lemma 6 and the proposition 1 it follows that $Alt = Ass\circ L = L\circ Ass$. The theorem is proved.

3. THE COMPOSITION STRUCTURE OF MALCEV ALGEBRAS.

Now we will construct the element $m = m(x,y) = x\bullet(x\bullet y)\bullet(x\bullet y)\bullet x \in Mal[X]$ which is an analog of $l = l(x,y)$ for Malcev algebras.

We have

$$4[x,[x,y]^2]\circ x \underset{(3)}{=} 4[x,x\circ[x,y]^2] = 4[x,[x,y]D_{[x,y],x}]+4[x,([x,y]\circ x)\circ[x,y]]$$

$$\underset{(3)}{=}[x,[[x,y],[[x,y],x]]]+4[x,[x,y\circ x]\circ[x,y]],$$

where $aD_{b,c}=(a\circ b)\circ c-(a\circ c)\circ b$ is a Jordan differentiation.

Hence,

$$[x,[[x,y],[[x,y],x]]]=4[x,[x,y]^2]\circ x-4[x,[x,y\circ x]\circ[x,y]]\in T(l).$$

Set

$$m=m(x,y)=-x\bullet((x\bullet y)\bullet((x\bullet y)\bullet x))=x\bullet(x\bullet y)\bullet(x\bullet y)\bullet x\in Mal[X],$$

where $x,y\in X$, $M=Mal[X]$ is the free Malcev algebra.

Lemma 7. The algebra $M=Mal[X]$ satisfies the identities:

$$J(m(x,y),a,b)=0, \quad (52)$$

$$J(a,b,c)\frac{d}{dx}m(x,y)=0, \quad (53)$$

$$J(a,b,c)\frac{d}{dy}m(x,y)=0. \quad (54)$$

PROOF. It is easy to check that

$$J(x\bullet(x\bullet y)\bullet(xy),x,a)\underset{(12)}{=}0. \quad (55)$$

Hence

$$J(m(x,y),a,b)\underset{(11),(55)}{=}0.$$

The identities (53), (54) can be proved in the same manner as (52). The lemma is proved.

Remark. In difference from an alternative case, the identities (52)-(54) are easily checked by computer.

From the lemma 7 and the proposition 1 it follows the theorem 3, 4.

Theorem 3. The element $m=m(x,y)=x\bullet(x\bullet y)\bullet(x\bullet y)\bullet x\in Z(Mal[X])$ and $\pi(m)\in T(l)$.

Theorem 4. $Mal=Lie\circ M=M\circ Lie$.

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