# Commuting $U$-Operators and Nondegenerate Imbeddings of Jordan Systems

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**Abstract**: Over an arbitrary ring of scalars, we build a Jordan algebra $J$ having two elements $x,y\in J$ such that $x\circ y=0$, but their $U$-operators $U_x,U_y$ do not commute. This shows that nondegeneracy is a necessary condition in the main theorem of "Commuting $U$-Operators in Jordan Algebras" by J. A Anquela, T. Cortés, and H. P. Petersson (2014, *Trans. Amer.Math. Soc.* **366**, 5877-5902). As a consequence, we obtain examples of Jordan systems over arbitrary rings of scalars that cannot be imbedded in nondegenerate systems.

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## Introduction

In [5], the following result is proved: if $J$ is a nondegenerate Jordan algebra and $x, y \in J$ satisfy $x \circ y = 0$, then the $U$-operators $U_x$ and $U_y$ commute. The interest of this question mainly comes from the connection between Moufang sets and (quadratic) Jordan division rings established by De Medts and Weiss [10] in 2006.

Recently, in [24], the need of nondegeneracy in the main result of [5] is shown by providing an example of linear Jordan algebra, having two elements with zero product and non-commuting $U$-operators.

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On the other hand, the problem of regular imbeddings of algebraic systems has a long tradition in the literature. In 1930, Van der Waerden [25, Section 13], poses the question of whether a non commutative ring without zero divisors imbeds in a skew field. This question is answered negatively by Malcev [16], while Cohn [9] shows that any ring without zero divisors imbeds in a simple ring without zero divisors. In 1963, Bokut [7] proves that every ring whose additive group is torsion free or of prime exponent can be imbedded in a simple ring. In a similar fashion [2, 3], associative systems with a sufficiently regular module structure are shown to imbed in primitive systems with simple heart.

The possibility of imbedding Jordan systems in primitive systems with simple heart is studied in [4]. Using the corresponding results in the associative setting [2, 3, 7], imbedding theorems are established for special Jordan systems, but there exist exceptional Jordan systems which cannot be imbedded in merely nondegenerate systems. The examples exhibited in [4] are either based on Jacobson Counterexample [11, ex. 3, p. 12], or in theorems on the existence of absolute zero divisors in free Jordan systems which are due to Medvedev [21] and Zelmanov [26, 27, 28]. In the first case, the ring of scalars $\Phi$ should have characteristic two in the sense that $2\Phi = 0$, and, in the second case, $\Phi$ should be a field of characteristic either zero or quite big.

In this paper, we extend the construction of [24] to arbitrary rings of scalars, and, as a consequence, we find examples of Jordan systems which cannot be imbedded in a nondegenerate system without restrictions on the rings of scalars.

The paper is divided into four sections, plus a preliminary one devoted to listing some basic notions and properties. In the first section we study the free special Jordan algebra inside the Jordan algebra of hermitian elements of the free associative algebra. The information here obtained on the rank of certain submodules of multilinear elements is used in the second section to construct, over an arbitrary ring of scalars, a Jordan algebra $J$ having two elements $x, y \in J$ with $x \circ y = 0$ and $U_x U_y \neq U_y U_x$. In the third section, we show that the example built in the previous one cannot be imbedded in a nondegenerate Jordan algebra. We also give similar examples of Jordan pairs and Jordan triple systems. Finally, in the fourth section, we show how the construction given in Section 2 fits in the general framework of the problem on associative and Jordan ideals studied by Zelmanov in [29], which leads us to some open problems.

## 0. Preliminaries

**0.1** We will deal with associative and Jordan systems (algebras, pairs, and triple systems) over an arbitrary ring of scalars $\Phi$. In particular, we will NOT assume $1/2 \in \Phi$.

The reader is referred to [12, 15, 19, 20] for basic results, notation, and terminology, though we will stress some notions.

— When dealing with an associative system, the (associative) products will be denoted by juxtaposition.

— Given a Jordan algebra $J$, its products will be denoted by $x^2$, $U_xy$, for $x, y \in J$. They are quadratic in $x$ and linear in $y$ and have linearizations denoted by $x \circ y = V_xy$ and $U_{x,z}y = \{x, y, z\} = V_{x,y}z$, respectively.

— For a Jordan pair $V = (V^+, V^-)$, we have products $Q_xy \in V^\sigma$, for any $x \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$, with linearizations $Q_{x,z}y = \{x, y, z\} = D_{x,y}z$.

— A Jordan triple system $J$ is given by its products $P_xy$, for any $x, y \in J$, with linearizations denoted by $P_{x,z}y = \{x, y, z\} = L_{x,y}z$.

**0.2** (i) A Jordan algebra gives rise to a Jordan triple system by simply forgetting the squaring and letting $P = U$. By doubling any Jordan triple system $T$ one obtains the *double Jordan pair* $V(T) = (T, T)$ with products $Q_xy = P_xy$, for any $x, y \in T$. From a Jordan pair $V = (V^+, V^-)$ one can get a (*polarized*) Jordan triple system $T(V) = V^+ \oplus V^-$ by defining $P_{x^+ \oplus x^-}(y^+ \oplus y^-) = Q_{x^+}y^- \oplus Q_{x^-}y^+$ [15, 1.13, 1.14].

(ii) An associative system $R$ gives rise to a Jordan system $R^{(+)}$ by *symmetrization*: over the same $\Phi$-module (the same pair of $\Phi$-modules, in the pair case), we define $x^2 = xx$, $U_xy = xyx$, for any $x, y \in R$ in the case of algebras, $P_xy = xyx$ in the case of triple systems, and $Q_{x^\sigma}y^{-\sigma} = x^\sigma y^{-\sigma}x^\sigma$, $\sigma = \pm$, in the pair case.

**0.3** A Jordan system $J$ is called *special* if it is a subsystem of $R^{(+)}$ for some associative system $R$. Otherwise $J$ is said to be *exceptional*.

**0.4** An *absolute zero divisor* of a Jordan algebra (resp., triple system) $J$ is an element $x \in J$ such that $U_xJ = 0$ (resp., $P_xJ = 0$). An absolute zero divisor in a Jordan pair $(V^+, V^-)$ is any element $x \in V^\sigma$ such that $Q_xV^{-\sigma} = 0$. A Jordan system is said to be *nondegenerate* if it has no nonzero absolute zero divisors.

**0.5** We recall that the *McCrimmon* or *nondegenerate radical* (also called *small radical* in [15, 4.5]) $\mathrm{Mc}(J)$ of a Jordan system $J$ is the smallest ideal of $J$ which produces a nondegenerate quotient. A Jordan system $J$ is said to be *McCrimmon radical* if $J = \mathrm{Mc}(J)$.

**0.6** We recall the following identities valid for arbitrary Jordan algebras which will be needed in the sequel:

(i) $(x \circ y) \circ z = \{x, y, z\} + \{y, x, z\}$,

(ii) $U_xy \circ z = \{x \circ z, y, x\} - U_x(y \circ z)$,

(iii) $\{x, y, z\} \circ t = \{x \circ t, y, z\} - \{x, y \circ t, z\} + \{x, y, z \circ t\}$,

(iv) $\{x^2, y, z\} = \{x, x, y \circ z\} - \{x^2, z, y\}$,

(v) $\{x \circ y, z, t\} = \{x, y, z \circ t\} + \{y, x, z \circ t\} - \{x \circ y, t, z\}$.

Indeed, (i, ii, iv) follow from Macdonald's Theorem [14], (iii) is the linearization of (ii), whereas (v) is the linearization of (iv).

**0.7** A Jordan algebra $J$ is said to be *unital* if it contains an element $1 \in J$, called the *unit* of $J$, such that

$$x^2 = U_x 1, \quad U_1 x = x, \tag{1}$$

for any $x \in J$. In a unital algebra, (1) determines the unit univocally.

**0.8** Given a Jordan algebra $J$ over $\Phi$, let $\Phi 1$ be a free $\Phi$-module of rank one spanned by 1. Then the direct sum $\widehat{J} := \Phi 1 \oplus J$ becomes a unital Jordan algebra with unit 1 called the (*free*) *unital hull* or (*free*) *unitization* of $J$ with squaring

$$(\lambda 1 + x)^2 = \lambda^2 1 + \big(2\lambda x + x^2\big)$$

and $U$-operator

$$U_{\lambda 1 + x}(\mu 1 + y) = \lambda^2 \mu 1 + (U_x y + \mu x^2 + 2\lambda\mu x + \lambda(x \circ y) + \lambda^2 y),$$

for any $\lambda, \mu \in \Phi$, $x, y \in J$. The algebra $J \subseteq \widehat{J}$ becomes an ideal of $\widehat{J}$ [18].

**0.9** *Local algebras* of Jordan systems are introduced in [22]:

— Given a Jordan triple system $J$, the *homotope* $J^{(a)}$ of $J$ at $a \in J$ is the Jordan algebra over the same $\Phi$-module as $J$ with products $x^{(2,a)} = x^2 = P_x a$, $U_x^{(a)} y = U_x y = P_x P_a y$, for any $x, y \in J$. The subset $\mathrm{Ker} a = \mathrm{Ker}_J a = \{x \in J \mid P_a x = P_a P_x a = 0\}$ is an ideal of $J^{(a)}$ and the quotient $J_a = J^{(a)}/\mathrm{Ker} a$ is called *the local algebra of $J$ at $a$*. When $J$ is nondegenerate, $\mathrm{Ker} a = \{x \in J \mid P_a x = 0\}$.

— Given a Jordan pair $V$, the *homotope* $V^{\sigma(a)}$ of $V$ at $a \in V^{-\sigma}$ ($\sigma = \pm$) is the Jordan algebra over the same $\Phi$-module as $V^\sigma$ with products $x^{(2,a)} = x^2 = Q_x a$, $U_x^{(a)} y = U_x y = Q_x Q_a y$, for any $x, y \in J$. The subset $\mathrm{Ker} a = \mathrm{Ker}_V a = \{x \in V^\sigma \mid Q_a x = Q_a Q_x a = 0\}$ is an ideal of $V^{\sigma(a)}$ and the quotient $V_a^\sigma = V^{\sigma(a)}/\mathrm{Ker} a$ is called *the local algebra of $V$ at $a$*. When $V$ is nondegenerate, $\mathrm{Ker} a = \{x \in V^{-\sigma} \mid Q_a x = 0\}$.

**0.10** Since $\Phi$ is an associative, commutative, unital ring, given a free $\Phi$-module $W$ of finite rank $n$, all bases of $W$ have cardinality $n$, and, moreover, any spanning set $S \subseteq W$ with cardinality less than or equal to $n$, has necessarily cardinality $n$ and is indeed a basis of $W$ [13, Prop. 7.20].

## 1. The Free Special Jordan Algebra

**1.1** Given a set $X$, FAss$[X]$ will denote the free associative algebra over $X$. It is a free $\Phi$-module with a basis consisting of the *associative algebra monomials* or *words* $x_{i_1} \cdots x_{i_n}$, for arbitrary $x_{i_1}, \ldots, x_{i_n} \in X$. The algebra FAss$[X]$ is $\mathbf{Z}$-graded by the *degree* or *length* of words, and also $\mathbf{Z}^X$-graded by the *composition* of words.

**1.2** The free associative algebra FAss$[X]$ is equipped with a natural or standard involution $*$, defined by fixing all the elements in $X$, hence "reversing" words, i.e.,

$$(x_{i_1} \cdots x_{i_n})^* = x_{i_n} \cdots x_{i_1}.$$

For an element $u \in \mathrm{FAss}[X]$, the *trace* of $u$ is given by $\{u\} = u + u^*$. The set $H(\mathrm{FAss}[X], *)$ of $*$-symmetric elements of FAss$[X]$ is a subalgebra of $\mathrm{FAss}[X]^{(+)}$ containing $X$.

Notice that, when dealing with symmetric elements, $a, b, c \in H(\mathrm{FAss}[X], *)$, the trace $\{abc\} = abc + (abc)^* = abc + c^*b^*a^* = abc + cba$ of the associative product $abc$ coincides with the Jordan product $\{a, b, c\}$ defined in (0.1) and (0.2)(ii).

**1.3** It is not difficult to prove that $H(\mathrm{FAss}[X], *)$ is a free $\Phi$-module with basis consisting of all $*$-symmetric associative words together with the traces of associative words which are not $*$-symmetric. When $1/2 \in \Phi$, a basis of $H(\mathrm{FAss}[X], *)$ is given just by the set of the traces of all associative words ($*$-symmetric or not).

**1.4** Following [20], a trace of an associative monomial of length $n$ in $X$ will be called an *$n$-tad* (*tetrad* when $n = 4$) in $X$.

**1.5** The *free special Jordan algebra* FSJ$[X]$ on $X$ is the subalgebra of $\mathrm{FAss}[X]^{(+)}$ generated by $X$. By (1.2),

$$\mathrm{FSJ}[X] \subseteq H(\mathrm{FAss}[X], *).$$

**1.6** Notice that both FSJ$[X]$ and $H(\mathrm{FAss}[X], *)$ are graded submodules of FAss$[X]$.

From now on, in this section, we will assume that $X$ is a set of four elements, namely $x_1$, $x_2$, $x_3$, $x_4$, and study the $\Phi$-submodule ML(FSJ$[X]$) of FSJ$[X]$ consisting of all multilinear elements of degree four.

For an arbitrary $\Phi$-submodule $W$ of FAss$[X]$, $\equiv_W$ will denote congruence modulo $W$ in FAss$[X]$ ($u \equiv_W v \Leftrightarrow u - v \in W$).

**1.7** Lemma. *The $\Phi$-module* ML(FSJ$[X]$) *is spanned by the elements of the form* $\{a \circ b, c, d\}$, *and* $\{a, b \circ c, d\}$, *where $a, b, c, d$ are pairwise different elements in $X$.*

Proof: Just notice that

$$((a\circ b)\circ c)\circ d =_{(0.6)(\mathrm{i})} \{a\circ b, c, d\} + \{c, a\circ b, d\},$$

$$(a\circ b)\circ(c\circ d) =_{(0.6)(\mathrm{i})} \{a, b, c\circ d\} + \{b, a, c\circ d\} = \{c\circ d, b, a\} + \{c\circ d, a, b\},$$

$$\begin{aligned}\{a, b, c\}\circ d &=_{(0.6)(\mathrm{iii})} \{a\circ d, b, c\} - \{a, b\circ d, c\} + \{a, b, c\circ d\}\\ &= \{a\circ d, b, c\} - \{a, b\circ d, c\} + \{c\circ d, b, a\}. \blacksquare\end{aligned}$$

Let $M$ be the $\Phi$-submodule of $\mathrm{FSJ}[X]$ spanned by all elements of the form $\{a\circ b, c, d\}$, where $a, b, c, d$ are pairwise different elements in $X$.

**1.8** Lemma. *The $\Phi$-module $M$ is spanned by a set $S_M \subseteq M$ of cardinality less than or equal to 9.*

Proof: Taking into account that $a\circ b = b\circ a$ by (0.1), $M$ is spanned by a set $S$ of at most 12 elements $\{a\circ b, c, d\}$ given by choosing arbitrary $c$ and $d$ in $X$ with $c \neq d$.

By (0.6)(v) and the fact the $\{x, y, z\} = \{z, y, x\}$ (0.1), we have that

$$\{a\circ b, c, d\} = \{c\circ d, b, a\} + \{c\circ d, a, b\} - \{a\circ b, d, c\},$$

which allows us to express $\{x_1\circ x_2, x_3, x_4\}$, $\{x_1\circ x_3, x_2, x_4\}$, and $\{x_1\circ x_4, x_2, x_3\}$ as a $\Phi$-linear combination (with coefficients 1 or $-1$) of the remaining elements of $S$. Thus, we just need to take

$$S_M = S \setminus \big\{\{x_1\circ x_2, x_3, x_4\}, \{x_1\circ x_3, x_2, x_4\}, \{x_1\circ x_4, x_2, x_3\}\big\}. \blacksquare$$

**1.9** Proposition. *The $\Phi$-module* $\mathrm{ML}(\mathrm{FSJ}[X])$ *is spanned by a set* $S_{\mathrm{ML}(\mathrm{FSJ}[X])}$ *of cardinality less than or equal to 11.*

Proof: In the following equality, we will make calculations in $H(\mathrm{FAss}[X], *) \subseteq \mathrm{FAss}[X]$, allowing elements outside $\mathrm{FSJ}[X]$:

$$\begin{aligned}\{a, b\circ c, d\} &=_{(0.6)(\mathrm{iii})} \{a\circ b, c, d\} + \{a, c, b\circ d\} - \{a, c, d\}\circ b\\ &\equiv_M -\{a, c, d\}\circ b = -\{acdb\} - \{dcab\}\\ &= -\{a\circ c, d, b\} + \{cadb\} - \{d\circ c, a, b\} + \{cdab\}\\ &\equiv_M \{cadb\} + \{cdab\} = \{c, a\circ d, b\} = \{b, a\circ d, c\}.\end{aligned} \tag{1}$$

Using (1.7), (1.8), (1) and the fact that, the expression $\{a, b\circ c, d\}$ remain the same when we exchange $a$ and $d$, or $b$ and $c$, the $\Phi$-module $\mathrm{ML}(\mathrm{FSJ}[X])$ is spanned by the set

$$S = S_M \cup \big\{\{x_1, x_2\circ x_3, x_4\}, \{x_1, x_2\circ x_4, x_3\}, \{x_1, x_3\circ x_4, x_2\}\big\}.$$

Let $S_{\mathrm{ML(FSJ}[X])} = S_M \cup \big\{\{x_1, x_2 \circ x_3, x_4\}, \{x_1, x_2 \circ x_4, x_3\}\big\} = S \setminus \big\{\{x_1, x_3 \circ x_4, x_2\}\big\}$, and

$$N = \Phi < S_{\mathrm{ML(FSJ}[X])} >= M + \Phi < \{x_1, x_2 \circ x_3, x_4\}, \{x_1, x_2 \circ x_4, x_3\} > . \qquad (2)$$

We just need to prove that $\{x_1, x_3 \circ x_4, x_2\} \in N$, i.e., $\{x_1, x_3 \circ x_4, x_2\} \equiv_N 0$, but

$$\begin{aligned}
&\{x_1, x_3 \circ x_4, x_2\} = \{x_1x_3x_4x_2\} + \{x_1x_4x_3x_2\}\\
&\quad = \{x_1 \circ x_3, x_4, x_2\} - \{x_3x_1x_4x_2\} + \{x_1 \circ x_4, x_3, x_2\} - \{x_4x_1x_3x_2\}\\
&\quad \equiv_N -\{x_3x_1x_4x_2\} - \{x_4x_1x_3x_2\}\\
&\quad = -\{x_3, x_1 \circ x_4, x_2\} + \{x_3x_4x_1x_2\} - \{x_4, x_1 \circ x_3, x_2\} + \{x_4x_3x_1x_2\}\\
&\quad \equiv_{N\ (1),(2)} - \{x_1, x_2 \circ x_3, x_4\} + \{x_3x_4x_1x_2\} - \{x_1, x_2 \circ x_4, x_3\} + \{x_4x_3x_1x_2\}\\
&\quad \equiv_N \{x_3x_4x_1x_2\} + \{x_4x_3x_1x_2\} = \{x_3 \circ x_4, x_1, x_2\} \equiv_N 0. \blacksquare
\end{aligned}$$

**1.10** Recall that the $\Phi$-submodule $\mathrm{ML}(H(\mathrm{FAss}[X], *))$ of multilinear elements of degree 4 of $H(\mathrm{FAss}[X], *)$ is a free module of rank 12 with basis $B$ consisting of the 12 different multilinear tetrads in the elements of $X$:

$$\begin{aligned}
B = \big\{&\{x_1x_2x_3x_4\}, \{x_1x_2x_4x_3\}, \{x_1x_3x_2x_4\}, \{x_1x_3x_4x_2\},\\
&\{x_1x_4x_2x_3\}, \{x_1x_4x_3x_2\}, \{x_2x_1x_3x_4\}, \{x_2x_1x_4x_3\},\\
&\{x_3x_1x_2x_4\}, \{x_3x_1x_4x_2\}, \{x_4x_1x_2x_3\}, \{x_4x_1x_3x_2\}\big\}.
\end{aligned}$$

**1.11** Given a tetrad $\{x_{i_1}x_{i_2}x_{i_3}x_{i_4}\}$ and a permutation $\sigma \in S_4$, we have that

$$\{x_{i_1}x_{i_2}x_{i_3}x_{i_4}\} \equiv_{\mathrm{ML(FSJ}[X])} \pm\{x_{i_{\sigma(1)}}x_{i_{\sigma(2)}}x_{i_{\sigma(3)}}x_{i_{\sigma(4)}}\}. \qquad (1)$$

Indeed, we can proceed as in the proof of [11, Cohn's Theorem on page 8]: the assertion is true for $\sigma = (1,2), (2,3), (3,4)$ since

$$\{abcd\} + \{bacd\} = \{a \circ b, c, d\} \in \mathrm{ML(FSJ}[X]),$$

$$\{abcd\} + \{acbd\} = \{a, b \circ c, d\} \in \mathrm{ML(FSJ}[X]),$$

$$\{abcd\} + \{abdc\} = \{a, b, c \circ d\} \in \mathrm{ML(FSJ}[X]),$$

for any pairwise different $a, b, c, d \in X$; but this implies (1) since $(1,2), (2,3), (3,4)$ generate the whole symmetric group $S_4$.

**1.12** PROPOSITION. *The* $\Phi$*-module* $\mathrm{ML(FSJ}[X])$ *is free of rank 11. Hence. any spanning set of 11 elements of* $\mathrm{ML(FSJ}[X])$ *is a basis of* $\mathrm{ML(FSJ}[X])$.

PROOF: Let $S_{\mathrm{ML(FSJ}[X])}$ be a spanning set of ML(FSJ$[X]$) of cardinality at most 11 (1.9). By (1.10) and (1.11), the set $C = S_{\mathrm{ML(FSJ}[X])} \cup \big\{\{x_1x_2x_3x_4\}\big\}$ is a spanning set for ML($H$(FAss$[X], *$)). But ML($H$(FAss$[X], *$)) is a free $\Phi$-module of rank 12 (1.10), and $C$ has cardinality at most 12, hence we can use (0.10) to obtain that $C$ has exactly 12 elements and it is a basis of ML($H$(FAss$[X], *$)). In particular, the elements of $S_{\mathrm{ML(FSJ}[X])}$ are linearly independent, and $S_{\mathrm{ML(FSJ}[X])}$ is a basis of ML(FSJ$[X]$), which is then a free module of rank 11. The last assertion follows from (0.10). ■

**1.13** In the proof of (1.12), the fact that $C$ is a basis implies, in particular, that

$$\{x_1x_2x_3x_4\} \notin \Phi < S_{\mathrm{ML(FSJ}[X])} >= \mathrm{ML(FSJ}[X]).$$

Moreover, using (1.11), for any $\sigma \in S_4$,

$$\{x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)}\} \notin \Phi < S_{\mathrm{ML(FSJ}[X])} >= \mathrm{ML(FSJ}[X]).$$

## 2. A Counter-Example

Throughout this section we fix $X$ to be the set of three elements $x$, $y$, $z$.

The following results are devoted to extending the construction given in [24, Theorem 1] to arbitrary rings of scalars.

By straightforward computation we have:

**2.1** LEMMA. *In* FAss$[X]$,

$$[U_y, U_x]z = U_yU_xz - U_xU_yz = \{(x \circ y)zxy\} - U_{x\circ y}z. \ ■$$

**2.2** Let $I$ be the ideal of FSJ$[X]$ generated by $x \circ y$, and $J = \mathrm{FSJ}[X]/I$. For an element $u \in \mathrm{FSJ}[X]$, we will write $\bar{u} = u + I \in J$.

**2.3** THEOREM. *The elements $\bar{x}, \bar{y} \in J$ satisfy $\bar{x} \circ \bar{y} = 0$, but $U_{\bar{x}}U_{\bar{y}} \neq U_{\bar{y}}U_{\bar{x}}$.*

PROOF: Clearly $\bar{x} \circ \bar{y} = \overline{x \circ y} = 0$, since $x \circ y \in I$. Now, by (2.1), it is enough to prove that $\{(x \circ y)zxy\} \notin I$.

Let us assume, on the contrary, that

$$\{(x \circ y)zxy\} \in I. \tag{1}$$

By [23, 1.9], $I$ is the outer hull of $\Phi < x \circ y > + U_{x\circ y}\widehat{\mathrm{FSJ}[X]}$, where $\widehat{\ }$ denotes the unital hull (0.8). Thus, (1) implies that we can find a Jordan polynomial $f(x, y, z, t) \in$

FSJ$[Y]$, where $Y$ is the set of four elements $x$, $y$, $z$, $t$, such that $\{(x \circ y)zxy\} = f(x, y, z, x \circ y)$, and all the Jordan monomials of $f$ contain the variable $t$. By degree considerations (1.6), we can write $f = g + h$, where $g, h \in$ FSJ$[Y]$, $g$ is multilinear, and $h(x, y, z, t)$ is a linear combination of $U_t z$ and $z \circ t^2$. Now we will argue as in [11, Theorem 1.2]. Notice that $g \in$ FSJ$[Y] \subseteq H(\text{FAss}[Y], *)$, and again by (1.6) and the fact that $z$ occupies inside positions in the associative monomials of $\{(x \circ y)zxy\}$, $g$ is a linear combination of

$$\{xzyt\}, \quad \{xzty\}, \quad \{tzxy\},$$

$$\{tzyx\}, \quad \{yztx\}, \quad \{yzxt\},$$

and $h$ is a scalar multiple of $U_t z$. Hence $f$ has the form

$$\begin{aligned} f(x, y, z, t) =& \alpha_1\{xzyt\} + \alpha_2\{xzty\} + \alpha_3\{tzxy\} + \\ & \alpha_4\{tzyx\} + \alpha_5\{yztx\} + \alpha_6\{yzxt\} + \\ & \alpha_7 tzt. \end{aligned} \tag{2}$$

Therefore

$$\begin{aligned} \{(x \circ y)zxy\} =& \alpha_1\{xzy(x \circ y)\} + \alpha_2\{xz(x \circ y)y\} + \alpha_3\{(x \circ y)zxy\} + \\ & \alpha_4\{(x \circ y)zyx\} + \alpha_5\{yz(x \circ y)x\} + \alpha_6\{yzx(x \circ y)\} + \\ & \alpha_7(x \circ y)z(x \circ y). \end{aligned} \tag{3}$$

We just need to compare the coefficients of the associative monomials in (3) as in [11, Theorem 1.2], to obtain

$$\alpha_1 = \alpha_2 = \alpha_5 = \alpha_6 = 0$$

$$\alpha_3 = \lambda + 1, \quad \alpha_4 = \lambda, \quad \alpha_7 = -2\lambda$$

for some $\lambda \in \Phi$. Going back to (2),

$$\begin{aligned} f &= (\lambda + 1)\{tzxy\} + \lambda\{tzyx\} - 2\lambda tzt = \{tzxy\} + \lambda\{tz(x \circ y)\} - 2\lambda U_t z \\ &= \{tzxy\} + \lambda\{t, z, (x \circ y)\} - 2\lambda U_t z, \end{aligned}$$

so that $\{tzxy\} \in$ FSJ$[Y]$, but $\{tzxy\}$ is multilinear, hence $\{tzxy\} \in$ ML(FSJ$[Y]$), which contradicts (1.13). ■

## 3. Nondegenerate Imbeddings

We start with a direct consequence of [5, 9.5].

**3.1** Proposition. *Let $J$ be a Jordan algebra which imbeds in a nondegenerate Jordan algebra. If $x, y \in J$ satisfy $x \circ y = 0$, then $U_x$ and $U_y$ commute.*

Proof: We may assume that $J$ is a subalgebra of $\tilde{J}$, where $\tilde{J}$ is nondegenerate. If $x, y \in J$ satisfy $x \circ y = 0$, we can apply [5, 9.5] to $\tilde{J}$ to obtain that the $U$-operators $U_x$ and $U_y$, defined in $\tilde{J}$, commute. Hence so do their restrictions to $J$, i.e., the $U$-operators $U_x$ and $U_y$ in $J$ commute. ■

Taking into account the above result, we get a Jordan algebra that enriches the list of "pathological" examples of [4, Section 2].

**3.2** Corollary. *The Jordan algebra $J$ of (2.2) does not imbed in a nondegenerate Jordan algebra.* ■

Next, we show how use the algebra of (2.2) to obtain analogous examples of Jordan pairs and triple systems.

**3.3** Let $J$ be the Jordan algebra defined in (2.2). Let $\widehat{J}$ be the free unitization of $J$ (0.8), $T$ the Jordan triple system obtained by forgetting the squaring of $\widehat{J}$, and $V := V(T)$ the Jordan pair obtained by doubling $T$ (0.2)(i).

If $T$ imbeds in a nondegenerate Jordan triple system $\widetilde{T}$, so that we may assume that $T \subseteq \widetilde{T}$, then we have the map $\varphi : J \longrightarrow \widetilde{T}_1$, given by

$$\varphi(x) = x + \mathrm{Ker}_{\widetilde{T}} 1,$$

for any $x \in J$, where $\widetilde{T}_1$ is the local algebra of $\widetilde{T}$ at the element 1 (0.9). But $\varphi$ is an algebra homomorphism since, for any $x, y \in J$,

$$\begin{aligned}\varphi(x^2) &=_{((0.7)(1) \text{ applied to } \widehat{J})} \varphi(U_x 1) = U_x 1 + \mathrm{Ker}_{\widetilde{T}} 1 \\ &=_{(0.2)(\mathrm{i})} P_x 1 + \mathrm{Ker}_{\widetilde{T}} 1 =_{(0.9)} (x + \mathrm{Ker}_{\widetilde{T}} 1)^2 = (\varphi(x))^2,\end{aligned}$$

and

$$\begin{aligned}\varphi(U_x y) &=_{((0.7)(1) \text{ applied to } \widehat{J})} \varphi(U_x U_1 y) = U_x U_1 y + \mathrm{Ker}_{\widetilde{T}} 1 \\ &=_{(0.2)(\mathrm{i})} P_x P_1 y + \mathrm{Ker}_{\widetilde{T}} 1 =_{(0.9)} U_{x+\mathrm{Ker}_{\widetilde{T}} 1}(y + \mathrm{Ker}_{\widetilde{T}} 1) = U_{\varphi(x)} \varphi(y).\end{aligned}$$

Moreover, $\varphi$ is injective, since $\varphi(x) = 0$ implies $x \in \mathrm{Ker}_{\widetilde{T}} 1$, hence

$$x =_{(0.7)(1)} U_1 x =_{(0.2)(\mathrm{i})} P_1 x = 0.$$

But this is impossible by (3.2) since $\widetilde{T}_1$ is a nondegenerate Jordan algebra by [1, 3.1(i)].

Similar arguments apply to the Jordan pair $V$, so that we have the following result.

**3.4** Corollary. (i) *The Jordan triple system $T$ of (3.3) does not imbed in a nondegenerate Jordan triple system.*

(ii) *The Jordan pair $V$ of (3.3) does not imbed in a nondegenerate Jordan pair.* ■

## 4. Some Open Problems Inside the Counter-Example

**4.1** Let $\mathrm{FJ}[X]$ denote the free Jordan algebra over a set $X$. Given a Jordan algebra $J$ and an element $f \in \mathrm{FJ}[X]$, we will say that $f = 0$ in $J$ if every evaluation of $f$ vanishes in $J$, i.e., $\mu(f) = 0$ for any algebra homomorphism $\mu : \mathrm{FJ}[X] \longrightarrow J$.

**4.2** Given a set of variables $X$. Let $\Psi : \mathrm{FJ}[X] \longrightarrow \mathrm{FSJ}[X]$ be the natural Jordan algebra epimorphism fixing all the elements in $X$. Let us consider $f, g \in \mathrm{FJ}[X]$ satisfying

(i) $\Psi(g) \in \mathrm{Id}_{\mathrm{FAss}[X]}(\Psi(f)) \cap \mathrm{FSJ}[X]$, where $\mathrm{Id}_{\mathrm{FAss}[X]}(\Psi(f))$ is the ideal of $\mathrm{FAss}[X]$ generated by $\Psi(f)$.

(ii) $\Psi(g) \notin \mathrm{Id}_{FSJ[X]}(\Psi(f))$, where $\mathrm{Id}_{\mathrm{FSJ}[X]}(\Psi(f))$ denotes the ideal of $\mathrm{FSJ}[X]$ generated by $\Psi(f)$.

(iii) For any nondegenerate Jordan algebra $\widetilde{J}$, and any Jordan algebra homomorphism $\mu : \mathrm{FJ}[X] \longrightarrow \widetilde{J}$, $\mu(f) = 0$ implies $\mu(g) = 0$.

We claim that

(iv) the algebra $J = \mathrm{FSJ}[X]/\,\mathrm{Id}_{FSJ[X]}(\Psi(f))$ cannot imbed in a nondegenerate algebra.

Otherwise, if $J$ imbeds in a nondegenerate Jordan algebra $\widetilde{J}$, we have the homomorphism $\mu : \mathrm{FJ}[X] \longrightarrow \widetilde{J}$ given by $\mu(h) = \Psi(h) + \mathrm{Id}_{FSJ[X]}(\Psi(f))$, for any $h \in \mathrm{FJ}[X]$, contradicting (iii).

**4.3** The algebra $J$ of (2.2) fits in the general situation described above when $X$ is the set of three elements $x$, $y$, $z$, and we take $f = x \circ y$, and $g = [U_y, U_x]z$. Here, (4.2)(i) is established in (2.1) and lies underneath [5, 4.1], (4.2)(ii) is proved in (2.3), and (4.2)(iii) in [5, 9.5].

**4.4** (i) The algebra $J = \mathrm{FSJ}[X]/\,\mathrm{Id}_{FSJ[X]}(\Psi(f))$ of (4.2) is exceptional by Cohn's Lemma [8, page 255; 11, Lemma 1 on page 10; 17, Corollary to Cohn's Criterion on page 763] without assuming (4.2)(iii).

(ii) In the algebra $J = \mathrm{FSJ}[X]/\operatorname{Id}_{FSJ[X]}(\Psi(f))$ of (4.2) we can find the ideal

$$M := (\operatorname{Id}_{\mathrm{FAss}[X]}(\Psi(f)) \cap \mathrm{FSJ}[X])/\operatorname{Id}_{FSJ[X]}(\Psi(f))$$

which is nonzero by (4.2)(i) and (4.2)(ii). In the case when $\Phi$ is a field of characteristic not two, we can apply Zelmanov's result [29, Theorem 1] to show that $M$ is McCrimmon radical, so that, in particular, $J$ is degenerate again without assuming (4.2)(iii).

**4.5** This leads us to the following problem that can be viewed as a general frame in which [5] fits:

(I) Does condition (4.2)(i) imply (4.2)(iii)?

The above can be weakened to an alternative problem:

(II) Assuming condition (4.2)(i), is it true that if $f = 0$ in a nondegenerate Jordan algebra $\widetilde{J}$, then $g = 0$ in $\widetilde{J}$?

Notice that in (I) we are just assuming that a particular evaluation of $f$ vanishes in $J$, i.e., $\mu(f) = f(a_1, \ldots, a_n) = 0$, for particular elements $a_1, \ldots, a_n \in J$, and wonder whether the same evaluation $\mu(g) = g(a_1, \ldots, a_n)$ vanishes. In (II) we assume that every evaluation of $f$ vanishes and wonder if the same holds for $g$ (see (4.1)).

There is always the difficulty that condition (4.2)(i) is not given for $f$ and $g$ but for their images $\Psi(f)$ and $\Psi(g)$ in $\mathrm{FSJ}[X]$. So it is hopeless to expect positive answers to (I) and (II) for arbitrary $f$ and $g$ satisfying (4.2)(i). In that case, given $f$ and $g$ satisfying (4.2)(i) (for example, $f = g$, for an arbitrary $f \in \mathrm{FJ}[X]$), we could replace $g$ by $g' = g + h$, where $h$ is any $s$-identity, so that $\Psi(g) = \Psi(g')$, and $f$, $g'$ also satisfy (4.2)(i). As a consequence, if (II) had a positive answer, and $f = 0$ in a nondegenerate Jordan algebra $\widetilde{J}$, then both $g = 0$ and $g' = 0$ in $\widetilde{J}$, hence $h = 0$ too, concluding that $\widetilde{J}$ would be an $i$-special Jordan algebra, which is not true in general.

We can reformulate the above problems as follows (see also [24]):

(I)' Assume condition (4.2)(i) for $f$ and $g$. Can there be found $g'$ such that $g - g'$ is an $s$-identity and, for any nondegenerate Jordan algebra $\widetilde{J}$, and any Jordan algebra homomorphism $\mu : \mathrm{FJ}[X] \longrightarrow \widetilde{J}$, $\mu(f) = 0$ implies $\mu(g') = 0$.

(II)' Assume condition (4.2)(i) for $f$ and $g$. Can there be found $g'$ such that $g - g'$ is an $s$-identity, so that $f = 0$ in a nondegenerate Jordan algebra $\widetilde{J}$ implies $g' = 0$ in $\widetilde{J}$?

Next we give a partial positive answer to the above problems when we restrict to special Jordan algebras regardless their nondegeneracy.

**4.6** PROPOSITION. *If $f$ and $g$ satisfy (4.2)(i), then, for any special Jordan algebra $\widetilde{J}$, and any Jordan algebra homomorphism $\mu : \mathrm{FJ}[X] \longrightarrow \widetilde{J}$, $\mu(f) = 0$ implies $\mu(g) = 0$.*

PROOF: If $\widetilde{J}$ is special and $R$ is an associative algebra such that $J$ is a subalgebra of $R^{(+)}$, then we have the inclusion maps $j : \widetilde{J} \longrightarrow R$ and $k : \mathrm{FSJ}[X] \longrightarrow \mathrm{FAss}[X]$, and, for any Jordan algebra homomorphism $\mu : \mathrm{FJ}[X] \longrightarrow \widetilde{J}$, we can find an associative algebra homomorphism $\overline{\mu} : \mathrm{FAss}[X] \longrightarrow R$ such that $\overline{\mu} k \Psi = j\mu$ (apply the fact that $\mathrm{FAss}[X]$ and $\mathrm{FJ}[X]$ are free objects to the restriction of $j\mu$ to $X$). Now,

$$\mu(g) = j\mu(g) = \overline{\mu} k \Psi(g) = \overline{\mu}\Psi(g).$$

but $\Psi(g) \in \mathrm{Id}_{\mathrm{FAss}[X]}(\Psi(f))$ by (4.2)(i), hence $\Psi(g)$ has the form

$$\Psi(g) = \lambda\Psi(f) + \sum_i a_i \Psi(f) + \sum_j \Psi(f) b_j + \sum_k c_k \Psi(f) d_k,$$

where $a_i, b_j, c_k, d_k \in \mathrm{FAss}[X]$, $\lambda \in \Phi$, and

$$\mu(g) = \lambda\overline{\mu}\big(\Psi(f)\big) + \sum_i \overline{\mu}(a_i)\overline{\mu}\big(\Psi(f)\big) + \sum_j \overline{\mu}\big(\Psi(f)\big)\overline{\mu}(b_j) + \sum_k \overline{\mu}(c_k)\overline{\mu}\big(\Psi(f)\big)\overline{\mu}(d_k) \tag{1}$$

since $\overline{\mu}$ is an associative algebra homomorphism. Therefore, if $\mu(f) = 0$, then $\overline{\mu}\big(\Psi(f)\big) = \overline{\mu} k \Psi(f) = j\mu(f) = 0$, which implies $\mu(g) = 0$ by (1). ■

The above result enables us to use the strategy of [5, Proof of 9.5] when dealing with problems (I), (II), (I)' and (II)', so that they now reduce to study the particular case when $\widetilde{J}$ is an Albert algebra.

On the other hand, Zelmanov's result [29, Theorem 1] can be used to take a first step towards the solutions to problems (I)' and (II)'.

**4.7** PROPOSITION. *Let $f$ and $g$ satisfy (4.2)(i), and assume that the ring of scalars $\Phi$ is a field of characteristic not two. There exists a positive integer $n$ and an s-identity $h$ such that $\widetilde{g} = g^n + h$ lies in the ideal of $\mathrm{FJ}[X]$ generated by $f$. In particular, for any Jordan algebra $J$, and any Jordan algebra homomorphism $\mu : \mathrm{FJ}[X] \longrightarrow J$, $\mu(f) = 0$ implies $\mu(\widetilde{g}) = 0$.*

PROOF: By using [29, Theorem 1], the quotient

$$M := (\mathrm{Id}_{\mathrm{FAss}[X]}(\Psi(f)) \cap \mathrm{FSJ}[X]) / \mathrm{Id}_{FSJ[X]}(\Psi(f))$$

is McCrimmon radical. Thus $M$ is locally nilpotent by [6, 3.11], hence nil, and there exists a positive integer $n$ such that

$$\Psi(g^n) = \Psi(g)^n \in \mathrm{Id}_{FSJ[X]}(\Psi(f)). \tag{1}$$

But $\Psi$ is surjective, hence $\mathrm{Id}_{FSJ[X]}(\Psi(f)) = \Psi\big(\mathrm{Id}_{FJ[X]}(f)\big)$, and (1) implies that there exists $h \in \operatorname{Ker}\Psi$ such that $g^n + h \in \mathrm{Id}_{FJ[X]}(f)$. ■

**4.8** Finally, taking into account (4.4)(ii), we may forget about (4.2)(iii) and consider the following problem:

(III) Do (4.2)(i)(ii) imply (4.2)(iv)?

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## REFERENCES

[1] A. D'Amour, K. McCrimmon, "The Local Algebras of Jordan Systems", *J. Algebra.* **177** (1995) 199-239.

[2] J. A. Anquela, T. Cortés, "Jordan cubes and Associative Powers", *J. Pure Appl. Algebra* **181** (2003), 131-139.

[3] J. A. Anquela, T. Cortés, "Growing Hearts in Associative Systems", *J. Algebra Appl.* **11** (5) (2012) 1250083 (11 pages).

[4] J. A. Anquela, T. Cortés, K. McCrimmon, "Imbedding Jordan Systems in Primitive Systems", *J. Algebra.* **449** (2016) 175-183.

[5] J. A. Anquela, T. Cortés, H. Petersson, "Commuting $U$-Operators in Jordan Algebras", *Trans. Amer. Soc.* **366** (2014) 5877-5902.

[6] J. A. Anquela, T. Cortés, E. Zelmanov, "Local Nilpotency of the McCrimmon Radical of a Jordan System", *Proc. Steklov Inst. Math.* **292** (2016) 1-9.

[7] L. A. Bokut, "Some Imbedding Theorems for Rings and Semigroups. I", *Sibirsk Mat. Z.* **4** (1963) 500-518.

[8] P. M. Cohn, "On Homomorphic Images of Special Jordan Algebras", *Canadian J. Math.* **6** (1954) 253-264.

[9] P. M. Cohn, "Simple Rings without Zero Divisors, and Lie Division Rings", *Mathematika* **6** (1959) 14-18.

[10] T. De Medts and R. M. Weiss, "Moufang Sets and Jordan Division Algebras", *Math. Ann.* **335** (2006) (2) 415-433.

[11] N. Jacobson, *Structure and Representations of Jordan Algebras*, Amer.

Math. Soc. Colloq. Publ., vol. 39, Providence, 1968.

[12] N. Jacobson, *Structure Theory of Jordan Algebras*, The University of Arkansas Lecture Notes in Mathematics, University of Arkansas, Fayetteville 1981.

[13] N. Jacobson, *Basic Algebra II (Second Edition)*, W. H Freeman and Company, New York, 1989.

[14] R. Lewand, K. McCrimmon, "Macdonald's Theorem for Quadratic Jordan Algebras", *Pacific J. Math.* **35** (1970) 681-707.

[15] O. Loos, *Jordan Pairs.* Lecture Notes in Math., Vol. 460, Springer-Verlag, Berlin, 1975.

[16] A. Malcev, "On the Immersion of an Algebraic Ring into a Field", *Math. Ann.* **113** (1) (1937) 686-691.

[17] K. McCrimmon, "Speciality of Quadratic Jordan Algebras", *Pacific J. Math.* **36** (1971) 761-773.

[18] K. McCrimmon, "Quadratic Jordan Algebras and Cubing Operations", *Trans. Amer. Math. Soc.* **153** (1971) 265-278.

[19] K. McCrimmon, *A taste of Jordan Algebras*, Universitext, Springer-Verlag, New York, 2004.

[20] K. McCrimmon, E. Zelmanov, "The Structure of Strongly Prime Quadratic Jordan Algebras", *Adv. Math.* **69** (1988) 133-222.

[21] Yu. A. Medvedev, "Nilpotent Elements of a Free Jordan Algebra", *Siberian Math. J.* **26** (1985), 271-277.

[22] K. Meyberg, *Lectures on Algebras and Triple Systems*, Lecture Notes, University of Virginia, Charlottesville, 1972.

[23] N. S. Nam, K. McCrimmon, "Minimal Ideals in Quadratic Jordan Algebras", *Proc. Amer. Math. Soc.* **88** (4) (1983) 579-583.

[24] I. Shestakov, "On Commuting $U$-Operators in Jordan Algebras", *Non Associative & Non Commutative Algebras and Operator Theory* (Cheikh Thiecoumbe Gueye and Mercedes Siles Molina, eds.), Springer Proceedings in Mathematics and Statistics, vol. 160, 2016, 105-115.

[25] B.L. Van der Waerden, *Modern Algebra*, vol. I (Second Editiion), Frederik Ungar Publishing Co., New York, 1949 (translation from the German version of 1937).

[26] E. I. Zelmanov, "On Prime Jordan Algebras", *Siberian Math. J.* **18** (2) (1979), 103-111.

[27] E. I. Zelmanov, "On Prime Jordan Algebras II", *Siberian Math. J.* **24** (1) (1983), 73-85.

[28] E. I. Zelmanov, "Prime Jordan Triple Systems", *Siberian Math. J.* **24** (5) (1983), 509-520.

[29] E. I. Zelmanov, "Ideals in Special Jordan Algebras", *Nova J. Algebra Geom.* **1** (1992) 59-71.