These notes are divided into three parts.

The first part is based on material developed for inclusion in Serre’s lecture notes in [GMS03], but was finally omitted. I learned most of that material from Serre. This part culminates with the determination of the invariants of $PGL_p \mod p$ (for $p$ prime) and the invariants of Albert algebras (equivalently, groups of type $F_4$) mod 3.

The second part describes a general recipe for finding a subgroup $N$ of a given semisimple group $G$ such that the natural map $H^1_{\text{fppf}}(\ast, N) \to H^1(\ast, G)$ is surjective. It is a combination of two ideas: that parabolic subgroups lead to representations with open orbits, and that such representations lead to surjective maps in Galois cohomology. I learned the second idea from Rost [Ros99b], but both ideas seem to have been discovered and re-discovered many times. We bring the two ideas together here, apparently for the first time. Representation theorists will note that our computations of stabilizers $N$ for various $G$ and $V$ — summarized in Table 21a — are somewhat more precise than the published tables, in that we compute full stabilizers and not just identity components. The surjectivities in cohomology are used to describe the mod 3 invariants of the simply connected split $E_6$ and split $E_7$’s.

The last two sections of this part describe a construction of groups of type $E_8$ that is “surjective at 5”, see Prop. 13.7. We use it to determine the mod 5 invariants of $E_8$ and to give new examples of anisotropic groups of that type. These examples have already been applied in [PSZ06].

The third part describes the mod 2 invariants of the groups $\text{Spin}_n$ for $n \leq 12$ and $n = 14$. It may be viewed as a fleshed-out version of Markus Rost’s unpublished notes [Ros99b] and [Ros99c]. A highlight of this part is Rost’s Theorem 19.3 on 14-dimensional quadratic forms in $I^3$.

There are also two appendices. The first uses cohomological invariants to give new examples of anisotropic groups of types $E_7$, answering a question posed by Kirill Zainoulline. The second appendix—written by Detlev W. Hoffmann—proves a generalization of the “common slot theorem” for 2-Pfister quadratic forms. This result is used to construct invariants of $\text{Spin}_{12}$ in §18.

These are notes for a series of talks I gave in a “mini-cours” at the Université d’Artois in Lens, France, in June 2006. Consequently, some material has been included in the form of exercises. Although this is a convenient device to avoid going into tangential details, no substantial difficulties are hidden in this way. The exercises are typically of the “warm up” variety. On the other end of the spectrum, I have included several open problems. “Questions” lie somewhere in between.

Acknowledgements. It is a pleasure to thank J-P. Serre and Markus Rost (both for things mentioned above and for their comments on this note), Detlev Hoffmann for providing Appendix B, and Pasquale Mamone for his hospitality during my stay in Lens. Gary Seitz and Philippe Gille both gave helpful answers to questions. I thank also R. Pari-mala, Zinovy Reichstein, and Adrian Wadsworth for their comments.
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Part I. Invariants, especially modulo an odd prime

1. Definitions and notations

1.1. Definition of cohomological invariant. We assume some familiarity with the notes from Serre’s lectures from [GMS03], which we refer to hereafter as S. A reader seeking a more leisurely introduction to the notion of invariants should see pages 7–11 of those notes.

We fix a base field $k_0$ and consider functors $A: \text{Fields}_{/k_0} \to \text{Sets}$ and $H: \text{Fields}_{/k_0} \to \text{Abelian Groups}$, where $\text{Fields}_{/k_0}$ denotes the category of field extensions of $k_0$. In practice, $A(k)$ will be the Galois cohomology set $H^1(k, G)$ for $G$ a linear algebraic group over $k_0$.

In S, various functors $H$ were considered (e.g., the Witt group), but here we only consider abelian Galois cohomology.

An invariant of $A$ (with values in $H$) is a morphism of functors $a: A \to H$, where we view $H$ as a functor with values in $\text{Sets}$. Unwinding the definition, an invariant of $A$ is a collection of functions $a_k: A(k) \to H(k)$, one for each $k \in \text{Fields}_{/k_0}$, such that for each morphism $\phi: k \to k'$ in $\text{Fields}_{/k_0}$, the diagram

$$
\begin{array}{ccc}
A(k) & \xrightarrow{a_k} & H(k) \\
\downarrow A(\phi) & & \downarrow H(\phi) \\
A(k') & \xrightarrow{a_k'} & H(k')
\end{array}
$$

commutes.

1.2. Examples. (1) Fix a natural number $n$ and write $\mathcal{S}_n$ for the symmetric group on $n$ letters. The set $H^1(k, \mathcal{S}_n)$ classifies étale $k$-algebras of degree $n$ up to isomorphism. The sign map $\text{sgn}: \mathcal{S}_n \to \mathbb{Z}/2\mathbb{Z}$ is a homomorphism of algebraic groups and so defines a morphism of functors—an invariant—$\text{sgn}: H^1(*, \mathcal{S}_n) \to H^1(*, \mathbb{Z}/2\mathbb{Z})$. The set $H^1(k, \mathbb{Z}/2\mathbb{Z})$ classifies quadratic étale $k$-algebras, i.e., separable quadratic field extensions together with the trivial class corresponding to $k \times k$, and $\text{sgn}$ sends a degree $n$ algebra to its discriminant algebra.

This example is familiar in the case where the characteristic of $k_0$ is not 2. Given a separable polynomial $f \in k[x]$, one can consider the étale $k$-algebra $K := k[x]/(f)$. The discriminant algebra of $K$—here, $\text{sgn}(K)$—is $k[x]/(x^2 - d)$, where $d$ is the usual elementary notion of discriminant of $f$, i.e., the product of squares of differences of roots of $f$.

(For a discussion in characteristic 2, see [Wat87].)

One of the main results of S is that $\mathcal{S}_n$ only has “mod 2” invariants, see S24.12.

(2) Let $G$ be a semisimple algebraic group over $k_0$. It fits into an exact sequence

$$
1 \longrightarrow C \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1,
$$

\[\text{We systematically refer to specific contents of S by S followed by a reference number. For example, Proposition 16.2 on page 39 will be referred to as S16.2.}\]

\[\text{Below, we only consider algebraic groups that are linear.}\]
where $\tilde{G}$ is simply connected and $C$ is finite and central in $\tilde{G}$. This gives a connecting homomorphism in Galois cohomology

$$H^1(k, G) \xrightarrow{\delta} H^2(k, C)$$

that defines an invariant $\delta: H^1(*, G) \to H^2(*, C)$.

(3) The map $e_n$ that sends the $n$-Pfister quadratic form $\langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle$ (over a field $k$ of characteristic $\neq 2$) to the class $(a_1) \cdot (a_2) \cdots (a_n) \in H^n(k, \mathbb{Z}/2\mathbb{Z})$ defines an invariant $\delta$ depending only on the isomorphism class of the quadratic form. (Compare §18 of S.)

The Milnor Conjecture (now a theorem, see [Voe03, 7.5] and [OVV, 4.1]) states that $e_n$ extends to a well-defined additive map

$$e_n : I^n \to H^n(k, \mathbb{Z}/2\mathbb{Z})$$

that is zero on $I^{n+1}$ and induces an isomorphism $I^n/I^{n+1} \xrightarrow{\sim} H^n(k, \mathbb{Z}/2\mathbb{Z})$. (Here $I^n$ denotes the $n$-th power of the ideal $I$ of even-dimensional forms in the Witt ring of $k$.)

(4) For $G$ a quasi-simple simply connected algebraic group, there is an invariant $r_G : H^1(*, G) \to H^3(*, \mathbb{Q}/\mathbb{Z}(2))$ called the Rost invariant. It is the main subject of [Mer03]. When $G$ is Spin$_n$, i.e., a split simply connected group of type $B$ or $D$, the Rost invariant amounts to the invariant $e_3$ in (3) above, cf. [Mer03, 2.3].

The Rost invariant has the following useful property: If $G'$ is also a quasi-simple simply connected algebraic group and $\rho : G' \to G$ is a homomorphism, then the composition

$$H^1(*, G') \xrightarrow{\rho} H^1(*, G) \xrightarrow{r_G} H^3(*, \mathbb{Q}/\mathbb{Z}(2))$$

equals $n_\rho r_{G'}$ for some natural number $n_\rho$, called the Rost multiplier of $\rho$, see [Mer03, p. 122].

(5) Suppose that $k_0$ contains a primitive 4-th root of unity. The trace quadratic form on a central simple algebra $A$ of dimension $4^2$ is Witt-equivalent to a direct sum $q_2 \oplus q_4$ where $q_i$ is an $i$-Pfister form, see [RST06]. The maps $f_i : A \mapsto e_i(q_i)$ define invariants $H^1(*, \text{PG}L_1) \to H^i(*, \mathbb{Z}/2\mathbb{Z})$ for $i = 2$ and 4. Rost-Serre-Tignol prove that $f_2(A)$ is zero if and only if $A \otimes \text{A}$ is a matrix algebra and $f_4(A)$ is zero if and only if $A$ is cyclic.\(^c\)

(For the case where $k_0$ has characteristic 2, see [Tig06].)

1.3. Let $C$ be a finite Gal$(k_0)$-module of exponent not divisible by the characteristic of $k_0$. We define a functor $M$ by setting

$$M^d(k, C) := H^d(k, C(d-1))$$

where $C(d-1)$ denotes the $(d-1)$-st Tate twist of $C$ as in S7.8 and

$$M(k, C) := \bigoplus_{d \geq 0} M^d(k, C).$$

We are mainly interested in

$$M(k, \mathbb{Z}/n\mathbb{Z}) = H^0(k, \text{Hom}(\mu_n, \mathbb{Z}/n\mathbb{Z})) \oplus \bigoplus_{d \geq 1} H^d(k, \mu_n^{\otimes (d-1)}).$$

Many invariants take values in $M(*, \mathbb{Z}/n\mathbb{Z})$, for example:

\(^c\)The term "cyclic" is defined in 5.4 below.
(2bis) For $G = \text{PGL}_n$, the invariant $\delta$ in Example 1.2.2 is

$$\delta : H^1(\ast, \text{PGL}_n) \to H^2(\ast, \mu_n) \subset M(\ast, \mathbb{Z}/n\mathbb{Z}).$$

We remark that $H^2(k, \mu_n)$ can be identified with the $n$-torsion in the Brauer group of $k$ via Kummer theory.

(4bis) Let $G$ be a group as in 1.2.4. Write $i$ for the Dynkin index of $G$ as in [Mer03, p. 130] and put $n := i$ if char $k_0 = 0$ or $i = p'^n$ for $n$ not divisible by $p$ if char $k_0$ is a prime $p$. The Rost invariant maps

$$H^1(\ast, G) \to H^3(\ast, \mu_n^{\otimes 2}) \subset M(\ast, \mathbb{Z}/n\mathbb{Z}).$$

(6) If the characteristic of $k_0$ is different from 2, then $\mathbb{Z}/2\mathbb{Z}$ equals $\mathbb{Z}/2\mathbb{Z}$ for every $d$, and $M(k, \mathbb{Z}/2\mathbb{Z})$ is the mod 2 cohomology ring $H^\ast(k, \mathbb{Z}/2\mathbb{Z})$.

So most of the cohomological invariants considered in S take values in the functor $k \mapsto M(k, \mathbb{Z}/2\mathbb{Z})$.

We remark that for $n$ dividing 24, $\mu_n^{\otimes 2}$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ [KMRT98, p. 444, Ex. 11]. In that case,

$$M^d(k, \mathbb{Z}/n\mathbb{Z}) \cong \begin{cases} H^0(k, \text{Hom}(\mu_n, \mathbb{Z}/n\mathbb{Z})) & \text{if } d = 0 \\ H^d(k, \mathbb{Z}/n\mathbb{Z}) & \text{if } d \text{ is odd} \\ H^d(k, \mu_n) & \text{if } d \text{ is even and } d \neq 0 \end{cases}$$

The other main target for invariants is $M(\ast, \mu_n)$, characterized by

$$M(k, \mu_n) = \mathbb{Z}/n\mathbb{Z} \oplus \bigoplus_{d \geq 1} H^d(k, \mu_n^{\otimes d}).$$

This is naturally a ring, and we write $R_n(k)$ for $M(k, \mu_n)$ when we wish to view it as such. (This ring is a familiar one: the Bloch-Kato Conjecture asserts that it is isomorphic to the quotient $K^M_\ast(k)/n$ of the Milnor $K$-theory ring $K^M_\ast(k)$.)

When $C$ is $n$-torsion, the abelian group $M(k, C)$ is naturally an $R_n(k)$-module.

For various algebraic groups $G$ and Galois-modules $C$, we will determine the invariants $H^1(\ast, G) \to M(\ast, C)$. We abuse language by calling these “invariants of $G$ with values in $C$”, “$C$-invariants of $G$”, etc. We write $\text{Inv}(G, C)$ or $\text{Inv}_{k_0}(G, C)$ for the collection of such invariants.\(^d\)

For example, the invariants in (2bis) and (4bis) above belong to $\text{Inv}(\text{PGL}_n, \mathbb{Z}/n\mathbb{Z})$ and $\text{Inv}(G, \mathbb{Z}/n\mathbb{Z})$ respectively. Note that $\text{Inv}(G, C)$ is an abelian group for every algebraic group $G$, and, when $G$ is $n$-torsion, $\text{Inv}_{k_0}(G, C)$ is an $R_n(k_0)$-module.

1.4. Constant and normalized. Fix an element $m \in M(k_0, C)$. For every group $G$, the collection of maps that sends every element of $H^1(k, G)$ to the image of $m$ in $M(k, C)$ for every extension $k/k_0$ is an invariant in $\text{Inv}(G, C)$. Such invariants are called constant.

An invariant $a \in \text{Inv}(G, C)$ is normalized if $a$ sends the neutral class in $H^1(k, G)$ to zero in $M(k, C)$ for every extension $k/k_0$. We write $\text{Inv}_{\text{norm}}(G, C)$ for the normalized invariants in $\text{Inv}(G, C)$.

The reader can find a typical application of cohomological invariants in Appendix A.

\(^d\)Strictly speaking, this notation disagrees with the notation defined on page 11 of S. But there is no essential difference, because in S the target $C$ is nearly always taken to be $\mathbb{Z}/2\mathbb{Z}$, and $\mathbb{Z}/2\mathbb{Z}(d)$ is canonically isomorphic to $\mathbb{Z}/2\mathbb{Z}$ for all $d$. 
2. Invariants of $\mu_n$

Fix a natural number $n$ not divisible by the characteristic of the field $k_0$. In this section, we determine the invariants of $\mu_n$ with values in $\mu_n$, along with a small variation.

There are two obvious invariants of $\mu_n$:

1. The constant invariant (as in 1.4) given by the element $1 \in \mathbb{Z}/n\mathbb{Z} \subset M(k_0, \mu_n)$.
2. The invariant $\text{id}$ that is the identity map

$$H^1(k, \mu_n) \rightarrow H^1(k, \mu_n) \subset M(k, \mu_n)$$
for every $k/k_0$.

2.1. Proposition. $\text{Inv}_{k_0}(\mu_n, \mu_n)$ is a free $R_n(k_0)$-module with basis $1, \text{id}$.

This proposition can easily be proved by adapting the proof of S16.2. Alternatively, it is [MPT03, Cor. 1.2]. In the interest of exposition, we give an elementary proof in the case where $k_0$ is algebraically closed.

We need the following lemma, which is a special case of S12.3.

2.2. Lemma. If invariants $a, a' \in \text{Inv}_{k_0}(\mu_n, \mu_n)$ agree on $(t) \in H^1(k_0(t), \mu_n)$, then $a$ and $a'$ are equal.

Proof. Replacing $a, a'$ with $a - a', 0$ respectively, we may assume that $a'$ is identically zero.

Fix an extension $E$ of $k_0$ and an element $y \in E^\times$. Write $M$ for the functor $M(\cdot, \mu_n)$ as in 1.3 and consider the commutative diagram

$$
\begin{array}{cccc}
H^1(E, \mu_n) & \xrightarrow{a_E} & H^1(E((t - y)), \mu_n) & \xleftarrow{a_{E(t - y)}} H^1(k_0(t), \mu_n) \\
M(E) & \xrightarrow{a_{E(t - y)}} & M(E((t - y))) & \xleftarrow{a_{k_0(t)}} M(k_0(t))
\end{array}
$$

The polynomial $x^n - y/t$ in $E((t - y))[x]$ has residue $x^n - 1$ in $E[x]$, which has a simple root, namely $x = 1$. Therefore $x^n - y/t$ has a root over $E((t - y))$ by Hensel’s Lemma, and the images of $(y) \in H^1(E, \mu_n)$ and $(t) \in H^1(k_0(t), \mu_n)$ in $H^1(E((t - y)), \mu_n)$ agree. The commutativity of the diagram implies that the image of $(y)$ in $M(E((t - y)))$ is the same as the image of $a_{k_0(t)}(t)$, i.e., zero. But the map $M(E) \rightarrow M(E((t - y)))$ is an injection by S7.7, so $a_{E}(y)$ is zero. This proves the lemma.

Proof of Prop. 2.1. We assume that $k_0$ is algebraically closed. Fix an invariant $a \in \text{Inv}_{k_0}(\mu_n, \mu_n)$, and consider the torsor class $(t) \in H^1(k_0(t), \mu_n)$. We claim that $a(t)$ is unramified away from $\{0, \infty\}$. Indeed, any other point on the affine line over $k_0$ is an ideal $(t - y)$ for some $y \in k_0^\times$ because $k_0$ is algebraically closed. Consider the diagram (2.3) with the $E$’s replaced with $k_0$’s. As in the proof of Lemma 2.2, the images of $(y) \in H^1(k_0, \mu_n)$ and $(t) \in H^1(k_0(t), \mu_n)$ agree in $H^1(k_0((t - y)), \mu_n)$ by Hensel’s Lemma, hence the image of $(t)$ in $M(k_0((t - y)))$ comes from $M(k_0)$. That is, $a(t)$ is unramified at $(t - y)$. This proves the claim, and by S9.4 we have:

$$a(t) = \lambda_0 + \lambda_1 \cdot (t)$$

for uniquely determined elements $\lambda_0, \lambda_1 \in M(k_0)$.
Put \( a' := \lambda_0 \cdot 1 + \lambda_1 \cdot \text{id} \). Since the invariants \( a, a' \) agree on \((t)\), the two invariants are the same by Lemma 2.2. This proves that \( 1, \text{id} \) span \( \text{Inv}_{k_0}(\mu_n, \mu_n) \).

As for linear independence, suppose that the invariant \( \lambda_0 \cdot 1 + \lambda_1 \cdot \text{id} \) is zero. Then \( \lambda_0 \)—the value of \( a \) on the trivial class—is zero. The other coefficient, \( \lambda_1 \), is the residue at \( t = 0 \) of \( a(t) \) in \( M(k_0) \).

Recall from S4.5 that every invariant can be written uniquely as \((\text{constant}) + \) (normalized). Clearly, the proposition proves that \( \text{Inv}_{k_0}^{\text{norm}}(\mu_n, \mu_n) \) is a free \( R_n(k_0) \)-module with basis \( \text{id} \).

Really, the proof of Prop. 2.1 given above is the same as the proof of S16.2 in the case where \( k_0 \) is algebraically closed, except that we have unpacked the references to S11.7 and S12.3 (which are both elaborations of the Rost Compatibility Theorem) with the core of the Rost Compatibility Theorem that is sufficient in this special case.

2.4. Remark. The argument using Hensel’s Lemma in the proof of Lemma 2.2 has real problems when the characteristic of \( k \) divides \( n \). For example, when the characteristic of \( k \) (and hence \( E \)) is a prime \( p \), the element \( t/y \) has no \( p \)-th root in \( E((t - y)) \) for every \( y \in E^\times \). Speaking very roughly, this is the reason for the global assumption that the characteristic of \( k_0 \) does not divide the exponent of \( C \).

2.5. \( \mu_n \) INVARIANTS OF \( \mu_{sn} \). Let \( s \) be a positive integer not divisible by the characteristic of \( k_0 \). The \( s \)-th power map (the natural surjection) \( s: \mu_{sn} \to \mu_n \) fits into a commutative diagram

\[
\begin{array}{cccccc}
1 & \longrightarrow & \mu_{sn} & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathbb{G}_m & \longrightarrow & 1 \\
& & s \downarrow & & s \downarrow & & \| & & \| \\
1 & \longrightarrow & \mu_n & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathbb{G}_m & \longrightarrow & 1 \\
\end{array}
\]

(2.6)

It induces an invariant \( s: H^1(*, \mu_{sn}) \to H^1(*, \mu_n) \). A diagram chase on (2.6) shows that for each \( k, \xi \) is the surjection

\[
k^\times/k^{\times sn} \to k^\times/k^{\times n} \text{ given by } xk^{\times sn} \to xk^{\times n}.
\]

The proof of Prop. 2.1 with obvious modifications gives:

**Proposition.** \( \text{Inv}_{k_0}(\mu_{sn}, \mu_n) \) is a free \( R_n(k_0) \)-module with basis \( 1, \xi \). \( \square \)

We will apply this in 15.8 and §18 below in the case \( s = n = 2 \). In that case, we will continue to write \( \xi \) instead of the more logical \( 2 \).

2.7. Exercise. Let \( C \) be a finite Gal\((k_0)\)-module whose order is a power of \( n \), and suppose that \( k_0 \) contains a primitive \( n \)-th root of unity. For \( x \in H(k_0, C(-1)) \) such that \( nx = 0 \), define a cup product

\[
- \ast x: H^1(k, \mu_n) \to H^1(k, C)
\]

mimicking §23 of S. Prove that every normalized invariant \( H^1(*, \mu_n) \to H(*, C) \) can be written uniquely as \( \text{id} \ast x \) for such an \( x \).

2.8. Exercise (char \( k_0 \neq 2 \)). Consider the group \( SL(Q) \) whose \( k \)-points are the norm 1 elements of \( Q \otimes_k k \) for some quaternion algebra \( Q \) over \( k_0 \). The center of this group is \( \mu_2 \) and the natural map

\[
H^1(k, \mu_2) \to H^1(k, SL(Q))
\]

(2.9)
is surjective for every $k/k_0$. The Rost invariant of $SL(Q)$ (as defined in Example 1.2.4) takes values in $H^3(k, \mu_{2}^{\otimes 2})$, and its composition with (2.9) is the cup product $x \mapsto x \cdot [Q]$.

Prove that the Rost invariant generates $\text{Inv}_{k_0}^{\text{form}}(SL(Q), \mathbb{Z}/2\mathbb{Z})$ as an $R_2(k_0)$-module.

3. Quasi-Galois extensions and invariants of $\mathbb{Z}/p\mathbb{Z}$

3.1. Let $p_1, p_2, \ldots, p_r$ be the distinct primes dividing the exponent of $C$. There is a canonical identification $C = \prod_{i=1}^{r} p_i C$, where $p_i C$ denotes the submodule of $C$ consisting of elements of order a power of $p_i$. This gives an identification

$$\text{Inv}_{k_0}(G, C) = \prod_{i=1}^{r} \text{Inv}_{k_0}(G, p_i C)$$

that is functorial with respect to changes in the field $k_0$ and the group $G$.

3.2. Lemma. If $k_1$ is a finite extension of $k_0$ of dimension relatively prime to the exponent of $C$, then the natural map

$$\text{Inv}_{k_0}(G, C) \to \text{Inv}_{k_1}(G, C)$$

is an injection.

Proof. By 3.1, we may assume that the exponent of $C$ is a power of a prime $p$.

Let $a$ be an invariant in the kernel of the displayed map. Fix an extension $E/k_0$ and an element $x \in H^1(E, G)$; we show that $a(x)$ is zero in $M(E, C)$, hence $a$ is the zero invariant.

First suppose that $k_1/k_0$ is separable. The tensor product $E \otimes_{k_0} k_1$ is a direct product of fields $E_1 \times E_2 \times \cdots \times E_r$ (since $k_1$ is separable over $k_0$), and at least one of them—say, $E_1$—has dimension over $E$ not divisible by $p$ (because $p$ does not divide $[k_1 : k_0]$). We have

$$\text{res}_{E_1/E} a(x) = a(\text{res}_{E_1/E} x) = 0$$

because $k_1$ injects into $E_1$. But the dimension $[E_1 : E]$ is not divisible by $p$, so $a(x)$ is zero in $M(E, C)$.

If $k_1/k_0$ is purely inseparable, then there is a compositum $E_1$ of $E$ and $k_1$ such that the dimension of $E_1/E$ is a power of the characteristic, which (by global hypothesis) is not $p$. As in the previous paragraph, $a(x)$ is zero in $M(E, C)$.

In the general case, let $k_s$ be the separable closure of $k_0$ in $k_1$. The map displayed in the lemma is the composition

$$\text{Inv}_{k_0}(G, C) \to \text{Inv}_{k_s}(G, C) \to \text{Inv}_{k_1}(G, C),$$

and both arrows are injective by the preceding two paragraphs. Hence the composition is injective. \qed

3.3. Suppose that $k_1/k_0$ is finite of dimension relatively prime to the exponent of $C$ as in 3.2, and suppose further that $k_1/k_0$ is quasi-Galois (= normal), i.e., $k_1$ is the splitting field for a collection of polynomials in $k_0[x]$. The separable closure $k_s$ of $k_0$ in $k_1$ is a Galois extension of $k_0$. (See [Bou Alg, §V.11, Prop. 13] for the general structure of $k_1/k_0$.) We write $\text{Gal}(k_1/k_0)$ for the group of $k_0$-automorphisms of $k_1$.

The group $\text{Gal}(k_1/k_0)$ acts on $H^1(k_1, G)$ as follows. An element $g \in \text{Gal}(k_1/k_0)$ sends a 1-cocycle $b$ to a 1-cocycle $g \ast b$ defined by

$$(g \ast b)_s = g b_{g^{-1} s}.$$  

The Galois group acts similarly acts on $M(k_1, C)$, see e.g. [Wei69, Cor. 2-3-3].
Lemma. If $k_1/k_0$ is finite quasi-Galois and $[k_1 : k_0]$ is relatively prime to the exponent of $C$, then the restriction map

$$M(k_0, C) \rightarrow M(k_1, C)$$

identifies $M(k_0, C)$ with the subgroup of $M(k_1, C)$ consisting of elements fixed by $\text{Gal}(k_1/k_0)$. \hfill \square

Proof. Write $k_i$ for the maximal purely inseparable subextension of $k_1/k_0$; the extension $k_1/k_i$ is Galois. It is standard that the restriction map $M(k_i, C) \rightarrow M(k_1, C)$ identifies $M(k_i, C)$ with the $\text{Gal}(k_1/k_i)$-fixed elements of $M(k_1, C)$. To complete the proof, it suffices to note that restriction identifies $\text{Gal}(k_1/k_i)$ with $\text{Gal}(k_1/k_0)$ and $M(k_0, C)$ with $M(k_i, C)$, because $k_i/k_0$ is purely inseparable. \hfill \square

3.4. INVARIANTS UNDER QUASI-GALOIS EXTENSIONS. Continue the assumption that $k_1$ is a finite quasi-Galois extension of $k_0$. For every extension $E$ of $k_0$, there is—up to $k_0$-isomorphism—a unique compositum $E_1$ of $E$ and $k_1$; the field $E_1$ is quasi-Galois over $E$ and $\text{Gal}(E_1/E)$ is identified with a subgroup of $\text{Gal}(k_1/k_0)$. We say that an invariant $a \in \text{Inv}_{k_1}(G, C)$ is Galois-fixed if for every $E/k_0$, $x \in H^1(E_1, G)$, and $g \in \text{Gal}(E_1/E)$, we have

$$g \ast a(g^{-1} \ast x) = a(x) \in M(E_1, C).$$

Proposition. If $k_1/k_0$ is finite quasi-Galois and $[k_1 : k_0]$ is relatively prime to the exponent of $C$, then the restriction map

$$\text{Inv}_{k_0}(G, C) \rightarrow \text{Inv}_{k_1}(G, C)$$

identifies $\text{Inv}_{k_0}(G, C)$ with the subgroup of Galois-fixed invariants in $\text{Inv}_{k_1}(G, C)$. \hfill \square

Proof. The restriction map is an injection by Lemma 3.2.

Fix an invariant $a_1 \in \text{Inv}_{k_1}(G, C)$. If $a_1$ is the restriction of an invariant defined over $k_0$, then $a_1$ commutes with every morphism in $\text{Aut}_{\text{Fields}_E}(E_1)$ for every extension $E/k_0$, i.e., $a_1$ is Galois-fixed.

To prove the converse, suppose that $a_1$ is Galois-fixed. For $x \in H^1(E, G)$ and $g \in \text{Gal}(E_1/E)$, we have

$$g \ast a_1(\text{res}_{E_1/E} x) = a_1(g \ast \text{res}_{E_1/E} x) = a_1(\text{res}_{E_1/E} x) \in M(E_1, C)$$

since $a_1$ is Galois-fixed. Lemma 3.3 gives that $a_1(\text{res}_{E_1/E} x)$ is the restriction of a unique element $a_0(x)$ in $M(E, C)$. In this way, we obtain a function $H^1(E, G) \rightarrow M(E, C)$. It is an exercise to verify that this defines an invariant $a_0 : H^1(\ast, G) \rightarrow M(\ast, C)$. Clearly, the restriction of $a_0$ to $k_1$ is $a_1$. \hfill \square

3.5. Continue the assumption that $k_1/k_0$ is finite quasi-Galois and $[k_1 : k_0]$ is relatively prime to the exponent of $C$.

We fix a natural number $n$ not divisible by the characteristic of $k_0$ such that $nC = 0$, and we suppose that $\text{Inv}_{k_0}^\text{norm}(G, C)$ contains $a_1, a_2, \ldots, a_r$ whose restrictions form an $R_n(k_0)$-basis of $\text{Inv}_{k_1}^\text{norm}(G, C)$. We find:

Corollary. $a_1, a_2, \ldots, a_r$ is an $R_n(k_0)$-basis of $\text{Inv}_{k_0}^\text{norm}(G, C)$.

[Clearly, the corollary also holds if one can replaces $\text{Inv}_{k_0}^\text{norm}$ with $\text{Inv}$ throughout.]
Proof. Since \( k_1 \) is finite quasi-Galois over \( k_0 \), restriction identifies \( R_n(k_0) \) with the \( \text{Gal}(k_1/k_0) \)-fixed elements in \( R_n(k_1) \) (by Lemma 3.3 with \( C = \mu_n \)) and the natural map

\[
\text{Inv}_{k_0}(G, C) \rightarrow \text{Inv}_{k_1}(G, C)
\]

is an injection by Prop. 3.4.

Let \( \lambda_1, \lambda_2, \ldots, \lambda_r \in R_n(k_0) \) be such that \( \sum \lambda_i a_i \) is zero in \( \text{Inv}_{k_0}(G, C) \). Every \( \lambda_i \) is killed by \( k_1 \), hence \( \lambda_i \) is zero in \( R_n(k_0) \) for all \( i \). This proves that the \( a_i \) are linearly independent over \( k_0 \).

As for spanning, let \( a \) be in \( \text{Inv}_{k_0}(G, C) \). The restriction of \( a \) to \( k_1 \) equals \( \sum \lambda_i a_i \) for some \( \lambda_i \in R_n(k_1) \). But \( a \) is fixed by \( \text{Gal}(k_1/k_0) \), hence so are the \( \lambda_i \), i.e., \( \lambda_i \) is the restriction of an element of \( R_n(k_0) \) which we may as well denote also by \( \lambda_i \). Since \( a - \sum \lambda_i a_i \) is zero over \( k_1 \), it is zero over \( k_0 \). This proves that the \( a_i \) span over \( k_0 \). \( \square \)

3.7. Proposition. If \( p \) is a prime not equal to the characteristic of \( k \), then \( \text{Inv}^\text{norm}_{k_0}(\mathbb{Z}/p^r\mathbb{Z}, \mathbb{Z}/p^r\mathbb{Z}) \) is a free \( R_p(k_0) \)-module with basis \( \mathbf{id} \).

[The reader may wonder why we have switched to describing the normalized invariants, whereas in the proposition above and in S, the full module of invariants was described. The difficulty is that here the invariants are taking values in \( H^0(\ast, \text{Hom}(\mu_p, \mathbb{Z}/p\mathbb{Z})) \oplus H^1(\ast, \mathbb{Z}/p\mathbb{Z}) \oplus H^2(\ast, \mu_p) \oplus \cdots \), and it is not clear how to specify a basis for the constant invariants.]

Proof. If \( k_0 \) contains a primitive \( p \)-th root of unity, then we may use it to identify \( \mathbb{Z}/p\mathbb{Z} \) with \( \mu_p \), and apply Prop. 2.1.

For the general case, take \( k_1 \) to be the extension obtained by adjoining a primitive \( p \)-th root of unity; it is a Galois extension of degree not divisible by \( p \), and the proposition holds for \( k_1 \) by the previous paragraph. Cor. 3.5 finishes the proof. \( \square \)

3.8. Exercise. Extend Prop. 3.7 by describing \( \text{Inv}^\text{norm}_{k_0}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \), where \( n \) is square-free and not divisible by the characteristic.

3.9. Exercise (mod \( p \) Bockstein). Let \( p \) be a prime not equal to the characteristic of \( k_0 \). The natural exact sequence

\[
1 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 1
\]

leads to a connecting homomorphism

\[
\delta_k : H^1(k, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^2(k, \mathbb{Z}/p\mathbb{Z})
\]

for each extension \( k/k_0 \). This is a normalized invariant of \( \mathbb{Z}/p\mathbb{Z} \), and arguments similar to those above show that it is of the form

\[
\delta_k(x) = c \cdot x
\]

for a uniquely determined \( c \in H^1(k_0, \mathbb{Z}/p\mathbb{Z}) \). Compute \( c \).

[In case \( p = 2 \), the answer is well-known to be the class of \(-1 \in k_0^2/k_0^2 \). In general, \( c \) can be expressed in terms of the cyclotomic character \( \text{Gal}(k_0) \rightarrow \mathbb{Z}_p^\times \) and a homomorphism \( \mathbb{Z}_p^\times \rightarrow \mathbb{Z}/p\mathbb{Z} \).]

3.10. Exercise. Let \( k_0 \) be a field of characteristic zero. What are the mod 2 invariants of the dihedral group \( G \) of order 8? That is, what is \( \text{Inv}^\text{norm}_{k_0}(G, \mathbb{Z}/2\mathbb{Z}) \)?

[Note that \( G \) is the Weyl group of a root system of type \( B_2 \), so one may apply S25.15: an invariant of \( G \) is determined by its restriction to the elementary abelian 2-subgroups of \( G \).]
4. Restricting invariants

4.1. Let $A$ and $A'$ be functors $\text{Fields}_{k_0} \to \text{Sets}$, and fix a morphism $\phi: A' \to A$. (For example, a homomorphism of algebraic groups $G' \to G$ induces such a morphism of functors $H^1(\ast, G') \to H^1(\ast, G)$.) We are interested in the following condition:

For every extension $k_1/k_0$ and every $x \in A(k_1)$ there is a finite extension $k_2$ of $k_1$ such that

\begin{equation}
(4.2)
\begin{align*}
& (1) \text{res}_{k_2/k_1}(x) \in A(k_2) \text{ is } \phi(x') \text{ for some } x' \in A'(k_2) \text{ and } \\
& (2) \text{the dimension } [k_2 : k_1] \text{ is relatively prime to the exponent of } C.
\end{align*}
\end{equation}

(In the case where $[k_2 : k_1]$ can always be chosen to be not divisible by a prime $p$, we say that $\phi$ is surjective at $p$.)

**Lemma.** If (4.2) holds, then the restriction map

$$\phi^*: \text{Inv}_{k_0}(A, C) \to \text{Inv}_{k_0}(A', C)$$

induced by $\phi$ is an injection.

We will strengthen this result in Section 6.

**Proof.** $\phi^*$ is a group homomorphism, so it suffices to prove that the kernel of $\phi^*$ is zero; let $a$ be in the kernel of $\phi^*$. Fix an extension $k_1$ of $k_0$ and a class $x \in A(k_1)$, and let $k_2$ be as in (4.2). By the assumption on $a$, the class $a(x) \in M(k_1, C)$ is killed by $k_2$. But the map $M(k_1, C) \to M(k_2, C)$ is injective by 4.2.2, so $a(x)$ is zero in $M(k_1, C)$. That is, $a$ is the zero invariant. □

4.3. Killable classes. Suppose that there is a natural number $e$ such that every element of $H^1(k, G)$ is killed by an extension of $k$ of degree dividing $e$ for every extension $k$ of $k_0$. This happens, for example, when:

1. $G = PGL_e$, a standard result from the theory of central simple algebras
2. $G$ is a finite constant group and $e = |G|$, because every 1-cocycle is a homomorphism $\varphi: \text{Gal}(k_1) \to G$ and $\varphi$ is killed by the extension $k_2$ of $k_1$ fixed by $\ker \varphi$. The dimension of $k_2$ over $k_1$ equals the size of the image of $\varphi$, which divides the order of $G$. (Compare S15.4.)

Applying Lemma 4.1 with $G'$ the group with one element gives: If the exponent of $C$ is relatively prime to $e$, then $\text{Inv}_{k_0}^\text{norm}(G, C)$ is zero.

4.4. Example. Suppose that $k_0$ is algebraically closed of characteristic zero, $G$ is a connected algebraic group, and the exponent of $C$ is relatively prime to the order of the Weyl group of a Levi subgroup of $G$. Then $\text{Inv}(G, C)$ is zero. Indeed, the paper [CGR06] gives a finite constant subgroup $S$ of $G$ such that the exponent of $C$ is relatively prime to $|S|$ and the map $H^1(k, S) \to H^1(k, G)$ is surjective for every extension $k$ of $k_0$. (We remark that the existence of such a subgroup $S$ answers the question implicit in the final paragraph of S22.10.) As a consequence of the surjectivity, the restriction map $\text{Inv}(G, C) \to \text{Inv}(S, C)$ is an injection. Hence $\text{Inv}(G, C)$ is zero by 4.3.

4.5. The previous example gives a “coarse bound” in the case where $G$ is simple. For $G$ simple of type $E_8$, the order of the Weyl group is $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$. So—roughly speaking—the previous example shows there are no nonconstant cohomological invariants mod $p$ for $p \neq 2, 3, 5, 7$. Tits [Tit92] showed that every $E_8$-torsor is split.
by an extension of degree dividing $2^9 \cdot 3^3 \cdot 5$, hence by 4.3 there are no nonconstant
invariants mod $p$ also for $p = 7$.

In Table 4, for each type of exceptional group $G$ and prime $p$, we give a reference
for classification results regarding the invariants $\text{Inv}^{\text{norm}}(G, C)$ where the exponent
of $C$ is a power of $p$. The preceding argument shows that $\text{Inv}^{\text{norm}}(G, C)$ is zero for all exceptional $G$ and $p \neq 2, 3, 5$.

<table>
<thead>
<tr>
<th>type of $G$</th>
<th>$p = 2$</th>
<th>$p = 3$</th>
<th>$p = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_2$</td>
<td>S18.4</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>$F_4$</td>
<td>S22.5</td>
<td>Th. 7.4</td>
<td>X</td>
</tr>
<tr>
<td>inner type $E_6$</td>
<td>Exercise 22.9 in S</td>
<td>Th. 10.9</td>
<td>X</td>
</tr>
<tr>
<td>outer type $E_6$</td>
<td>?</td>
<td>Exercise 10.10</td>
<td>X</td>
</tr>
<tr>
<td>$E_7$</td>
<td>?</td>
<td>12.2, Exercise 12.3</td>
<td>X</td>
</tr>
<tr>
<td>$E_8$</td>
<td>?</td>
<td>?</td>
<td>Th. 14.1</td>
</tr>
</tbody>
</table>

Table 4. References for results on $\text{Inv}^{\text{norm}}(G, C)$ where $G$ is ex-
ceptional and the exponent of $C$ is a power of a prime $p$

For entries marked with an X, $\text{Inv}^{\text{norm}}(G, C)$ is zero. For conjectures re-
garding the question marks, see Problems 12.4 and 14.3.

4.6. Example (Groups of square-free order). Suppose now that $k_0$ is algebraically
closed, $G$ is a finite constant group, and $|G|$ is square-free and not divisible by the
characteristic of $k_0$. Then every normalized invariant $H^1(\ast, G) \to H^d(\ast, C)$ is zero
for $d > 1$. Indeed, by 3.1 and 4.3, we may assume that $C$ is a power of prime $p$
dividing $|G|$. For $G'$ a $p$-Sylow subgroup of $G$, S15.4 (a more powerful version of
Lemma 4.1 that is specifically for finite groups) says the restriction map

$$
\text{Inv}^{\text{norm}}(G, C) \to \text{Inv}^{\text{norm}}(G', C)
$$

is injective. As $G'$ is isomorphic to $\mu_p$, Exercise 2.7 says that every normalized
invariant $H^1(\ast, \mu_p) \to H^{d}(\ast, C)$ can be written uniquely as $\text{id} \bullet x$ for some $p$-torsion
element $x \in H^{d-1}(k_0, C(-1))$. But this last set is zero because $k_0$ is algebraically
closed.

4.7. Ineffective bounds for essential dimension. Recall from S5.7 that the essential dimension of an algebraic group $G$ over $k_0$—written $\text{ed}(G)$—is the minimal transcendence degree of $K/k_0$, where $K$ is the field of definition of a versal $G$-torsor. Cohomological invariants can be used to prove lower bounds on $\text{ed}(G)$: If $k_0$ is algebraically closed and there is a nonzero invariant $H^1(\ast, G) \to H^d(\ast, C)$, then $\text{ed}(G) \geq d$, see S12.4.

But this bound need not be sharp, as Example 4.6 shows. Indeed, in that example
we find the lower bound $\text{ed}(G) \geq 1$ for $G$ not the trivial group. But when $k_0$ has
characteristic zero and $G$ is neither cyclic nor dihedral of order $2 \cdot (\text{odd})$, $\text{ed}(G) \geq 2$
by [BR97, Th. 6.2].

Another example is furnished by the alternating group $A_6$. The bound provided
by cohomological invariants is $\text{ed}(A_6) \geq 2$ but the essential dimension cannot be 2
(Serre, unpublished).
5. Mod $p$ invariants of $\text{PGL}_p$

In this section, we fix a prime $p$ not equal to the characteristic of $k_0$. Our goal is to determine the invariants of $\text{PGL}_p$ with values in $\mathbb{Z}/p\mathbb{Z}$.

The short exact sequence

$$1 \longrightarrow \mu_n \longrightarrow \text{SL}_n \longrightarrow \text{PGL}_n \longrightarrow 1$$

gives a connecting homomorphism $\delta : H^1(k, \text{PGL}_n) \rightarrow H^2(k, \mu_n)$, cf. [Ser02, Ch. III], [KMRT98, p. 386], or Example 1.2.2. We remark that $\delta$ gives a connecting homomorphism and an element $z$.

Let $\text{char } k$ be a natural number not divisible by $\text{char } k_0$ and fix a primitive $n$-th root of unity $\zeta$ in some separable closure of $k_0$. Let $e_i$ denote the $i$-th standard basis vector of $k_0^n$. Define $u, v \in \text{GL}_n$ to be the matrices such that

$$u(e_i) = \zeta^i e_i \text{ and } v(e_i) = \begin{cases} e_{i+1} & \text{for } 1 \leq i < n \\ e_1 & \text{for } i = n. \end{cases}$$

The maps $\mathbb{Z}/n\mathbb{Z} \rightarrow \text{GL}_n$ and $\mu_n \rightarrow \text{GL}_n$ given by $i \mapsto v^i$ and $j \mapsto u^j$ are defined over $k_0$. Since $uv = \zeta vu$, there is a map

$$c : \mathbb{Z}/n\mathbb{Z} \times \mu_n \rightarrow \text{PGL}_n$$

defined over $k_0$ given by $(i, \zeta^j) \mapsto \overline{v}^i \overline{\tau}^j$ for $\overline{u}, \overline{v}$ the images of $u, v$ in $\text{PGL}_n$.

The sets $H^1(k, \mathbb{Z}/n\mathbb{Z})$ and $H^1(k, \text{PGL}_n)$ classify cyclic extensions $k'$ of $k$ and central simple $k$-algebras of degree $n$ respectively. Recall that $H^1(k, \mu_n) = k^\times/k^{\times n}$.

The map

$$c_* : H^1(k, \mathbb{Z}/n\mathbb{Z}) \times H^1(k, \mu_n) \rightarrow H^1(k, \text{PGL}_n)$$

sends the cyclic extension $k'$ and $\alpha \in k^\times/k^{\times n}$ to the class of the cyclic algebra $(k', \alpha)$, see Exercise 5.4 below.

5.4. Exercise. The cyclic algebra $(k', \alpha)$ is defined to be the $k$-algebra generated by $k'$ and an element $z$ such that $z^\ell = \rho(\ell)z$ for all $\ell \in k'$ and $\rho$ a fixed generator of $\text{Gal}(k'/k)$. Justify the italicized claim in 5.2.

[One possible solution: Fix a separable closure $k_{\text{sep}}$ of $k$. The image of $k'$ and $\alpha$ under $c$ define a 1-cocycle in $H^1(k, \text{PGL}_n)$, which defines a twisted Galois action on $M_n(k_{\text{sep}}).$ A 1-cocycle determining $k'$ also determines a preferred generator of $\text{Gal}(k'/k)$; fix an element $\rho \in \text{Gal}(k_{\text{sep}}/k)$ which restricts to this preferred generator. Prove that the map $f : k' \rightarrow M_n(k_{\text{sep}})$ given by $f(\beta)e_i = \rho(\beta)e_i$ is defined over $k$. Fix an $n$-th root $\alpha$ of $\alpha$. Prove that the element $z = au^{-1}$ in $M_n(k_{\text{sep}})$ is $F$-defined. Conclude that the fixed subalgebra of $M_n(k_{\text{sep}})$ is isomorphic to $(k', \alpha).$]

5.5. Remark. The composition $\delta c_*$ is a map

$$H^1(k, \mathbb{Z}/n\mathbb{Z}) \times H^1(k, \mu_n) \rightarrow H^2(k, \mu_n).$$

There is another such map given by the cup product; they are related by

$$\delta c_*(k', \alpha) = -(k') \cdot (\alpha),$$
see [KMRT98, pp. 397, 415].

5.6. Lemma. If $A$ is a central simple algebra over $k$ of dimension $p^2$, there is a finite extension $k'/k$, of degree prime to $p$, over which $A$ becomes cyclic.

That is, the map (5.3) is surjective at $p$.

Proof. This is well known. Recall the proof. We may assume that $A$ is a division algebra, in which case it contains a field $L$ that is a separable extension of $k$ of degree $p$. Let $E$ be the smallest Galois extension of $k$ containing $L$ (in some algebraic closure of $k$); the Galois group $\Gamma$ of $E/k$ is a transitive subgroup of the symmetric group $S_p$; a $p$-Sylow subgroup $S$ of $\Gamma$ is thus cyclic of order $p$. Take for $k'$ the subfield of $E$ fixed by $S$. We have $E = Lk'$. Hence $E$ is a cyclic extension of $k'$ of degree $p$ which splits $A$ over $k'$. $\square$

5.7. Invariants of a product. Suppose we have algebraic groups $G$ and $G'$ such that

\begin{equation}
\text{(5.8)} \quad \text{There is a set } \{a_i\} \subset \text{Inv}^{\text{norm}}_{k_0}(G,C) \text{ that is an } R_n(k)-\text{basis of } \text{Inv}^{\text{norm}}_{k}(G,C) \text{ for every extension } k/k_0, \text{ and }
\end{equation}

\begin{equation}
\text{(5.9)} \quad \text{There is an } R_n(k_0)-\text{basis } \{b_j\} \text{ of } \text{Inv}_{k_0}(G',\mu_n),
\end{equation}

where $n$ is the exponent of $C$. The cup product

\[ H^{d_1}(*,C(d_1-1)) \times H^{d_2}(*,\mathbb{Z}/n\mathbb{Z}(d_2)) \to H^{d_1+d_2}(*,C(d_1+d_2-1)) \]

induces an $R_n(k_0)$-module homomorphism

\begin{equation}
\text{(5.10)} \quad \text{Inv}^{\text{norm}}_{k_0}(G,C) \otimes_{R_n(k_0)} \text{Inv}_{k_0}(G',\mu_n) \to \text{Inv}^{\text{norm}}_{k_0}(G \times G',C).
\end{equation}

Lemma. The map (5.10) is injective. Its image $I$ is the set of normalized invariants whose restriction to $H^1(*,G')$ is zero. The images of the $a_i \otimes b_j$ form a basis for $I$ as an $R_n(k_0)$-module.

This is a slight variation of Exercise 16.5 in S. We give a proof because we will use this result repeatedly later.

Proof. Let $c$ be a normalized invariant of $G \times G'$ with values in $C$ that vanishes on $H^1(*,G')$. For a given $k$-$G'$-torsor $T'$, the map

\[ c_{T'}: T \mapsto c(T \times T') \]

is an invariant of $G$ with values in $C$. As $c$ vanishes on $H^1(*,G')$, $c_{T'}$ is normalized. By (5.8), $c_{T'}$ is the map $T \mapsto \sum \lambda_{i,T'} a_i(T)$ for uniquely determined $\lambda_{i,T'} \in R_n(k)$. The maps $T' \mapsto \lambda_{i,T'}$ are invariants of $G'$ and belong to $\text{Inv}_{k_0}(G',\mu_n)$, which by (5.9) can be written uniquely as $\sum \lambda_{i,j} b_j$ for $\lambda_{i,j} \in R_n(k_0)$. This proves that $c$ is the image of $\sum \lambda_{i,j} a_i \otimes b_j$, hence that the image of (5.10) includes every normalized invariant whose restriction to $H^1(*,G')$ is zero. As the reverse inclusion is trivial, we have proved the second sentence in the lemma.

The proof of the first sentence is similar. Suppose that the invariant

\[ T \times T' \mapsto \sum_{i,j} \lambda_{i,j} b_j(T') a_i(T) \]

of $G \times G'$ is zero, where the $\lambda_{i,j}$ are in $R_n(k_0)$. For each $k$-$G'$-torsor $T'$, we find that $\sum_j \lambda_{i,j} b_j(T')$ is zero by (5.8), hence the invariant $\sum_j \lambda_{i,j} b_j$ is zero. By (5.9), $\lambda_{i,j}$ is zero for all $i,j$.
Because $\text{Inv}^\text{norm}_{k_0}(G, C)$ and $\text{Inv}_{k_0}(G', \mu_n)$ are free $R_n(k_0)$-modules, the third sentence in the lemma follows from the first two.

In the examples below, the set $\{b_j\}$ is a basis of $\text{Inv}_k(G', \mu_n)$ for every extension $k/k_0$. This implies that the lemma holds when $k_0$ is replaced with $k$.

We can now prove Prop. 5.1.

Proof of Prop. 5.1. Combining Lemmas 5.6 and 4.1, we find that the map

\[(5.11) \quad c^*: \text{Inv}^\text{norm}_{k_0}(\text{PGL}_p, \mathbb{Z}/p\mathbb{Z}) \to \text{Inv}^\text{norm}_{k_0}(\mathbb{Z}/p\mathbb{Z} \times \mu_p, \mathbb{Z}/p\mathbb{Z})\]

induced by $c$ is an injection.

It follows from Remark 5.5 or [KMRT98, 30.6] that $c_*(x, 1)$ and $c_*(1, y)$ are the neutral class in $H^1(k, \text{PGL}_p)$ for every extension $k/k_0$, every $x \in H^1(k, \mathbb{Z}/p\mathbb{Z})$, and every $y \in H^1(k, \mu_p)$. In particular the image of (5.11) is contained in the submodule $I$ of invariants that are zero on $H^1(*, \mu_p)$.

Fix a normalized invariant $a$ in $\text{Inv}_{k_0}(\text{PGL}_p, \mathbb{Z}/p\mathbb{Z})$. By Lemma 5.7, Prop. 3.7, and Prop. 2.1, its image $c^*a$ under (5.11) is of the form

$$(x, y) \mapsto \lambda_1 \cdot x + \lambda_2 \cdot y \cdot x \quad (x \in H^1(k, \mathbb{Z}/p\mathbb{Z}), y \in H^1(k, \mu_p))$$

for uniquely determined $\lambda_1, \lambda_2 \in R_p(k_0)$. But

$$(c^*a)(x, 1) = a(c_*(x, 1)) = a(M_p(k)) = 0$$

for every $x \in H^1(k, \mathbb{Z}/p\mathbb{Z})$ and every extension $k$, so $\lambda_1$ is zero. Therefore,

$$(c^*a)(x, y) = \lambda_2 \cdot y \cdot x.$$ 

Since $c^*a$ is a $R_p(k_0)$-multiple of $c^*\delta$, we conclude that $\delta$ spans $\text{Inv}^\text{norm}(\text{PGL}_p, \mathbb{Z}/p\mathbb{Z})$.

\[\square\]

5.12. Remark. A versal torsor for $\mathbb{Z}/p\mathbb{Z} \times \mu_p$ gives a $\text{PGL}_p$-torsor $T$. The injectivity of (5.11) combined with S12.3 shows that invariants $a, a'$ of $\text{PGL}_p$ that agree on $T$ are the same. One may view $T$ as a “$p$-versal torsor” (appropriately defined) for $\text{PGL}_p$.

5.13. Open problem. (Reichstein-Youssin [RY00, p. 1047]) Let $k_0$ be an algebraically closed field of characteristic zero. Is there a nonzero invariant $H^1(*, \text{PGL}_p) \to H^{2r}(*, \mathbb{Z}/p\mathbb{Z})$?

[For $p = r = 2$, one has the Rost-Serre-Tignol invariant described in Example 1.2.5.]

5.14. Question. Let $k_0$ be an algebraically closed field of characteristic zero. What are the mod 2 invariants of $\text{PGL}_4$? That is, what is $\text{Inv}^\text{norm}_{k_0}(\text{PGL}_4, \mathbb{Z}/2\mathbb{Z})$?

[This is a “question” and not an “exercise” because there are central simple algebras of dimension $4^2$ that are neither cyclic nor tensor products of two quaternion algebras [Alb33].]

6. Extending invariants

6.1. Fix functors $A$ and $A'$ mapping $\text{Fields}_{k_0} \to \text{Sets}$ and a morphism $\phi: A' \to A$. When can an invariant $a': A' \to M(*, C)$ be extended to an invariant $a: A \to \text{Cohomological invariants 17}$. 


$M(\ast, C)$? That is, when is there an invariant $a$ that makes the diagram

$$
\begin{array}{ccc}
A' & \xrightarrow{a'} & M(\ast, C) \\
\phi \downarrow & & \downarrow \\
A & \xleftarrow{a} &
\end{array}
$$

commute?

Clearly, we must have

$$
(6.2) \quad \text{For every extension } k/k_0 \text{ and every } x, y \in A'(k):
\phi(x) = \phi(y) \implies a'(x) = a'(y)
$$

**Proposition.** If $\phi$ satisfies (4.2), then the restriction

$$
\phi^* : \text{Inv}_{k_0}(A, C) \to \text{Inv}_{k_0}(A', C)
$$

defines an isomorphism of $\text{Inv}_{k_0}(A, C)$ with the invariants $a'$ of $A'$ satisfying (6.2).

That is, assuming (4.2), condition (6.2) is sufficient as well as necessary.

Note that the proposition gives a solution to Exercise 22.9 in $S$ as a corollary.

That is, if $\phi$ satisfies (4.2) and $\phi$ is injective, then the restriction map

$$
\phi^* : \text{Inv}_{k_0}(A, C) \to \text{Inv}_{k_0}(A', C)
$$

is an isomorphism.

The rest of this section is a proof of the proposition. The homomorphism $\phi^*$ is injective by Lemma 4.1, so it suffices to prove that every invariant $a'$ of $A'$ satisfying (6.2) is in the image. As in 3.1, we may assume that the exponent of $C$ is the power of a prime $p$.

6.3. For each perfect field $k/k_0$ and each $x \in A(k)$, we define an element $a(x) \in M(k, C)$ as follows. Fix an extension $k_2$ of $k$ as in (4.2), i.e., such that there is an $x' \in A'(k_2)$ such that $\phi(x')$ is the restriction of $x$.

**Lemma A.** $a'(x')$ is the restriction of the unique element of $M(k, C)$.

We define $a(x)$ to be the unique element of $M(k, C)$ such that $\text{res}_{k_2/k} a(x)$ is $a'(x')$. For the proof of this lemma and Lemma B below, we fix a separable closure $k_{\text{sep}}$ of $k$ (hence also of $k$).

**Proof.** Uniqueness is easy, so we prove that $a'(x')$ is defined over $k$.

For each finite extension $k_3$ of $k_2$ in $k_{\text{sep}}$ and every $\sigma \in \text{Gal}(k_{\text{sep}}/k)$ such that $\sigma(k_3) \supseteq k_2$, we claim that

$$
(6.4) \quad \sigma_* \text{res}_{k_3/k_2} a'(x') = \text{res}_{\sigma(k_3)/k_2}(a'(x'))
$$

in $M(\sigma(k_3), C)$, i.e., that $a'(x')$ is “stable” in $M(k_2, C)$. The invariant $a'$ commutes with $\sigma_*$ and res. By (6.2), Equation (6.4) is equivalent to

$$
\phi(\sigma_* \text{res}_{k_3/k_2} x') = \phi(\text{res}_{\sigma(k_3)/k_2} x').
$$

The morphism $\phi$ also commutes with $\sigma_*$ and res, so this equation is equivalent to

$$
\sigma_* \text{res}_{k_3/k_2} x = \text{res}_{\sigma(k_3)/k_2} x,
$$

which holds because $x$ is defined over $k$. This proves (6.4).

Combining (6.4) with the double coset formula for the composition res $\circ$ cor as in [AM04, Th. II.6.6] shows that $a'(x')$ is the restriction of an element of $M(k, C)$. □
Lemma B. The element \( a(x) \in M(k, C) \) depends only on \( x \) (and not on the choice of \( k_2 \) and \( x' \)).

Proof. Let \( \ell_2 \) be a finite extension of \( k \) in \( k_{\text{sep}} \) such that \( \text{res}_{\ell_2/k} x \) is the image of some \( y' \in A'(\ell_2) \) and the prime \( p \) does not divide \( [\ell_2 : k] \). (I.e., \( \ell_2 \) is an extension as provided by (4.2), and it is separable because \( k \) is perfect.) We prove that \( a'(x') \in M(k_2, C) \) and \( a'(y') \in M(\ell_2, C) \) are restrictions of the same element in \( M(k, C) \).

Case 1: \( \ell_2 \) is a conjugate of \( k_2 \). Suppose that there is a \( \sigma \in \text{Gal}(k_{\text{sep}}/k) \) such that \( \sigma(\ell_2) \) equals \( k_2 \). One quickly checks that \( \phi(\sigma y') \) equals \( \phi(x') \) in \( A(k_2) \), hence \( a'(x') \) equals \( a'(\sigma y') \) by (6.2), i.e., \( a'(x') \) is \( \sigma a'(y') \). The lemma follows in this special case.

Case 2. Suppose that the compositum \( K \) of \( k_2 \) and \( \ell_2 \) in \( k_{\text{sep}} \) has degree \( [K : k] \) not divisible by the prime \( p \). Since \( \phi(\text{res}_{K/k_2} x') \) and \( \phi(\text{res}_{K/\ell_2} y') \) equal \( \text{res}_{K/k}(x) \), the restriction of \( a'(x') \) and \( a'(y') \) in \( K \) agree. By the hypothesis on the degree \( [K : k] \), the lemma holds in this special case.

Case 3: general case. Let \( S \) be a \( p \)-Sylow in \( \text{Gal}(k_{\text{sep}}/k) \) fixing \( k_2 \) elementwise. There is a \( \sigma \in \text{Gal}(k_{\text{sep}}/k) \) such that \( \sigma(\ell_2) \) is also fixed elementwise by \( S \). It follows that the compositum of \( \sigma(\ell_2) \) and \( k_2 \) has degree over \( K \) not divisible by \( p \). A combination of cases 1 and 2 gives the lemma in the general case. \( \square \)

6.5. For an arbitrary extension \( k \) of \( k_0 \), write \( k_2 \) for the “perfect closure” of \( k \). Since \( M(k, C) \) is canonically isomorphic to \( M(k_p, C) \), we define \( a(x) \) to be the element \( a(\text{res}_{k_2/k} x) \in M(k_p, C) \) defined in 6.3 above.

For every extension \( k \) of \( k_0 \), we have defined a function \( a_k \) making the diagram

\[
\begin{array}{ccc}
A'(k) & \xrightarrow{a'_k} & M(k, C) \\
\phi_k \downarrow & & \downarrow a_k \\
A(k) & & 
\end{array}
\]

commute. We leave the proof that this defines a morphism of functors \( A \to M(*, C) \) to the reader.

7. Mod 3 invariants of Albert algebras

In this section, we assume that \( k_0 \) has characteristic \( \neq 2,3 \) and classify the normalized mod 3 invariants of Albert algebras. Recall that Albert \( k \)-algebras are 27-dimensional exceptional Jordan algebras—see [SV00, Ch. 5], [PR94a], or [KMRT98, Ch. IX]—and we write \( \text{Alb} \) for the functor such that \( \text{Alb}(K) \) is the isomorphism classes of Albert \( K \)-algebras. We compute \( \text{Inv}^\text{norm}(\text{Alb}, \mathbb{Z}/3\mathbb{Z}) \).

The automorphism group of the “split” Albert algebra is a split algebraic group of type \( F_4 \), and by Galois descent we have an isomorphism of functors \( H^1(*, F_4) \cong \text{Alb}(*) \), see [KMRT98, p. 517]. This isomorphism identifies \( \text{Inv}^\text{norm}(\text{Alb}, \mathbb{Z}/3\mathbb{Z}) \) with \( \text{Inv}^\text{norm}(F_4, \mathbb{Z}/3\mathbb{Z}) \).

7.1. Example. Let \( M = M_3(k) \) be the algebra of 3-by-3 matrices over \( k \). On the 27-dimensional space \( J = M \times M \times M \), define a cubic form \( N \) by

\[
N(a, b, c) = \det(a) + \det(b) + \det(c) - \text{tr}(abc).
\]
Write 1 for the element $(1,0,0)$ in $J$. The “Springer construction” endows $J$ with the structure of an Albert $k$-algebra induced by $N$ and the choice of the element $1$, see [McC69, §5]. It is the split Albert algebra and its automorphism group $F_4$ is the subgroup of $GL(J)$ consisting of elements that fix 1 and $N$ [Jac59, Th. 4].

If $(g,z)$ is a point of $PGL_3 \times \mu_3$, let $t(g,z)$ be the element of $GL(J)$ defined by

$$(a,b,c) \mapsto (i_g(a), z \cdot i_g(b), z^2 \cdot i_g(c)),$$

where $i_g$ is the inner automorphism of $M$ defined by $g$. Since $(g,z)$ fixes both 1 and $N$, it belongs to the group $F_4$. This gives an inclusion $t: PGL_3 \times \mu_3 \to F_4$ and a corresponding map

$$(7.2) \quad t_*: H^1(\ast, PGL_3) \times H^1(\ast, \mu_3) \to H^1(\ast, F_4) \cong \text{Alb}(\ast).$$

The image of a pair $(A, \alpha)$ is often denoted by $J(A, \alpha)$; such algebras are known as first Tits constructions, cf. [KMRT98, §39.A].

Every Albert $k$-algebra is a first Tits construction or becomes one over a quadratic extension of $k$—see, e.g., [KMRT98, 39.19]—so the map in (7.2) satisfies (4.2) when $C$ has odd exponent. In particular, it is surjective at 3, hence the restriction map

$$t^*: \text{Inv}^\text{norm}(F_4, \mathbb{Z}/3\mathbb{Z}) \to \text{Inv}^\text{norm}(PGL_3 \times \mu_3, \mathbb{Z}/3\mathbb{Z})$$

is injective.

7.4. Invariants of $F_4$ mod 3. Consider the invariant

$$g_3: H^1(\ast, PGL_3) \times H^1(\ast, \mu_3) \to H^3(\ast, \mu_3^{\otimes 2})$$

defined by $g_3(A, \alpha) = \delta(A) \cdot (\alpha)$ for $\delta$ as defined in §5. We now give two arguments that $g_3$ is the restriction of an invariant of $F_4$.

**Proof #1.** The meat of [PR96] is their Lemma 4.1, which says that $g_3$ “factors through” the image of (7.2) in $H^1(\ast, F_4)$. That is, if the first Tits constructions $J(A, \alpha)$ and $J(A', \alpha')$ are isomorphic, then $g_3(A, \alpha)$ equals $g_3(A', \alpha')$. Prop. 6.1 gives that $g_3$ extends to an invariant of $F_4$. □

**Proof #2.** The Dynkin index of $F_4$ is 6 [Mer03, 16.9], so the mod 3 portion of the Rost invariant gives a nonzero invariant

$$g_3': H^1(\ast, F_4) \to H^3(\ast, \mu_3^{\otimes 2}).$$

Applying Lemma 5.7, we conclude that $t^*g_3'$ equals $\lambda g_3$ for some fixed $\lambda \in R_3(k_0)$. Since the image of $g_3'$ under $\text{Inv}^\text{norm}_{k_0}(G, \mathbb{Z}/3\mathbb{Z}) \to \text{Inv}^\text{norm}_K(G, \mathbb{Z}/3\mathbb{Z})$ is nonzero for every extension $K/k_0$, $t^*g_3'$ is nonzero over every $K$, and we conclude that $\lambda = \pm 1$, i.e., $t^*g_3'$ is $\pm g_3$. □

We abuse notation by writing $g_3$ also for the invariant $(t^*)^{-1}(g_3)$ of $F_4$. This invariant was originally constructed in [Ros91].

**Proposition.** $\text{Inv}^\text{norm}_{k_0}(F_4, \mathbb{Z}/3\mathbb{Z})$ is a free $R_3(k_0)$-module with basis $g_3$.

**Proof.** We imitate the proof of Prop. 5.1, with the role of $\mathbb{Z}/p\mathbb{Z} \times \mu_p$ played by $PGL_3 \times \mu_3$. For every central simple algebra $A$ over every extension $k/k_0$ and every $\alpha \in k^\times$, the algebra $J(A, \alpha)$ is “split”, i.e., $t_*(A, \alpha)$ is the neutral class in $H^1(k, F_4)$, if and only if $\alpha$ is the reduced norm of an element of $A^\times$ by [Jac68, p. 416, Th. 20] or [McC69, Th. 6]. In particular, $t_*(M_3(k), \alpha)$ is the neutral class
for every \( \alpha \), and Lemma 5.7 gives that the restriction of a normalized invariant in \( \text{Inv}_{k_0}(F_4, \mathbb{Z}/3\mathbb{Z}) \) to \( \text{PGL}_3 \times \mu_3 \) can be written as

\[
(A, \alpha) \mapsto \lambda_1 \cdot [A] + \lambda_2 \cdot \langle \alpha \rangle \cdot [A]
\]

for uniquely determined \( \lambda_1, \lambda_2 \in R_3(k_0) \). But the algebra \( J(A, 1) \) is also split for every \( A \). It follows that \( \lambda_1 \) is zero. This proves that \( g_3 \) spans \( \text{Inv}_{k_0}^\text{norm}(F_4, \mathbb{Z}/3\mathbb{Z}) \). □

Combining the proposition with the classification of the invariants mod 2 in S22.5, we have found just three interesting invariants of \( F_4 \), namely \( g_3 \), \( f_3 \), and \( f_5 \).

Perhaps the outstanding open problem in the theory of Albert algebras is:

7.5. **Open problem.** (Serre [Ser95, §9.4], [PR94a, Q. 1, p. 205]) Is the map

\[
g_3 \times f_3 \times f_5 : H^1(\ast, F_4) \to H^3(\ast, \mathbb{Z}/3\mathbb{Z}) \times H^3(\ast, \mathbb{Z}/2\mathbb{Z}) \times H^3(\ast, \mathbb{Z}/2\mathbb{Z})
\]

injective? That is, is an Albert algebra \( J \) determined up to isomorphism by its invariants \( g_3(J), f_3(J), \) and \( f_5(J) \)?

[The map is injective on the kernel of \( g_3 \) [SV00, 5.8.1], i.e., “reduced Albert algebras are classified by their trace form”. Also, Rost has an unpublished result on this problem, see [Ros02]. Note that it is still unknown if the map is injective on the kernel of \( f_3 \times f_5 \), i.e., for first Tits constructions.]

7.6. **Symbols.** We now drop the assumption that the characteristic of \( k_0 \) is \( \neq 2, 3 \), and instead assume that it does not divide some fixed natural number \( n \). We call an element \( x \in H^d(k, \mu_n^{(d-1)}) = M^d(k, \mathbb{Z}/n\mathbb{Z}) \) (for \( d \geq 2 \)) a **symbol** if it is in the image of the cup product map

\[
H^1(k, \mathbb{Z}/n\mathbb{Z}) \times H^1(k, \mu_n) \times \cdots \times H^1(k, \mu_n) \to H^d(k, \mu_n^{(d-1)})
\]

\( d-1 \) copies

In particular, the zero class is always a symbol. In the usual identification of \( H^2(k, \mu_n) \) with the \( n \)-torsion in the Brauer group of \( k \), symbols are identified with cyclic algebras of dimension \( n^2 \) as defined in §5.

7.7. **Example.** In the case \( n = 2 \), \( M^d(k, \mathbb{Z}/2\mathbb{Z}) \) is just \( H^d(k, \mathbb{Z}/2\mathbb{Z}) \), and it is isomorphic to \( I^d/I^{d-1} \) as in 1.2.3. Symbols in \( M^d(k, \mathbb{Z}/2\mathbb{Z}) \) correspond to the (equivalence classes of) \( d \)-Pfister quadratic forms. Further, one has the following nice property: If there is an odd-degree extension \( K/k \) such that \( \text{res}_{K/k}(x) \) is a symbol in \( H^d(k, \mathbb{Z}/2\mathbb{Z}) \), then \( x \) is itself a symbol by [Ros99a, Prop. 2].

In the case \( n = 3 \) (and char \( k_0 \neq 3 \)), we have the following weaker property, mentioned in [Ros99a]:

7.8. **Lemma.** Fix \( x \in H^2(k, \mu_3) \). If there is an extension \( K/k \) such that \( 3 \) does not divide the dimension \( [K : k] \) and \( \text{res}_{K/k}(x) \) is a symbol in \( H^2(K, \mu_3) \), then \( x \) is itself a symbol.

**Proof.** We identify \( H^2(k, \mu_3) \) and \( H^2(K, \mu_3) \) with the 3-torsion in the Brauer group of \( k \) and \( K \) respectively. We assume that \( x \) is nonzero, hence that it corresponds to a central division \( k \)-algebra \( A \) of dimension \( (3^r)^2 \) for some positive \( r \). By hypothesis, \( A \otimes K \) is isomorphic to \( M_r(B) \) for a cyclic \( K \)-algebra \( B \) of dimension \( 3^2 \). But as \( 3 \) does not divide \( [K : k] \), the index of \( A \) and \( A \otimes K \) agree [Dra83, §9, Th. 12]. It follows that \( A \) is a division algebra of dimension \( 3^2 \) over \( k \), hence by Wedderburn’s Theorem [KMRT98, 19.2] \( A \) is cyclic, i.e., \( x \) is a symbol. □
Returning to groups of type $F_4$, the image of the invariant
\[ g_3: H^1(k, F_4) \to H^3(k, \mu_3^\otimes 2) \]
consists of symbols by [Tha99, p. 303]. For an alternative proof, combine [KMRT98, 40.9] with Lemma 7.8.
Part II. Surjectivities and invariants of $E_6$, $E_7$, and $E_8$

8. Surjectivities: internal Chevalley modules

Consider the following well-known example.

8.1. Example. Let $q$ be a nondegenerate quadratic form on a vector space $V$ over a field $k$ of characteristic $\neq 2$. Fix an anisotropic vector $v \in V$. Over a separable closure $k_{\text{sep}}$ of $k$, the orthogonal group $O(q)(k_{\text{sep}})$ acts transitively on the open subset of $\mathbb{P}(V)$ consisting of anisotropic vectors by Witt’s Extension Theorem. The stabilizer of an anisotropic line $[v]$ in $O(q)$ is isomorphic to $\mu_2 \times O(v^\perp)$. It follows from [Ser02, §I.5.5, Prop. 37] that the natural map

$$H^1(k, \mu_2 \times O(v^\perp)) \to H^1(k, O(q))$$

is surjective. Repeating this procedure, we find a surjection

$$\bigoplus \dim V H^1(k, \mu_2) \to H^1(k, O(q)).$$

Since $H^1(k, \mu_2)$ is the same as $k^*/k^{*2}$, this surjection can be viewed as reflecting the fact that quadratic forms can be diagonalized.

8.3. Let $G$ be an algebraic group over $k$. Roughly speaking, we now abstract the example by finding a subgroup $N$ of $G$ such that the natural map $H^1(*, N) \to H^1(*, G)$ is surjective. We suppose that $k$ is infinite and that $G$ has a representation $V$ such that there is an open $G$-orbit in $\mathbb{P}(V)$ over an algebraic closure of $k$. As $k$ is infinite, there is a $k$-point $[v]$ in the open orbit.

**Theorem.** The natural map

$$H^1_{\text{fppf}}(k, N) \to H^1(k, G)$$

is surjective, where $N$ is the scheme-theoretic stabilizer of $[v]$ in $G$.

We write $H^1_{\text{fppf}}(k, N)$ for the pointed set of $k$-$N$-torsors relative to the fppf topology as in [DG70]. When $N$ is smooth, this group agrees with the usual Galois cohomology set $H^1(k, N)$ [DG70, p. 406, III.5.3.6], so the reader who wishes to avoid flat cohomology may simply add hypotheses that various groups are smooth or—more restrictively—only consider fields of characteristic zero.

In the case where $N$ is smooth, a concrete proof of the theorem can be found in [Gar01a, 3.1] or by applying [Ser02, §III.2.1, Exercise 2] with $B, C, D$ replaced by $G, N, GL(V)$.

**Proof.** Write $\mathcal{O}$ for the $G$-orbit of $[v]$ in $\mathbb{P}(V)$ (equivalently, $G/N$). For $z \in H^1(k, G)$, there is an inclusion of twisted objects $\mathcal{O}_z \to \mathbb{P}(V)_z$. As $G$ acts on $\mathbb{P}(V)$ through $GL(V)$, the twisted variety $\mathbb{P}(V)_z$ is isomorphic to $\mathbb{P}(V)$ and the $k$-points are dense in $\mathbb{P}(V)_z$ (because $k$ is infinite). Moreover, $\mathcal{O}_z$ is open in $\mathbb{P}(V)_z$ because $\mathcal{O}$ is open in $\mathbb{P}(V)$. Hence $\mathcal{O}_z$ has a $k$-point and the map $H^1_{\text{fppf}}(k, N) \to H^1(k, G)$ is surjective [DG70, p. 373, Prop. III.4.4.6b].

---

*This hypothesis is harmless. In the examples, $G$ will be connected, so $H^1(k, G)$ will be zero when $k$ is finite.*
8.4. Example (char $k = 0$). Let $G$ be a semisimple group. The adjoint representation $V$ of $G$ has an open orbit in $\mathbb{P}(V)$ if and only if $G$ has absolute rank 1, i.e., $G$ is of type $A_1$. Indeed, if $G$ is of rank 1, then the regular semisimple elements in $V$ form an open orbit in $\mathbb{P}(V)$, because $G$ acts transitively on the collection of maximal toral subalgebras of $V$ [Hum80, 16.4]. Conversely, if there is an open $G$-orbit in $\mathbb{P}(V)$, it contains a regular semisimple element $v$. The stabilizer of $[v]$ normalizes the centralizer of $v$ in the Lie algebra, i.e., normalizes a maximal toral subalgebra $t$ of $V$ containing $v$. Hence $N$ normalizes the maximal torus $T$ of $G$ with Lie algebra $t$. As $T$ fixes $v$, we have:

$$\text{rank } G = \dim N = \dim G - \dim \mathbb{P}(V) = 1.$$ 

8.5. A non-example is furnished by a representation $V$ of $G$ on which $G$ acts trivially. If $\dim V = 1$, then $\mathbb{P}(V)$ is a point, $N$ equals $G$, and the conclusion of Th. 8.3 is uninteresting. If $\dim V$ is at least 2, then $\mathbb{P}(V)$ does not have an open orbit (exercise).

8.6. Example (Reducible representations). Let $V$ be a representation of $G$ as in 8.3, and suppose that there is a proper $G$-invariant subspace $W$ of $V$. The quotient map $V \to V/W$ gives a $G$-equivariant rational surjection $f: \mathbb{P}(V) \to \mathbb{P}(V/W)$. If $[v]$ is in the open $G$-orbit in $\mathbb{P}(V)$, then $f$ is defined at $[v]$ and the orbit of $f([v])$ is dense in $\mathbb{P}(V/W)$, hence open.

8.7. Example (char $k = 0$). A group $G$ of type $E_8$ has no nontrivial representations $V$ with an open $G$-orbit in $\mathbb{P}(V)$. Indeed, by Example 8.6, it suffices to prove that no faithful irreducible representation $V$ of $G$ has an open orbit in $\mathbb{P}(V)$. By the following exercise, the constraint $\dim G \geq \dim \mathbb{P}(V)$ leaves the adjoint representation as the only possibility, and there is no open $G$-orbit in that case by Example 8.4.

8.8. Exercise (char $k = 0$). Check that the split group of type $E_8$ has unique irreducible representations of dimensions 1, 248 (adjoint), 3875, 27000, and 30380, and no others of dimension $< 10^5$.

[Compare [Gar].]

8.9. Internal Chevalley modules. How to find groups and representations that satisfy the hypotheses of Theorem 8.3? We now give a mechanism from representation theory that produces such.

Let $\bar{G}$ be a semisimple algebraic group that is defined and isotropic over $k$. We fix a maximal $k$-torus $T$ in $\bar{G}$ that contains a maximal $k$-split torus $\bar{T}_d$. Fix also a set $\Delta$ of simple roots of $\bar{G}$ with respect to $\bar{T}$. We suppose that there is some $\pi \in \Delta$ that is fixed by the Galois group (under the $*$-action, which permutes $\Delta$) and is not constant on $\bar{T}_d$. (In the notation of Tits’s classification paper [Tit66], the vertex $\pi$ in the Dynkin diagram is circled and the circle does not include any other vertices.) Finally, we assume that $k$ has characteristic $\neq 2$ if $\bar{G}$ is of type $B$, $C$, or $F_4$ and $\neq 2, 3$ if $\bar{G}$ has type $G_2$. This concludes our list of assumptions.

We define $G$ to be the semisimple subgroup of $\bar{G}$ that is generated over a separable closure $k_{\text{sep}}$ of $k$ by the 1-dimensional unipotent subgroups $U_{\alpha}$ of $\bar{G}$ as $\alpha$ varies over the roots of $\bar{G}$ with $\pi$-coordinate zero. The Dynkin diagram of $G$ is the diagram of $\bar{G}$ with the vertex $\pi$ deleted. If $\bar{G}$ is simply connected, then so is $G$ by [SS70, 5.4b]. The reader can find a list of Dynkin diagrams in Table 8 below and
Table 8. Extended Dynkin diagrams.

Vertices are numbered as in [Bou Lie]. The unlabeled vertex corresponds to the negative $-\tilde{\alpha}$ of the highest root, and omitting this vertex leaves the usual Dynkin diagram. Type $A$ is omitted entirely.

Over $k_{\text{sep}}$, there is a cocharacter $\lambda: \mathbb{G}_m \to \widetilde{T}$ such that $\pi(\lambda)$ is negative and $\alpha(\lambda)$ is zero for $\alpha \in \Delta \setminus \{\pi\}$. The cocharacter $\lambda$ is even defined over $k$ by [BT65, 6.7, 6.9]; its image is in $T_d$. We take $P$ (respectively, $L$) to be the parabolic subgroup of $G$ (resp., Levi subgroup of $P$) picked out by $\lambda$ in the sense of [Spr98, 13.4.1], i.e., the subgroup generated over $k_{\text{sep}}$ by $\widetilde{T}$ and the $U_\alpha$ where $\alpha$ varies over the roots of $G$. A list of concrete examples of groups $G$ that we will consider by looking ahead at Tables 21a or 11 below.
of $\tilde{G}$ with non-positive $\pi$-coordinate (resp., $\pi$-coordinate zero). Note that $G$ is the derived subgroup of $L$.

The Levi subgroup $L$ acts on the unipotent radical $Q$ of $\tilde{P}$. We fix a positive integer $i$ and write $Q(i)$ for the subgroup of $Q$ spanned by the $U_\alpha$ where the $\pi$-coordinate of $\alpha$ is $\leq -i$. We put $V := Q(1)/Q(2)$; it is a representation of $L$ and there is an open $L$-orbit in $V$ over an algebraic closure of $k$ [ABS90, Th. 2]. The representation $V$ is called an internal Chevalley module. It is irreducible with highest weight $-\pi$ [ABS90, Th. 2].

8.10. Remarks. (1) The addition on $V$ comes from the multiplication in $\tilde{G}$. What is the scalar multiplication that turns $V$ into a $k$-vector space? Suppose that $\tilde{T}$ is split. Number the roots of $\tilde{G}$ with $\pi$-coordinate $-1$ arbitrarily as $\rho_1, \rho_2, \ldots, \rho_s$. The product map

$$U_{\rho_1} \times U_{\rho_2} \times \cdots \times U_{\rho_s} \longrightarrow V$$

is an isomorphism by [Bor91, Prop. 14.4(2)]. The group $U_\alpha$ is the image of a homomorphism $x_\alpha: G_\alpha \to \tilde{G}$ and the scalar multiplication is the naive one: For $\lambda \in k^\times$ and $u \in k$, we have

$$\lambda \cdot m \left( \prod x_{\rho_i}(u_i) \right) = m \left( \prod x_{\rho_i}(\lambda u_i) \right),$$

see [ABS90, p. 554].

(2) If, instead of the parabolic $\tilde{P}$, we chose the “opposite” parabolic, then everything would work out the same except that the highest weight of $V$ would be the highest positive root with $\pi$-coordinate $1$—something that is more difficult to read off of the Dynkin diagrams. The resulting $V$ would be the $L$-module that is dual to the one we consider here.

(3) In most of the examples considered below, the vertex $\pi$ of the Dynkin diagram is adjacent to only one other vertex—call it $\delta$—and the two vertices are joined by a single bond, so the highest weight of $V$ is the fundamental weight corresponding to $\delta$.

(4) Although one could consider the modules $Q(i)/Q(i + 1)$ for various $i$, no real generality is gained, see [Roh93c, 1.8].

(5) Although the root $\pi$ is fixed by the Galois group under the $\ast$-action (and the cocharacter $\lambda$ is $k$-defined), $\pi$ need not be fixed by the usual Galois action. Indeed, [Gar98] gives a concrete construction of groups of type $\tilde{3}D_4$ with $k$-rank $1$ where the root $\pi := \alpha_2$ is fixed by the $\ast$-action (and is non-constant on the split torus), but the usual Galois action interchanges $\alpha_2$ and $\alpha_1 \pm \alpha_2 + \alpha_3 + \alpha_4$.

(6) In case $\tilde{G}$ is split, there is an open $\tilde{P}$-orbit in the unipotent radical $Q$ whose elements are known as “Richardson elements”. Clearly, any Richardson element with $\pi$-coordinate $-1$ maps to an element of the open $L$-orbit in $V$. For $\tilde{G}$ of classical type, the reader can find concrete examples of Richardson elements in [Bau06].

8.11. We maintain the assumptions from 8.9, and we further assume—as in 8.3—that the field $k$ is infinite. We fix a $k$-point $[v]$ in the open $L$-orbit in $\mathbb{P}(V)$.

Theorem. The natural map

$$H^1_{fppf}(k, N) \to H^1(k, G)$$
is surjective, where \( N \) is the scheme-theoretic stabilizer of \([v] \in \mathbb{P}(V)\) in \( G \).

**Proof.** Write \( \mathcal{O} \) for the \( L \)-orbit of \([v] \in \mathbb{P}(V)\). Note that since the \( L \)-orbit of \( v \) is dense in \( V \), \( \mathcal{O} \) is dense in \( \mathbb{P}(V) \), hence open in \( \mathbb{P}(V) \) because orbits are locally closed.

As \( V \) is an irreducible representation of \( L \), the torus \( S \) in the center of \( L \) acts on \( V \) by scalar multiplication. But \( G \) and \( S \) generate \( L \), so the \( G \)- and \( L \)-orbits in \( \mathbb{P}(V) \) coincide. That is, the \( G \)-orbit of \([v] \in \mathbb{P}(V)\) is open. Theorem 8.3 completes the proof. \( \Box \)

Note that Theorems 8.3 and 8.11 give surjections \( H^1_{\text{pro}}(K, N) \rightarrow H^1(K, G) \) for every extension \( K/k \), where \( N \) is the scheme-theoretic stabilizer of a \( k \)-point in the open orbit. We summarize this by saying that the morphism of functors \( H^1_{\text{pro}}(\ast, N) \rightarrow H^1(\ast, G) \) is surjective or that the inclusion \( N \subset G \) induces a surjection on \( H^1 \)'s.

8.12. Example \((F_4 \times \mu_3 \subset E_6)\). The natural inclusion of root systems leads to an inclusion of split simply connected groups \( E_6 \subset E_7 \). We take these groups as \( G \) and \( \tilde{G} \) respectively in the notation of 8.11, so that \( e \) is the root \( \alpha_7 \) of \( E_7 \). We number the simple roots as in Table 8. The representation \( V \) of \( G \) is irreducible and 27-dimensional with highest weight \( \omega_6 \).

There is a split group of type \( F_4 \) inside of \( E_6 \), and we denote it also by \( F_4 \). Writing \( x_\alpha : \mathbb{G}_a \rightarrow E_6 \) for the generators of \( E_6 \) as in [Ste68], \( F_4 \) is generated by the maps

\[
(8.13) \quad x_{\alpha_2}, \quad x_{\alpha_1}, \quad u \mapsto x_{\alpha_3}(u) x_{\alpha_5}(u), \quad u \mapsto x_{\alpha_4}(u) x_{\alpha_6}(u),
\]

e tc., where the displayed maps correspond to the roots \( \alpha_1, \alpha_2, \alpha_3, \) and \( \alpha_4 \) respectively in \( F_4 \), cf. [Spr98, §10.3]. We claim that \( N \) is the direct product of \( F_4 \) with the center \( Z \) of \( E_6 \), which is isomorphic to \( \mu_3 \).

Restricting the representation \( V \) of \( E_6 \) to \( F_4 \), we find that \( V \) is a direct sum of an \( F_4 \)-invariant line \([v] \) (for some \( v \)) and an indecomposable 26-dimensional representation \( W \) (which is even irreducible if the characteristic of \( k_0 \) is not 3 [GS88, p. 412]). We take \( v \) to be a generator of the line.

Note that the maximal proper parabolics of \( L \) have Levi subgroups of type

\[ D_5, \quad A_1 \times A_4, \quad A_1 \times A_2 \times A_2, \quad A_5, \]

and these semisimple parts all have dimension strictly less than 52, the dimension of \( F_4 \). Therefore \( F_4 \) is not contained in a proper parabolic subgroup of \( L \). By [Röh93a, Prop. 3.5], it follows that \( v \) belongs to the open \( L \)-orbit in \( V \).

Clearly, \( F_4 \) is contained in the stabilizer \( N \) of \([v] \) in \( E_6 \), and by dimension count it is the identity component of \( N \). Since \( Z \) is also contained in \( N \), it suffices to prove that \( F_4 \) and \( Z \) generate the normalizer of \( F_4 \) in \( E_6 \). But every automorphism of \( F_4 \) is inner and \( F_4 \) has trivial center, so the normalizer of \( F_4 \) in \( E_6 \) is the product \( F_4 \times C \), where \( C \) is the centralizer of \( F_4 \) in \( E_6 \). Therefore it suffices to prove that the center \( Z \) is all of \( C \).

Write \( T_4 \) for the maximal torus \((F_4 \cap T)^o\) of \( F_4 \). The centralizer of \( T_4 \) in \( E_6 \) contains \( T \), is reductive [Bor91, 13.17, Cor. 2a], and is generated by \( T \) and the images of the \( x_\gamma \)'s, where \( \gamma \) varies over the roots of \( E_6 \) whose inner product with \( \alpha_2, \alpha_4, \alpha_3 + \alpha_5 \), and \( \alpha_1 + \alpha_6 \) is zero. Such a root \( \gamma \) is a \( \mathbb{Q} \)-linear combination of the
weights
\[ \omega_3 - \omega_5 = \frac{1}{3}(\alpha_1 + 2\alpha_3 - 2\alpha_5 - \alpha_6) \quad \text{and} \quad \omega_1 - \omega_6 = \frac{1}{3}(2\alpha_1 + \alpha_3 - \alpha_5 - 2\alpha_6). \]
But such a \( \gamma \) would have disconnected support,\(^{\dagger}\) which is impossible by [Bou Lie, §VI.1.6, Cor. 3a to Prop. 19]. So the centralizer of \( T_4 \) in \( E_6 \) is \( T \), and in particular \( C \) is contained in \( T \). But \( C \) commutes with the images of the maps in (8.13), hence with the image of \( x_\alpha \), for \( 1 \leq i \leq 6 \). That is, \( C \) is contained in the center \( Z \) of \( E_6 \).

This completes the proof that \( N \) equals \( F_4 \times Z \).

Combining this example with Th. 8.11 gives that every \( k \-E_6 \) torsor can be written (not necessarily uniquely) as a pair \((J, \beta)\), where \( J \) is an Albert \( k \) -algebra and \( \beta \) belongs to \( k^\times /k^\times 3 \). For a classical proof of this in characteristic \( \neq 2,3 \), see [Spr62]. For an application, see [GH06, §5] or 10.9 below.

**Context.** The representations appearing in 8.3 are nearly the same as the prehomogeneous vector spaces appearing in [SK77]. Recall that a **prehomogeneous vector space** is a representation \( V \) of an algebraic group \( G \) such that there is a \( G \)-orbit in \( V \). These too lead to surjections in cohomology, by the same proof as in 8.3. However, we are interested in the case where \( G \) is semisimple (and not merely reductive), for which there are not enough prehomogeneous vector spaces.

Continuing the comparison of \( G \)-orbits in \( V \) and \( \mathbb{P}(V) \), we note that in the examples of 8.11 considered below (listed in Table 21a), the \( G \)-orbit of \( v \) in the affine space \( V \) is a hypersurface, more specifically a level set of a homogeneous \( G \)-invariant polynomial on \( V \). However, this need not be true, as considering \( G = \text{Spin}_{10} \) shows: Viewing \( G \) as a subgroup of \( E_6 \), the recipe of 8.11 gives that \( V \) is a half-spin representation, and the \( G \)-orbit of \( v \) in that case is dense in \( V \) [Igu70, Prop. 2]. (In Example 15.8, we view \( G \) as a subgroup of \( \text{Spin}_{12} \), the resulting \( V \) is the 10-dimensional vector representation, and the \( G \)-invariant polynomial on \( V \) is the quadratic form.)

In the setup for Th. 8.11, we cited [ABS90] because it is a convenient reference, but the core idea can certainly be found in other, earlier references, e.g., [Vin76].

### 9. New Invariants from Homogeneous Forms

A **(homogeneous) form of degree** \( d \) on a \( k_0 \)-vector space \( V \) is a nonzero element of the \( d \)-th symmetric power \( S^d(V^*) \). Equivalently, fixing a basis \( x_1, x_2, \ldots, x_n \) for the dual space \( V^* \), it is a homogeneous polynomial of degree \( d \) in \( k_0[x_1, x_2, \ldots, x_n] \).

In this section, we give a mechanism for constructing new invariants of a group \( G \) from \( G \)-invariant forms.

Suppose that \( V \) is a representative of an algebraic group \( G \) and that \( V \) supports a \( G \)-invariant form \( f \). Each \( y \in H^1(k, G) \) defines a twisted form \( f_y \) on \( V \otimes k \).

We are concerned with the case where \( f \) is a form of degree \( d \) such that \( dC = 0 \). For an invariant \( a \in \text{Inv}_{k_0}(G, C) \) and \( v \in V \otimes k \) such that \( f_y(v) \) is not zero, we consider the element
\[
(9.1) \quad a(y) \cdot (f_y(v)) \in M(k, C).
\]
(We view \((f_y(v))\) as an element of \( H^1(k, \boldsymbol{\mu}_d) \).) The following proposition is adapted from [Ros99c, Prop. 5.2].

\(^{\dagger}\)Recall that every root \( \gamma \) can be written uniquely as an integral linear combination of the simple roots. The **support of** \( \gamma \) is the set of those simple roots whose coefficient is nonzero in this expression.
9.2. Proposition. If \( a(y) \) is zero whenever \( f_y \) has a nontrivial zero, then the element \((9.1)\) depends only on \( y \) (and not on the choice of \( v \)) and the map \( y \mapsto a(y) \cdot (f_y(v)) \) defines an invariant of \( G \) over \( k_0 \).

Recall that \( f_y \) is said to have a nontrivial zero if there is some nonzero \( v \in V \otimes k \) such that \( f_y(v) = 0 \). (Obviously, \( f_y \) always has the “trivial” zero \( f_y(0) = 0 \).)

9.3. Example. In the “smallest” case, when \( V \) is 1-dimensional, we can see the proposition directly. If we fix a dual basis \( x \) for \( V \), then \( f \) is \( \alpha x^d \) for some \( \alpha \in k_0^\times \).

The action of \( G \) on \( V \) is given by a homomorphism \( \chi: G \to \mu_d \) and this defines an invariant \( \chi: H^1(\ast, G) \to H^1(\ast, \mu_d) \). For \( y \in H^1(k, G) \), \( f_y \) is the form \( \alpha \chi(y)x^d \), and for nonzero \( v \in V \otimes k \), we have \( (f_y(v)) = (\alpha) + (\chi(y)) \). In particular, \((9.1)\) is the value of the invariant \( (\alpha) \cdot a + \chi \cdot a \) at \( y \).

Proof of the proposition. By the example, we may assume the dimension of \( V \) is at least 2. Fix a basis for \( V^* \) as above. Writing \( f_y \) (viewed as an element of \( k[V] \)) in terms of this basis is equivalent to evaluating it at the generic point of \( V \). Put

\[
\omega := a(y) \cdot (f_y) \in M(k(V), C).
\]

We claim that \( \omega \) is the restriction of some \( \omega_0 \in M(k, C) \). By S10.1, it suffices to check that \( \omega \) is unramified at every discrete valuation of \( k(V)/k \) that corresponds to an irreducible hypersurface in \( V \). Such a hypersurface is defined by some irreducible \( \pi \in k[V] \). If \( \pi \) does not divide \( f_y \) (i.e., the hypersurface is not a component of the variety \( \{f_y = 0\} \)), then \( \omega \) is unramified on the hypersurface by definition. If \( \pi \) does divide \( f_y \) in \( k[V] \), we write \( f_y = \pi^\varepsilon \) for some \( \varepsilon \) not divisible by \( \pi \), so

\[
\omega = a(y) \cdot (\varepsilon) + n a(y) \cdot (\pi).
\]

The residue of the first term is zero and the residue of the second is a multiple of \( \text{res}_{k(\pi)/k} a(y) \). The form \( f_y \) is zero on the sum of the vectors in the dual basis in \( V \otimes k(\pi) \), and this is a nontrivial zero because the dimension of \( V \) is not 1. It follows that \( \omega \) has residue zero. This proves the claim.

Specializing the generic point to \( v \in V \otimes k \) maps \( f_y \mapsto f_y(v) \) and \( \omega \mapsto (a(y)) \cdot (f_y(v)) \), but does not change \( \omega_0 \). This proves that \( a(y) \cdot (f_y(v)) \) does not depend on the choice of \( v \). The remainder of the proposition is clear.

9.4. Example. The invariants produced by the lemma need not be interesting. In the following examples, we consider the case where \( f \) is a quadratic form.

(1) Suppose that \( a \) is an invariant as in the lemma. Applying the proposition once produces an invariant \( a' \), and this new invariant also satisfies the hypothesis of the proposition. Applying the proposition again, we obtain an invariant

\[
a'' : y \mapsto a(y) \cdot (f_y(v)) \cdot (f_y(v)) = a'(y) \cdot (-1).
\]

That is, \( a'' = (-1) \cdot a' \).

(2) Suppose that—in the situation of the proposition—the form \( f_y \) is Witt-equivalent to an \( n \)-Pfister form for every \( y \in H^1(k, G) \) and every \( k/k_0 \), and \( a \) is the invariant \( y \mapsto e_n(f_y) \). (For example, take \( G \) to be the split group of type \( G_2 \) and \( a \) to be the invariant from S18.4, i.e., the Rost invariant.) Then for each \( y \), \( f_y \) is Witt-equivalent to some \( \otimes_{i=1}^n (1, -\alpha_i) \) and the invariant \( a' \) given by the proposition satisfies

\[
a'(y) = [((\alpha_1) \cdot (\alpha_2) \cdots (\alpha_n)) \cdot (-\alpha_n)] = 0.
\]
That is, $a'$ is the zero invariant.

10. **MOD 3 INVARIANTS OF SIMPLY CONNECTED $E_6$**

In this section we assume that the characteristic of $k_0$ is different from 2 and 3.

10.1. **INVARIANTS OF THE SPLIT $E_6$**. We compute the invariants of the simply connected split group of type $E_6$, which we denote simply by $E_6$. The mod 2 invariants were computed in Exercise 22.9 in S. We note that $E_6$ has no invariants modulo primes $\neq 2, 3$ by the remarks in 4.5.

As in Example 8.12, we have an inclusion

$$i: F_4 \times \mu_3 \hookrightarrow E_6$$

that identifies $\mu_3$ with the center of $E_6$ such that the induced map

$$i_\ast: H^1(\ast, F_4 \times \mu_3) \rightarrow H^1(\ast, E_6)$$

is a surjection. Two classes $(J, \beta)$ and $(J', \beta')$ have the same image in $H^1(k, E_6)$ if and only if there is a vector space isomorphism $f: J \rightarrow J'$ such that $\beta N_J = \beta' N_{J'} f$, where $N_J$ and $N_{J'}$ denote the cubic norms on $J$ and $J'$, see [Gar01b, 2.8(2)].

10.3. **Exercise.** Albert algebras $J, J'$ are isotopic (see [Jac68] for a definition) if and only if their norm forms are similar, i.e., $i_\ast(J, \beta) = i_\ast(J', \beta')$ for some $\beta, \beta' \in k^\times$, see pages 242–244 of [Jac68]. Prove that $J$ and $J'$ are isotopic if and only if their norms are isomorphic, i.e., $i_\ast(J, 1) = i_\ast(J', 1)$. Prove also that $i_\ast(J, 1) = i_\ast(J, \beta)$ if and only if $\beta$ is the norm of an element of $J$.

Composing (10.2) and (7.2) gives a functor

$$H^1(\ast, (PGL_3 \times \mu_3) \times \mu_3) \rightarrow H^1(\ast, E_6)$$

where the $PGL_3 \times \mu_3$ in parentheses is the subgroup of $F_4$ from §7. This functor is surjective at 3 because every Albert algebra is in the image of (7.2) after an extension of the base field of degree 1 or 2. Therefore the restriction map

$$\text{Inv}^{\text{norm}}(E_6, \mathbb{Z}/3\mathbb{Z}) \rightarrow \text{Inv}^{\text{norm}}(PGL_3 \times \mu_3 \times \mu_3, \mathbb{Z}/3\mathbb{Z})$$

is injective.

10.5. **AN INVARIANT OF DEGREE 3.** Consider the invariant $g_3$ of $PGL_3 \times \mu_3 \times \mu_3$ defined by

$$g_3: (A, \alpha, \beta) \mapsto [A] \cdot (\alpha) \in H^3(k, \mu_3^{\otimes 2})$$

for $(A, \alpha, \beta)$ defined over $k$. We now give two arguments to show that it is in the image of (10.4).

**Proof #1.** If $(A, \alpha, \beta)$ and $(A', \alpha', \beta')$ have the same image in $H^1(k, E_6)$, then the Albert algebras $J(A, \alpha), J(A', \alpha')$ have similar norms. But as they are first Tits constructions, this implies that the algebras are isomorphic [PR84, 4.9], hence $[A] \cdot (\alpha)$ equals $[A'] \cdot (\alpha')$ as in 7.4. That is, $g_3$ satisfies (6.2) and so extends uniquely to an invariant of $E_6$. □

**Proof #2.** The Dynkin index of $E_6$ is 6 [Mer03, 16.6], so the mod 3 portion of the Rost invariant defines a nonzero invariant

$$g'_3: H^1(\ast, E_6) \rightarrow H^3(\ast, \mu_3^{\otimes 2}).$$
As the inclusion $F_4 \hookrightarrow E_6$ has Rost multiplier one [Gar01a, 2.4], the restriction of $g_3'$ to $H^1(\ast, F_4)$ is $\epsilon g_3$ for $\epsilon = \pm 1$ and $g_3$ the invariant from 7.4. The composition

$$H^1(k, F_4) \times H^1(k, \mu_3) \to H^1(k, E_6) \xrightarrow{\beta \mapsto} H^3(k, \mu_3^{\otimes 2})$$

sends an Albert $k$-algebra $J$ and a $\beta \in k^\times /k^3$ to the element $\epsilon g_3(J)$. (When $\beta$ is 1, this is clear. In general, one uses a twisting argument as in [GQ06, Remark 2.5(i)].) The invariant $\epsilon g_3'$ restricts to the map $g_3$ from (10.6).

We abuse notation and write also $g_3$ for the invariant of $E_6$ that restricts to the $g_3$ from (10.6). Note that the image of this invariant of $E_6$ consists of symbols in $H^3(k, \mu_3^{\otimes 2})$, because the same is true for the invariant $g_3$ of $F_4$.

10.7. An invariant of degree 4. Define an invariant $g_4$ of $PGL_3 \times \mu_3 \times \mu_3$ by putting

$$g_4 : (A, \alpha, \beta) \mapsto [A] \cdot (\alpha) \cdot (\beta) \in H^4(k, \mu_3^{\otimes 3}).$$

We give two proofs of the fact that $g_4$ extends to an invariant $H^1(\ast, E_6) \to H^4(\ast, \mu_3^{\otimes 3})$.

Proof #1. We check (6.2). Suppose that $(A, \alpha, \beta)$ and $(A', \alpha', \beta')$ have the same image in $H^1(k, E_6)$. As in 10.5, $J(A, \alpha)$ and $J(A', \alpha')$ are isomorphic and $[A] \cdot (\alpha)$ equals $[A'] \cdot (\alpha')$. Further, $\beta / \beta'$ is a similarity of the norm of $J(A, \alpha)$. By Exercise 10.3, $\beta / \beta'$ is a norm from $J(A, \alpha)$, hence $J(A, \alpha)$ is isomorphic to $J(A'', \alpha'')$ for some central simple algebra $A''$ such that $\beta / \beta'$ is reduced norm from $A''$ [PR84, 4.2]. We conclude that

$$[A] \cdot (\alpha) \cdot (\beta) - [A'] \cdot (\alpha') \cdot (\beta') = [A] \cdot (\alpha) \cdot (\beta / \beta') = [A''] \cdot (\alpha'') \cdot (\beta / \beta') = 0.$$

This verifies (6.2), hence $g_4$ extends to an invariant of $H^1(\ast, E_6)$. □

Proof #2 (sketch). Observe that $H^1(k, E_6)$ classifies cubic forms that become isomorphic to the norm of an Albert algebra over a separable closure of $k$. The statement “$i_\ast$ is surjective” says that such a cubic form is a scalar multiple—say, $\beta . N_J$—of the norm on an Albert $k$-algebra $J$. Moreover, $g_3(J)$ is zero whenever the norm $N_J$ has a nontrivial zero (i.e., whenever $J$ is reduced), so Prop. 9.2 gives that the map

$$g_4 : \beta . N_J \mapsto g_3(J) \cdot (\beta)$$

is a well-defined invariant of $E_6$. □

As usual, we write also $g_4$ for the invariant of $E_7$ that restricts to give the $g_4$ defined in (10.8).

10.9. Proposition. $\text{Inv}_{k_0}^{\text{norm}}(E_6, \mathbb{Z}/3\mathbb{Z})$ is a free $R_3(k_0)$-module with basis $g_3, g_4$.

Proof. We imitate the proofs of Propositions 5.1 and 7.4. The restriction map

$$i^* : \text{Inv}_{k_0}^{\text{norm}}(E_6, \mathbb{Z}/3\mathbb{Z}) \to \text{Inv}_{k_0}^{\text{norm}}(F_4 \times \mu_3, \mathbb{Z}/3\mathbb{Z})$$

is an injection.

The center of $E_6$ is contained in a maximal split torus, hence the image of the map $H^1(\ast, \mu_3) \to H^1(\ast, E_6)$ is zero. Applying 5.7 and Propositions 2.1 and 7.4, we find that $g_3$ and $g_4$ span $\text{Inv}_{k_0}^{\text{norm}}(E_6, \mathbb{Z}/3\mathbb{Z})$. □
10.10. Exercise (Mod 3 invariants of the quasi-split $E_6$). For $K$ a quadratic field extension of $k_0$, write $E_6^K$ for the simply connected quasi-split group of type $E_6$ associated with the extension $K/k$. Describe the “mod 3” invariants of $E_6^K$.

10.11. Open problem. [PR94a, p. 205, Q. 4] Let $J,J'$ be Albert $k$-algebras. If $J$ and $J'$ have similar norms, then their images in $H^1(k,E_6)$ are the same, hence they have the same Rost invariant. In the notation §7 of this note and §22 of S, $f_3(J) = f_3(J')$ and $g_3(J) = g_3(J')$. Does the converse hold? That is, if $f_3(J) = f_3(J')$ and $g_3(J) = g_3(J')$, are the norms of $J$ and $J'$ necessarily similar? [If $J$ and $J'$ are reduced, the answer is “yes”, see [Jac68, p. 369, Th. 2].]

11. Surjectivities: the highest root

We now describe a general situation where — in the setting of 8.11 — we can describe the identity component $N^\circ$ of the stabilizer. We will use this to apply Th. 8.11 to the simply connected group of type $E_7$.

11.1. Let $\tilde{G}$ be a simply connected split algebraic group not of type $A$. The highest root $\tilde{\alpha}$ is connected to a unique simple root—see Table 8—which we take to be $\pi$ in the notation of 8.11. This situation was studied by Röhrle in [Röh93b], and for convenience of reference, we adopt the hypotheses of his main theorem. Namely, we assume that $\pi$ is long (equivalently, $\tilde{G}$ is not of type $C$), the rank of $\tilde{G}$ is at least 4, and the characteristic is $\neq 2$.

As $-\tilde{\alpha}$ is joined to $\pi$ by a single bond, $\tilde{\alpha}$ is the fundamental weight corresponding to the simple root $\pi$, i.e., for every root $\beta$, the integer $\langle \tilde{\alpha}, \beta \rangle$ is the coordinate of $\pi$ in $\beta$. For example, the $\pi$-coordinate of $\tilde{\alpha}$ is $\langle \tilde{\alpha}, \tilde{\alpha} \rangle = 2$. For $w_0$ the longest element of the Weyl group of $\tilde{G}$, clearly $w_0(\tilde{\alpha}) = -\tilde{\alpha}$, hence $w_0(\pi) = -\pi$.

We take $V$ to be $Q(1)/Q(2)$, where $Q$ is the unipotent radical of the parabolic subgroup opposite to the one chosen in 8.11, so that $Q$ is generated over $k_{\text{sep}}$ by the $U_\alpha$ where $\alpha$ has positive $\pi$-coordinate. We do this both to agree with Röhrle’s notation and for the convenience of working with positive roots. As mentioned in Remark 8.10.1, this $V$ is the dual of the irreducible $L$-module with highest weight $\tilde{\alpha}$, meaning it has highest weight $-w_0(\tilde{\alpha}) = -\tilde{\alpha}$ also. Changing from the parabolic in 8.11 to its opposite has not changed the isomorphism class of $V$.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$G$</th>
<th>$V$</th>
<th>dim $V$</th>
<th>$(N^\circ)^{ss}$</th>
<th>dim $Z(N^\circ)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_4$</td>
<td>$(SL_2)^\times 3$</td>
<td>$k^2 \otimes k^2 \otimes k^2$</td>
<td>8</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$F_4$</td>
<td>$Sp_6$</td>
<td>14</td>
<td>$SL_3$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$E_6$</td>
<td>$SL_6$</td>
<td>$\wedge^3 k^6$</td>
<td>20</td>
<td>$SL_3 \times SL_3$</td>
<td>0</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$Spin_{12}$</td>
<td>half-spin</td>
<td>32</td>
<td>$SL_6$</td>
<td>0</td>
</tr>
<tr>
<td>$E_8$</td>
<td>$E_7$</td>
<td>minuscule</td>
<td>56</td>
<td>$E_6$</td>
<td>0</td>
</tr>
<tr>
<td>$Spin_d$ ($d \geq 9$)</td>
<td>$SL_2 \times Spin_{d-4}$</td>
<td>$k^2 \otimes$ vector</td>
<td>$2d - 8$</td>
<td>$Spin_{d-6}$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 11. Internal Chevalley modules corresponding to the highest root

The last line of the table combines the cases where $\tilde{G}$ is of type $B_n$ ($n \geq 4$) or $D_n$ ($n \geq 5$).

Table 11 describes the possibilities we consider. The last two columns will be explained below. Readers who know some nonassociative algebra will immediately
recognize that $V$ must be a Freudenthal triple system. Some convenient comparisons are [Mey68, (8.4)] and [Kru, Table 1].

The top five rows of the table are “sisters”: The groups $\tilde{G}$ from these rows form the bottom row of Freudenthal’s “magic square”, resp. the $G$’s form the next-to-the-bottom row. The representations $V$ are the “preferred representations” from the bottom row of the magic triangle in [DG02, Table 2]. These rows of the table are related to triple systems coming from cubic Jordan algebras of dimension 3, 6, 9, 15, and 27 respectively. The little one (with $\dim V = 8$) has appeared in Bhargava’s work on higher reciprocity laws [Bha04, p. 220] and in Gopal Prasad’s solution [Pra05] to the Kneser-Tits Problem for rank 1 groups of type $3\mathcal{D}_4$ and $6\mathcal{D}_4$.

Note that in Prasad’s case, the group $G$ is not split, but $\pi$ is circled in the index of $G$. In that case, the representation $V$ is defined and there is an open $G$-orbit in $\mathbb{P}(V)$ as in 8.11. The stabilizer of a $k$-point in the open orbit is a “twisted form” of the $N$ we now compute.

Write $\tilde{\Phi}$ for the set of roots of $\tilde{G}$. We view $\tilde{G}$ as defined by generators and relation as in [Ste68]. In particular, for each root $\alpha \in \tilde{\Phi}$, the unipotent subgroup $U_\alpha$ is the image of a homomorphism $x_\alpha : G_a \to G$.

We put $v := x_\pi(r)x_{\tilde{\alpha} - \pi}(s)U_{\tilde{\alpha}} \in V$ for some $r, s \in k^\times$. It belongs to the open $L$-orbit in $V$ [Röh93b, 4.4], and we define $N$ to be the scheme-theoretic stabilizer of $[v]$ in $G$.

11.2. Lemma (char $k_0 \neq 2$). The identity component $N^0$ of $N$ is reductive. Its semisimple part is simply connected and generated by the subgroups $U_\beta$ as $\beta$ varies over the roots in $\tilde{\Phi}$ whose support contains neither $\pi$ nor any root adjacent to $\pi$. The rank of its central torus equals $\deg \pi - 1$.

The notation $\deg \pi$ denotes the degree of the vertex $\pi$ of the Dynkin diagram, i.e., the number of simple roots that are distinct from and not orthogonal to $\pi$. The lemma says that the Dynkin diagram of $N$ is obtained from the Dynkin diagram of $\tilde{G}$ by deleting $\pi$ and every vertex adjacent to $\pi$.

Proof. Let $\beta \in \tilde{\Phi}$ be as in the statement of the lemma. The support of $\pi \pm \beta$ has two connected components—the support of $\pi$ and $\beta$—so $\pi \pm \beta$ is not a root of $\tilde{G}$. For sake of contradiction, suppose that $\tilde{\alpha} - \pi \pm \beta$ is a root of $\tilde{G}$. It has $\pi$-coordinate 1, hence

$$\langle \tilde{\alpha}, \tilde{\alpha} - \pi \pm \beta \rangle = 1 \quad \text{and} \quad s_{\tilde{\alpha} - \pi \pm \beta}(\tilde{\alpha}) = \pi \mp \beta.$$  

(Here and below we write $s_\beta$ for the reflection defined by a root $\beta$.) That is, $\pi \mp \beta$ is a root of $\tilde{G}$, a contradiction.

The previous paragraph is summarized by saying: $\beta$ is strongly orthogonal to $\pi$ and to $\tilde{\alpha} - \pi$. It follows that the subgroup $H$ of $\tilde{G}$ generated by the $U_\beta$’s fixes $v$ and so is a subgroup of $N$. The type of $H$ is listed in the next-to-the-last column of Table 11, and $H$ is simply connected by [SS70, 5.4b]. We note that, line-by-line in the table, $H$ has dimension

$$0, 8, 16, 35, 78, \frac{d^2 - 13d}{2} + 21.$$

Next consider the largest subtorus $T_Z$ of $\tilde{T}$ on which $\pi, \tilde{\alpha}$, and the simple roots belonging to $H$ vanish. This torus belongs to $N$, commutes with $H$, and has
dimension

\[ \text{rank } \tilde{G} - \text{rank } H - 2 = \deg \pi - 1. \]

This number is listed in the last column of Table 11.

The subgroup \( H.T_Z \) of \( G \) is connected and reductive with derived subgroup \( H \). To complete the proof of the lemma, it suffices to check that \( H.T_Z \) and \( N \) have the same dimension, i.e., to check the equation

\[ (11.3) \quad \dim H + \dim T_Z = \dim G - \dim V + 1. \]

The dimension of \( G \), line-by-line in the table, is

\[ 9, 21, 35, 66, 133, \frac{d^2 - 9d}{2} + 13, \]

so equation (11.3) holds in each case. \( \square \)

11.4. Orthogonal long roots in \( \tilde{\Phi} \).

We put \( \tilde{\Phi}_j \) for the roots whose \( \pi \)-coordinate is \( j \). For a positive root \( \beta \in \tilde{\Phi} \), the \( \pi \)-coordinate of \( \beta \) is 0, 1, or 2, and it is 2 if and only if \( \beta \) equals \( \tilde{\alpha} \), see [Bou Lie, §VI.1.8, Prop. 25(iv)]. That is, \( \tilde{\Phi}_j \) is nonempty only for \( j = 0, \pm 1, \pm 2 \) and \( \tilde{\Phi}_2 \) is the singleton \( \{ \tilde{\alpha} \} \).

As in [Röh93b, p. 145], there is a sequence \( \mu_1, \mu_2, \mu_3, \mu_4 \) of pairwise orthogonal long roots in \( \tilde{\Phi}_1 \).

Lemma. The roots \( \mu_1, \mu_2, \mu_3, \mu_4 \) are pairwise strongly orthogonal.

Proof. If \( \mu_i + \mu_j \) is a root, then it has \( \pi \)-coordinate 2, hence it equals \( \tilde{\alpha} \). But

\[ 0 = \langle \mu_i, \mu_j \rangle = \langle \tilde{\alpha} - \mu_j, \mu_j \rangle = -1, \]

a contradiction. Further, \( \mu_i \) and \( \mu_j \) are orthogonal, so since \( \mu_i + \mu_j \) is not a root, neither is \( \mu_i - \mu_j \). \( \square \)

11.5. Strongly orthogonal roots in \( G \).

The Weyl group of \( G \) acts transitively on the roots in \( \tilde{\Phi}_1 \) of the same length [ABS90, §2, Lemma 1], so we may assume that \( \mu_1 \) equals \( \pi \). For \( j = 2, 3, 4 \), we set:

\[ \gamma_j := \tilde{\alpha} - \pi - \mu_j. \]

Lemma. \( \gamma_2, \gamma_3, \gamma_4 \) are pairwise strongly orthogonal long roots of \( G \). For various \( x, y \), the value of \( \langle x, y \rangle \) is given by the table:

<table>
<thead>
<tr>
<th></th>
<th>( \tilde{\alpha} )</th>
<th>( \pi )</th>
<th>( \mu_j )</th>
<th>( \gamma_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( \mu_j )</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>( \gamma_j )</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>2</td>
</tr>
</tbody>
</table>

Proof. The top row of the table is the \( \pi \)-coordinate of \( y \), and we know these already. We calculate that \( \gamma_j \) equals \( s_\pi s_{\mu_j} (\tilde{\alpha}) \), so in particular, \( \gamma_j \) has the same length as \( \tilde{\alpha} \); long. As all the roots in table have the same length, the table is symmetric. As for \( \pi \) and \( \mu_j \), they are orthogonal by construction. The entries for \( \langle \gamma_j, \pi \rangle \) and \( \langle \gamma_j, \mu_j \rangle \) are straightforward computations.

Similarly, for \( i \neq j \) we have \( \langle \gamma_j, \mu_i \rangle = 1 \), hence \( \langle \gamma_i, \gamma_j \rangle = 0 \). Further, \( \gamma_i - \gamma_j = \mu_i - \mu_j \) is not a root. As in the proof of Lemma 11.4, \( \gamma_i + \gamma_j \) is not a root. \( \square \)
We note that \( \mu_2 + \mu_3 + \mu_4 = 2\tilde{\alpha} - \pi \) [Röh93b, 1.4], so

\[
\gamma_2 + \gamma_3 + \gamma_4 = \tilde{\alpha} - 2\pi.
\]

11.7. A COPY OF SL₂. Define 1-parameter subgroups \( x, y : \mathbb{G}_a \to G \) via

\[
x(u) := \prod_{j=2}^{4} x_{\gamma_j}(u) \quad \text{and} \quad y(u) := \prod_{j=2}^{4} x_{-\gamma_j}(u).
\]

Since the \( \gamma_j \)'s are strongly orthogonal, the images of the \( x_{\gamma_j} \) commute [Ste68, p. 30, (R2)], i.e., it does not matter in what order the displayed products are written. The images of \( x \) and \( y \) generate a copy of \( SL₂ \) in \( G \) which we denote simply by \( SL₂ \). For \( t \in \mathbb{G}_m \), we set

\[
w(t) := x(t)y(-t^{-1})x(t) \quad \text{and} \quad h(t) := w(t)w(-1).
\]

The map \( h \) is a homomorphism and its image is a maximal torus in \( SL₂ \). How does \( SL₂ \) act on \( V \)? Identity (R8) from [Ste68, p. 30] says that for roots \( \beta, \delta \), we have:

\[(R8) \quad h_\beta(t)x_\delta(u) = x_\delta(t^{\langle \delta, \beta \rangle})h_\beta(t) \]

where \( h_\beta : \mathbb{G}_m \to \mathbb{T} \) is the cocharacter corresponding to the coroot \( \check{\beta} \). In particular,

\[
h(t)x_\pi(u) = x_\pi(t^{-3}u)h(t) \quad \text{and} \quad h(t)x_{-\pi}(u) = x_{-\pi}(t^3u)h(t)
\]

Moreover, we have:

**Lemma.** There exists a \( c \in \{ \pm 1 \} \) such that

\[
w(t)x_\pi(u) = x_{-\pi}(ct^3u)w(t) \quad \text{and} \quad w(t)x_{-\pi}(u) = x_\pi(-ct^{-3}u)w(t)
\]

for all \( t \in \mathbb{G}_m \) and \( u \in \mathbb{G}_a \).

**Proof.** Steinberg gives the formula [Ste68, p. 67, Lemma 37a]:

\[
w_\beta(t)x_\delta(u) = x_{s_\beta c(c(\beta, \delta)t^{\langle \delta, \beta \rangle}u)w_\beta(t)},
\]

where \( w_\beta(t) \) is defined to be \( x_\beta(t)x_{-\beta}(-t^{-1})x_\beta(t) \) and \( c(\beta, \delta) = \pm 1 \) depends only on \( \beta \) and \( \delta \). Applying this with \( \delta = \pi \) and successively with \( \beta = \gamma_2, \gamma_3, \gamma_4 \), we find \( c \in \{ \pm 1 \} \) such that

\[
w(t)x_\pi(u) = x_{-\pi}(ct^3u)w(t).
\]

(For the exponent of \( t \), note e.g. that \( \langle s_{\gamma_2} \pi, \gamma_3 \rangle = \langle \pi, s_{-\gamma_2} \gamma_3 \rangle = \langle \pi, \gamma_3 \rangle = -1 \).) Similarly, we obtain

\[
w(t)x_{-\pi}(u) = x_\pi(c't^{-3}u)w(t)
\]

for some \( c' \in \{ \pm 1 \} \).

The equations (11.8) give \( h(-1)x_\pi(u) = x_\pi(u)h(-1) \) and since \( h(-1) = w(-1)^2 \), we have:

\[
x_\pi(-u)h(-1) = w(-1)^2x_\pi(u) = x_\pi(cc'u)h(-1).
\]

So \( c' = -c \). \qed
11.9. Remark. We can describe this copy of $SL_2$ concretely in the notation of Dynkin [Dyn57b, Ch. III]. For simplicity, we consider the cases where $\tilde{G}$ is simply laced, so we may identify roots and coroots by defining all roots to have length 2 with respect to the Weyl-invariant inner product $(\ , \ )$. By (11.6), the intersection of the maximal torus $\tilde{T}$ of $\tilde{G}$ with $SL_2$ is the image of the cocharacter $h_{\bar{\alpha}_2 - 2\pi}$. For $\delta$ a simple root of $G$, the inner product $(\bar{\alpha} - 2\pi, \delta)$ is 2 if $\delta$ is adjacent to $\pi$ and 0 otherwise. (Recall that $\pi$ is not a root of $G$.) That is, Dynkin would denote the corresponding copy of $\mathfrak{sl}_2$ in the Lie algebra of $G$ by attaching a 2 to the vertices of the Dynkin diagram of $G$ that are adjacent to $\pi$.

11.10. We take $N$ to be the scheme-theoretic stabilizer of

$$v := x_\pi(1)x_{\bar{\alpha}_1 - \pi}(c)U_{\bar{\alpha}} \in V$$

for $c$ as in Lemma 11.7. For a primitive 4-th root of unity $i$, we have $w(i)v = iv$. (See Remark 8.10 for the vector space structure on $V$.) The map $i \mapsto w(i)$ defines an injection $\mathfrak{mu}_4 \hookrightarrow N$, and we abuse notation by writing also $\mathfrak{mu}_4$ for the image in $N$.

So far, what we have written holds for the general setting of 11.1. We now specialize to the case where $G$ is $E_7$.

11.11. Lemma. In $E_7$, the centralizer $C$ of $E_6$ is the rank 1 torus from 11.9 and the normalizer of $E_6$ is the group generated by $C$, $E_6$, and the copy of $\mathfrak{mu}_4$ from 11.10.

Proof. Write $T_6$ and $T_7$ for the maximal tori in $E_6$ and $E_7$ respectively, obtained by intersecting with the maximal torus $\tilde{T}$ of $\tilde{E}_6$. We argue along the lines of Example 8.12. First note that the centralizer of $T_6$ in $E_7$ contains $T_7$, is reductive, and is generated by root subgroups $U_\gamma$ of $E_7$ for roots $\gamma$ of $E_7$ whose inner product with the simple root $\alpha_i$ is zero for $1 \leq i \leq 6$. Such a $\gamma$ is a multiple of the fundamental weight $\omega_7$ with integer coefficients, i.e., an integer multiple of $2\omega_7$. However, $2\omega_7$ has height 27 and the highest root of $E_7$ has height 17, so no such $\gamma$ exists. Therefore the centralizer of $T_6$ in $E_7$ is $T_7$.

It follows that the centralizer of $E_6$ in $E_7$ is the subgroup of $T_7$ formed by intersecting the kernels of the roots of $E_6$. This is a computation in terms of root systems: the character group of this centralizer is the quotient of the $E_7$ weight lattice by the sublattice generated by the $\alpha_i$ for $1 \leq i \leq 6$; this quotient is free of rank 1. Therefore the centralizer is a rank 1 torus in $T_7$. To prove the first claim in the lemma, it suffices to observe that the inner product $(\bar{\alpha} - 2\pi, \delta)$ is zero for every root $\delta$ of $E_6$, which is clear because $\bar{\alpha} - 2\pi$ equals $2\omega_7$.

The quotient group of “outer automorphisms” (automorphisms modulo inner automorphisms) of $E_6$ is $\mathbb{Z}/2\mathbb{Z}$, so to prove the claim about the normalizer it suffices to show that conjugation by a generator $w(i)$ of $\mathfrak{mu}_4(k_{sep})$ gives an outer automorphism of $E_6$. As $\mathfrak{mu}_4$ belongs to $N$, it normalizes the identity component $E_6$ of $N$. Further, conjugation by $w(i)$ inverts elements of the maximal torus $C$ of $SL_2$. But $C$ contains the center of $E_6$ by the previous paragraph, so conjugation by $w(i)$ is an outer automorphism of $E_6$.  

11.12. Remark. The torus $C$ appearing above is the image of the cocharacter $h_{2\omega_7}: \mathbb{G}_m \to \tilde{T}$, which maps

$$t \mapsto h_{\alpha_1}(t^2)h_{\alpha_2}(t^3)h_{\alpha_3}(t^4)h_{\alpha_4}(t^6)h_{\alpha_5}(t^5)h_{\alpha_6}(t^4)h_{\alpha_7}(t^3).$$
Restricting this homomorphism to $\mu_3$ and $\mu_2$ respectively, we find

$$\zeta \mapsto h_{\alpha_1}(\zeta^2)h_{\alpha_2}(\zeta)h_{\alpha_3}(\zeta^2)h_{\alpha_6}(\zeta) \quad \text{and} \quad \varepsilon \mapsto h_{\alpha_2}(\varepsilon)h_{\alpha_3}(\varepsilon)h_{\alpha_7}(\varepsilon).$$

The images of these maps are the centers of $E_6$ and $E_7$ respectively, see [GQ06, 8.2, 8.1].

11.13. Example ($E_6 \times \mu_4 \subset E_7$). We now show that the inclusion $E_6 \times \mu_4 \subset E_7$ from 11.10 induces a surjection

\begin{equation}
H^1(k, E_6 \times \mu_4) \to H^1(k, E_7)
\end{equation}

for every extension $k/k_0$. By Th. 8.11, it suffices to show that $E_6 \times \mu_4$ is the stabilizer $N$ of $[v] \in \mathbb{P}(V)$ for $v$ as in 11.10. The subgroup of the torus $C$ stabilizing $[v]$ is the image of $\mu_3$ by (11.8), which is the subgroup of $C$ generated by the center of $E_6$ and the copy of $\mu_3$ in $\mu_4$. Combining Lemma 11.11 and the fact that $E_6$ and $\mu_4$ belong to $N$, we conclude that $N$ equals $E_6 \times \mu_4$.

The surjectivity of (11.14) can be interpreted as a statement about Freudenthal triple systems; see [Gar01b, 4.15] for a precise statement and an algebraic proof.

12. Mod 3 invariants of $E_7$

The goal of this section is to compute the invariants of a split group of type $E_7$ (simply connected or adjoint) with values in $\mathbb{Z}/3\mathbb{Z}$. We write $E_7$ for the simply connected split group of that type, and we assume throughout this section that the characteristic of $k_0$ is $\neq 2, 3$. (Roughly speaking, we avoid characteristic 2 in order to use the results of the previous section, and we avoid characteristic 3 because we wish to describe the invariants mod 3, cf. Remark 2.4.) The “heavy lifting” was already done in the previous section.

Recall that the split group $F_4$ of that type can be viewed as a subgroup of $E_6$ as in Example 8.12.

12.1. Lemma. The inclusion $F_4 \subset E_7$ gives a morphism

$$H^1(\ast, F_4) \to H^1(\ast, E_7)$$

that is surjective at 3.

Proof. We have inclusions

$$F_4 \times \mu_3 \subset E_6 \subset E_6 \times \mu_4 \subset E_7.$$ 

The first and third of these induce surjections on $H^1$ by Example 8.12 and 11.13. The second inclusion gives a morphism that is surjective at 3.

The image of $\mu_3$ in $E_7$ is the center of $E_6$ as in Remark 11.12, so the inclusion $F_4 \times \mu_3 \subset E_7$ factors through the subgroup $F_4 \times C$. As $H^1(k, C)$ is zero for every $k/k_0$, the images of $H^1(k, F_4)$ and $H^1(k, F_4 \times \mu_3)$ in $H^1(k, E_7)$ agree. The claim follows.

12.2. Mod 3 invariants of $E_7$. We now give two proofs that the invariant $g_3$ of $F_4$ defined in 7.4 extends to an invariant of $E_7$, which we will also denote by $g_3$.

Proof #1. Let $J, J' \in H^1(k, F_4)$ be Albert algebras whose images in $H^1(k, E_7)$ agree. Then $J$ and $J'$ have similar norms by [Fer72, 6.8] and $g_3(J)$ equals $g_3(J')$ by 10.5. Lemma 6.1 gives that $g_3$ extends to an invariant of $E_7$. 

□
Proof #2. The Rost invariant of $E_7$ has order 12 [Mer03, 16.7], so the mod 3 portion defines a nonzero invariant of $E_7$ with values in $\mathbb{Z}/3\mathbb{Z}$. Moreover, the inclusion $F_4 \subset E_7$ has Rost multiplier 1, so the restriction of this invariant of $E_7$ to $F_4$ is — up to sign — the $g_3$ from 7.4.

Combining Lemmas 4.1 and 12.1, we conclude that the restriction

$$\text{Inv}_{k_0}^\text{norm}(E_7, \mathbb{Z}/3\mathbb{Z}) \rightarrow \text{Inv}_{k_0}^\text{norm}(F_4, \mathbb{Z}/3\mathbb{Z})$$

is injective; by the above and Th. 7.4, it is an isomorphism. We conclude:

**Theorem.** $\text{Inv}_{k_0}^\text{norm}(E_7, \mathbb{Z}/3\mathbb{Z})$ is a free $R_3(k_0)$-module with basis $g_3$.  

12.3. **Exercise (Mod 3 invariants of adjoint $E_7$).** Write $E_7^{\text{adj}}$ for the split adjoint group of type $E_7$. Prove that the invariant $g_3$ of $E_7$ induces an invariant $g_3^{\text{adj}}: H^1(\ast, E_7^{\text{adj}}) \rightarrow H^3(\ast, E_7^{\text{adj}})$ and that $\text{Inv}_{k_0}^\text{norm}(E_7, \mathbb{Z}/3\mathbb{Z})$ is a free $R_3(k_0)$-module with basis $g_3^{\text{adj}}$.

For the mod 2 invariants of $E_7$, the situation is much less clear.

12.4. **Open problem.** (Reichstein-Youssin [RY00, p. 1047]) Let $k_0$ be an algebraically closed field of characteristic zero. Is there a nonzero invariant

$$H^1(\ast, E_7^{\text{adj}}) \rightarrow H^8(\ast, \mathbb{Z}/2\mathbb{Z})?$$

[Some readers have expressed skepticism about the precise degree — 8 — suggested above. Nonetheless, the core of the question remains: What are the invariants of degree > 3?]

13. **Construction of groups of type $E_8$**

Write $E_8$ for the split algebraic group of that type over $k_0$. Since it is adjoint and every automorphism is inner, the set $H^1(k, E_8)$ is identified with the group of isomorphism classes of groups of type $E_8$ over $k$. (This same phenomenon occurs with groups of type $G_2$ and $F_4$.) Here we describe a construction of groups of type $E_8$ that is analogous to the first Tits construction of groups of type $F_4$ (equivalently, Albert algebras) from 7.1. The fruit of this construction will appear in the next section. As the results of §8 do not apply to $E_8$ by Example 8.7, we take a new approach here. We assume throughout this section that the characteristic of $k_0$ is $\neq 5$.

13.1. **A subgroup $H$ of $E_8$.** Let $G$ be split of type $E_8$. We write $H$ for the subgroup of $G$ generated by the root subgroups $U_{\pm \alpha}$ and $U_{\pm 2\alpha}$, for $i \neq 5$. This subgroup is of type $A_4 \times A_4$. We identify the first component of $H$ — generated by $U_{\pm \alpha}$, for $i = 1, 2, 3, 4$ — with $SL_5$ via an irreducible representation whose highest weight is 1 on $\alpha_1$ and 0 on $\alpha_2, \alpha_3$, and $\alpha_4$. We identify the second component of $H$ with $SL_5$ via an irreducible representation whose highest weight is 1 on $\alpha_6$ and 0 on $\alpha_7, \alpha_8$, and $-\alpha$.

Write $\varrho$ for the homomorphism $\mu_5 \rightarrow E_8$ defined by

$$\varrho: \zeta \mapsto h_1(\zeta) h_2(\zeta^4) h_3(\zeta^2) h_4(\zeta^3).$$

Applying the method described in [GQ06, §8], one finds that the image of $\varrho$ is the center of both copies of $SL_5$ in $E_8$. More precisely, the canonical identification of the center of $SL_5$ with $\mu_5$ is the map $\varrho$ for the first component of $H$ and $\varrho^3$ for the second component of $H$. That is, we have identified $H$ with the quotient of $SL_5 \times SL_5$ by the subgroup generated by $(\zeta, \zeta^2)$ for $\zeta \in \mu_5$. 


For $i = 1, 2$, write $\pi_i : H \to PGL_5$ for the projection on the $i$-th factor.

13.3. Lemma. For $\eta \in H^1(k, H)$, write $A_1$ for the central simple $k$-algebra of degree 5 defined by $\pi_1(\eta)$. Then $2[A_1] = [A_2]$ in the Brauer group of $k$.

The twisted group $H_\eta$ is isomorphic to $(SL(A_1) \times SL(A_2))/\mu_5$.

Proof. Consider the diagram with exact rows

$$
\begin{array}{cccccc}
1 & \longrightarrow & \mu_5 \times \mu_5 & \longrightarrow & SL_5 \times SL_5 & \longrightarrow & PGL_5 \times PGL_5 & \longrightarrow & 1 \\
\downarrow & & \downarrow q & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mu_5 & \longrightarrow & H & \longrightarrow & PGL_5 \times PGL_5 & \longrightarrow & 1 \\
\end{array}
$$

where $q$ is given by $(x, y) \mapsto y/x^2$. The diagram commutes because $y/x^2 = x(y/x^3)$ and $(y/x^3)^2 = y(y/x^3)^2$. We obtain a commutative diagram with exact rows:

$$
\begin{array}{cccc}
1 & \longrightarrow & H^1(k, PGL_5 \times PGL_5) & \longrightarrow & Br_5 \times Br_5 \\
\downarrow & & \downarrow & & \downarrow q \\
H^1(k, H) & \underset{\pi_1 \times \pi_2}{\longrightarrow} & H^1(k, PGL_5 \times PGL_5) & \underset{\delta}{\longrightarrow} & Br_5. \\
\end{array}
$$

In the Brauer group, we find the equation:

$$0 = \delta(\pi_1 \times \pi_2)(\eta) = q(\pi_1(\eta), \pi_2(\eta)) = -2[A_1] + [A_2].$$

13.4. The subgroup $C$ of $H$. Write $C$ for the group $\mathbb{Z}/5\mathbb{Z} \times \mu_5$. We define a homomorphism $t : C \times \mu_5 \to H$ such that $t$ restricted to $\mu_5$ is the map $\rho$ from (13.2) and the restriction of $t$ to $C$ is given by

$$t|_C(i, \zeta) = (v^i w^j, v^i u^{2j})$$

in the notation of 5.2, where $\zeta$ is a fixed primitive 5-th root of unity. The formula $uv = \zeta vu$ shows that $t$ is indeed a group homomorphism.

For each extension $k/k_0$, there is an induced function

$$(13.5) \quad t_* : H^1(k, C \times \mu_5) \to H^1(k, E_8).$$

Because $H^1(k, E_8)$ classifies groups of type $E_8$ over $k$, we view $t_*$ as a construction of groups of type $E_8$ via Galois descent.

13.6. Example. We now compute $t_*(\gamma, z)$ for some $\gamma \in H^1(k, C)$ and $z \in H^1(k, \mu_5)$. Put $\eta := t_*(\gamma, 1)$ and write $A_1$ for $\pi_1(\eta)$ as in Lemma 13.3. Twisting $SL_5$ and $E_8$ by $\eta$, we find a subgroup $SL(A_1)$ of $(E_8)_\eta$.

The diagram

$$
\begin{array}{ccc}
C & \stackrel{t}{\longrightarrow} & H \\
\downarrow c & & \downarrow \pi_1 \\
PGL_5
\end{array}
$$

commutes for $c$ as in 5.2, so $\pi_1(\eta) = c_\ast(\gamma)$.

(1) Suppose that $c_\ast(\gamma)$ is zero, i.e., $A_1$ is split. Then $H_\eta$ is split. Since $\eta$ and $t_*(\gamma, z)$ — viewed as elements of $H^1(k, H)$ — differ by a central cocycle $t_*(1, z)$, the twisted group $H_{t_*(\gamma, z)}$ is also split. Hence $E_8$ twisted by $t_*(\gamma, z)$ is split. We conclude that $t_*(\gamma, z)$ is zero in $H^1(k, E_8)$. 
(2) Suppose now that \( z_1, z_2 \in H^1(k; \mu_5) \) differ by a reduced norm from \( A \). The inclusion \( g \) of \( \mu_5 \) into \( H \subset E_8 \) is unaffected by twisting by \( \eta \), and we obtain a map \( H^1(k; \mu_5) \rightarrow H^1(k, (E_8)_\eta) \). The composition
\[
H^1(k, \mu_5) \xrightarrow{\psi} H^1(k, (E_8)_\eta) \xrightarrow{\omega} H^1(k, E_8),
\]
where \( \tau_\eta \) is the twisting isomorphism, sends \( z_i \) to \( t_\ast(\gamma, z_i) \). However, the first arrow factors through \( H^1(k, SL(A_1)) \), hence \( z_1 \) and \( z_2 \) have the same image in \( H^1(k, E_8) \).

13.7. Proposition. The morphism
\[
t_\ast : H^1(*, C \times \mu_5) \rightarrow H^1(*, E_8)
\]
is surjective at 5.

Proof. Step 1. We first show that — for \( H \) the subgroup of \( E_8 \) defined in 13.1 — the morphism \( H^1(*, H) \rightarrow H^1(*, E_8) \) is surjective at 5.

Let \( \xi \) be in \( Z^1(k, E_8) \). Fix a maximal and \( k \)-split torus \( T \) of \( E_8 \) and a \( k \)-defined maximal torus \( T' \) in the twisted group \( (E_8)_\xi \). There is some \( g \in E_8(k_{\text{sep}}) \) such that \( g^{-1}T'g = T \), and replacing \( \xi \) with \( \sigma \rightarrow g^{-1} \xi, g \), we may assume that \( \xi, \sigma(T) = T \) for every \( \sigma \in \text{Gal}(k_{\text{sep}}/k) \), i.e., that \( \xi \) takes values in \( N_{E_8}(T) \).

The Galois group acts trivially on the Weyl group \( N_{E_8}(T)/T \), so the image \( \bar{\xi} \in Z^1(k, N_{E_8}(T)/T) \) is a continuous homomorphism \( \bar{\xi} : \text{Gal}(k_{\text{sep}}/k) \rightarrow N_{E_8}(T)/T \). Fix a 5-Sylow subgroup \( S \) of \( N_{E_8}(T)/T \). Take \( K \) to be the subfield of \( k_{\text{sep}} \) fixed by \( \bar{\xi}^{-1}(S) \); it is an extension of \( K \) of dimension not divisible by 5.

Because \( S \) and a 5-Sylow in \( N_H(T)/T \) both have order \( 5^2 \), there is a \( \bar{w} \in (N_{E_8}(T)/T)(K) \) such that the image of the map \( \sigma \rightarrow \bar{w}^{-1} \xi, \bar{w} \) is contained in \( N_H(T)/T \). Further, \( T \) is \( K \)-split, so there is some \( w \in N_{E_8}(T)(K) \) such that \( w \) maps to \( \bar{w} \). Replacing \( \xi \) with \( \sigma \rightarrow w^{-1} \xi, w \), we may assume that \( \text{res}_{K/k}(\xi) \in H^1(K, E_8) \) is in the image of \( H^1(K, H) \).

Step 2. We now show that the morphism \( H^1(*, C \times \mu_5) \rightarrow H^1(*, H) \) is surjective at 5. Fix \( \eta \in Z^1(K, H) \) and let \( A \) be the central simple algebra of degree 5 representing \( \pi_1(\eta) \in H^1(K, PGL_5) \). By Lemma 5.6, there is an extension \( L/K \) of dimension not divisible by 5 such that \( A \otimes L \) is cyclic, i.e., equals \( c_{\gamma}(\gamma) \) for some \( \gamma \in H^1(L, C) \). We have a commutative diagram with exact rows:
\[
\begin{array}{cccccc}
1 & \longrightarrow & \mu_5 & \longrightarrow & C \times \mu_5 & \longrightarrow & C & \longrightarrow & 1 \\
& & \| & & \| & & \| & & \\
1 & \longrightarrow & \mu_5 & \xrightarrow{\psi} & H & \xrightarrow{\pi_1 \times \pi_2} & PGL_5 \times PGL_5 & \longrightarrow & 1.
\end{array}
\]
By Lemma 13.3, \( \gamma \) and \( \eta \) have the same image in \( H^1(k, PGL_5 \times PGL_5) \), namely the class of \( (\pi_1(\eta), \pi_2(\eta)) \). It follows that \( \eta \) and \( t_\ast(\gamma) \) are in the same \( H^1(k, \mu_5) \)-orbit. Fixing a \( \lambda \in H^1(k, \mu_5) \) such that \( \eta = \lambda \cdot t_\ast(\gamma) \), we have:
\[
t_\ast(\lambda \cdot \gamma) = \lambda \cdot t_\ast(\gamma) = \eta,
\]
as desired. \( \square \)

13.8. The Rost Invariant. We now compute the composition
\[
H^1(k, C \times \mu_5) \xrightarrow{t_\ast} H^1(k, E_8) \xrightarrow{r_{E_8}} H^3(k, \mathbb{Q}/\mathbb{Z}(2))
\]
for every extension $k/k_0$. As the Dynkin index of $E_8$ is $60 = 5 \cdot 12$ [Mer03, 16.8], 4.3 says that the image of the composition is 5-torsion, hence lies in $H^3(k, \mu_5^{(2)})$. Recall that $C$ is $\mathbb{Z}/5\mathbb{Z} \times \mu_5$. We have:

13.10. **Lemma.** There is a uniquely determined $\lambda \in H^1(k_0, \mu_5)$ and a natural number $m$ not divisible by 5 such that the composition (13.9) is given by

$$(x, y, z) \mapsto \lambda \cdot x \cdot y + m x \cdot y \cdot z$$

for every $x \in H^1(k, \mathbb{Z}/5\mathbb{Z})$ and $y, z \in H^1(k, \mu_5)$ and every $k/k_0$.

**Proof.** We first prove the claim in the case where $z$ is zero and $k_0$ contains a primitive 5-th root of unity, which we use to identify $\mu_5$ with $\mathbb{Z}/5\mathbb{Z}$. If $x$ or $y$ is zero, the class $t_*(x, y, 1)$ is zero in $H^1(k, E_8)$ by Example 13.6.1. Applying Lemma 5.7, we conclude that the composition (13.9) is $(x, y, 1) \mapsto \lambda \cdot x \cdot y$ for a unique $\lambda \in R_5(k_0)$. That is, the claim holds in this case.

We now consider the case where $z$ is zero, but the extension $k_1$ obtained by adjoining a primitive 5-th root of unity to $k_0$ may be proper. By the previous paragraph, the restriction of (13.9) to $H^1(\mu, C)$ and viewed as an invariant $\text{Fields}_{/k_1} \to \text{Abelian Groups}$ is given by

$$(x, y) \mapsto \lambda_1 \cdot x \cdot y$$

for a uniquely determined $\lambda_1 \in H^1(k_1, \mu_5)$. Write $\lambda_0$ for the unique class in $H^1(k_0, \mu_5)$ whose restriction to $k_1$ is $\lambda_1$. Since the invariants (13.9) and $(x, y) \mapsto \lambda_0 \cdot x \cdot y$ agree over every extension $k/k_1$, Lemma 3.2 proves the claim.

Finally, we consider the general case. Put $\eta := c_s(x, y)$ and consider the diagram

$$
\begin{array}{ccc}
H^1(k, \mu_5) & \longrightarrow & H^1(k, (E_8)_\eta) \\
\tau_*(x, y, z) & \cong & \tau_\eta \\
& \downarrow \tau_\eta & \downarrow \tau_\eta \\
& r_{E_8} & H^3(k, \mathbb{Q}/\mathbb{Z}(2))
\end{array}
$$

where $\tau_\eta$ is the twisting isomorphism. The triangle obviously commutes and the square commutes by [Gil00, p. 76, Lemma 7]. The image of $z \in H^1(k, \mu_5)$ in the lower right corner going counterclockwise is $r_{E_8}(t_*(x, y, z))$, i.e., the image of $(x, y, z$) under (13.9). The arrow in the upper left factors as $H^1(k, \mu_5) \xrightarrow{\phi} H^1(k, SL(A_1)) \to H^1(k, (E_8)_\eta)$. Since the inclusion of $SL(A_1)$ in $(E_8)_\eta$ has Rost multiplier 1, the composition on the top row is $z \mapsto m x \cdot y \cdot z$ for some natural number $m$ not divisible by 5. This proves the claim.

13.11. **A twisted morphism.** Fix a 1-cocycle $\mu \in Z^1(k, \mu_5)$ such that $\mu = -m^* \lambda$ in $H^1(k, \mu_5)$, where $m^*$ denotes a natural number such that $mnm^*$ is congruent to 1 mod 5, and $\lambda$ is as in Lemma 13.10. Define $i$ to be the composition

$$i: H^1(\mu, C \times \mu_5) \xrightarrow{m+2} H^1(\mu, C \times \mu_5) \xrightarrow{t_*} H^1(\mu, E_8).$$

Since $t_*(x, y, z) = i(x, y, z - \mu)$, Prop. 13.7 holds with $t_*$ replaced by $i$. Furthermore, by Lemma 13.10 we have:

$$r_{E_8} i(x, y, z) = r_{E_8} t_*(x, y, \mu + z) = m x \cdot y \cdot z.$$
13.13. Theorem. Suppose that $k$ is perfect. For $x \in H^1(k, \mathbb{Z}/5\mathbb{Z})$ and $y, z \in H^1(k, \mu_5)$, we have: $i(x, y, z)$ is zero in $H^1(k, E_8)$ if and only if $r_{E_8}i(x, y, z)$ is zero.

**Proof.** The “only if” direction is a basic property of the Rost invariant, so we suppose that $r_{E_8}i(x, y, z)$ is zero, i.e., that $x \cdot y \cdot z$ is zero in $H^3(k, \mu_5^{\otimes 2})$. By the Merkurjev-Suslin Theorem, $z$ is a reduced norm from the cyclic algebra $c_8(x, y)$, so by Example 13.6.1 we have:

$$i(x, y, z) = t_*(x, y, z + \mu) = t_*(x, y, \mu) = i(x, y, 1).$$

Now consider the class $i(x, u, 1)$ in $H^1(k(u), E_8)$ for $u$ an indeterminate. Note that this class is split by the cyclic extension of degree 5 defined by $x$ and it has $r_{E_8}i(x, u, 1) = 0$ by (13.12). The proof of [Gil02a, 1.4] shows that — for every completion $K$ of $k(u)$ with respect to a discrete valuation trivial on $k$ — the image of $i(x, u, 1)$ in $H^1(K, E_8)$ is the image of some element of $H^1(k, E_8)$, i.e., $i(x, u, 1)$ is ramified on $K_{E_8}$. (This argument uses Bruhat-Tits theory in particular the hypothesis that $k$ is perfect.) We conclude that $i(x, u, 1) \in H^1(k(u), E_8)$ is also the image of a class in $H^1(k, E_8)$. By specialization, the value of $i(x, y, 1)$ does not depend on $y$. In particular, we have:

$$i(x, y, 1) = i(x, 1, 1) = t_*(x, 1, \mu) \quad \text{for } y \in H^1(k, \mu_5).$$

But $c_8(x, 1)$ is the matrix algebra $M_5(k)$, so $t_*(x, 1, \mu)$ is zero by Example 13.6.1. \hfill \Box

14. Mod 5 invariants of $E_8$

We now derive consequences of the construction in the previous section. We classify the invariants mod 5 of the split group $E_8$ of that type, recover Chernousov’s result on the kernel of these invariants, and give new examples of anisotropic groups of type $E_8$ over a broad class of fields. We continue the assumption that the characteristic of $k_0$ is $\neq 5$.

As in 13.4, the 5-torsion in $H^1(k, \mathbb{Q}/\mathbb{Z}(2))$ is identified with $H^3(k, \mu_5^{\otimes 2})$. Composing the Rost invariant $r_{E_8}$ with the projection on 5-torsion, we find a normalized invariant

$$h_3: H^1(\ast, E_8) \to H^3(\ast, \mu_5^{\otimes 2}).$$

14.1. Theorem. Inv$_{k_0}^{\text{norm}}(E_8, \mathbb{Z}/5\mathbb{Z})$ is a free $R_5(k_0)$-module with basis $h_3$.

**Proof.** By Cor. 3.5, we may assume that $k_0$ is perfect and that it contains a primitive 5-th root of unity, which we use to identify $\mu_5$ with $\mathbb{Z}/5\mathbb{Z}$. Because $i$ is surjective at 5, the restriction map

$$i^*: \text{Inv}_{k_0}^{\text{norm}}(E_8, \mathbb{Z}/5\mathbb{Z}) \to \text{Inv}_{\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}, \mathbb{Z}/5\mathbb{Z}}^{\text{norm}}$$

is an injection. By the same proof as for $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in S16.4, we see that every normalized invariant of $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ is of the form

$$(x, y, z) \mapsto \lambda_x \cdot x + \lambda_y \cdot y + \lambda_z \cdot z + \lambda_{xy} \cdot x \cdot y + \lambda_{xz} \cdot x \cdot z + \lambda_{yz} \cdot y \cdot z + \lambda_{xyz} \cdot x \cdot y \cdot z$$

for uniquely determined $\lambda$’s in $R_5(k_0)$. However, if $x$, $y$, or $z$ is zero, then $x \cdot y \cdot z$ is zero, hence $i(x, y, z)$ is zero in $H^1(k, E_8)$ by Th. 13.13. It follows that the image of (14.2) is contained in the span of the invariant

$$(x, y, z) \mapsto \lambda_{xyz} \cdot x \cdot y \cdot z.$$
But this $R_5(k_0)$-submodule of $\text{Inv}^{\text{norm}}(\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}, \mathbb{Z}/5\mathbb{Z})$ is also the submodule spanned by the restriction of $h_3$ by (13.12), so the theorem is proved. □

14.3. Open problem. (Reichstein-Youssin [RY00, p. 1047]) Let $k_0$ be an algebraically closed field of characteristic zero. Do there exist nonzero invariants mapping $H^1(*, E_8)$ into $H^9(*, \mathbb{Z}/2\mathbb{Z})$ and $H^5(*, \mathbb{Z}/3\mathbb{Z})$?

14.4. Comparison with groups of type $F_4$. There are tantalizing similarities between the behavior of groups of type $F_4$ relative to the prime 3 and groups of type $E_6$ relative to the prime 5, see e.g. [Gil02a, 3.2] or compare Theorems 7.4 and 14.1. We now investigate these similarities. For $E_8$, the morphism $i$ from 13.11 plays the role of the first Tits construction of Albert algebras.

As groups of type $F_4$ and $E_8$ have trivial centers and only inner automorphisms, the groups $H^1(k, F_4)$ and $H^1(k, E_8)$ are in bijection with isomorphism classes of split groups of type $F_4$ and $E_8$ respectively. Using this bijection, it makes sense to write $g_3(G)$ when $G$ is of type $F_4$ and $g_3$ denotes the invariant from 7.4, as well as to write $h_3(G)$ for $G$ of type $E_8$ and $h_3$ as defined above.

14.5. Splitting by extensions prime to $p$. Every group $G$ of type $F_4$ over $k$ is of the form $\text{Aut}(J)$ for some Albert $k$-algebra $J$. If $g_3(G) \in H^3(k, \mu_3^{\otimes 2})$ is nonzero, then clearly $G$ cannot be split by an extension of degree not divisible by 3. Conversely, if $g_3(G)$ is zero, then $J$ is reduced, i.e., constructed from an octonion algebra $O$ and a 2-Pfister form. In that case, every quadratic extension of $k$ that splits $O$ also splits $J$ and $G$ [Jac68, p. 369, Th. 2].

For groups of type $E_8$, the analogous result is the following. It is due to Chernousov, see [Che95].

**Proposition.** An algebraic group $G$ of type $E_8$ over $k$ is split by an extension of $k$ of dimension not divisible by 5 if and only if $h_3(G) = 0$.

**Proof.** As $h_3$ is normalized, the “only if” direction is clear. So assume that $h_3(G)$ is zero. After replacing $k$ by an extension of dimension not divisible by 5, we may assume that $k$ is perfect and that $G$ equals $i(x, y, z)$ for $i$ the map defined in 13.11 and some $x, y, z$. Since $r_E, i(x, y, z)$ equals $h_3(G)$, Th. 13.13 gives the claim. □

14.6. Anisotropy. For a group $G = \text{Aut}(J)$ of type $F_4$, one knows that $G$ is isotropic if and only if $J$ has nonzero nilpotents [CG06, 9.1]. If $g_3(G)$ is nonzero, then $J$ has no zero divisors (i.e., is not reduced), see [Ros91] or [PR96], and in particular $G$ is anisotropic.

We now prove the corresponding result for $E_8$.

**Proposition.** If a group $G$ of type $E_8$ has $h_3(G) \neq 0$, then $G$ is anisotropic.

**Proof.** If $G$ is split, then clearly $h_3(G) = 0$. So suppose that $G$ is isotropic but not split. According to the list of possible indexes in [Tit66, p. 60], the semisimple anisotropic kernel $A$ of $G$ is a strongly inner group of type $D_4$, $D_6$, $D_7$, $E_6$, or $E_7$. That is, $A$ is obtained by twisting a split simply connected group $S$ of one of these types by a 1-cocycle $\eta \in Z^3(k, S)$. Tits’s Witt-type Theorem [Spr98, 16.4.2] implies that $G$ is isomorphic to $E_8$ twisted by $\eta$. 
The inclusion of $S$ in $E_8$ comes from the obvious inclusion of Dynkin diagrams, so has Rost multiplier one. That is, the diagram

$$H^1(k, S) \xrightarrow{r_S} H^3(k, \mathbb{Q}/\mathbb{Z}(2))$$

commutes. However, for each of the possibilities for $S$, the Dynkin index is 2, 2, 2, 6, or 12 respectively by [Mer03, 15.4, 16.6, 16.7], so the mod 5 portion of $r_{E_8}(\eta)$, namely $h_3(G)$, is zero. □

14.7. Anisotropic groups split by extensions of degree $p$. If $G = \text{Aut}(J)$ is a group of type $F_4$ over $k$ where $J$ is a first Tits construction that is not split, then $G$ is anisotropic over $k$ but split by a cubic extension of $k$. That is, nonzero symbols in $H^3(k, \mu_5^{\otimes 2})$ give anisotropic groups of type $F_4$ that are split by a cubic extension.

The analogous statement for $E_8$ is the following:

**Corollary** (of Prop. 14.6). If $H^3(k, \mu_5^{\otimes 2})$ contains a nonzero symbol, then $k$ supports an anisotropic group of type $E_8$ that is split by a cyclic extension of dimension 5.

**Proof.** Fix $x \in H^1(k, \mathbb{Z}/5\mathbb{Z})$ and $y, z \in H^1(k, \mu_5)$ such that $x \cdot y \cdot z$ is not zero in $H^3(k, \mu_5^{\otimes 2})$. Then $h_3i(x, y, z)$ is not zero by (13.12), so the group obtained by twisting $E_8$ by $i(x, y, z)$ is anisotropic by Prop. 14.6. It is split by the cyclic extension of $k$ of dimension 5 determined by $x$ by Example 13.6. □

The interesting part of the corollary is that the groups are split by an extension of degree 5, and not merely that the groups are anisotropic. Indeed, anisotropic groups of type $E_8$ abound. For example, a number field $k$ supports an anisotropic group $G$ of type $E_8$ if and only if $k$ has a real embedding by the Hasse Principle [PR94b, p. 286, Th. 6.6]. But the Hasse Principle also implies that $G$ cannot be split by an odd-degree extension of $k$.

As a concrete illustration of the corollary, fix a number field $k$. It supports a cyclic division algebra $A$ of dimension $5^2$. (One can specify $A$ by local data, see [Rei75, §32].) For $t$ an indeterminate, the symbol $[A \cdot (t)]$ is nonzero in $H^3(k(t), \mu_5^{\otimes 2})$.

14.8. Failure of the analogy. If $G = \text{Aut}(J)$ is a group of type $F_4$ such that the Rost invariant $r_{F_4}(G)$ is 3-torsion (i.e., $r_{F_4}(G)$ equals $g_3(G)$), then it is a result of Petersson and Racine that $J$ is a first Tits construction [KMRT98, 40.5]. In this case, the analogy between first Tits constructions and the map $i$ fails. Gille [Gil02h, App.] has given an example of a group $G$ of type $E_8$ over a particular field $k$ such that $r_{E_8}(G)$ is zero but $G$ is not split. By Th. 13.13, such a $G$ cannot be in the image of $i$.

14.9. Exercise (prime-to-5-closed fields). Suppose that $k$ is a field such that every finite separable extension of $k$ has degree a power of 5. Prove that every group of type $E_8$ over $k$ is split or anisotropic.

[The assumption on $k$ is stronger than necessary; it suffices to assume that the group $H^3(k, \mathbb{Z}/6\mathbb{Z}(2))$ defined in [Mer03, App. A] is zero.]
Part III. Spin groups

15. Surjectivities: Spin\textsubscript{n} for n \leq 12

We continue the examples of internal Chevalley modules as defined in 8.11, focusing on the case where G is Spin\textsubscript{n} for n \leq 12. We assume throughout this section that the characteristic of k\textsubscript{0} is different from 2.

15.1. Example (Spin\textsubscript{2n-1} \cdot Z \subset Spin\textsubscript{2n}). Taking \tilde{G} to be the split simply connected group of type D\textsubscript{n+1} and \pi to be \alpha\textsubscript{1}, we find that G is the split simply connected group Spin\textsubscript{2n} of type D\textsubscript{n}, and V is the vector representation.

There is a G-invariant quadratic form on V and we fix an anisotropic vector v. The stabilizer of v in SO(V) is easily seen to be \mu\textsubscript{2}. SO(v\perp) (using that V is even-dimensional), hence the stabilizer of [v] in G is the compositum Z. Spin\textsubscript{2n-1}, where the center Z of G meets Spin\textsubscript{2n-1} in a copy of \mu\textsubscript{2} that is the kernel of the vector representation. By dimension count, the orbit of [v] is the open orbit in P(V).

Theorem 8.11 gives that the induced map

\begin{equation}
H^1(k, \text{Spin}_{2n-1} \cdot Z) \rightarrow H^1(k, \text{Spin}_{2n})
\end{equation}

is surjective for every k\textsubscript{0}. But we can say a little more. Since Z is central, the multiplication map Spin\textsubscript{2n-1} \times Z \rightarrow Spin\textsubscript{2n-1} \cdot Z is a group homomorphism, and composing this with (15.2) gives a map

\begin{equation}
H^1(k, \text{Spin}_{2n-1}) \times H^1(k, Z) \rightarrow H^1(k, \text{Spin}_{2n})
\end{equation}

and this map is also surjective. Indeed, the intersection Spin\textsubscript{2n-1} \cap Z is the center of Spin\textsubscript{2n-1}, i.e., \mu\textsubscript{2}, and there is an exact sequence

\begin{equation}
1 \rightarrow \text{Spin}_{2n-1} \rightarrow \text{Spin}_{2n-1} \cdot Z \rightarrow \mu\textsubscript{2} \rightarrow 1.
\end{equation}

The center Z of Spin\textsubscript{2n} satisfies

\[
Z \cong \begin{cases} 
\mu\textsubscript{4} & \text{if } n \text{ is odd,} \\
\mu\textsubscript{2} \times \mu\textsubscript{2} & \text{if } n \text{ is even}
\end{cases}
\]

and in either case the restriction of q to Z yields a surjection H\textsuperscript{1}(k, Z) \rightarrow H\textsuperscript{1}(k, \mu\textsubscript{2}). (For surjectivity in the n odd case, see 2.5.) A twisting argument combined with the exactness of (15.4) now gives that the map

\[
H^1(k, \text{Spin}_{2n-1}) \times H^1(k, Z) \rightarrow H^1(k, \text{Spin}_{2n-1} \cdot Z)
\]

is surjective, hence that (15.3) is surjective, as claimed.

Attempting to do the same for groups of type B (equivalently, odd-dimensional quadratic forms) gives a stabilizer that is less attractive.

15.5. Example (G\textsubscript{2} \times \mu\textsubscript{2} \subset Spin\textsubscript{7}). Take \tilde{G} to be the split group of type F\textsubscript{4} and \pi := \alpha\textsubscript{4}. The subgroup G is the split simply connected group Spin\textsubscript{7} of type B\textsubscript{3} and V is its spin representation.

Write G\textsubscript{2} for the split group of that type. The irreducible representation W with highest weight \omega\textsubscript{1} is 7-dimensional (in characteristic \neq 2 [GS88, p. 413]) and supports a G\textsubscript{2}-invariant nonsingular quadratic form q. It gives an embedding of G\textsubscript{2} in Spin\textsubscript{7}. We claim that N may be taken to be the direct product of G\textsubscript{2} with the center \mu\textsubscript{2} of Spin\textsubscript{7}. 
As a representation of $G_2$, $V$ is a direct sum of $W$ and a 1-dimensional representation, say $k v$. As in Example 8.12, dimension considerations imply that $[v]$ belongs to the open $L$-orbit in $\mathbb{P}(V)$ and $G_2$ is the identity component of the stabilizer $N$ of $[v]$.

As the kernel $\mu_2$ of the map $\text{Spin}_7 \to \text{SO}(W)$ clearly belongs to $N$, we may compute $N$ by determining its image in $GL(W)$. Since $W$ is an irreducible representation of $G_2$ and every automorphism of $G_2$ is inner, the normalizer of $G_2$ in $GL(W)$ consists of scalar matrices. It follows that $N$ is contained in $G_2 \times \mu_2$, hence $N$ equals $G_2 \times \mu_2$.

For a version of this example over the reals, see [Var01, Th. 3].

Combining this example with Th. 8.11 gives that every 8-dimensional form in $I^3$ that represents 1 is the norm quadratic form of an octonion algebra, hence every 8-dimensional form in $I^3$ is similar to a 3-Pfister form. This is a special case of the general theorem: a 2$n$-dimensional form in $I^n$ is similar to an $n$-Pfister form [Lam05, X.5.6].

15.6. Exercise. Prove: If $g$ is an 8-dimensional quadratic form over $k$ such that $C_0(g)$ is isomorphic to $M_8(K)$ for some quadratic étale $k$-algebra $K$, then $g$ is similar to $(1) \oplus (\alpha)q_0$ for $\alpha \in k^\times$ such that $K = k[x]/(x^2 - \alpha)$ and a uniquely determined 7-dimensional form $q_0$ such that $(1) \oplus q_0$ is a 3-Pfister form.

[This can be proved using standard quadratic form theory, or by combining Examples 15.1 and 15.5.]

In the examples above, we have used internal Chevalley modules as in 8.11 to produce representations with open orbits. For the cases where $G$ is $\text{Spin}_9$ or $\text{Spin}_{11}$, such arguments are somewhat more complicated than the naive setup in 8.11. (See [Rub04, 4.3(3), 5.1] for details.) Instead, we refer to Igusa’s paper [Igu70]; he proves the existence of an open orbit using concrete computations in the Clifford algebra.

15.7. Example ($\text{Spin}_7 \times \mu_2 \subset \text{Spin}_9$). As in [Igu70, p. 1017], there are inclusions

$$\text{Spin}_7 \to \text{Spin}_9 \to \text{Spin}_9$$

such that $\text{Spin}_9$ has an open orbit in $\mathbb{P}(V)$ for $V$ its (16-dimensional) spin representation, and $\text{Spin}_7$ is the stabilizer of a $v \in V$ whose image in $\mathbb{P}(V)$ is in the open orbit. Recall that there are three non-conjugate embeddings of $\text{Spin}_7$ in $\text{Spin}_9$, distinguished by which copy of $\mu_2$ in the center of $\text{Spin}_9$ they contain, cf. [Dyn57a, Th. 6.3.1] or [Var01, Th. 5]. The $\mu_2$ in this $\text{Spin}_7$ is not in the kernel of the map $\text{Spin}_9 \to \text{SO}_9$, i.e., is not the center of $\text{Spin}_9$.

Write $Z$ for the copy of $\mu_2$ that is the center of $\text{Spin}_9$; the element $-1 \in Z$ sends $v$ to $-v$. But $v$ is an anisotropic vector for the $\text{Spin}_9$-invariant quadratic form on $V$, hence $Z \times \text{Spin}_7$ is the stabilizer of the line $[v]$ in $\text{Spin}_9$.

15.8. Example ($G_2 \times \mu_4 \subset \text{Spin}_{10}$). Example 15.1 gives a surjection

$$H^1(k, \text{Spin}_9 \cdot \mu_4) \to H^1(k, \text{Spin}_{10}).$$

Example 15.7 gives an inclusion

$$\text{Spin}_7 \times \mu_2 \subset \text{Spin}_9$$

that induces a surjection on $H^1$'s, i.e., the map

$$H^1(k, \text{Spin}_7 \times \mu_2) \times H^1(k, \mu_4) \to H^1(k, \text{Spin}_{10})$$
is surjective. The copy of $\mu_2$ here is the center of Spin$_3$, which is contained in $\mu_4$. So combining all the previous statements we obtain an inclusion

$$\text{Spin}_7 \times \mu_4 \subset \text{Spin}_{10}$$

that gives a surjection on $H^1$’s.

In terms of quadratic forms, we view Spin$_{10}$ as the spin group of the quadratic form $q := \langle 1, -1 \rangle \oplus 4\langle 1, -1 \rangle$, where Spin$_7$ acts on the second summand. Therefore, we have proved that the image of the map

$$(15.9) \quad H^1(k, \text{Spin}(q)) \to H^1(k, \text{SO}(q))$$

consists of isotropic quadratic forms. On the other hand, the image of (15.9) is precisely the collection of 10-dimensional forms in $I^n$, so we have recovered Pfister’s result—see [Pf66, p. 123] or [Lam05, XII.2.8]—that such forms are isotropic. (Pfister’s proof used quadratic form theory. Tits gave a characteristic-free proof using algebraic groups in [Tit90, 4.4.1(ii)]. We remark that this theorem has been generalized by Hoffmann, Vishik, and Karpenko: There are no anisotropic forms in $I^n$ of dimension $d$ such that $2^n < d < 2^n + 2^{n-1}$ for $n \geq 2$, see e.g. [EKM].)

We can find a subgroup of Spin$_{10}$ that is smaller than Spin$_7 \times \mu_4$ and yet still gives a surjection on $H^1$’s. As in the remarks at the end of Example 15.5, everything in the image of (15.9) is in the image of

$$(15.10) \quad H^1(k, G_2 \times \mu_4) \to H^1(k, \text{Spin}(q)) \to H^1(k, \text{SO}(q)),$$

where $G_2$ is a subgroup of Spin$_7$ in the natural way. Said differently, everything in $H^1(k, \text{Spin}(q))$ is in the $H^1(k, \mu_4)$-orbit of something in the image of $H^1(k, G_2)$, i.e., the first map in (15.10) is surjective.

Instead of starting with Example 15.1, we could have viewed Spin$_{10}$ as a subgroup of $E_6$, in which case the representation $V$ given by 8.11 is a half-spin representation. However, this gives an ugly stabilizer, see [Igu70, Prop. 2].

The following exercise gives an example of a useful surjection on cohomology.

15.11. Exercise. Recall that for every quadratic étale $k_0$-algebra $k_1$, there is a surjective functor $\text{Quad}_{k_1} \to \text{Herm}_{k_1/k_0, n}$ that sends a quadratic form $q$ to a $k_1/k_0$-hermitian form $q_H$ ("hermitian forms can be diagonalized"). That is, in the commutative diagram

\[
\begin{array}{ccc}
H^1(\ast, O(q)) & \longrightarrow & H^1(\ast, U(q_H)) \\
\uparrow & & \uparrow \\
H^1(\ast, SO(q)) & \longrightarrow & H^1(\ast, SU(q_H)),
\end{array}
\]

the top arrow is a surjection. Prove that the bottom arrow is also a surjection.

15.12. Example (SO(6) $\times$ $\mu_4 \subset$ Spin$_{12}$). Take $\tilde{G}$ to be the split simply connected group of type $E_7$ and $\pi := \alpha_1$. The subgroup $G$ is the split simply connected group Spin$_{12}$ of type $D_6$ and $V$ is a half-spin representation. Speaking concretely, we view Spin$_{12}$ as the spin group of the symmetric bilinear form $b$ on the space with basis $e_1, e_2, \ldots, e_{12}$ such that

\[b(e_i, e_j) = b(e_{6+i}, e_{6+j}) = 0 \quad \text{and} \quad b(e_i, e_{6+j}) = \delta_{ij} \quad (1 \leq i, j \leq 6)\]

Our $b$ is the same bilinear form used by Igusa in [Igu70]; as he did, we write $e_L := e_1 e_2 \cdots e_6$. The element $v := 1 + e_L \in V$ belongs to the open orbit in $\mathcal{P}(V)$, and the stabilizer of $v$ in Spin$_{12}$ is isomorphic to $SL_6$ by [Igu70, Prop. 3] such that
the copy of $\mu_2$ in the center of $SL_6$ is the kernel of the (half-spin) representation on $V$.

We identify $SL_6$ with its image in $Spin_{12}$. As $SL_6$ does not meet the kernel of the vector representation, it is identified with its image in $SO(b)$. With respect to our fixed basis, $b$ is the form

$$b(x, y) = x^t \begin{pmatrix} 0 & I_6 \\ I_6 & 0 \end{pmatrix} y$$

and $SL_6$ sits inside $GL_{12}$ as the matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} (a \in SL_6)$$

We claim that the stabilizer $N$ of $[v]$ is isomorphic to $SL_6 \rtimes \mu_4$. Fix a primitive 4-th root of unity $\zeta$ (in some algebraic closure of $k_0$) and put

$$s := \zeta (e_1 + \zeta e_7)(e_2 + \zeta e_8) \cdots (e_6 + \zeta e_{12}).$$

This element belongs to $Spin_{12}$ and satisfies $s \cdot v = \zeta v$, and $s^2$ is the element $-1$ in the Clifford algebra, i.e., the nontrivial element in the kernel of the vector representation of $Spin_{12}$. As $V$ supports a $Spin_{12}$-invariant quartic form [Igu70, Prop. 3], it follows that $N$ is generated by $SL_6$ and $s$, hence $N$ is isomorphic to $SL_6 \rtimes \mu_4$.

We now compute the action of $\mu_4$ on $SL_6$. Write $\chi : Spin(b) \to SO(b)$ for the vector representation of $Spin_{12}$. Then

$$\chi(s) = \begin{pmatrix} 0 & \zeta \\ -\zeta & 0 \end{pmatrix} \text{ and } \chi(sas^{-1}) = \chi(a^{-t})$$

for $a \in SL_6$. Hence $sas^{-1}$ equals $a^{-t}$ in $Spin_{12}$.

As in Th. 8.11 (or Th. 11.10) we have a surjection

$$H^1(\ast, SL_6 \rtimes \mu_4) \to H^1(\ast, Spin_{12}).$$

Write $SO(6)$ for the special orthogonal group of the dot product on $k_0^6$. It is a subgroup of $SL_6$ and is fixed elementwise by the map $g \mapsto g^{-t}$, so there is a natural inclusion

$$SO(6) \times \mu_4 \hookrightarrow SL_6 \rtimes \mu_4$$

that is the identity on $\mu_4$. The induced map

$$H^1(k, SO(6) \times \mu_4) \to H^1(k, SL_6 \rtimes \mu_4)$$

is a surjection for every extension $k/k_0$. (To see that a given class in $\eta \in H^1(k, SL_6 \rtimes \mu_4)$ is in the image, twist by the image of $\eta$ in $H^1(k, \mu_4)$ and then apply Exercise 15.11.) It follows that the inclusion $SO(6) \times \mu_4 \to Spin_{12}$ induces a surjection on $H^1$'s.

Concretely, this says that every 12-dimensional quadratic form in $I^3$ is isomorphic to $(1, -a)/q$ for some $a \in k^x$ and some 6-dimensional quadratic form $q$ with determinant 1, a result due to Pfister [Pfi66, pp. 123, 124]. Hoffmann has conjectured [Hof98, Conj. 2] a generalization of this statement for forms of dimension $2^n + 2^{n-1}$ in $I^n$ with $n \geq 4$.

15.13. Example (SO(5) $\times \mu_4 \subset Spin_{11}$). We view $Spin_{12}$ as the spin group of the bilinear form $b$ from Example 15.12. In this way, we see $Spin_{11}$ as a subgroup of $Spin_{12}$ consisting of elements that fix the vector

$$\varepsilon_1 := e_6 - e_{12}$$
in the space underlying \( b \). The image of \( \text{Spin}_{11} \) under the vector representation 
\( \chi: \text{Spin}_{12} \to \text{SO}(b) \) is the special orthogonal group of \( b \) restricted to the subspace with basis

\[ e_0 := e_6 + e_{12}, e_1, e_2, \ldots, e_5, e_7, e_9, \ldots, e_{11}. \]

(This is the description of \( \text{Spin}_{2n-1} \subset \text{Spin}_{2n} \) used on pages 1000, 1001 of [Igu70].)

The spin representation of \( \text{Spin}_{11} \) is the restriction of the half-spin representation \( V \) of \( \text{Spin}_{12} \) from the previous example, and \( v := 1 + e_L \) is again a representation of an open orbit in \( \mathbb{P}(V) \) [Igu70, Prop. 6]. The stabilizer \( N \) of \( [v] \) in \( \text{Spin}_{11} \) is the intersection of \( \text{Spin}_{11} \) with \( SL_6 \rtimes \mu_4 \).

We first compute the intersection of \( \chi(\text{Spin}_{11}) \) with \( \chi(SL_6 \rtimes \mu_4) \) in \( \text{SO}(b) \), i.e., we find the subgroup of \( \chi(SL_6 \rtimes \mu_4) \) that fixes \( e_1 \). The intersection is \( SL_5 \rtimes \mu_2 \), where \( SL_5 \) is viewed as the subgroup of matrices

\[
\begin{pmatrix}
\begin{array}{cc}
a & I_1 \\
I_1 & a^{-t}
\end{array}
\end{pmatrix}
\]

\((a \in SL_5)\)

of \( GL_{12} \) and the nontrivial element of \( \mu_2 \) is

\[
\chi\left(\begin{pmatrix}
\zeta^{-1}I_5 & I_1 \\
-I_1 & -I_5
\end{pmatrix}
\right) = \begin{pmatrix}
I_5 \\
-I_1
\end{pmatrix}.
\]

It follows that the stabilizer \( N \) of \( [v] \) in \( \text{Spin}_{11} \) is \( SL_5 \rtimes \mu_4 \). As in the previous example, the composition

\[
\text{SO}(5) \times \mu_4 \to SL_5 \rtimes \mu_4 \to \text{Spin}_{11}
\]

induces a surjection on \( H^1 \)'s.

16. Invariants of \( \text{Spin}_n \) for \( n \leq 10 \)

We now determine the invariants of \( \text{Spin}_n \) (for \( n \leq 10 \)) with values in \( \mathbb{Z}/2\mathbb{Z} \). We assume throughout this section that the field \( k_0 \) has characteristic \( \neq 2 \).

16.1. Invariants of \( \text{Spin}_8 \). Combining Examples 15.5 and 15.1, we find an inclusion

\[
i: G_2 \times Z \to \text{Spin}_8
\]

such that the induced map \( i_* \) on \( H^1 \)'s is surjective, where \( Z \) is the center of \( \text{Spin}_8 \) and is isomorphic to \( \mu_2 \times \mu_2 \). As \( \text{Spin}_8 \) is split, the image of \( H^1(k, Z) \) in \( H^1(k, \text{Spin}_8) \) is zero. Applying Lemma 5.7, \( i_* \) identifies \( \text{Inv}^{\text{norm}}_{k_0}(\text{Spin}_8, \mathbb{Z}/2\mathbb{Z}) \) with an \( R_2(k_0) \)-submodule of the free module \( I \) with basis the invariants

\[ e_3 \cdot 1, \quad e_3 \cdot \text{id}, \quad e_3 \cdot 1 \cdot \text{id}, \quad e_3 \cdot \text{id} \cdot \text{id} \]

of \( G_2 \times Z \), where \( 1 \) and \( \text{id} \) are as defined in §2. Fix inequivalent 8-dimensional representations \( \chi_1, \chi_2: \text{Spin}_8 \to \text{SO}_8 \). They restrict to characters \( \chi_j: Z \to \mu_2 \) which induce invariants \( \chi_j: H^1(*, Z) \to H^1(*, \mu_2) \). Clearly, the invariants

\[
e_3, \quad e_3 \cdot \chi_1, \quad e_3 \cdot \chi_2, \quad e_3 \cdot \chi_1 \cdot \chi_2
\]

are also an \( R_2(k_0) \)-basis for the module \( I \). We prove that each of the invariants in (16.2) is the restriction of an invariant of \( \text{Spin}_8 \).
Let \((C, \zeta)\) be a class in \(H^1(k, G_2 \times Z)\), where \(C\) is an octonion algebra. Abusing notation, we write \(\chi_j(\zeta)\) for the corresponding element of \(k^\times/k^\times 2\). The composition

\[
\begin{align*}
H^1(k, G_2 \times Z) & \longrightarrow H^1(k, \text{Spin}_8) \\
& \longrightarrow H^1(k, \text{SO}_8)
\end{align*}
\]

sends \((C, \zeta)\) to the quadratic form \(\langle \chi_j(\zeta) \rangle N_C\). Composing (16.3) with the Arason invariant \(e_3\) defined in Example 1.2.3 sends \((C, \zeta)\) to \(e_3(C)\). That is, the invariant \(e_3\) from (16.2) is the restriction of an invariant of \(\text{Spin}_8\).

Of course, \(e_3(C)\) is zero whenever \(\langle \chi_j(\zeta) \rangle N_C\) is isotropic. Applying Prop. 9.2 to the representations \(\chi_1\) and \(\chi_2\), we find that \(e_3 \cdot \chi_1\) and \(e_3 \cdot \chi_2\) are also restrictions of invariants of \(\text{Spin}_8\). Finally, \(e_3 \cdot \chi_1\) is zero whenever \(\langle \chi_2(\zeta) \rangle N_C\) is isotropic, and applying Prop. 9.2 again gives that \(e_3 \cdot \chi_1\) is the restriction of an invariant of \(\text{Spin}_8\).

We have proved that \(\text{Inv}_{k_0}^{\text{norm}}(\text{Spin}_8, Z/2Z)\) is a free \(R_2(k_0)\)-module of rank 4 with generators of degree 3, 4, 4, 5.

16.4. Invariants of \(\text{Spin}_7\). By Example 15.5, there is a subgroup \(G_2 \times \mu_2\) of \(\text{Spin}_7\) such that the induced map

\[
i_* : H^1(*, G_2 \times \mu_2) \rightarrow H^1(*, \text{Spin}_8)
\]

is surjective. Combined with the inclusion \(\text{Spin}_7 \hookrightarrow \text{Spin}_8\) obtained by viewing \(\text{Spin}_7\) as the identity component of the stabilizer of a vector of length 1, we have maps

\[
\text{Inv}_{k_0}^{\text{norm}}(G_2 \times \mu_2, Z/2Z) \hookrightarrow \text{Inv}_{k_0}^{\text{norm}}(\text{Spin}_7, Z/2Z) \hookrightarrow \text{Inv}_{k_0}^{\text{norm}}(\text{Spin}_8, Z/2Z).
\]

As in 16.1, the image in \(\text{Inv}_{k_0}^{\text{norm}}(G_2 \times \mu_2, Z/2Z)\) is contained in the free \(R_2(k_0)\)-module \(I\) with basis \(e_3, e_3 \cdot \text{id}\) and the invariant \(e_3\) is the restriction of the Arason invariant on \(\text{SO}_8\). Similarly, the copy of \(\mu_2\) in \(\text{Spin}_7\) is the kernel of a representation \(\text{Spin}_8 \rightarrow \text{SO}_8\), say \(\chi_2\). The invariant \(e_3 \cdot \chi_1\) of \(\text{Spin}_8\) restricts to the invariant \(e_3 \cdot \chi_1\) of \(G_2 \times \mu_2\). This proves that \(\text{Inv}_{k_0}^{\text{norm}}(\text{Spin}_7, Z/2Z)\) is a free \(R_2(k_0)\)-module of rank 2 with basis elements of degrees 3 and 4.

16.5. Invariants of \(\text{Spin}_{10}\). From Example 15.8 we have an inclusion \(i : G_2 \times \mu_4 \rightarrow \text{Spin}_{10}\) such that the induced map \(i_*\) on \(H^1\)'s is surjective. For a Cayley \(k\)-algebra \(C\) and \(\alpha \in k^\times/k^\times 4\), define

\[
a_3(C, \alpha) = e_3(C) \quad \text{and} \quad a_4(C, \alpha) = e_3(C) \cdot s(\alpha)
\]

in \(\text{Inv}_{k_0}^{\text{norm}}(G_2 \times \mu_4, Z/2Z)\). (The invariant \(s\) is defined in 2.5.) As for \(\text{Spin}_8\), \(i_*\) identifies \(\text{Inv}_{k_0}^{\text{norm}}(\text{Spin}_{10}, Z/2Z)\) with a submodule of the free \(R_2(k_0)\)-module with basis \(a_3, a_4\). The image of a pair \((C, \alpha)\) in \(H^1(k, \text{SO}_{10})\) corresponds to the quadratic form \(\langle 1, -1 \rangle \oplus \langle \alpha \rangle N_C\), so \(a_3\) and \(a_4\) are obviously restrictions of invariants of \(\text{Spin}_{10}\).

16.6. Invariants of \(\text{Spin}_9\). We view \(\text{Spin}_8 \subset \text{Spin}_9 \subset \text{Spin}_{10}\) as the spin groups of the quadratic forms

\[
4(1, -1), \quad \langle -1 \rangle \oplus 4(1, -1), \quad \text{and} \quad \langle 1, -1 \rangle \oplus 4(1, -1)
\]

in the obvious manner. Combining Examples 15.5 and 15.7 and putting \(Z = \mu_2 \times \mu_2\), we find an inclusion of \(G_2 \times Z\) in \(\text{Spin}_9\) that gives a surjection

\[
H^1(*, G_2 \times Z) \rightarrow H^1(*, \text{Spin}_9)
\]

and identifies \(\text{Inv}_{k_0}^{\text{norm}}(\text{Spin}_9, Z/2Z)\) with a submodule of \(\text{Inv}_{k_0}^{\text{norm}}(G_2 \times Z, Z/2Z)\), contained in the free \(R_2(k_0)\)-module with basis (16.2).
For the sake of fixing notation, suppose that the restriction of the vector representation of Spin$_9$ to Spin$_8$ is the direct sum of $\chi_1$ (as opposed to $\chi_2$ or $\chi_3$) and a 1-dimensional trivial representation. The image of a pair $(C, \zeta) \in H^1(k, G_2 \times Z)$ under the maps

$$H^1(k, G_2 \times Z) \to H^1(k, \text{Spin}_9) \to H^1(k, \text{Spin}_{10})$$

is $\langle 1, -1 \rangle \oplus \langle \chi_1(\zeta) \rangle N_C$. Thus the invariants $a_3$ and $a_4$ of Example 16.5 restrict to invariants $e_3$ and $e_3 \cdot \chi_1$ of $G_2 \times Z$ from (16.2).

We can also view Spin$_9$ as the subgroup of the automorphism group of the split Albert algebra $J$ consisting of the algebra automorphisms that fix a primitive idempotent in $J$ [Jac68, IX.3]. Restricting $J$ to a representation of Spin$_8$, we find a direct sum of a 3-dimensional trivial representation and the three inequivalent irreps $\chi_1, \chi_2, \chi_3$: Spin$_8 \to SO_8$. The invariant

$$f_5: H^1(\ast, \text{Aut}(J)) \to H^5(\ast, \mathbb{Z}/2\mathbb{Z})$$

defined in §22 of S restricts to be nonzero on $G_2 \times Z$. If $-1$ is a square in $k_0$, then its restriction is the invariant $e_3 \cdot \chi_1 \cdot \chi_2$ on $G_2 \times Z$ from (16.2). (The assumption on $-1$ is here only for the convenience of ignoring various factors of $-1$.)

Finally we claim that the invariant $\lambda \cdot e_3 \cdot \chi_2$ of $G_2 \times Z$, for every nonzero $\lambda \in R_2(k_0)$, is not the restriction of an invariant of Spin$_9$. Let $k$ be the extension of $k_0$ obtained by adjoining indeterminates $x, y, z, w$, and write $C$ for the Cayley $k$-algebra with $e_3(C)$ equal to $(x) \cdot (y) \cdot (z)$. Fix a $\zeta \in H^1(k, Z)$ such that $\chi_1(\zeta) = (1)$ and $\chi_2(\zeta) = (w)$. The invariant $e_3 \cdot \chi_2$ takes different values on $(C, 1)$ and $(C, \zeta)$, namely $0$ and $\lambda \cdot (x) \cdot (y) \cdot (z) \cdot (w)$. However, the two classes have the same image in $H^1(k, SO_9)$, the form $-1 \oplus N_C$. As this form is isotropic, its spinor norm map is onto and the fiber of

$$H^1(k, \text{Spin}_9) \to H^1(k, SO_9)$$

over $-1 \oplus N_C$ is a singleton. That is, $(C, 1)$ and $(C, w)$ have the same image in $H^1(k, \text{Spin}_9)$, proving the claim.

In the case where $-1$ is a square in $k_0$, this determines Inv$_{\text{norm}}^{SO_9}(\text{Spin}_9, \mathbb{Z}/2\mathbb{Z})$: it is free of rank 3 with basis elements of degree 3, 4, 5.

17. Divided squares in the Grothendieck-Witt ring

In this section, we define a function $P_3: I^n \to I^{2n}$ in the Witt ring that will be used to construct invariants of Spin$_n$ for $n = 11, 12, 14$. It can also be used to give bounds on the symbol length of a class in $H^d(k, \mathbb{Z}/2\mathbb{Z})$, cf. Example A.3.

Recall the Grothendieck-Witt ring $\widehat{W}$ (denoted WGr in S) over a field $k$ of characteristic $\neq 2$: it is the ring of formal differences of (nondegenerate) quadratic forms over $k$. It is a $\lambda$-ring in the sense of Grothendieck, see e.g. S27.1. For a quadratic form $q = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $0 < p \leq n$, we have

$$\lambda^p q = \oplus_{i_1 < i_2 < \cdots < i_p} (\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_p}).$$

In particular, $\lambda^0 q = (1)$ and $\lambda^1 q = q$.

17.1. Example. Writing $\mathcal{H}$ for a hyperbolic plane, we have:

$$\lambda^2(n\mathcal{H}) \cong (n^2 - n)\mathcal{H} \oplus n(-1).$$

17.2. Exercise. Prove: The Killing form on the Lie algebra $so(q)$ is $\langle -2 \rangle (\dim q - 2)\lambda^2 q$. 
We will only make use of $\lambda^2$. Here are a few useful identities in $\widehat{W}$, where $x$ and $y$ denote quadratic forms:

\begin{align}
(17.3) \quad \lambda^2(x + y) &= \lambda^2 x + xy + \lambda^2 y \\
(17.4) \quad \lambda^2(\langle c \rangle x) &= \lambda^2 x \\
(17.5) \quad \lambda^2(x - y) &= \lambda^2 x - y(x - y) - \lambda^2 y = \lambda^2 x - xy + \dim y + \lambda^2 y \\
(17.6) \quad \lambda^2(xy) &= x^2\lambda^2 y + y^2\lambda^2 x - 2(\lambda^2 x)(\lambda^2 y)
\end{align}

17.7. Example. For a quadratic form $z$ and a natural number $n$, the image of $\lambda^2(z - n\mathcal{H})$ in the Witt ring is $n + \lambda^2 z$, as can be seen by combining (17.5) and Example 17.1.

17.8. Lemma. For every $n$-Pfister form $\phi$ with $n \geq 1$, we have: $\lambda^2 \phi \cong 2^{n-1} \phi'$.

Proof. By induction on $n$. As $\lambda^2(\langle 1, -\alpha \rangle)$ is isomorphic to $\langle -\alpha \rangle$, the case $n = 1$ holds. For $\phi$ an $n$-Pfister form with $n > 1$, we may write $\phi = \langle \alpha \rangle \psi$ for some $\alpha \in k^\times$ and $(n-1)$-Pfister $\psi$. In $\widehat{W}$, we have

$$\lambda^2 \phi = \langle \alpha \rangle \langle \alpha \rangle^2 \lambda^2 \psi + \langle -\alpha \rangle \psi^2 - 2 \langle -\alpha \rangle \lambda^2 \psi$$

by (17.6). In the Witt ring, $\langle \alpha \rangle^2 - 2 \langle -\alpha \rangle$ equals 2, and

$$\lambda^2 \phi = 2\lambda^2 \psi + (-\alpha) \psi^2,$$

which by the induction hypothesis is

$$2^{n-1} \psi' + (-\alpha) \psi^2 = 2^{n-1} (\psi' + (-\alpha) \psi) = 2^{n-1} \langle \alpha \rangle \psi'.$$

Since $\lambda^2 \phi$ equals $2^{n-1} \phi'$ in the Witt ring and both have dimension $2^{n-1}(2^n - 1)$, the conclusion follows.

For $q$ an even-dimensional quadratic form, there is a canonical lift $\hat{q}$ to the Grothendieck-Witt ring $\widehat{W}$, namely

$$\hat{q} := q - r\mathcal{H}, \quad \text{where dim } q = 2r.$$  

Note that $\hat{q} \in \widehat{W}$ only depends on $q$ up to Witt-equivalence. (This is just a restatement of the fact that the quotient map $\widehat{W} \to W$ restricts to an isomorphism $\widehat{I} \sim I$, where $\widehat{I}$ is ideal of zero-dimensional virtual forms; $\hat{q}$ is the inverse image of $q$ under this isomorphism.) For $n \geq 1$, we define

$$P_n : I \to W \quad \text{via} \quad P_n(x) := \lambda^2 \hat{x} - 2^{n-1} x,$$

where we conflate $\lambda^2 \hat{x}$ with its image in the Witt ring. We remark that the device of replacing $x$ with $\hat{x}$ is necessary, as $\lambda^2$ is not well-behaved with respect to Witt-equivalence. (For example, the dimensions of $\lambda^2 \mathcal{H}$ and $\lambda^2(2\mathcal{H})$ are not even congruent mod 2.)

Using Example 17.7, it is easy to check that

\begin{align}
(17.10) \quad P_n(x + y) &= P_n(x) + xy + P_n(y) \\
(17.11) \quad P_n(\langle c \rangle x) &= P_n(x) + 2^{n-1} \langle c \rangle x
\end{align}

hold, for $x, y \in I$ and $c \in k^\times$.

\footnote{Here and below we write $\langle \alpha_1, \ldots, \alpha_n \rangle$ for the $n$-Pfister form $\otimes_{i=1}^n (1, -\alpha_i)$.}
17.12. **Proposition.** For \( n \geq 1 \):

1. \( P_n \) is zero on \( n \)-Pfister forms.
2. \( P_n \) restricts to a map \( I^n \to I^{2n} \).

If \( n \geq 2 \) and \(-1\) is a square in \( k \):

3. \( P_n \) induces a map \( I^n/I^{n+1} \to I^{2n}/I^{2n+1} \).
4. For \( c_i \in k^\times \) and \( n \)-Pfister forms \( \phi_i \), we have:
   \[
   P_n \left( \sum_i \langle c_i \rangle \phi_i \right) = \sum_{i<j} \langle c_i c_j \rangle \phi_i \phi_j
   \]

**Proof.** Combining Example 17.7 and Lemma 17.8 gives (1). For (2), we use that every element of \( I^n \) is a sum of elements of the form \( \langle c \rangle \phi \), where \( \phi \) is an \( n \)-Pfister form and \( c \) is in \( k^\times \). By (17.10) and (17.11), it suffices to prove that \( P_n(\phi) \) belongs to \( I^{2n} \), which is true by (1).

Both (3) and (4) rest on the fact that \( 2^{n-1} = 0 \) in the Witt ring because \( n \) is at least 2 and \(-1\) is a square. We prove (3). Let \( x, y \in I^n \) be such that \( z := x - y \) belongs to \( I^{n+1} \). Then \( \hat{z} = \hat{x} - \hat{y} \) in \( \hat{W} \), and we have
\[
P_{n+1}(z) = \lambda^2 \hat{x} - \lambda^2 \hat{y} - yz - 2^n z
\]
by (17.5). So:
\[
P_n(x) - P_n(y) = P_{n+1}(z) + yz + 2^{n-1} z.
\]
All three summands on the right belong to \( I^{2n+1} \). For the first term, this is (2). For the last term, it is because \( 2^{n-1} = 0 \).

As for (4), under our special hypotheses, Equation (17.11) takes the nice form:
\[
P_n(\langle c \rangle x) = P_n(x).
\]
Applying (17.10) and (1) gives (4). □

For the remainder of this section, we maintain the hypotheses that \(-1\) is a square in \( k_0 \) and \( n \) is at least 2. Applying the map \( e_{2n} \) from Example 1.2.3 to Prop. 17.12.4 gives:
\[
e_{2n} \left( P_n \left( \sum_i \langle c_i \rangle \phi_i \right) \right) = \sum_{i<j} e_n(\phi_i) e_n(\phi_j).
\]

17.14. **Example** (Invariants of \( \text{SO}(6) \)). We write the invariants of \( \text{SO}(6) \) (the special orthogonal group of the dot product) in terms of the maps \( e_n \) and \( P_n \). By S20.6, the normalized invariants of \( \text{SO}(6) \) with values in \( \mathbb{Z}/2\mathbb{Z} \) form a free \( R_2(k_0) \)-module with basis \( w_2, w_4, b \), where \( b \) satisfies
\[
b(\langle \alpha_1, \alpha_2, \ldots, \alpha_5, \alpha_6 \rangle) = \langle \alpha_1 \rangle \langle \alpha_2 \rangle \cdots \langle \alpha_5 \rangle.
\]
(In S20.1, this \( b \) was denoted "\( b_1 \)\), where 1 is the nonzero element of \( H^0(k_0, \mathbb{Z}/2\mathbb{Z}) \), i.e., the identity element of \( R_2(k_0) \).)

An element of \( H^1(k, \text{SO}(6)) \) corresponds to a 6-dimensional form \( q \) in \( I^2 \). Such a form is isomorphic to \( \langle \beta \rangle (\phi_1 + (-1)\phi_2) \) for some \( \beta \in k^\times \) and 2-Pfister forms \( \phi_1, \phi_2 \). Direct computation gives
\[
w_2(q) = e_2(q),
\]
\[
w_4(q) = w_2(\langle \beta \rangle \phi_1) \cdot w_2(\langle \beta \rangle \phi_2) = e_2(\phi_1) \cdot e_2(\phi_2) = e_4(P_2(q))
\]
by (17.13), and
\[
b(q) = \langle \beta \rangle w_4(q).
\]
17.15. Prop. 17.12.4 makes \( P_n \) look like a “divided square”, meaning a squaring operation from a divided power structure. We remark that—still assuming that \(-1\) is a square—there are also divided square operations on Milnor \( K\)-theory \( P_n^M : K_n^M / 2 \to K_{2n}^M / 2 \), see [Kah00, App. A]. For \( n \geq 2 \), the diagram
\[
\begin{array}{ccc}
K_n^M / 2 & \xrightarrow{P_n^M} & K_{2n}^M / 2 \\
\downarrow & & \downarrow \\
I_n / I_{n+1} & \xrightarrow{P_n} & I_{2n} / I_{2n+1}
\end{array}
\]
commutes, where the vertical arrows are the natural surjections that send the symbol \( \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \) to the class of the Pfister form \( \langle \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \rangle \).

18. Invariants of \( \text{Spin}_{11} \) and \( \text{Spin}_{12} \)

We now determine the invariants of \( \text{Spin}_{12} \) and \( \text{Spin}_{11} \) with values in \( \mathbb{Z}/2\mathbb{Z} \). We assume throughout this section that the field \( k_0 \) has characteristic \( \neq 2 \). We begin with some results on quadratic forms.

18.1. Lemma. Let \( x, y \) be quadratic forms of the same dimension and fix \( c \in k^X \).

If \( \langle \langle c \rangle \rangle(x - y) \) is zero in the Witt ring, then \( \langle \langle c \rangle \rangle \lambda^2(x - y) \in \tilde{W} \) maps to zero in the Witt ring.

Proof. Replacing \( x, y \) with \( x \oplus (-1)y \), \( (\dim y) \mathcal{H} \) respectively does not change \( \langle \langle c \rangle \rangle(x - y) \) nor the image of \( \lambda^2(x - y) \) in the Witt ring. Therefore, we may assume that \( x \) has even dimension \( 2r \) and \( y = r \mathcal{H} \).

The hypothesis on \( \langle \langle c \rangle \rangle(x - y) \) says that the quadratic form \( \langle \langle c \rangle \rangle x \) is hyperbolic, so by [EL73, 2.2], \( x \) is isomorphic to a sum \( \oplus_{i=1}^r \langle \langle c_i \rangle \rangle \langle \langle n_i \rangle \rangle \) such that \( c_i \in k^X \) and \( n_i \) is a norm from \( k(\sqrt{c}) \). Then by Example (17.7) and (17.3), \( \langle \langle c \rangle \rangle \lambda^2(x - r \mathcal{H}) \) maps to
\[
\begin{align*}
r \langle \langle c \rangle \rangle + \langle \langle c \rangle \rangle \sum_{i=1}^r (-n_i) + & \sum_{1 \leq i < j \leq r} \langle \langle c, n_i, n_j \rangle \rangle & \text{ in } W.
\end{align*}
\]
Because the \( n_i \)'s are norms, the middle term equals \(-r \langle \langle c \rangle \rangle \) and each of the forms \( \langle \langle c, n_i, n_j \rangle \rangle \) is hyperbolic. That is, (18.2) is zero. \( \square \)

18.3. Proposition. For \( x \in I^2 \), the class of \( \langle \langle c \rangle \rangle \lambda^2 \hat{x} \) in the Witt ring depends only on the isomorphism class of \( \langle \langle c \rangle \rangle x \) (and not on \( c \) or \( x \)).

(See (17.9) for a definition of \( \hat{x} \).)

Proof. Write \( \dim x = 2r \) and suppose that \( \langle \langle c \rangle \rangle x \) is isomorphic to \( \langle \langle d \rangle \rangle y \) for some \( d \in k^X \) and \( 2r \)-dimensional form \( y \). We must show that \( \langle \langle c \rangle \rangle \lambda^2(x - r \mathcal{H}) \) and \( \langle \langle d \rangle \rangle \lambda^2(y - r \mathcal{H}) \) have the same image in the Witt ring. Let \( \tau \) be the \( 2r \)-dimensional form provided by Cor. B.5 such that
\[
\langle \langle c \rangle \rangle x = \langle \langle c \rangle \rangle \tau = \langle \langle d \rangle \rangle \tau = \langle \langle d \rangle \rangle y.
\]
We first prove
\[
\langle \langle c \rangle \rangle \lambda^2(x - r \mathcal{H}) = \langle \langle c \rangle \rangle \lambda^2(\tau - r \mathcal{H}) \quad \text{in } W.
\]
In view of Example 17.7, it suffices to prove that \( \langle c \rangle (\lambda^2 x - \lambda^2 \tau) \) is zero in the Witt ring. Applying (17.5), we find:

\[
\langle c \rangle (\lambda^2 x - \lambda^2 \tau) = \langle c \rangle (\lambda^2 (x - \tau) + (x - \tau) \tau) \quad \text{in } \widehat{W}.
\]

Since \( \langle c \rangle x \) and \( \langle c \rangle \tau \) are isomorphic, \( \langle c \rangle (x - \tau) \) is zero in the Witt ring, and Lemma 18.1 gives that \( \langle c \rangle \lambda^2 (x - \tau) \) is hyperbolic. We conclude that (18.5) is zero and hence that (18.4) holds. By symmetry, we also have (18.4) where \( c \) and \( x \) are replaced by \( d \) and \( y \).

It remains to prove

\[
\langle c \rangle \lambda^2 (\tau - r \mathcal{H}) = \langle d \rangle \lambda^2 (\tau - r \mathcal{H}) \quad \text{in } W.
\]

Since

\[
\langle c \rangle \tau = \langle c \rangle \tau \quad \text{in } W
\]

and \( \langle c \rangle \tau \) is isomorphic to \( \langle d \rangle \tau \), the form \( \langle c \rangle \lambda^2 \tau \) is hyperbolic. Applying Lemma 18.1, we find that \( \langle c \rangle \lambda^2 (\tau - r \mathcal{H}) \) is zero in the Witt ring. That is, (18.6) holds.

From here until the end of this section, we assume that \( -1 \) is a square in \( k_0 \). We now construct invariants \( a_5 \) and \( a_6 \) of Spin\(_{12} \) as in Rost’s paper [Ros99c].

18.7. Definition of \( a_5 \). For \( \eta \in H^1(k, \text{Spin}_{12}) \), Example 15.12 says that the corresponding quadratic form \( q_\eta \in H^1(k; \text{SO}_{12}) \) is isomorphic to \( \langle c \rangle x \) for some \( c \in k^\times \) and 6-dimensional form \( x \) of determinant 1. As \( -1 \) is a square in \( k \), the form \( x \) belongs to \( I^7 \) and \( \langle c \rangle \langle P_2(x) \rangle \) is in \( I^5 \). We define \( a_5(\eta) \in H^2(k, \mathbb{Z}/2\mathbb{Z}) \) to be \( e_5(\langle c \rangle \langle P_2(x) \rangle) \), equivalently, \( (c) \cdot e_4(P_2(x)) \). Prop. 18.3 shows that \( a_6(\eta) \) is well defined: As \( -1 \) is a square in \( k \), we have \( 2 = 0 \) in the Witt ring, so \( P_2(x) = \lambda^2 (x - 3 \mathcal{H}) \).

18.8. Example. The invariant \( a_5 \) is not zero. Indeed, let \( k \) be the field obtained by adjoining indeterminates \( u, w \) and \( v_1, v_2, v_3, v_4 \) to \( k_0 \). The 12-dimensional form \( q := \langle u \rangle \langle u \rangle \langle v_1, v_2 \rangle - \langle v_3, v_4 \rangle \) belongs to \( I^3 \), hence it is of the form \( q_\eta \) for some class \( \eta \in H^1(k, \text{Spin}_{12}) \). By (17.13), we find:

\[
a_5(\eta) = (u) \cdot (v_1) \cdot (v_2) \cdot (v_3) \cdot (v_4).
\]

18.9. Lemma. If \( q_\eta \) is isotropic, then \( a_5(\eta) \) is zero.

Proof. If \( q_\eta \) is isotropic, then it is Witt-equivalent to a 10-dimensional form in \( I^3 \), hence by Example 15.8 it is isomorphic to \( \langle d \rangle \langle c \rangle \mathcal{H} + 2 \mathcal{H} \) for some \( c, d \in k^\times \) and 2-Pfister form \( \mathcal{H} \), equivalently, is isomorphic to \( \langle c \rangle \langle d \rangle \mathcal{H} \). As \( -1 \) is a square in \( k_0 \), we have:

\[
a_5(\eta) = (c) \cdot e_4(P_2((d) \mathcal{H})) = 0.
\]

18.10. Definition of \( a_6 \). Prop. 9.2 applied to \( a_5 \) gives an invariant \( a_6 \) of \( \text{Spin}_{12} \) defined by setting

\[
a_6(\eta) = a_5(\eta) \cdot (\alpha),
\]

where \( \alpha \) is a nonzero element of \( k^\times \) represented by \( q_\eta \).

In Example 18.8, the form \( q_\eta \) represents \( uv_3 \), so

\[
a_6(\eta) = a_5(\eta) \cdot (uv_3) = (u) \cdot (v_1) \cdot (v_2) \cdot (v_3) \cdot (v_4) \cdot (w).
\]

In particular, \( a_6 \) is not the zero invariant.

18.11. Proposition. \( (\sqrt{-1} \in k_0) \text{ Inv}^\text{norm}_{k_0}(\text{Spin}_{12}, \mathbb{Z}/2\mathbb{Z}) \) is a free \( R_2(k_0) \)-module with basis \( e_3 \) (the Rost invariant), \( a_5, a_6 \).
Proof. Recall from S20.6 or Example 17.14 that $\text{Inv}^\text{norm}_{k_0}(SO(6), \mathbb{Z}/2\mathbb{Z})$ is a free $R_2(k_0)$-module with basis $w_2, w_4, b$. By Example 15.12, the inclusion $SO(6) \times \mu_4 \rightarrow \text{Spin}_{12}$ induces a surjection on $H^1$'s. The image of $SO(6)$ in $\text{Spin}_{12}$ sits in a copy of $SL_6$, so $H^1(\ast, SO(6)) \rightarrow H^1(\ast, \text{Spin}_{12})$ is the zero map. Applying Lemma 5.7 with $G = \mu_4$ and $G' = SO(6)$, we find that restricting invariants of $\text{Spin}_{12}$ to $SO(6) \times \mu_4$ identifies $\text{Inv}^\text{norm}_{k_0}(\text{Spin}_{12}, \mathbb{Z}/2\mathbb{Z})$ with a submodule of the free $R_2(k_0)$-module with basis $1 \cdot \mathfrak{s}, \ w_2 \cdot \mathfrak{s}, \ w_4 \cdot \mathfrak{s}, \ b \cdot \mathfrak{s}$.

The last three are invariants of $\text{Spin}_{12}$ by Example 17.14, e.g., $b \cdot \mathfrak{s}$ is a restriction of $a_6$. However, $\lambda \cdot 1 \cdot \mathfrak{s}$ is not such a restriction for any nonzero $\lambda \in R_2(k_0)$. To see this, one argues as in 16.6, comparing the images of the trivial class and an indeterminate $(t) \in H^1(k_0(t), \mu_4)$ in $H^1(k_0(t), \text{Spin}_9)$. □

18.12. Invariants of $\text{Spin}_{11}$. There are two invariants of $\text{Spin}_{11}$ with values in $\mathbb{Z}/2\mathbb{Z}$ that we can find without doing any work. As always, one has the Rost/Arason invariant $e_3 : H^1(\ast, \text{Spin}_{11}) \rightarrow H^3(\ast, \mathbb{Z}/2\mathbb{Z})$. On the other hand, the inclusion of $\text{Spin}_{11}$ in $\text{Spin}_{12}$ from Example 15.13 leads to an invariant of $\text{Spin}_{11}$ of degree 5 via the composition

$$H^1(\ast, \text{Spin}_{11}) \rightarrow H^1(\ast, \text{Spin}_{12}) \xrightarrow{a_5} H^5(\ast, \mathbb{Z}/2\mathbb{Z}).$$

We denote this invariant also by $a_5$. (Note that restricting $a_6$ to $\text{Spin}_{11}$ gives the zero invariant. Indeed, the image of $H^1(\ast, \text{Spin}_{11})$ in $H^1(\ast, SO_{12})$ consists of those forms that represent 1.)

Proposition. $(\sqrt{-1} \in k_0)$ $\text{Inv}^\text{norm}_{k_0}(\text{Spin}_{12}, \mathbb{Z}/2\mathbb{Z})$ is a free $R_2(k_0)$-module with basis $e_3, a_5$.

Proof. As in the proof of Prop. 18.11, we restrict the invariants of $\text{Spin}_{11}$ to the subgroup $SO(5) \times \mu_4$. Recall from S19.1 that $\text{Inv}^\text{norm}_{k_0}(SO(5), \mathbb{Z}/2\mathbb{Z})$ is a free $\mathbb{Z}/2\mathbb{Z}$-module with basis $1, w_2, w_4$. Therefore, the set of normalized invariants of $\text{Spin}_{11}$ with values in $\mathbb{Z}/2\mathbb{Z}$ is identified with a subspace of the free $R_2(k_0)$-module with basis $1 \cdot \mathfrak{s}, \ w_2 \cdot \mathfrak{s}, \ w_4 \cdot \mathfrak{s}$.

We have a commutative diagram

$$
\begin{array}{ccc}
H^1(k, SO(5)) \times H^1(k, \mu_4) & \longrightarrow & H^1(k, SO(6)) \times H^1(k, \mu_4) \\
\downarrow & & \downarrow \\
H^1(k, \text{Spin}_{11}) & \longrightarrow & H^1(k, \text{Spin}_{12})
\end{array}
$$

The inclusion $SO(5) \rightarrow SO(6)$ is given by $g \mapsto ( \begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix} )$, so the arrow $H^1(k, SO(5)) \rightarrow H^1(k, SO(6))$ sends a 5-dimensional quadratic form $q$ to $q \oplus (1)$. The restriction of $w_j : H^1(k, SO(6)) \rightarrow H^1(k, \mathbb{Z}/2\mathbb{Z})$ to $SO(5)$ is

$$w_j(q \oplus (1)) = (1) \cdot w_{j-1}(q) + w_j(q) = w_j(q),$$

so the invariants $e_3$ and $a_5$ of $\text{Spin}_{12}$ restrict to $w_2 \cdot \mathfrak{s}$ and $w_4 \cdot \mathfrak{s}$ on $H^1(k, SO(5) \times \mu_4)$. As in the proof of Prop. 18.11, one checks that $\lambda \cdot 1 \cdot \mathfrak{s}$ is not the restriction of an invariant of $\text{Spin}_{11}$ for any nonzero $\lambda \in R_2(k_0)$. □
Throughout this section, we assume that the field $k_0$ has characteristic different from 2. We fix a primitive 4-th root of unity $i$ in a separable closure of $k_0$.

19. Surjectivities: $\text{Spin}_{14}$

19.1. Example ($\left( G_2 \times G_2 \right) \rtimes \mu_8 \subset \text{Spin}_{14}$). Returning to the methods of 8.11, we take $\tilde{G}$ to be the split group of type $E_8$ and we omit the root $\pi := \alpha_1$. The semisimple subgroup $G$ is simply connected of type $D_7$—i.e., it is isomorphic to $\text{Spin}_{14}$—and the representation $V$ is a half-spin representation.

Fix a 7-dimensional quadratic form $q$ such that $(1) \oplus q$ is hyperbolic. We view $\text{Spin}_{14}$ as the spin group of the quadratic form $q \oplus -q$, which gives a homomorphism $\text{Spin}(q) \times \text{Spin}(-q) \rightarrow \text{Spin}_{14}$. We may identify the vector spaces underlying the form $q$ and underlying the 7-dimensional fundamental representation of $G_2$ (which we call the standard representation of $G_2$) so that $q$ is $G_2$-invariant. (Note that the standard representation of $G_2$ is irreducible since the characteristic is different from 2 [GS88, p. 413].) This gives an embedding of $G_2$ in $\text{Spin}(q)$, hence of $G_2 \times G_2$ in $\text{Spin}_{14}$.

We now argue as in Example 8.12. The restriction of the representation $V$ to $\text{Spin}(q) \times \text{Spin}(-q)$ is the tensor product of the (8-dimensional) spin representations of $\text{Spin}(q)$ and $\text{Spin}(-q)$. As in Example 15.5, each of these restricts to be a direct sum of the 7-dimensional irreducible representation of $G_2$ and a 1-dimensional trivial representation. We take $v$ to be a tensor product of nonzero vectors that are fixed by the two $G_2$'s.

To see that $G_2 \times G_2$ is not contained in a proper parabolic subgroup of $L$, we note that $G_2 \times G_2$ has no faithful representations of dimension $< 14$ [KL90, 5.4.13], so it cannot be contained in a group of type $D_n$ for $n < 7$ or $A_n$ for $n < 13$. (Popov [Pop80, p. 225, Prop. 11] gives a proof that $G_2 \times G_2$ is the identity component of $N$ using concrete computations in the half-spin representation in the style of Igusa's paper [Igu70].)

We conclude that $G_2 \times G_2$ is the identity component of the stabilizer $N$ of $[v]$ in $\text{Spin}_{14}$. Rather than computing the full stabilizer $N$, we compute instead the normalizer of $G_2 \times G_2$ in $\text{Spin}_{14}$, which contains $N$.

Write $W$ for the 14-dimensional vector space underlying $q \oplus -q$. The image of $G_2 \times G_2$ in $GL(W)$ has normalizer
\[(19.2) \quad \left( (G_2, G_m) \times (G_2, G_m) \right) \times \mathbb{Z}/2\mathbb{Z},\]
where the nonidentity element in $\mathbb{Z}/2\mathbb{Z}$ is the matrix $\left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$. The normalizer of $G_2 \times G_2$ in $\text{SO}(W)$ is the intersection of (19.2) with $\text{SO}(W)$, namely $(G_2 \times G_2) \rtimes \mu_4$, where a primitive 4-th root of unity $i$ in $\mu_4$ is identified with the matrix
\[
\left( \begin{array}{cc} 0 & i \\ i & 0 \end{array} \right) \in \text{SO}(W).
\]

Fix orthogonal bases $\{x_j\}$ and $\{y_j\}$ of the two standard representation of $G_2$ in $W$ such that $q(x_j) = -q(y_j) = \pm 1$ for all $j$. The element
\[
s := \prod_{j=1}^{7} \frac{1 + ix_jy_j}{\sqrt{2}}
\]
in the even Clifford algebra belongs to $\text{Spin}_{14}$, has order 8 since $s^2 = \prod ix_jy_j$, and maps to $\left( \begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix} \right)$ in $\text{SO}(W)$. Therefore, the normalizer of $G_2 \times G_2$ in $\text{Spin}_{14}$ is $(G_2 \times G_2) \rtimes \mu_8$, where the copy of $\mu_8$ is generated by $s$. 
Th. 8.11 says that the inclusion 
\[(G_2 \times G_2) \rtimes \mu_8 \to \text{Spin}_{14}\]
induces a surjection on \(H^1\)'s.

We now interpret this result in terms of quadratic forms. Fix a quadratic extension \(K := k(\sqrt{d})\) of \(k\). The \textit{trace} \(\text{tr}_*(q)\) of a quadratic form \(q\) over \(K\) is a quadratic form over \(k\) of dimension \(2 \dim q\). It is defined by viewing the \(K\)-vector space \(V\) underlying \(q\) as a vector space over \(k\) and taking the bilinear form
\[V \times V \xrightarrow{\text{bilinearization}} k,\]
In other words, \(\text{tr}_*(q)\) is the Scharlau transfer of \(q\) via the linear map \(\text{tr}_{K/k}\), see e.g. [Lam05, §VII.1].

The goal of this section is to prove:

19.3. \textbf{Theorem.} (Rost [Ros99b]) Every 14-dimensional form in \(I^3k\) is of (at least) one of the following two types:

1. \(\langle a \rangle (\phi_1' - \phi_2')\) for some \(a \in k^\times\) and \(\phi_1, \phi_2\) 3-Pfister forms over \(k\).
2. \(\text{tr}_*(\sqrt{d}\phi')\) for some nonsquare \(d \in k^\times\) and \(\phi\) a 3-Pfister form over \(k(\sqrt{d})\).

[Here we have written \('\) for the pure part of a Pfister form, so for example \(\phi\) equals \(\langle 1 \rangle \oplus \phi'\) in (2).]

We remark that a 14-dimensional form in \(I^3k\) is as in (1) if and only if it contains a subform similar to a 2-Pfister form, see [HT98, 2.3] or [IK00, 17.2]. The two papers just cited give concrete examples of 14-dimensional forms that cannot be written as in (1), see [HT98, p. 211] and [IK00, 17.3]. Izhboldin and Karpenko applied Th. 19.3 to give a concrete description of 8-dimensional forms in \(I^2k\) whose Clifford algebra has index 4, see [IK00, 16.10].

19.4. First, we compute that trace of a 1-dimensional form. Directly from the definition, we find:

(19.5) For \(\ell \in K^\times\), the 2-dimensional quadratic form \(\text{tr}_*(\sqrt{d}\ell)\) represents \(\text{tr}_{K/k}(\sqrt{d}\ell) \in k\) and has determinant \(-N_{K/k}(\ell) \in k^\times/k^\times 2\).

That is,
\[
\text{tr}_*(\sqrt{d}\ell) \cong \begin{cases} 
\text{hyperbolic plane} & \text{if } \ell \in k^\times, \text{ i.e., if } \text{tr}_{K/k}(\sqrt{d}\ell) = 0; \\
(\text{tr}_{K/k}(\sqrt{d}\ell))(1, -N_{K/k}(\ell)) & \text{otherwise.}
\end{cases}
\]

To see that this isomorphism holds, it suffices by [Lam05, I.5.1] to observe that the forms on either side of the isomorphism have the same determinant and represent \(\text{tr}_{K/k}(\sqrt{d}\ell)\), which follows from (19.5).

Next we compute a toy example.

19.6. \textbf{Example.} Write \(V\) for the vector space \(k^2\) endowed with the quadratic form \(q: (x, y) \mapsto x^2 - y^2\). Map the group \((\mu_2 \times \mu_2) \rtimes \mu_4\) into the orthogonal group \(O(q)\) of \(q\) by sending
\[
(\varepsilon_1, \varepsilon_2, r) \mapsto \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}^r
\]
for \(\varepsilon_1, \varepsilon_2 \in \{\pm 1\}\) and \(r \in \mathbb{Z}\). The set \(H^1(k, O(q))\) classifies 2-dimensional quadratic forms over \(k\) and we ask: Given a class \(\eta \in H^1(k, (\mu_2 \times \mu_2) \rtimes \mu_4)\), what is the 2-dimensional quadratic form \(q_\eta\) deduced from it?
The quotient map \((\mu_2 \times \mu_2) \times \mu_4 \to \mu_4\) sends \(q\) to an element \(\eta \in H^1(k, \mu_3)\), i.e., some \(dk^\times 4 \in k^\times /k^\times 4\).

If \(d\) is a square in \(k\), then \(q\) comes from \(H^1(k, \mu_2 \times \mu_2 \times \mu_2)\), i.e., \(q\) corresponds to a triple \((\alpha, \beta, \gamma) \in (k^\times/k^\times 2)^3\). The 2-dimensional \(k\)-subspace of \(V \otimes_k k_{sep}\) fixed by \(\eta_\sigma\sigma\) for all \(\sigma\) in the Galois group of \(k\) is spanned by

\[
\begin{pmatrix}
\sqrt{\alpha \gamma} \\
0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 \\
\sqrt{\beta \gamma}
\end{pmatrix}.
\]

The quadratic form \(q_\eta\) is the restriction of \(q\) to this subspace, i.e., \(q_\eta\) is isomorphic to \(\langle \gamma \rangle \langle \alpha, -\beta \rangle\).

Suppose now that \(d\) is not a square in \(k\). Fix a 4-th root \(\delta\) of \(d\) such that \(\eta_\sigma\sigma(\delta) = \delta\). Note that

\[
\eta_\sigma\sigma(\delta^3) = \begin{cases}
\delta^3 & \text{if } \sigma \text{ is the identity on } K \\
-\delta^3 & \text{otherwise}.
\end{cases}
\]

If we twist \(\mu_2 \times \mu_2\) by \(\eta\), we find the transfer \(\tilde{R}_{K/k}(\mu_2)\) for \(K := k(\sqrt{d})\). Moreover, \(\eta\) is in the image of the map

\[
K^\times/K^\times 2 = H^1(k, R_{K/k}(\mu_2)) \xrightarrow{\sim} H^1(k, (\mu_2 \times \mu_2)\eta) \to H^1(k, (\mu_2 \times \mu_2) \times \mu_4),
\]

i.e., \(\eta\) is the image of a class \(\ell K^\times 2 \in K^\times/K^\times 2\). Write \(\ell\) for the image of \(\ell\) under the nonidentity \(k\)-automorphism of \(K\) and fix square roots \(\sqrt{\ell}, \sqrt{\ell} \in k_{sep}\). Then \(\eta\) is the image of the 1-cocycle

\[
\sigma \mapsto \begin{cases}
(\sigma(\sqrt{\ell})^{-1} \sqrt{\ell}, \sigma(\sqrt{\ell}^{-1} \sqrt{\ell}) & \text{if } \sigma \text{ is the identity on } K \\
(\sigma(\sqrt{\ell}^{-1} \sqrt{\ell}, \sigma(\sqrt{\ell}^{-1} \sqrt{\ell}) & \text{otherwise}
\end{cases}
\]

with values in \((\mu_2 \times \mu_2)\eta\). By considering separately the cases where \(\sigma\) is and is not the identity on \(K\), it is easy to check that \(\eta_\sigma\sigma\) fixes the vectors

\[
\begin{pmatrix}
\delta \sqrt{\ell} \\
\delta \sqrt{\ell}
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\delta^3 \sqrt{\ell} \\
-\delta^3 \sqrt{\ell}
\end{pmatrix}
\]

in \(V \otimes k_{sep}\); the quadratic form \(q_\eta\) is the restriction of \(q\) to the subspace they span. The value of \(q\) on the first vector is

\[
\delta^2(\ell - \overline{\ell}) = \text{tr}_{K/k}(\sqrt{d}\ell).
\]

The determinant of the restriction \(q_\eta\) of \(q\) to this subspace is

\[
\det
\begin{pmatrix}
\delta^2(\ell - \overline{\ell}) & \delta^4(\ell + \overline{\ell}) \\
\delta^4(\ell + \overline{\ell}) & \delta^6(\ell - \overline{\ell})
\end{pmatrix}
= -4d^2 N_{K/k}(\ell).
\]

As in 19.4, \(q_\eta\) is \(\text{tr}_*(\langle \sqrt{d}\ell \rangle)\).

\textit{Sketch of proof of Th. 19.3.} By Example 19.1, it suffices to describe the quadratic form deduced from a \((G_2 \times G_2) \times \mu_4\)-torsor, as that is the image of \((G_2 \times G_2) \times \mu_4\) in \(\text{SO}(W)\). Reasoning as in Example 8.1, one can reduce the descent computation to the case of a 2-dimensional quadratic form. This computation was done in Example 19.6. \(\square\)
20. Invariants of Spin\(_{14}\)

In this section, we exhibit some invariants of Spin\(_{14}\) with \(\mathbb{Z}/2\mathbb{Z}\) coefficients using results from \(\S 19\). The results here are all derived from [Ros99c]. We write \(k\) for a fixed base field of characteristic \(\neq 2\).

20.1. We define an invariant \(a_6\) of Spin\(_{14}\) to be the composition

\[
a_6: H^1(k, \text{Spin}_{14}) \to I^3 \xrightarrow{P_3} I^6 \cong H^6(k, \mathbb{Z}/2\mathbb{Z}),
\]

where \(P_3\) and \(e_6\) are the maps from \(\S 17\) and Example 1.2.3 respectively.

We argue that \(a_6\) is not the zero invariant. For a given base field \(k\), define \(k_1\) to be the field obtained by adjoining 6 indeterminates \(t_r\) for \(r = 1, 2\) and \(s = 1, 2, 3\), and—if it is not already in \(k\)—a square root of \(-1\). Put \(\phi_r := \langle t_{r_1}, t_{r_2}, t_{r_3} \rangle\) and take \(\eta \in H^1(k_1, \text{Spin}_{14})\) to have corresponding quadratic form \(q_{\eta} = \phi_1 - \phi_2\). By (17.13), we have:

\[
a_6(\eta) = e_3(\phi_1) \cdot e_3(\phi_2) = \prod_{r,s} (t_{r,s}) \neq 0.
\]

20.2. Proposition. Fix \(\eta \in H^1(k, \text{Spin}_{14})\) and write \(q_{\eta}\) for the quadratic form deduced from it. Suppose that \(-1\) is a square in \(k\).

1. If \(q_{\eta}\) is isotropic, then \(a_6(\eta)\) is zero.
2. If \(3\) is a square in \(k\) and \(k\) has characteristic \(\neq 3\) (and \(\neq 2\)), then \(a_6(\eta)\) is a symbol.

Proof. (1): If \(q_{\eta}\) is isotropic, then it is Witt-equivalent to a 12-dimensional form in \(I^3\). By Example 15.12, \(q_{\eta}\) is isomorphic to \(\langle c \rangle x + H\) for some \(c \in k^\times\) and some 6-dimensional form \(x\) of determinant \(1\). As \(-1\) is a square in \(k\), \(x\) is an Albert form, i.e., \(x = \langle d \rangle (\psi'_1 - \psi'_2)\) for 2-Pfister forms \(\psi_1, \psi_2\) and some \(d \in k^\times\) [Lam05, XII.2.13]. In the Witt ring,

\[
g_{\eta} = \langle d \rangle (\langle c \rangle \psi_1 - \langle c \rangle \psi_2).
\]

Equation (17.13) gives:

\[
a_6(\eta) = (c \cdot (c)) \cdot e_2(\psi_1) \cdot e_2(\psi_2).
\]

As \(-1\) is a square in \(k\), \((c \cdot (c))\) is zero, proving (1).

We now prove (2). Computing in the Witt ring, \(P_3(q_{\eta})\) is \(7 + \lambda^2 q_{\eta}\) by Example 17.7, which equals \(\lambda^2 q_{\eta} + 1\).

Now the Lie algebra \(\mathfrak{so}(q_{\eta})\) contains a subalgebra of type \(G_2 \times G_2\) or the transfer of a \(G_2\) from a quadratic extension. The Killing form on a Lie algebra of type \(G_2\) associated with a 3-Pfister form \(\psi\) is \((-1, -3)\psi'\)—see e.g. S27.21—so it is hyperbolic, and contains a 7-dimensional totally isotropic subspace. Hence the Killing form \((-24)\lambda^2 q\) (see Exercise 17.2) contains a totally isotropic subspace of dimension at least 14. By the previous paragraph, the class of \(P_3(q_{\eta})\) in the Witt ring is represented by an anisotropic quadratic form of dimension at most

\[
\dim \lambda^2 q_{\eta} + 1 - 28 = 64.
\]

But \(P_3(q_{\eta})\) belongs to \(I^6\), so it is similar to a 6-Pfister form [Lam05, X.5.6].

In the proof of (2) above, we assumed that the characteristic was not 3 so that the Killing form of \(\mathfrak{so}(q_{\eta})\) was not identically zero.
20.3. Suppose now that $-1$ is a square in $k$. We define an invariant

$$a_7: H^1(\ast, \text{Spin}_{14}) \to H^7(\ast, \mathbb{Z}/2\mathbb{Z}) \text{ via } a_7(\eta) := a_6(\eta) \cdot (\alpha)$$

where $\alpha$ is any nonzero element of $k$ represented by $q_\eta$. By Propositions 20.2.1 and 9.2, this is a well-defined invariant of $\text{Spin}_{14}$.

20.4. Example. (Assuming $\sqrt{-1} \in k$.) Let $\eta \in H^1(k, \text{Spin}_{14})$ be such that $q_\eta$ equals $\langle c \rangle (\phi_1' - \phi_2')$ for some $c \in k^\times$ and $\phi_1', \phi_2'$ 3-Pfister forms. Write $\phi_1$ as $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$. we have

$$a_7(\eta) = (-c \alpha_1) \cdot e_3(\phi_1) \cdot e_3(\phi_2)$$

by (17.13). But $(-\alpha_1) \cdot e_3(\phi_1)$ is zero as in Example 9.4.2, hence

$$a_7(\eta) = (c) \cdot e_3(\phi_1) \cdot e_3(\phi_2).$$

As in 20.1, it is easy to see that $a_7$ is not the zero invariant.

21. Partial summary of results

**Surjectivities.** Table 21a summarizes the examples of surjectivities given above. The restrictions on the characteristic listed in the table should not be taken seriously. They only reflect the availability of easy-to-cite results in the literature.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$G$</th>
<th>char $k_0$</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Spin}_{2n-1} \times \mu_2$</td>
<td>$\text{Spin}_{2n}$</td>
<td>$\neq 2$</td>
<td>15.1</td>
</tr>
<tr>
<td>$G_2 \times \mu_2$</td>
<td>$\text{Spin}_7$</td>
<td>$\neq 2$</td>
<td>15.5</td>
</tr>
<tr>
<td>$G_2 \times \mu_2 \times \mu_2$</td>
<td>$\text{Spin}_8$</td>
<td>$\neq 2$</td>
<td>15.1 and 15.5</td>
</tr>
<tr>
<td>$\text{Spin}_7 \times \mu_2$</td>
<td>$\text{Spin}_9$</td>
<td>$\neq 2$</td>
<td>15.7</td>
</tr>
<tr>
<td>$G_2 \times \mu_4$</td>
<td>$\text{Spin}_{10}$</td>
<td>$\neq 2$</td>
<td>15.8</td>
</tr>
<tr>
<td>$\text{SO}(5) \times \mu_4$</td>
<td>$\text{Spin}_{11}$</td>
<td>$\neq 2$</td>
<td>15.13</td>
</tr>
<tr>
<td>$\text{SO}(6) \times \mu_4$</td>
<td>$\text{Spin}_{12}$</td>
<td>$\neq 2$</td>
<td>15.12</td>
</tr>
<tr>
<td>$(G_2 \times G_2) \times \mu_5$</td>
<td>$\text{Spin}_{14}$</td>
<td>$\neq 2$</td>
<td>19.1</td>
</tr>
<tr>
<td>$F_4 \times \mu_3$</td>
<td>$E_6$</td>
<td>any</td>
<td>8.12</td>
</tr>
<tr>
<td>$E_6 \times \mu_1$</td>
<td>$E_7$</td>
<td>$\neq 2$</td>
<td>11.13</td>
</tr>
</tbody>
</table>

**Table 21a.** Examples of inclusions for which $H^1_{\text{fppf}}(\ast, N) \to H^1(\ast, G)$ is surjective

This table is obviously not exhaustive. We have only considered a short list of internal Chevalley modules; the recipe in 8.11 gives others. For example, taking $\tilde{G}$ to be $E_6, E_7, E_8$ and $\pi = \alpha_2$, one finds that there is an open $GL_n$-orbit in $\wedge^3 k^n$ ("alternating trilinear forms") for $n = 6, 7, 8$, hence an open $SL_n$-orbit in $\mathbb{P}(\wedge^3 k^n)$. Alternatively, other examples where there is an open $G$-orbit in $\mathbb{P}(V)$ can be found by consulting the table at the end of [PV94] or the lists of prehomogeneous vector spaces in [SK77].
Table 21b. Invariants and essential dimension of Spin$_n$ for $n \leq 14$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\text{ed}(\text{Spin}_n)$</th>
<th>$\text{Inv}^\text{norm}_{k_0}(\text{Spin}_n, \mathbb{Z}/2\mathbb{Z})$</th>
<th>Restrictions on $k_0$?</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\leq 6$</td>
<td>0</td>
<td>$\emptyset$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>3, 4</td>
<td>16.4</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>3, 4, 4, 5</td>
<td>16.1</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>5</td>
<td>3, 4, 5</td>
<td>$\sqrt{-1} \in k_0$</td>
<td>16.6</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>3, 4</td>
<td></td>
<td>16.5</td>
</tr>
<tr>
<td>11</td>
<td>5</td>
<td>3, 5</td>
<td>$\sqrt{-1} \in k_0$</td>
<td>18.12</td>
</tr>
<tr>
<td>12</td>
<td>6</td>
<td>3, 5, 6</td>
<td>$\sqrt{-1} \in k_0$</td>
<td>18.11</td>
</tr>
<tr>
<td>13</td>
<td>6</td>
<td>?</td>
<td></td>
<td>[Ros99c, §10]</td>
</tr>
<tr>
<td>14</td>
<td>7</td>
<td>?</td>
<td>$\sqrt{-1} \in k_0$</td>
<td>20.3</td>
</tr>
</tbody>
</table>

All statements are under the global hypothesis that the characteristic of $k_0$ is $\neq 2$.

Invariants and essential dimensions of Spin groups. Table 21b summarizes the results on invariants of Spin$_n$ for $n \leq 14$. We remark that in the examples considered in S (O$_n$, SO$_n$, the symmetric group on $n$ letters, ...), the description of the invariants depended in a regular way on $n$; clearly, that is not the case here.

The values for the essential dimension given in the table are easily deduced from various results in Part III. For example, the table claims that the essential dimension of Spin$_7$ is 4. Since Spin$_7$ has a nontrivial cohomological invariant of degree 4, the essential dimension is $\geq 4$, cf. 4.7. (All lower bounds on essential dimension here are proved by constructing nonzero cohomological invariants. These bounds can also be obtained by less ad hoc means, see [CS].) On the other hand, the essential dimension of $G_2 \times \mu_2$ is 4, so the surjectivity from Example 15.5 shows that the essential dimension is $\leq 4$.

What of Spin$_{13}$, which we have not yet discussed? One knows that the essential dimension is at least 6 by [CS] or because the invariant $a_6$ of Spin$_{14}$ restricts to be nonzero on Spin$_{13}$. One cannot get an upper bound by imitating the methods of §15 to get a surjectivity in Galois cohomology because the spin representation $V$ does not have an open orbit in $\mathbb{P}(V)$. See [Ros99c] for a proof that the essential dimension is at most 6.

21.1. Open problem. (Reichstein-Youssin [RY00, p. 1047]) Let $k_0$ be an algebraically closed field of characteristic zero. Is there a nonzero invariant $H^1(\ast, \text{Spin}_n) \to H^{[n/2]+1}(\ast, \mathbb{Z}/2\mathbb{Z})$ when $n \equiv 0, \pm1 \mod 8$?

[For $n = 7, 8, 9$, one has the invariants described in Examples 16.4, 16.1, and 16.6 above.]
Appendixes

Appendix A. Examples of anisotropic groups of types $E_7$

We use cohomological invariants to give examples of algebraic groups of type $E_7$ that are anisotropic over “prime-to-2 closed” fields or are anisotropic but split by an extension of degree 2.

A.1. Groups of type $E_7$. Write $E_7$ for the split simply connected group of that type over a field $k$. The Rost invariant $r_{E_7}$, recalled in Example 1.2.4 maps

$$r_{E_7}: H^1(*, E_7) \to H^3(*, \mathbb{Z}/12\mathbb{Z}(2)),$$

see [Mer03, pp. 150, 154]. (In this appendix, the group $H^3(k, \mathbb{Z}/n\mathbb{Z}(2))$ is as defined in [Mer03].) If the characteristic of $k$ does not divide $n$, then $H^3(k, \mathbb{Z}/n\mathbb{Z}(2))$ is $H^3(k, \mu_n^{\otimes 2})$, as in the main body of the notes. In any case, it is $n$-torsion.) The group $H^3(k, \mathbb{Z}/12\mathbb{Z}(2))$ is 12-torsion, and its 4- and 3-torsion are identified with $H^3(k, \mathbb{Z}/4\mathbb{Z}(2))$ and $H^3(k, \mathbb{Z}/3\mathbb{Z}(2))$ respectively. We write $r'$ for the composition of $r_{E_7}$ with the projection of $H^3(k, \mathbb{Z}/12\mathbb{Z}(2))$ onto its 4-torsion, i.e.:

$$r': H^1(k, E_7) \xrightarrow{r_{E_7}} H^3(k, \mathbb{Z}/12\mathbb{Z}(2)) \to H^3(k, \mathbb{Z}/4\mathbb{Z}(2)).$$

Proposition. Suppose that, for $\eta \in H^1(k, E_7)$, the twisted group $(E_7)_\eta$ is isotropic. Then $2r'(\eta) = 0$. If furthermore $k$ contains a primitive 4-th root of unity, then $r'(\eta)$ has symbol length $\leq 2$ in $H^3(k, \mathbb{Z}/2\mathbb{Z})$.

The last sentence of the proposition warrants some comments. Note that the hypothesis implies that $k$ has characteristic $\neq 2$, so $H^3(k, \mathbb{Z}/4\mathbb{Z}(2))$ and $H^3(k, \mathbb{Z}/2\mathbb{Z}(2))$ are simply the Galois cohomology groups $H^3(k, \mathbb{Z}/4\mathbb{Z})$ and $H^3(k, \mathbb{Z}/2\mathbb{Z})$ respectively. It makes sense to speak of $r'(\eta)$ as belonging to $H^3(k, \mathbb{Z}/2\mathbb{Z})$, because the natural map $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ identifies $H^3(k, \mathbb{Z}/2\mathbb{Z})$ with the 2-torsion in $H^3(k, \mathbb{Z}/4\mathbb{Z})$. As for the symbol length, recall that every element of $H^3(k, \mathbb{Z}/2\mathbb{Z})$ can be written as a sum of symbols. The symbol length of a class in $z \in H^3(k, \mathbb{Z}/2\mathbb{Z})$ is the smallest natural number $n$ such that $z$ can be written as a sum of $n$ symbols.

Proof. We consult the list of the possible Tits indexes of groups of type $E_7$ from [Tit66, p. 59]. (See 2.3 in that paper for the definition of the Tits index.) In three of these indexes ($E_{7,1}^8$, $E_{7,2}^3$, and $E_{7,4}^2$), one of the summands of the semisimple anisotropic kernel is of the from $SL(Q)$ for some quaternion division algebra $Q$. However, $(E_7)_\eta$ has trivial Tits algebras, so by [Tit71, p. 211] the semisimple anisotropic kernel cannot have such a summand. In the remaining cases, the vertex 1 or 7 is circled, where the vertices are numbered as in Table 8. We refer to these possibilities as cases 1 and 7 respectively. If both vertices are circled, we arbitrarily say we are in case 1.

Fix a maximal split torus $T$ in the split group $E_7$. As $E_7$ is simply connected and all roots have the same length, the cocharacter group $T_\ast$ is identified with the root lattice. In case $c$, write $S$ for the image of the cocharacter corresponding to twice the fundamental weight $\omega_c$. Write $G$ for the derived subgroup of the centralizer of $S$ in $E_7$; it is simply connected and split; it has type $D_6$ in case 1 and type $E_6$ in case 7. For precision, we write $i$ for the inclusion $G \hookrightarrow E_7$ and $i_\ast$ for the induced map on $H^1$’s.

By Tits’s Witt-type theorem, $(E_7)_\eta$ is isomorphic to $(E_7)_{i_\ast, \tau}$ for some class $\tau$ in $H^1(k, G)$. It follows that $i_\ast\tau = \zeta \cdot \eta$ where $\zeta$ is a 1-cocycle taking values in the
As the Rost invariant is compatible with twisting [Gil00, p. 76, Lem. 7], we have
\[ r'(i \tau) = r'((\zeta \cdot \eta)) = r'((\zeta)) + r'((\eta)), \]
cf. [Gar01a, 7.1]. However, \(E_7\) is split, so the image of \(H^1(k, Z)\) in \(H^1(k, E_7)\) is zero. In particular, \(r'(\zeta)\) is zero and \(r'(i \tau) = r'((\eta))\). Replacing \(\eta\) with \(i \tau\), we may assume that \(\eta\) is the image of \(\tau\). Since the inclusion \(i\) of \(G\) in \(E_7\) has Rost multiplier one, \(r_{E_7}(\eta)\) equals \(r_G(\tau)\).

To prove the first claim in the proposition, it suffices to observe that the order of \(r_G\) is 2 in case 1 and 6 in case 7 by [Mer03, 15.4, 16.6]. In both cases, the 2-primary part is 2 and not 4.

We now prove the last claim. In case 7, the “mod 2” portion of the Rost invariant is a symbol over an odd-degree extension of \(k\) by S22.9, hence it is a symbol over \(k\) by [Ros99a], see Example 7.7. In case 1, \(G\) is the split group \(\text{Spin}_{12}\) and as in 18.7 and 18.8 the quadratic form \(q_\tau \in H^1(k, \text{SO}_{12})\) deduced from \(\tau\) is of the form \(q_\tau = (d, e)(\phi_1 \cdot \phi_2)\) for some \(c, d \in k^\times\) and 2-Pfister forms \(\phi_1, \phi_2\). The Rost invariant of \(\tau\) is
\[ r_G(\tau) = e_3(q_\tau) = (c) \cdot e_2(\phi_1) + (c) \cdot e_2(\phi_2), \]
a sum of two symbols.

Suppose that \(k\) is a prime-to-2 closed field, i.e., every finite extension of \(k\) has degree a power of 2. Every group of inner type \(E_6\) is isotropic. In case \(k = \mathbb{R}\), the unique anisotropic simply connected group of type \(E_7\) is not a strongly inner form of \(E_7\) (i.e., it has nontrivial Tits algebras). We now give an example of a prime-to-2 closed field \(k\) that supports an anisotropic strongly inner form of \(E_7\).

\[ \textbf{A.2. Example.}\ Rost [Gil00, p. 91, Prop. 8] gives an extension \(k_0\) of \(\mathbb{Q}\) and a class \(\eta \in H^1(k_0, E_7)\) such that \(2r'(\eta)\) is not zero. If we take \(k\) to be the extension of \(k_0\) fixed by a 2-Sylow subgroup of the absolute Galois group of \(k_0\), then every finite extension of \(k\) has degree a power of 2, yet \(k\) supports the strongly inner form of \(E_7\) obtained by twisting by \(\text{res}_{k/k_0}(\eta)\), and this group is anisotropic by the proposition.

In the preceding example, \(2r'(\eta)\) is not zero over \(k\), so a restriction/corestriction argument shows that the twisted group \((E_7)_\eta\) is not split by a quadratic extension of \(k\). We can use the second criterion in the proposition to give an example of a strongly inner form of \(E_7\) that is anisotropic but is split by a quadratic extension.

\[ \textbf{A.3. Example.}\ \text{Let } F \text{ be a field of characteristic zero containing a primitive 4-th root of unity. Let } k_0 \text{ be the field obtained by adjoining the indeterminates } t_1, t_2, \ldots, t_6 \text{ and put } k \text{ for the field } k_0(d), \text{ for } d \text{ an indeterminate. We construct a strongly inner form } G \text{ of } E_7 \text{ that is anisotropic over } k \text{ and split over the quadratic extension } K := k(\sqrt{d}). \text{ Let } H \text{ denote the quasi-split simply connected group of type } ^2E_6 \text{ associated with the quadratic extension } K/k; \text{ it is a subgroup of the split simply connected group } E_7 \text{ and the inclusion has Rost multiplier one. Chernousov [Che03, p. 321] gives a 1-cocycle } \eta \in H^1(K/k, H) \text{ whose image under } r' \text{ is}
\]
\[ \text{(A.4) } (d) \cdot \left[ (t_1) \cdot (t_3) + (t_2t_3t_5) \cdot (t_4) + (t_5) \cdot (t_6) \right] \in H^3(k, \mathbb{Z}/2\mathbb{Z}). \]

We take \(G\) to be \(E_7\) twisted by \(\eta\). As \(\eta\) is killed by \(K\), \(G\) is \(K\)-split. For sake of contradiction, suppose that \(G\) is isotropic over \(k\). Applying Prop. A.1, we note
that $r'(p)$ can be written as a sum of $\leq 2$ symbols in $H^3(k, \mathbb{Z}/2\mathbb{Z})$. It follows that the residue with respect to $d$, namely
\begin{equation}
(t_1) \cdot (t_3) + (t_2t_3t_5) \cdot (t_4) + (t_5) \cdot (t_6) \in H^2(k_0, \mathbb{Z}/2\mathbb{Z})
\end{equation}
can be written as a sum of $\leq 2$ symbols in $H^2(k_0, \mathbb{Z}/2\mathbb{Z})$. For $P_n$ as in Prop. 17.12.3, the image of (A.5) under the composition
\begin{equation}
H^2(k_0, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} I^2 / I^4 \xrightarrow{P_2} I^4 / I^5 \xrightarrow{\sim} c_4 \xrightarrow{c_4} H^4(k_0, \mathbb{Z}/2\mathbb{Z})
\end{equation}
is
\begin{equation}
(t_1) \cdot (t_3) \cdot (t_2t_5) \cdot (t_4) + (t_1) \cdot (t_3) \cdot (t_5) \cdot (t_6) + (t_2t_3) \cdot (t_4) \cdot (t_5) \cdot (t_6).
\end{equation}
By parts 1 and 4 of Prop. 17.12, this is a (possibly zero) symbol in $H^4(k_0, \mathbb{Z}/2\mathbb{Z})$. Taking residues with respect to $t_2$ and then $t_4$, we find
\begin{equation}
(t_1) \cdot (t_3) + (t_5) \cdot (t_6) \in H^3(F(t_1, t_3, t_5, t_6), \mathbb{Z}/2\mathbb{Z}).
\end{equation}
Our assumption implies that this is a symbol, which is impossible as the $t_i$’s are indeterminates. We conclude that $G$ is anisotropic over $k$.

Appendix B. A generalization of the common slot theorem

By Detlev W. Hoffmann

The purpose of this appendix is to prove Cor. B.5, which is used in the construction of the degree 5 invariant of Spin$_{12}$ in §18. The corollary as such is due to Rost, but his original argument had a small flaw. The version we present here is actually more general and can be considered as a generalization of the well known Common Slot Theorem, see, e.g., Lam [Lam05, III.4.13]. Recall that the Common Slot Theorem says that if $A = \left(\frac{a_2}{a_6}\right)$ and $B = \left(\frac{b_2}{b_6}\right)$ are quaternion algebras over a field $k$ with char($k$) $\neq 2$ such that $A \cong B$, then there exists $z \in k^*$ with $A \cong \left(\frac{az}{k}\right)$ and $B \cong \left(\frac{bz}{k}\right)$. Translated into Pfister forms, it means that if $\langle a, x \rangle \cong \langle b, y \rangle$ then
\begin{equation}
\langle a, x \rangle \cong \langle a, z \rangle \cong \langle b, z \rangle \cong \langle b, y \rangle
\end{equation}
for some $z \in k^*$. Furthermore, $z \in D_k(\langle ab \rangle)$, i.e., $z$ is represented by the form $\langle ab \rangle$ and hence is a norm of the extension $k(\sqrt{ab})/k$. Indeed, $\langle 1, -a, -z, az \rangle \cong \langle a, z \rangle \cong \langle b, z \rangle \cong \langle 1, -b, -z, bz \rangle$ implies after Witt cancellation and scaling that $\langle ab \rangle \cong \langle z \rangle \langle ab \rangle$.

In the sequel, all fields are assumed to be of characteristic different from 2. To state our version, we first recall the notion of linkage of Pfister forms introduced by Elman and Lam [EL72]. Let $\alpha$ and $\beta$ be Pfister forms over $k$ of folds $m$ and $n$, respectively. Then $\alpha$ and $\beta$ are called r-linked form some nonnegative integer $r \leq \min(n, m)$ if there exist Pfister forms $\rho, \sigma, \tau$ of folds $m - r$, $n - r$ and $r$, respectively, such that $\alpha \cong \rho \sigma \tau$ and $\beta \cong \sigma \tau$. In other words, $\alpha$ and $\beta$ are r-linked if they can be written with $r$ slots in common. It can be shown that $\alpha$ and $\beta$ are r-linked if and only if the Witt index of $\alpha \oplus \langle -1 \rangle \beta$ is $\geq 2^r$ (see [EL72, 4.4]).

If $m \geq n$, we call $\alpha$ and $\beta$ linked if they are $(n - 1)$-linked in the above sense, and we say that they are strictly linked if they are $(n - 1)$-linked but not $n$-linked (i.e., $\alpha$ is not similar to a subform of $\beta$). So if $n = m$, being (strictly) linked means that there exist an $(n - 1)$-fold Pfister form $\pi$ and $a, b \in k^*$ such that $\alpha \cong \langle a \rangle \pi$ and $\beta \cong \langle b \rangle \pi$ (and $\alpha \not\cong \beta$). Note that in this situation, we have in the Witt ring $W_k$ that $\alpha - \beta = \langle b \rangle \langle ab \rangle$.  


Recall also that a form $\phi$ is called \textit{round} if $\phi \cong \langle a \rangle \phi$ if and only if $a \in D_k(\phi)$, i.e., the group of similarity factors $G_k(\phi)$ coincides with the set $D_k(\phi)$ of nonzero elements represented by $\phi$. It is well known that Pfister forms are round. The following facts about round forms are also well known, see [WS77, Theorem 2] for a proof (or [EL72, 1.4] in case of Pfister forms).

**B.1. Lemma.** Let $\alpha$ and $q$ be forms over $k$ and assume that $\alpha$ is round.

1. If $x \in D_k(\alpha q)$, then there exists a form $q_1$ such that $\phi q \cong \alpha(\langle x \rangle \oplus q_1)$.
2. If $\phi$ is anisotropic and $aq$ isotropic, then there exists a form $q_2$ such that $aq \cong \alpha(H \oplus q_2)$.

The crucial ingredient in the proof of our result is the following theorem by Wadsworth and Shapiro [WS77, Theorem 3].

**B.2. Theorem.** Let $\alpha$ and $\beta$ be strictly linked Pfister forms over $k$ of folds $m$ and $n$, respectively, with $m \geq n \geq 1$. Let $q$ be an anisotropic form over $k$ and suppose that there exist forms $\phi$ and $\psi$ over $k$ with $q \cong \alpha \phi \cong \beta \psi$. Then there exist forms $q_i, \phi_i, \psi_i$, $1 \leq i \leq r$, such that

- $q \cong q_1 \oplus q_2 \oplus \cdots \oplus q_r$, and
- $\dim \phi_i = 2$, $\dim \psi_i = 2^{m-n+1}$ for each $i$, and
- $q_i \cong \alpha \phi_i \cong \beta \psi_i$ for each $i$.

Our result now reads as follows.

**B.3. Proposition.** Let $\alpha$ and $\beta$ be $n$-fold Pfister forms over $k$ that are strictly linked. Let $\pi$ be an $(n-1)$-fold Pfister form and $a, b \in k^*$ such that $\alpha \cong \pi \langle a \rangle$ and $\beta \cong \pi \langle b \rangle$, and let $\gamma \cong \pi \langle ab \rangle$.

If $\phi, \psi$ are forms over $k$ such that $\alpha \phi = \beta \psi$ in $W_k$, then there exists a form $\tau$ over $k$, a nonnegative integer $r$, $c_i \in k^*$ and $d_i \in D_k(\gamma)$ $(1 \leq i \leq r)$ such that $\tau \cong \bigoplus_{i=1}^r \langle c_i \rangle \langle d_i \rangle$, $\alpha \tau$ anisotropic and

$$\alpha \phi = \alpha \tau = \beta \tau = \beta \psi \in W_k .$$

**Proof.** Note that the assumption on $\alpha$ and $\beta$ being strictly linked implies that $\gamma$ is anisotropic.

By Lemma B.1(2), we may assume that $\alpha \phi$ and $\beta \psi$ are anisotropic and hence $\alpha \phi \cong \beta \psi$. We denote this anisotropic form by $q$ and apply Theorem B.2 to deduce that there exist forms $q_i, \phi_i, \psi_i$, $1 \leq i \leq r$ such that

- $q \cong q_1 \oplus q_2 \oplus \cdots \oplus q_r$, and
- $\dim \phi_i = \dim \psi_i = 2$ for each $i$, and
- $q_i \cong \alpha \phi_i \cong \beta \psi_i$ for each $i$.

But then, by Lemma B.1(1), there exist $c_i, x_i, y_i \in k^*$ such that $c_i \in D_k(q_i)$ and

$$q_i \cong \langle c_i \rangle \alpha \langle x_i \rangle \cong \langle c_i \rangle \beta \langle y_i \rangle .$$

Hence, $\alpha \langle x_i \rangle \cong \beta \langle y_i \rangle$, and with $\alpha \cong \pi \langle a \rangle$, $\beta \cong \pi \langle b \rangle$, $\gamma \cong \pi \langle ab \rangle$, we get in $W_k$ that

$$0 = \alpha \langle x_i \rangle - \beta \langle y_i \rangle = \alpha - \beta + \langle y_i \rangle \beta - \langle x_i \rangle \alpha$$

and therefore

$$\langle b \rangle \gamma = \langle x_i \rangle \alpha - \langle y_i \rangle \beta .$$
Comparing dimensions shows that \((x_i)\alpha \oplus \langle -y_i \rangle \beta\) is isotropic. Thus, there exists \(d_i \in D_k(\langle x_i \rangle \alpha) \cap D_k(\langle y_i \rangle \beta)\), and by Lemma B.1(1), we have \((x_i)\alpha \cong \langle d_i \rangle \alpha\) and \(\langle y_i \rangle \beta \cong \langle d_i \rangle \beta\). We conclude that \(\alpha \langle x_i \rangle \cong \alpha \langle d_i \rangle \) and \(\beta \langle y_i \rangle \cong \beta \langle d_i \rangle\), hence

\[
q_i \cong \langle c_i \rangle \langle d_i \rangle \alpha \cong \langle c_i \rangle \langle d_i \rangle \beta .
\]

The proof is now finished by putting \(\tau \cong \bigoplus_{i=1}^r \langle c_i \rangle \langle d_i \rangle\). \(\Box\)

B.4. Remarks. (i) One could relax the condition on being strictly linked by linked, and the above statement would still hold provided \(\dim \phi\) is even. But this doesn’t really yield anything new of interest. Indeed, if \(\alpha\) and \(\beta\) are linked but not strictly so, then this just means that \(\alpha \cong \beta\) which in turn implies that \(\gamma\) is hyperbolic. Hence, \(D_k(\gamma) = k^*\). By Lemma B.1, there exists a (necessarily even-dimensional) form \(\tau\) such that \((\alpha \gamma)_{\alpha m} \cong \alpha \tau\), and one can simply take \(\alpha\) orthogonal decomposition \(\tau \cong \bigoplus_{i=1}^r \langle c_i \rangle \langle d_i \rangle\).

(ii) Recall that a field \(k\) is called linked if any two Pfister forms over \(k\) are linked. In fact, it is not difficult to check that \(k\) is linked iff any two 2-fold Pfister forms over \(k\) are linked. This notion of a linked field has been coined in [EL73]. Well-known examples of linked fields are finite, local and global fields, fields of transcendence degree \(\leq 1\) over a real closed field or of transcendence degree \(\leq 2\) over an algebraically closed field.

Hence, if we assume the field \(k\) in Proposition B.3 to be linked, then the condition of the two Pfister forms being strictly linked can be replaced by the two Pfister forms being nonisometric.

Let us state the case \(n = 1\) for the above proposition explicitly.

B.5. Corollary. Let \(a, b \in k^*\) represent different nontrivial square classes. Let \(\ell = k(\sqrt{ab})\). If \(\phi, \psi\) are forms over \(k\) such that \(\langle a \rangle \phi = \langle b \rangle \psi\) in \(W_k\), then there exists a form \(\tau\) over \(k\), a nonnegative integer \(r\), \(c_i \in k^*\) and \(d_i \in D_k(\langle ab \rangle) = N_k(\ell^*)\) \((1 \leq i \leq r)\) such that \(\tau \cong \bigoplus_{i=1}^r \langle c_i \rangle \langle d_i \rangle\), \(\alpha \tau\) anisotropic and

\[
\alpha \phi = \alpha \tau = \beta \tau = \beta \psi \in W_k .
\]

Suppose that, as in the corollary, \(a\) and \(b\) represent different nontrivial square classes in \(k^*\). Let \(\phi\) be an anisotropic form over \(k\). If \(\langle a \rangle \phi\) is isotropic, it is well known and not difficult to see that there exists a 2-dimensional subform \(\phi'\) of \(\phi\) such that already \(\langle a \rangle \phi'\) is isotropic (and hence hyperbolic) as it is similar to a 2-fold Pfister form, cf. [EL73, 2.2].

Indeed, \(\langle a \rangle \phi \cong \phi \oplus (-a) \phi\) being isotropic clearly implies that there are nonzero vectors \(x, y\) in an underlying vector space \(V\) of \(\phi\) such that \(\phi(x) = a \phi(y)\). Since \(a\) is not a square, \(x\) and \(y\) span a 2-dimensional subspace \(W\) of \(V\). Then just take \(\phi'\) to be the restriction of \(\phi\) to \(W\).

Now let \(K = k(\sqrt{b})\) and suppose that \(\langle a \rangle \phi_K\) is isotropic (or possibly even hyperbolic, in which case \(\langle a \rangle \phi \cong \langle b \rangle \psi\) for some form \(\psi\), see [Lam05, VII.3.2]). By the above, we see that there exists over \(K\) a 2-dimensional subform \(\phi'\) of \(\phi_K\) such that \(\langle a \rangle \phi'\) is isotropic over \(K\). If, in this situation, one could always find a 2-dimensional subform \(\phi'\) of \(\phi\) already over \(k\) such that \(\langle a \rangle \phi'\) is isotropic (and hence hyperbolic) over \(K\), then one could use the Common Slot Theorem plus a straightforward induction on \(\dim \phi\) to easily deduce the above corollary. In fact, for \(\dim \phi = 2\), the above corollary is essentially nothing else but the Common Slot Theorem.
However, such a 2-dimensional subform $\phi'$ of $\phi$ over $k$ doesn’t exist in general as the following counterexamples will show for forms $\phi$ of dimension $n$ for any given $n > 2$.

B.6. Example. Recall that the Pythagoras number $p(k)$ of a field $k$ is defined to be the least positive integer $p$ (provided such an integer exists) such that each sum of squares in $k$ can be written as a sum of $\leq p$ squares. If no such integer exists, then we put $p(k) = \infty$.

Let $k$ be a formally real field with $p(k) = \infty$ (e.g., the rational function field over the reals in infinitely many variables, cf. [Lam05, IX.2.4]). Let $n \geq 3$ and $s$ be such that $2^s < n \leq 2^{s+1}$. Pick an element $-b$ that is a sum of $2^{s+1} + 2$ squares but not fewer. Note that this is always possible since $p(k) > 2^{s+1} + 1$. Now let $a = -1$, so $\langle a \rangle \cong (1,1)$, $\phi \cong (1,\ldots,1)$ (sum of $n$ squares), and let $K = k(\sqrt{b})$.

Then $\langle a \rangle \phi$ is a Pfister neighbor of $P \cong \langle -1,-1,\ldots,-1 \rangle$, a sum of $2^{s+2}$ squares. Now $P \cong \langle 1 \rangle \oplus P'$ with $P'$ a sum of $2^{s+2} - 1$ squares. In particular, $P'$ represents $-b$, and $P$ has therefore a subform $\langle 1,-b \rangle$ which becomes isotropic over $K$. Hence $P_K$ is hyperbolic and the Pfister neighbor $\langle a \rangle \phi_K$ is isotropic. Note that if $n = 2^{s+1}$ then in fact $\langle a \rangle \phi_K \cong P_K$ is hyperbolic.

Suppose now that $\phi$ contains a subform $\langle u,v \rangle$ over $k$ with $\langle a \rangle \langle u,v \rangle \cong \langle 1,1 \rangle \langle u,v \rangle$ isotropic over $K$. Note that both $u$ and $v$ are necessarily sums of $n \leq 2^{s+1}$ squares in $k$ as both are represented by $\phi$.

Let $w = uv$. Then $\langle 1,1,w,w \rangle \cong \langle -1,-w \rangle$ is similar to $\langle 1,1 \rangle \langle u,v \rangle$ and thus isotropic (and hence hyperbolic) over $K$. But then $b$ can be chosen as a slot of the Pfister form $\langle -1,-w \rangle$: $\langle -1,-w \rangle \cong \langle b,c \rangle$ for some $c \in k^*$ (cf. [Lam05, III.4.1]). By Witt cancellation, $\langle 1,w,w \rangle \cong \langle -b,c,bc \rangle$ and thus $-b$ is represented by $\langle 1,w,w \rangle$. In particular, there exist $x, y, z \in k^*$ with $-b = x^2 + w(y^2 + z^2)$.

Now $w(y^2 + z^2) = w(y^2 + z^2)$ is the product of three factors, each of which being a sum of at most $2^{s+1}$ squares. A famous result by Pfister states that, for each nonnegative integer $m$, the nonzero sums of $2^m$ squares in a field form a multiplicative group (see, e.g., [Lam05, X.1.9]). Hence, we have that $w(y^2 + z^2)$ can be expressed itself as a sum of at most $2^{s+1}$ squares. But then, $-b$ can be written as a sum of at most $2^{s+1} + 1$ squares, a contradiction!

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