CLIFFORD ELEMENTS IN LIE ALGEBRAS

JOSE BROX[1], ANTONIO FERNÁNDEZ LÓPEZ[2], AND MIGUEL GÓMEZ LOZANO[3]

Abstract. Let L be a Lie algebra over a field $\mathbb{F}$ of characteristic zero or $p > 3$. An element $c \in L$ is called Clifford if $\mathrm{ad}_c^3 = 0$ and its associated Jordan algebra L_c is the Jordan algebra $\mathbb{F} \oplus X$ defined by a symmetric bilinear form on a vector space X over $\mathbb{F}$. Roughly speaking, we prove in this note that c is a Clifford element if and only if there exists a centrally closed prime ring R with involution $*$ such that $c \in \mathrm{Skew}(R, *)$, $c^3 = 0$, $c^2 \neq 0$ and $c^2kc = ckc^2$ for all $k \in \mathrm{Skew}(R, *)$.

1. Introduction

Let L be a Lie algebra over a field $\mathbb{F}$ of characteristic not 2 or 3. An element $a \in L$ is called a *Jordan element* if $\mathrm{ad}_a^3 L = 0$. In [8], a Jordan algebra was attached to any Jordan element $a \in L$. This Jordan algebra, denoted by L_a, inherits most of the properties of the Lie algebra L, as well as the nature of the Jordan element in question is reflected in the structure of the attached Jordan algebra. For instance, if L is nondegenerate ($\mathrm{ad}_x^2 L = 0 \Rightarrow x = 0$) so is the Jordan algebra L_a, and in this case, L_a is unital if and only if a is *von Neumann regular* ($a \in \mathrm{ad}_a^2 L$).

By a *Clifford element* of L we mean a Jordan element $c \in L$ such that L_c is the Jordan algebra $J = \mathbb{F} \oplus X$ defined by a symmetric bilinear form on a vector space X over $\mathbb{F}$ (we do not discard the case $X = 0$, i.e., $J = \mathbb{F}1$). Suppose now that L is nondegenerate, $\mathrm{char}(\mathbb{F}) = 0$ or $p > 5$, and c is a Clifford element of L. Since L_c is then unital, c is von Neumann regular (see 2.6), and hence, by the Jacobson-Morozov Lemma (see [5, Proposition 1.18], L has a 5-grading $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ such that the Jordan pair $V = (L_{-2}, L_2)$

Key words and phrases. Lie algebra, ring with involution, Jordan algebra, inner ideal, Jordan element.
[1] Supported by the Spanish MEC through the FPU grant AP2009-4848.

[2,3] Supported by the Spanish MEC and Fondos FEDER, MTM2010-19482.

is isomorphic to the Clifford Jordan pair defined by the Jordan algebra L_c, whose Tits-Kantor-Koecher algebra $TKK(V)$ is a finitary orthogonal Lie algebra (see [6, 5.11]), that is, $TKK(V) \cong \mathrm{Skew}(R, *)$, where R is a simple ring coinciding with its socle and $*$ is an involution of the first kind and transpose type. Thus every Clifford element c actually lives in a ring, and in this associative context verifies $c^3 = 0$ and $c^2 \neq 0$, except for the case that R is the algebra of 2×2 matrices over a field with the transpose involution (see [7, Lemma 3.7(ii)]). In this paper we prove the following converse of the above result:

Let R be a centrally closed prime ring of characteristic zero or greater than three, let $*$ be an involution of R and let c be a Jordan element of the Lie algebra $K = \mathrm{Skew}(R, *)$ such that $c^3 = 0$ and $c^2 \neq 0$. Then $*$ is of the first kind and c is a Clifford element of K.

2. Preliminaries

Throughout this section Φ will denote a ring of scalars, i.e., a commutative ring with 1, and $\mathbb{F}$ will stand for a field. An *algebra over* Φ (in short, a Φ-algebra) is a Φ-module A with a product (bilinear operation). Thus no associativity condition is assumed; neither it is supposed the existence of a unit in A. According to this definition, a ring is an associative $\mathbb{Z}$-algebra.

Jordan algebras and Lie algebras.

2.1. Suppose that 2 is invertible in Φ. A (linear) *Jordan algebra* is a Φ-algebra J whose product, denoted by $\bullet$, is commutative and satisfies the identity $x^2 \bullet (y \bullet x) = (x^2 \bullet y) \bullet x$, for all $x, y \in J$, where $x^2 = x \bullet x$. For each $x \in J$, the U-operator $U_x : J \to J$, defined by $U_x y = 2x \bullet (x \bullet y) - x^2 \bullet y$, $y \in J$, satisfies the identity $U_{U_x y} = U_x U_y U_x$, for all $x, y \in J$. A Jordan algebra is said to be *nondegenerate* if $U_x = 0$ implies $x = 0$.

2.2. Suppose that 2 is invertible in Φ and that A is an associative Φ-algebra, whose product is denoted by juxtaposition. In the Φ-module A, we define a new product by $x \circ y := xy + yx$. The resulting algebra is a Jordan algebra denoted by A^+, with $U_x y = 2xyx$. Note that A is semiprime if and only if A^+ is nondegenerate. A Jordan algebra J is called *special* if it is isomorphic to a subalgebra of A^+ for some associative algebra A. As usual, we denote by A^- the Lie algebra defined in the Φ-module A by the bracket-product: $[x, y] = xy - yx$.

2.3. Let $\mathbb{F}$ be a field of characteristic not 2 and let X be an $\mathbb{F}$- vector space with a symmetric bilinear form $\langle\, ,\rangle$. Then the vector space $J = \mathbb{F} \oplus X$ is endowed with a structure of Jordan algebra by defining

$$(\alpha, x) \bullet (\beta, y) = (\alpha\beta + \langle x, y\rangle,\ \beta x + \alpha y),$$

for $\alpha, \beta \in \mathbb{F}$ and $x, y \in X$. This Jordan algebra is unital, with $(1, 0)$ as unit element, and special; in fact, it is isomorphic to a Jordan subalgebra of the Clifford (associative) algebra defined by $\langle\, ,\rangle$ (see [9, II.3]). For this reason, $J = \mathbb{F} \oplus X$ is sometimes called a *Clifford* Jordan algebra.

2.4. Let L be a Lie algebra over Φ, with $[x, y]$ denoting the product and ad_x the adjoint map determined by x (sometimes we will use capital letters instead, i.e., X by ad_x). An *inner ideal* of L is a Φ-submodule B of L such that $[[B, L], B] \subseteq B$. An *abelian inner ideal* is an inner ideal B which is also an abelian subalgebra, i.e., $[B, B] = 0$. For example, if $L = \bigoplus_{-n\le i\le n} L_i$ is a finite $\mathbb{Z}$-grading, then L_{-n} and L_n are easily checked to be abelian inner ideals of L. An element $a \in L$ is said to be a *Jordan element* whenever $ad_a^3 L = 0$; every element in an abelian inner ideal is easily shown to be a Jordan element, and conversely, if L is 3-torsion free and $a \in L$ is Jordan, then $B = \Phi a + \mathrm{ad}_a^2 L$ is an abelian inner ideal of L (see [2, Lemma 1.8]).

The following identities (see [2, Lemma 1.7]) will be used in what follows. Let L be a 3-torsion free Lie algebra and $a, x \in L$, where a is a Jordan element. Then:

(JE1) $A^2XA = AXA^2$,

(JE2) $\mathrm{ad}^2_{A^2x} = A^2X^2A^2$.

where $A = \mathrm{ad}_a$ and $X = \mathrm{ad}_x$.

2.5. Suppose that 2 and 3 are invertible in Φ. Let L be a Lie Φ-algebra and let $a \in L$ be a Jordan element. In the Φ-module L a new product is defined by $x \bullet y = [[x, a], y]$,

$x, y \in L$. Denote by $L^{(a)}$ the resulting algebra. Then $\mathrm{Ker}(a) := \{x \in L : \mathrm{ad}_a^2 x = 0\}$ is an ideal of $L^{a)}$ and the quotient algebra $L_a := L^{a)}/\mathrm{Ker}(a)$ is a Jordan algebra (with product $\bar{x} \bullet \bar{y} = \overline{[[x, a], y]}$, where $\bar{x}$ stands for the coset of x, for any $x \in L$), called the *Jordan algebra of L at a* (see [8, Theorem 2.4]).

2.6. If a is *von Neumann regular*, i.e., a is Jordan and $a \in \mathrm{ad}_a^2 L$, then L_a is unital with $\bar{b}$ as unit element for any $b \in L$ such that $a = [[a, b], a]$. In this case, L_a is isomorphic to the Jordan algebra $J(a, b)$ defined in the Φ-module $\mathrm{ad}_a^2 L$ by the product $x \bullet y = [[x, b], y]]$ for all $x, y \in \mathrm{ad}_a^2 L$. We provide here a proof of these results under conditions less restrictive than those required in [8].

Proof. (i) Proving that L_a is unital with $\bar{b}$ as unit element it is equivalent to show that $A^2[B, A] = A^2BA = A^2$ (since $A^3 = 0$). Now $a = [[a, b], a]$ implies

$$A = \mathrm{ad}_{[[a,b],a]} = [[A, B], A] = 2ABA - A^2B - BA^2,$$

and hence, by (JE1), $A^2 = 2A^2BA - ABA^2 = A^2BA$ (since $A^3 = 0$), as required.

(ii) Let us now show that the map $\varphi : L_a \to J(a, b)$ defined by $\varphi(\bar{x}) := -A^2x$ is an algebra-isomorphism. Clearly, φ is a linear isomorphism, and since both algebras are commutative and $\frac{1}{2} \in \Phi$, it suffices to check that $\varphi(\bar{x})^2 = \varphi(\bar{x}^2)$.

$$\varphi(\bar{x})^2 = [[A^2x, b], A^2x] = -\mathrm{ad}_{A^2x}^2 b = -A^2X^2A^2b = A^2X^2a = -A^2XAx = \varphi(\bar{x}^2),$$

where we have used (JE2) and $XAx = [x, [a, x]] = -X^2a$. □

Prime rings.

2.7. Let R be a prime ring. The *extended centroid* $\mathcal{C}$ of R (see ([1, Section 2.2]) is a field containing the centroid Γ if R, and the *central closure* $\mathcal{C}R$ of R is a prime associative algebra over the field $\mathcal{C}$. A prime ring R is *centrally closed* if it coincides with its central closure.

The following lemma (see [3, Theorem A.7]) will play a fundamental role in the proof of our main result.

Lemma 2.8 (**Martindale**). *Let R be a prime ring with extended centroid $\mathcal{C}$. Let $a_i, b_i \in R$ with $b_1 \neq 0$ be such that $\sum_{i=1}^{n} a_i x b_i = 0$ for every $x \in R$. Then $a_1 \in \sum_{i=2}^{n} \mathcal{C} a_i$.*

Involutions.

2.9. Let A be an associative Φ-algebra with an involution $*$, that is, $* : A \to A$ is a Φ-linear map satisfying $*^2 = \mathrm{Id}_A$ and $(ab)^* = b^* a^*$ for all $a, b \in A$. Denote by H (respectively by K) the set of the symmetric (respectively, skew-symmetric) elements of A, i.e., $H := \{x \in A : x = x^*\}$ and $K = \{x \in A : \ x = -x^*\}$. Then K is a subalgebra of the Lie Φ-algebra A^-, and if $\frac{1}{2} \in \Phi$, then H is a subalgebra of the Jordan Φ-algebra A^+ (so it is a special Jordan algebra) and $A = H \oplus K$.

2.10. Set $\kappa(x) := x - x^* \in K$ for every $x \in R$. Note that the mapping $x \mapsto \kappa(x)$ is Φ-linear and it satisfies $\kappa(axa^*) = a\kappa(x)a^*$ for all $a, x \in R$. Note also that for $h \in H, k \in K$, $h \circ k := hk + kh = hk - (hk)^* = \kappa(hk) \in K$, a simple identity that will show up frequently.

If M is a Φ-submodule of R which is $*$-*invariant*, i.e., $M^* = M$, then $\kappa(M) = \mathrm{Skew}(M, *)$, since if $k \in \mathrm{Skew}(M, *)$ then $k = \frac{1}{2}(k + k) = \frac{1}{2}(k - k^*) = \frac{1}{2}\kappa(k)$ and $\kappa(x) = x - x^* \in M \cap K = \mathrm{Skew}(M, *)$ for every $x \in M$. In particular $\kappa(R) = K$. If M is not $*$-invariant, then $\kappa(M) = \kappa(M^*)$ implies that $\kappa(M) = \kappa(M) + \kappa(M^*) = \kappa(M + M^*) = (M + M^*) \cap K$.

2.11. Let A be an associative Φ-algebra with involution $*$. If $a \in A$ is von Neumann regular, i.e, $a = axa$ for some $x \in A$, then, by replacing x by $b = xax$, we obtain $a = aba$ and $b = bab$. If a is also symmetric and $\frac{1}{2} \in \Phi$, then b can be chosen to be symmetric by replacing x by $\frac{1}{2}(x + x^*)$. The following lemma is a further step in the choice of b.

Lemma 2.12. *Let A be an associative Φ-algebra and let $c \in A$ be a von Neumann regular element such that $c^2 = 0$. Then there exists $d \in A$ such that $c = cdc$, $d = dcd$ and $d^2 = 0$. Moreover, if A has an involution $*$, $\frac{1}{2} \in \Phi$ and c is symmetric (skew-symmetric), then d can be chosen to be symmetric (respectively, skew-symmetric).*

Proof. Let c be a von Neumann regular element of A. By above, there exists $b \in A$ such that $cbc = c$ and $b = bcb$. We claim that $d := b - b^2 c$ satisfies the required properties. Indeed,

$d^2 = (b - b^2c)(b - b^2c) = b^2 - b^3c - b(bcb) + b(bcb)bc = b^2 - b^3c - b^2 - b^3c = 0,$

$cdc = c(b - b^2c)c = cbc = c$, and

$bcb = (b - b^2c)c(b - b^2c) = bc(b - b^2c) = bcb - (bcb)bc = b - b^2c = d.$

Suppose now that c is symmetric. Since $\frac{1}{2} \in \Phi$, we can take $b \in H$ such that $cbc = b$ and $b = bcb$. We claim that $d := b - \frac{1}{2}(cb^2 + b^2c) + \frac{1}{4}cb^3c$ satisfies the required properties. It is clear that $d^* = d$. Moreover, we have:

$$
\begin{aligned}
d^2 = {} & \left(b - \frac{1}{2}(cb^2 + b^2c) + \frac{1}{4}cb^3c\right)\left(b - \frac{1}{2}(cb^2 + b^2c) + \frac{1}{4}cb^3c\right) = b^2 - \frac{1}{2}(bcb)b \\
& - \frac{1}{2}b^3c + \frac{1}{4}(bcb)b^2c - \frac{1}{2}cb^3 + \frac{1}{4}cb(bcb)b + \frac{1}{4}cb^4c - \frac{1}{8}cb(bcb)b^2c - \frac{1}{2}b(bcb) \\
& + \frac{1}{4}b(bcb)bc + \frac{1}{4}cb^2(bcb) - \frac{1}{8}cb^2(bcb)bc = b^2 - \frac{1}{2}b^2 - \frac{1}{2}b^3c + \frac{1}{4}b^3c - \frac{1}{2}cb^3 \\
& + \frac{1}{4}cb^3 + \frac{1}{4}cb^4c - \frac{1}{8}cb^4c - \frac{1}{2}b^2 + \frac{1}{4}b^3c + \frac{1}{4}cb^3 - \frac{1}{8}cb^4c = 0,
\end{aligned}
$$

$cdc = c(b - \frac{1}{2}(cb^2 + b^2c))c = cbc = c$, and

$$
\begin{aligned}
dcd = {} & \left(b - \frac{1}{2}(cb^2 + b^2c\right))c\left(b - \frac{1}{2}(cb^2 + b^2c)\right) = (b - \frac{1}{2}cb^2)c(b - \frac{1}{2}b^2c) \\
= {} & bcb - \frac{1}{2}(bcb)bc - \frac{1}{2}cb(bcb) + \frac{1}{4}cb(bcb)bc = bcb - \frac{1}{2}b^2c - \frac{1}{2}cb^2 + \frac{1}{4}cb^3c = d.
\end{aligned}
$$

If c is skew-symmetric, then the same d works taking $b \in K$. □

2.13. Let R be a centrally closed prime ring with involution $*$ such that $\mathrm{char}(R) \neq 2$. Then $*$ naturally extends to an involution of the extended centroid $\mathcal{C}$ of R, also denoted by $*$. If $*$ acts trivially) on $\mathcal{C}$, then it is called *of the first kind*. In this case, K can be regarded as a Lie algebra over $\mathcal{C}$.

3. Clifford element of a prime ring

Throughout this section R will denote a centrally closed prime ring of characteristic not 2 or 3 which is endowed with an involution $*$. Then K, the set of skew-symmetric element of R, is a Lie algebra over the field $\mathrm{Sym}(\mathcal{C}, *)$. It follows from [4, Propostion 6.2] that if K is not abelian and $*$ is of the first kind, then any Jordan element a of K is zero-cube. This leads us to the following.

Definition 3.1. By a *Clifford element* of R we mean a Jordan element c of K such that $c^3 = 0$ and $c^2 \neq 0$.

The square of a Clifford element of R.

Proposition 3.2. *Let $c \in K$ be a Clifford element of R. Then:*

(1) $c^2kc = ckc^2$ for all $k \in K$
(2) $c^2Kc^2 = 0$.
(3) $(c^2xc^3)^* = c^2x^*c^2 = c^2xc^2$ for all $x \in R$.
(4) $c^2Rc^2 = \mathcal{C}c^2$.
(5) The involution $*$ is of the first kind.
(6) R has nonzero socle with division ring isomorphic to $\mathcal{C}$ and $*$ is of the transpose type.

Proof. (1) Since c is a Jordan element of K, for every $k \in K$ we have $0 = ad_c^3 k = c^3k - 3c^2kc + 3ckc^2 - kc^3 = -3(c^2kc - ckc^2)$. Since $\mathrm{char}(R) \neq 3$, this implies that $ckc^2 = c^2kc$.

(2) By (1), $c^2kc^2 = c(ckc^2) = c(c^2kc) = c^3kc = 0$.

(2) It follows from (2).

(4) Let $x, y \in R$. Since c^2 is symmetric, it follows from (3) that

$$c^2xc^2yc^2 = c^2(xc^2y)^*c^2 = (c^2y^*c^2)x^*c^2 = c^2y(c^2x^*c^2) = c^2yc^2xc^2.$$

Thus, fixed x, for every $y \in R$, we get $(c^2xc^2)y(c^2) - (c^2)y(c^2xc^2) = 0$, with $c^2 \neq 0$. Then, by Martindale's Lemma (2.8), for each $x \in R$ there is a $\lambda_x \in \mathcal{C}$ such that $c^2xc^2 = \lambda_x c^2$. Since $c^2 \neq 0$ and R is prime, $c^2Rc^2 \neq 0$ and hence $c^2Rc^2 = \mathcal{C}c^2$, since $\mathcal{C}$ is a field.

(5) by (4), given $\alpha \in \mathcal{C}$ there exists $x \in R$ such that $\alpha c^2 = c^2xc^2$. Then, by (3), $\alpha^*c^2 = c^2x * c^2 = c^2xc^2 = \alpha c^2$, so $\alpha^* = \alpha$, proving that $*$ is of the first kind.

(6) By (4), $c^2 = c^2ac^2$ for some $a \in R$, and hence $c^2R = eR$ where $e = c^2a$ is an idempotent of R. Then $eRe = c^2Rc^2a = \mathcal{C}c^2a = \mathcal{C}e$, which proves ([1, Proposition 4.3.3]) that eR is a minimal right ideal of R, so R has nonzero socle with associated division ring isomorphic to the field $\mathcal{C}$ ([1, Theorem 4.3.7]). Now it follows from Kaplansky's Theorem ([1, Theorem 4.6.8]) that the involution $*$ of R is either of transpose type or of symplectic type; but the latter cannot occur because c^2 is a symmetric rank-one element, so $*$ is of transpose type. □

3.3. Let c be a Clifford element of R. Since c^2 is a symmetric zero-square element which is also von Neumann regular 3.2(4), we have by (2.12) that there exists $d \in R$ such that

$$d^* = d,\ d^2 = 0,\ c^2dc^2 = c^2 \text{ and } d = dc^2d.$$

Such an element d will be called a *regular partner* of c^2. Then $e := dc^2$ is a $*$-*orthogonal idempotent*, i.e., $e^2 = e$ and $ee^* = e^*e = 0$.

Proposition 3.4. *Let c be a Clifford element of R, let d be a regular partner of c^2 and set $e := dc^2$. Then:*

(1) $dKd = 0$.
(2) $dRd = \mathcal{C}d$.
(3) $eRe = \mathcal{C}e$, $e^*Re = \mathcal{C}c^2$, $eRe^* = \mathcal{C}d$ and $eKe^* = e^*Ke = 0$.
(4) $ec = ce^* = 0$, $e^*c^2 = c^2e = c^2$ and $de^* = ed = d$.
(5) $[K, K] \neq 0$.
(6) $e + e^* \neq 1$ in the unital hull $\hat{R} = \mathcal{C}1 + R$ of R.

Proof. We will frequently use the fact that $c^2Mc^2 = \mathcal{C}c^2$ for any abelian subgroup M of R such that $c^2Mc^2 \neq 0$, which follows from 3.2(3).

(1) $dKd = dc^2(dKd)c^2d = 0$, where we have used 3.2(2) and the fact that dkd is skew-symmetric for every $k \in K$. Similarly, we have:

(2) $dRd = (dc^2d)R(dc^2d) = dc^2(dRd)c^2d = d\mathcal{C}c^2d = \mathcal{C}dc^2d = \mathcal{C}d$, since $c^2 = c^2dc^2$ and $d = dc^2d$ imply that $c^2(dRd)c^2 \neq 0$.

(3) $eRe = dc^2(Rd)c^2 = d\mathcal{C}c^2 = \mathcal{C}e$, since $c^2 = c^2(dc^2d)c^2 \in c^2(Rd)c^2$ and therefore the latter is nonzero. In a similar way it is proved that $e^*Re = \mathcal{C}c^2$ and $eRe^* = \mathcal{C}d$. Now $eKe^* = dc^2Kc^2d = 0$ by 3.2(2), and $e^*Ke = 0$ is obtained in a similar way.

(4) The identities of this item follow straightforward from the very definition of e.

(5) By (4), $[c, e - e^*] = ce + e^*c = cdc^2 + c^2dc \neq 0$. Otherwise $cdc^2 = -c^2dc$ would lead to the contradiction $c^2 = c^2dc^2 = -c^3dc = 0$. Since $[c, e - e^*] \in [K, K]$, $[K, K] \neq 0$.

(6) It follows from (3) and (4) that $(e + e^*)c(e + e^*) = 0$, so $e + e^* \neq 1$. □

Remark 3.5. Regular partners d for c^2 are not unique. In fact, the elements $d_\lambda := d + \lambda(dc - cd) - \lambda^2 cdc + \frac{1}{2}\lambda^2(dc^2 + c^2d) + \frac{1}{2}\lambda^3(c^2dc - cdc^2) + \frac{1}{4}\lambda^4 c^2$, where λ ranges in $\mathcal{C}$, are proved to be distinct regular partners for c^2.

As we have seen in the above proposition, any Clifford element c of R gives rise to two nonzero orthogonal elements e and e^*, associated to any regular partner d of c^2. Moreover, the idempotent $e + e^*$ is no complete (3.4(6)), i.e., the symmetric idempotent $g := 1 - e - e^*$ in the unital hull $\hat{R} = \mathcal{C}1 + R$ of R is nonzero. We next prove that the complete system $\{e, e^*, g\}$ induces a 3-grading in the Lie algebra K.

Proposition 3.6. *Let c be a Clifford element of R, $e = dc^2$ and $g = 1 - e - e^*$, where d is a regular partner of c^2. Then $K = K_{-1} \oplus K_0 \oplus K_1$ is a a 3-grading of K, with $K_{-1} = \kappa((1 - e)Ke) = \kappa((1 - e)Re) = \kappa(gRe)$, $K_0 = \kappa(eRe) \oplus gKg$ and $K_1 = \kappa(eK(1 - e)) = \kappa(eR(1 - e)) = \kappa(eRg)$.*

Proof. Consider the complete system $\{e_0 := e^*, e_1 := g, e_2 := e\}$ of orthogonal idempotents of $\hat{R}$ and set $R_i = \bigoplus_{m-n=i} e_m R e_n$, $-2 \leq i \leq 2$. Then (see [10, p.174] for instance), $R = \bigoplus_{-2 \leq i \leq 2} R_i$ is an (associative) 5-grading of R. Explicitly,

$$R = e^*Re \oplus (e^*Rg \oplus gRe) \oplus (e^*Re^* \oplus gRg \oplus eRe) \oplus (gRe^* \oplus eRg) \oplus eRe^*.$$

Since all the components R_i are $*$-invariant subspaces, $K = \bigoplus_{-2 \leq i \leq 2} K_i$, where $K_i := R_i \cap K = \mathrm{Skew}(R_i, *)$ for each index i and $[K_i, K_j] \subseteq [R_i, R_j] \cap [K, K] \subseteq R_{i+j} \cap K = K_{i+j}$. Thus

$K = \bigoplus_{-2\le i\le 2} K_i$ is (a priori) a 5-grading of the Lie algebra K. But $K_{-2} = \kappa(e^*Re) = e^*\kappa(R)e = e^*Ke = 0$ and similarly $K_2 = e^*Ke = 0$. Moreover, the i-th homogenous component k_i of any $k \in K$ coincides with $\bigoplus_{m-n=i} \kappa(e_mke_n)$, so $k \in K_{-1}$ if and only if

$$kgke + e^*kg = (1-e-e^*)ke + e^*k(1-e-e^*) = (1-e)ke + e^*k(1-e^*)$$
$$(1-e)ke - ((1-e)ke)^* = \kappa((1-e)ke),$$

since $e^*Ke = 0$ by 3.4(4), which proves that $K_{-1} = \kappa(gRe) = \kappa((1-e)Ke)$. Similarly, $K_1 = \kappa(eRg) = \kappa(eK(1-e))$. Therefore $K = \kappa((1-e)Ke) \oplus (\kappa(eRe) \oplus gKg) \oplus \kappa(eK(1-e))$ is a 3-grading of K. Now, for any $x \in R$,

$$\kappa(gxe) = \kappa((1-e)xe) - \kappa(e^*xe) = \kappa((1-e)xe) - e^*\kappa(x)e = \kappa((1-e)xe)$$

since $e^*\kappa(x)e \in e^*Ke = 0$, which proves that $K_{-1} = \kappa((1-e)Re)$. Similarly we obtain that $K_1 = \kappa(eR(1-e)$. □

Although the 3-grading of K has been defined by choosing a regular partner d of c^2, it will be seen now that the component $K_{-1} = \kappa((1-e)Ke)$ only depends on the Clifford element c.

Proposition 3.7. *Let c be a Clifford element of R, $e := dc^2$, where d is a a regular partner of c^2, and $B = \kappa((1-e)Ke)$. Then:*

(1) If $b \in B$ then $eb = 0$ and $b = e^*b + be$.
(2) $B = c^2 \circ K$.
(3) $c = e^*c + ce = c^2dc + cdc^2$.
(4) $c \in B$.
(5) $cKc = \mathfrak{C}c$.

Proof. (1) Let $b = (1-e)ke + e^*k(1-e^*) \in B$. Then $eb = e((1-e)ke + e^*k(1-e^*)) = 0$ and $e^*b = e^*k(1-e^*)$, since $e^*e = 0$ and $e^*Ke = 0$. Similarly, $be = (1-e)ke$. Thus $b = e^*b + be$.

(2) $c^2 \circ k = c^2k + kc^2 = \kappa(kc^2) = \kappa(ke^*c^2) - (eke^*)c^2) = \kappa((1-e)k(e^*c^2)) = \kappa((1-e)k(c^2e)) \in \kappa((1-e)Re) = B$. Conversely, let $b \in B$. Then $bc^2 = (1-e)bec^2 + e^*b(1-e^*)c^2 = 0$, since $ec^2 = dc^2c^2 = 0$ and $(1-e^*)c^2 = (1-e^*)e^*c^2 = 0$. Hence $c^2b = -(bc^2) = 0$. Now we have $b = e^*b + be = c^2db + bdc^2 = c^2(db + bd) + (bd + db)c^2 \in c^2 \circ K$.

(3) As in the proof of (3.6), set $g = 1 - e - e^*$. We have

$$c = (e + e^* + g)c(e + e^* + g) = e^*c + ce + gcg$$

since $ec = ce^* = 0$ by 3.4(4). Thus all we need to prove it is that $gcg = 0$. Set $z := gcg$, which is a skew-symmetric element and let $k \in K$. Recall that $c^2 = c^2e$ and e, e^*, g are orthogonal idempotents. As $c^2kc = ckc^2$, $(c^2kc)g = (ckc^2)g = ck(c^2e)g = (ckc^2)(eg) = 0$. But since $ec = 0$, $c^2 = c^2e$ and $eKe^* = 0$, we have $c^2kz = (c^2kg)gz = c^2k(1 - e - e^*)cg = c^2kcg - c^2k(ec)g - c^2ke^*cg = 0$. Hence $c^2Kz = 0$, and therefore $zKc^2 = (c^2Kz)^* = 0$. So $c^2xz = c^2x^*z$ and $zxc^2 = zx^*c^2$ for every $x \in R$. Now pick $x, y \in R$. Then $c^2\kappa(xzy)c^2 = 0$ since $c^2Kc^2 = 0$, so that $0 = c^2(xzy + y^*zx^*)c^2 = c^2xzyc^2 + c^2y^*zx^*c^2 = c^2xzyc^2 + c^2yzxc^2 = (c^2xz)y(c^2) + (c^2)y(zxc^2) = 0$, with $c^2 \neq 0$. By Martindale's Lemma (2.8), for every $x \in R$ there is $\lambda_x \in \mathcal{C}$ such that $c^2xz = \lambda_x c^2$. But since $z = gcg$ and $g = g(1-e)$, we have $c^2xz = c^2xz(1-e) = \lambda_x c^2(1-e) = 0$, so $c^2Rz = 0$. But R is prime and $c^2 \neq 0$; therefore $z = 0$. Thus $c = e^*c + ce = c^2dc + cdc^2$, as required.

(4) By (3), $c = c^2dc + cdc^2 = c^2(dc + cd) + (dc + cd)c^2 \in c^2 \circ K = B$ by (2). Another proof of this result: $ec = dc^3 = 0$ implies $c = e^*c + ce = (1-e)ce + e * c(1-e^*) \in \kappa((1-e)Ke = B$.

(5) We know that $c = ce + e^*c$, $e^*Ke = 0 = eKe^*$, $ckc \in K$ and $eRe = \mathcal{C}e$; moreover, for every $x \in R$ it is true that if $exe = \lambda_x e$, then $e^*xe^* = (ex^*e)^* = (\lambda_{x^*}e)^* = \lambda_{x^*}e^*$. Pick $k \in K$. Then we have $ckc = (ce + e^*c)k(ce + e^*c) = c(ekce) + c(eke^*)c + e^*(ckc)e + (e^*kce)^*c = \lambda_{kc}ce + \lambda_{(ck)^*}e^*c = \lambda_{kc}(ce + e^*c) = \lambda_{kc}c$, which proves that $cKc \subseteq \mathcal{C}c$. The equality follows because $c(cd + dc)c = c^2dc + cdc^2 = c$ by (3), with $cd + dc \in K$ since $c \in K$ and $d \in H$. □

The square root of d.

3.8. Given a Clifford element c of R and a regular partner d of c^2, we set $\sqrt{d} := cd + dc$. As will be seen now, the square-root notation is absolutely justified.

Proposition 3.9. *Let c be a Clifford element of R and let d be a regular partner for c^2. Then:*

(1) $\sqrt{d} \in K_1$ in the 3-grading of Theorem 3.6. In particular $\sqrt{d}$ is a Jordan element.

(2) $(\sqrt{d})^2 = d$.

(3) $(\sqrt{d})^3 = 0$.

(4) $\sqrt{d}K\sqrt{d} = \mathcal{C}\sqrt{d}$.

(5) $\sqrt{d}c\sqrt{d} = \sqrt{d}$.

(6) $c\sqrt{d}c = c$.

(7) $c^2 \circ \sqrt{d} = c$.

(8) $d \circ c = \sqrt{d}$.

(9) $[[c, \sqrt{d}], c] = c$.

(10) $[[\sqrt{d}, c], \sqrt{d}] = \sqrt{d}$.

(11) $[[c, \sqrt{d}], b] = b$ for every $b \in B_e$.

Proof. (1) Since $c \in K$ and $d \in H$, $\sqrt{d} = cd + dc \in K$. Now we have

$$\begin{aligned}\kappa(e\sqrt{d}(1-e)) &= e(cd+dc)(1-e) + (1-e^*)(dc+cd)e^* = edc(1-e)\\ &+ (1-e^*)cde^* = edc - edce + cde^* - e^*cde^* = (dc^2d)c\\ &- e(dcd)c^2 + c(dc^2d) - c^2(dcd)e^* = dc + cd = \sqrt{d},\end{aligned}$$

since $ec = dc^2c = dc^3 = 0$, $dc^2d = d$ and $dcd \in dKd = 0$. We have thus proved (see 3.6) that $\sqrt{d} \in \kappa(eK(1-e)) = K_1$. And since K_1 is an abelian inner ideal (because is the extreme of a finite grading), $\sqrt{d}$ is a Jordan element of K.

(2) $(\sqrt{d})^2 = (cd+dc)(cd+dc) = c(dcd) + cd^2c + dc^2d + (dcd)c = dc^2d = d$.

(3) $(\sqrt{d})^3 = (\sqrt{d})^2\sqrt{d} = d(cd+dc) = dcd + d^2c = 0$.

(4) If follows from (1), (2) and (3) that $\sqrt{d}$ is a Clifford element of R. Hence, by 3.7(3), $\sqrt{d}K\sqrt{d} = \mathcal{C}\sqrt{d}$.

(5) $\sqrt{d}c\sqrt{d} = (cd+dc)c(cd+dc) = c(dc^2d) + c(dcd)c + dc^3d + (dc^2d)c = cd + dc = \sqrt{d}$.

(6) $c\sqrt{d}c = c(cd+dc)c = c^2dc + cdc^2 = c$, by 3.7(1).

(7) $c^2 \circ \sqrt{d} = c^2(cd+dc) + (cd+dc)c^2 = c^2dc + cdc^2 = c$.

(8) $d \circ c = dc + cd = \sqrt{d}$.

(9) $[[c, \sqrt{d}], c] = 2c\sqrt{d}c - c^2 \circ \sqrt{d} = 2c - c = c$, by (6) and (7).

(10) $[[\sqrt{d}, c], \sqrt{d}] = 2\sqrt{d}c\sqrt{d} - (\sqrt{d})^2 \circ c = 2\sqrt{d} - \sqrt{d} = \sqrt{d}$, by (2), (5) and (8).

(11) $[[c, \sqrt{d}], b] = [[c, cd + dc], b] = [c^2d - dc^2, b] = [e^* - e, b] = e^*b + be = b$ by 3.7(1). □

4. The theorem

As in the previous section, R will denote a centrally closed prime ring of characteristic not 2 or 3 which is endowed with an involution $*$. We prove here that if c is a Clifford element of R, then the abelian inner ideal $c^2 \circ K = \kappa((1 - e)Ke)$ (see 3.7) can be endowed with a structure of Jordan algebra of Clifford type (see 2.3) and that this Jordan algebra is isomorphic to K_c. We begin by defining a linear form and a symmetric bilinear form on the $\mathcal{C}$-vector space K (recall that $*$ is of the first kind by 3.2(5)).

4.1. By Proposition 3.7(4), there exists a linear map $\mathrm{tr} : K \to \mathcal{C}$ such that

$$\mathrm{tr}(k)c = ckc$$

for every $k \in K$. Note that

(1) $\mathrm{tr}(\sqrt{d}) = 1$ since $c\sqrt{d}c = c$ by Proposition 3.9(6), and hence

(2) $K = \mathcal{C}\sqrt{d} \oplus \mathrm{Ker}(\mathrm{tr})$.

4.2. Since $c^2Rc^2 = \mathcal{C}c^2$ (3.2(4)) with $c^2k_1k_2c^2 = c^2k_2k_1c^2$ for all $k_1, k_2 \in K$ (3.2(2)), we have a symmetric bilinear form $\langle\, , \rangle : K \times K \to \mathcal{C}$ defined by

$$\langle k_1, k_2\rangle c^2 = c^2k_1k_2c^2$$

for all $k_1, k_2 \in K$.

Remarks 4.3. The trace can be realized from the bilinear form and vice versa. Let $k, k' \in K$:

(1) $\langle \sqrt{d}, k\rangle c^2 = c^2\sqrt{d}kc^2 = c^2(cd + dc)kc^2 = c^3dkc^2 + c^2dckc^2 = c^2dckc^2 = c^2d(ckc)c = \mathrm{tr}(k)c^2dc^2 = \mathrm{tr}(k)c^2$, since $c^3 = 0$ and $c^2dc^2 = c^2$. Thus $\mathrm{tr}(k) = \langle k, \sqrt{d}\rangle$.

(2) $\mathrm{tr}(\kappa(ckk'))c^2 = (c\kappa(ckk')c)c = c^2kk'c^2 + ck'kc^3 = c^2kk'c^2 = \langle k, k'\rangle c^2$. Thus $\langle k, k'\rangle = \mathrm{tr}(\kappa(ckk'))$.

Proposition 4.4. *Let c be a Clifford element of R and $B = c^2 \circ K$. Then:*

(1) $B = \mathcal{C}c \oplus X$, *where* $X := \{c^2 \circ k : \ k \in \mathrm{Ker}(\mathrm{tr})\}$.

(2) $B = \mathrm{ad}_c^2 K$.

Proof. (1) By 4.1(2), $K = \mathrm{Ker}(\mathrm{tr}) \oplus \mathcal{C}\sqrt{d}$. Hence

$$B = c^2 \circ K = c^2 \circ \mathrm{Ker}(\mathrm{tr}) + \mathcal{C}c^2 \circ \sqrt{d} = c^2 \circ \mathrm{Ker}(\mathrm{tr}) + \mathcal{C}c$$

since $c^2 \circ \sqrt{d} = c$ by 3.9(7). But this sum is direct since $c^2 \circ k_0 = \alpha c$, with $\mathrm{tr}(k_0) = 0$ and $\alpha \in \mathcal{C}$, implies $\alpha c^2 = c(c^2 k_0 + k_0 c^2) = (ck_0c)c = \mathrm{tr}(k_0)c = 0$, and hence $\alpha = 0$, since $c^2 \neq 0$ by the very definition of Clifford element.

(2) For any $k \in K$, we have $ad_c^2 k = c^2 k - 2ckc + kc^2 = c^2 \circ k - 2\mathrm{tr}(k)c \in B$. Conversely, let $c^2 \circ k_0 + \mu c \in B$, with $k_0 \in \mathrm{Ker}(\mathrm{tr})$ and $\mu \in \mathcal{C}$. Taking $k = k_0 - \mu\sqrt{d}$, we have

$$c^2 \circ k = c^2 \circ k_0 - \mu c^2 \circ \sqrt{d} = \mathrm{ad}_c^2 k_0 - \mu c = \mathrm{ad}_c^2 (k_0 + \mu\sqrt{d})$$

since $c^2 \circ \sqrt{d} = c$ by 3.9(7), $ck_0c = 0$ and $\mathrm{ad}_c^2 \sqrt{d} = -c$ by 3.9(9). □

Lemma 4.5. *The symmetric $\mathcal{C}$-bilinear form defined on X by*

$$\langle c^2 \circ k, c^2 \circ k' \rangle_0 := -\langle k, k' \rangle$$

is well defined.

Proof. Suppose that $c^2 \circ k_1 = c^2 \circ k_1'$. By multiplying the two members of this equality on the right by $k_2 c^2$, we obtain $c^2 k_1 k_2 c^2 = c^2 k_1' k_2 c^2$ since $c^2 K c^2 = 0$. This proves that $\langle\, , \rangle_0$ is well defined. □

Remarks 4.6. Consider the 3-grading $K = K_{-1} \oplus K_0 \oplus K_1$ due to $e := dc^2$ (3.6(3)), with $K_{-1} = B$, $K_0 = \kappa(eKe) \oplus gKg$ and $K_1 = \kappa(eKg)$.

(1) It follows from the symmetry of the previous theorem, that

$$K_1 = d \circ K = \{d \circ k : \ k \in K, \sqrt{d}k\sqrt{d} = 0\} \oplus \mathcal{C}\sqrt{d} = ad_{\sqrt{d}}^2 K.$$

(2) B_0 can be zero and therefore $B = \mathcal{C}c$. But this can only happen if R is 3-dimensional over $\mathcal{C}$. Let $X = H \oplus \mathbb{F}z$ be the orthogonal sum of a hyperbolic plane $H = \mathbb{F}x \oplus \mathbb{F}y$ and the line $\mathbb{F}z = H^\perp$ with z being an anisotropic vector, and let R the simple ring $\mathrm{End}(X)$ with the adjoint as involution. For any $u, v \in X$, let $u \otimes v$ be the linear map defined by $w(u \otimes v) = \langle w, u\rangle v$ for all $w \in X$. Then $(u \otimes v)^* = v \otimes u$ and hence $c := x \otimes z - z \otimes x$ is in the Lie algebra $K = \mathrm{Skew}(R, *)$. It is easy to check that c is a Clifford element of R such that $\mathrm{ad}_c^2 K = \mathbb{F}c$.

Theorem 4.7. *Let R be a centrally closed ring of characteristic not 2 or 3, let $*$ be an involution of R and let c be a Jordan element of the Lie algebra K such that $c^3 = 0$ and $c^2 \neq 0$. Then:*

(1) The involution $*$ is of the first kind.

(2) The $\mathcal{C}$-vector space $X = \{c^2 \circ k : \; ckc = 0\}$ is endowed with a symmetric bilinear form denoted by $\langle \, , \rangle_0$.

(3) K_c is isomorphic to the Clifford Jordan algebra $\mathcal{C} \oplus X$ defined by $\langle \, , \rangle_0$.

Proof. That the involution $*$ is of the first kind was proved in 3.2(5), and that $\langle \, , \rangle_0$ is a well defined symmetric bilinear form on the $\mathcal{C}$-vector space X follows from 4.5. Thus only the item (3) needs to be proved. But since $c = [[c, \sqrt{d}], c]$ (3.9(9)), we have by (2.6) that $K_c \cong J(c, \sqrt{d})$, the Jordan algebra defined on the $\mathcal{C}$-vector space $\mathrm{ad}_c^2 K = c^2 \circ K = B = \mathcal{C}c \oplus X$ (4.4) by the product $(\alpha_1 c + c^2 \circ k_1) \bullet (\alpha_2 c + c^2 \circ k_2) = [[\alpha_1 c + c^2 \circ k_1, \sqrt{d}], \alpha_2 c + c^2 \circ k_2]$, for all $\alpha_1, \alpha_2 \in \mathcal{C}$ and $k_1, k_2 \in K$ such that $ck_1c = ck_2c = 0$. Let us then see that the linear isomorphism $(\alpha c + c^2 \circ k) \mapsto (\alpha, c^2 \circ k)$ of $J(c, \sqrt{d})$ onto $\mathcal{C} \oplus X$ is actually an isomorphism of Jordan algebras. Since $\frac{1}{2} \in \Phi$, it suffices to check the identity

$$(\alpha c + c^2 \circ k)^2 = [[\alpha c + c^2 \circ k, \sqrt{d}], \alpha c + c^2 \circ k] = \alpha^2 c + \langle c^2 \circ k, c^2 \circ k\rangle_0 + 2\alpha(c^2 \circ k).$$

Using the bilinearity of the bracket-product reduces the checking to three products: (i) scalar by scalar, (ii) scalar by vector, and (iii) vector by vector.

(i) $[[\alpha c, \sqrt{d}], \alpha c] = \alpha^2 [[c, \sqrt{d}], c] = \alpha^2 c$, by 3.9(9).

(ii) $[[\alpha c, \sqrt{d}], c^2 \circ k] = \alpha[[c, cd + dc], c^2k + kc^2] = \alpha[c^2d - dc^2, c^2k + kc^2] = \alpha(c^2 \circ k)$, where we have used $c^2dc^2 = c^2$, $c^4 = 0$ and $c^2kc^2 = c^2(dk + kd)c^2 = 0$, the latter because $c^2Kc^2 = 0$ and $(dk + kd)^* = -(kd + dk)$, since $d^* = d$ and $k^* = -k$.

(iii) $[[c^2 \circ k, \sqrt{d}], c^2 \circ k] = 2(c^2 \circ k)\sqrt{d}(c^2 \circ k) - (c^2 \circ k)^2 \circ \sqrt{d}$,

with

$$(c^2 \circ k)\sqrt{d}(c^2 \circ k) = (c^2k + kc^2)(cd + dc)((c^2k + kc^2)) = (c^2kdc + kc^2dc)(c^2k + kc^2) = 0,$$

since $c^3 = 0$ and $ckc = 0$ ($\mathrm{tr}(k) = 0$), and

$$(c^2 \circ k)^2 \circ \sqrt{d} = c^2k^2c^2(cd+dc)+(cd+dc)c^2k^2c^2 = c^2k^2c^2dc+cdc^2k^2c^2 = \langle k, k\rangle(c^2dc+cdc^2) = \langle k, k\rangle c$$

since $c = c^2dc + cdc^2$ by 3.7(1). Therefore, $(c^2 \circ k) \bullet (c^2 \circ k) = -\langle k, k\rangle c = \langle c^2 \circ k, c^2 \circ k\rangle_0 c$, which completes the proof. □

Remark 4.8. Since $\sqrt{d}$ is a Clifford element of R (see 3.9), the theorem above also proves that $K_{\sqrt{d}}$ is a Clifford Jordan algebra with $\sqrt{d}$ as unit element and symmetric bilinear $\langle k, k'\rangle_d\, d := -dkk'd$ for every $k, k' \in K$.

References

[1] K.I. Beidar, W.S. Martindale 3rd and A.V. Mikhalev. *Rings with Generalized Identities.* Pure and Applied Mathematics; Marcel Dekker Inc., New York-Basel-Hong Kong, 1995.

[2] G. Benkart, *On inner ideals and ad-nilpotent elements of Lie algebras.* Trans. Amer. Math. Soc. **232** (1977), 61-81.

[3] M. Brešar, M.A. Chebotar and W.S. Martindale 3rd. *Functional Identities.* Frontiers in Mathematics; Birkhäuser Verlag, Basel-Boston-Berlin, 2007.

[4] J. Brox, A. Fernández López and M. Gómez Lozano. *Lie inner ideals of skew elements of prime rings with involution.* (Submitted).

[5] C. Draper, A. Fernández López, E. García and M. Gómez Lozano. *The socle of a nondegenerate Lie algebra.* J. Algebra **319** (2008), 2372–2394.

[6] A. Fernández López, E. García and M. Gómez Lozano. *The Jordan socle and finitary Lie algebras.* J. Algebra **280** (2004), 635–654.

[7] A. Fernández López, E. García and M. Gómez Lozano. *Inner ideals of finitary simple Lie algebras.* J. Lie Theory **16** (2006), 97–114.

[8] A. Fernández López, E. García and M. Gómez Lozano. *The Jordan algebras of a Lie algebra.* J. Algebra **308**(1) (2007), 164–177.

[9] N. Jacobson. *Structure and representations of Jordan algebras.* American Mathematical Society Colloquium Publications, Vol. XXXIX; merican Mathematical Society; Providence, R.I., 1968.

[10] O.N. Smirnov. *Simple associative algebras with finite Z-grading.* J. Algebra **196** (1997), 171–184.

Departamento de Álgebra, Geometría y Topología, Universidad de Málaga, 29071, Málaga, Spain

E-mail address: brox@agt.cie.uma.es[1]

E-mail address: emalfer@uma.es[2]

E-mail address: magomez@agt.cie.uma.es[3]