Jordan Algebras and Symmetric Manifolds

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Jordan algebras correspond to a class of Lie algebras. While the role of Lie algebras in geometry is universally recognised, the same cannot be said about Jordan algebras. We explain in this article the close connection between Jordan algebras and symmetric manifolds.

Jordan and Lie algebras

We are familiar with the role of Lie algebras in geometry, for instance, we know that the smooth vector fields (Figure 1) of a smooth manifold form a Lie algebra. Jordan algebras are close relatives of Lie algebras, but less famous and perhaps even lesser known is their connection to geometry. The revelation below, it is hoped, may help to ameliorate this unfavourable state of affairs and generate wider interest in Jordan algebras.

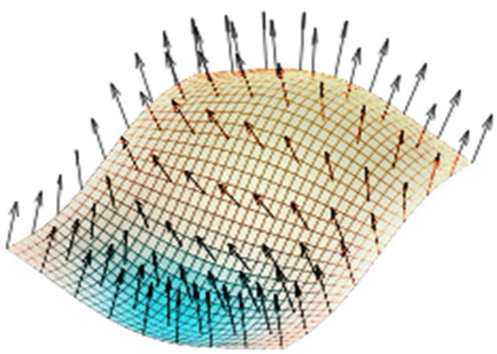

Figure 1. A vector field

Let us begin by introducing Jordan algebras and explain their relationship with Lie algebras. We refer to [7] for a more informative sketch of Jordan algebras. In what follows, all vector spaces are either real or complex. A *Jordan algebra* A is a vector space equipped with a bilinear product

$$(a,b) \in A \times A \mapsto a \circ b \in A$$

which is commutative and satisfies the *Jordan identity*

$$a \circ (b \circ a^2) = (a \circ b) \circ a^2 \qquad (a,b \in A).$$

We do not assume associativity of the product. We call A *unital* if it has an identity.

One can turn any associative algebra A into a Jordan algebra by defining a new product

$$a \circ b = \frac{1}{2}(ab + ba) \qquad (a,b \in A)$$

with which A becomes a Jordan algebra, where the product on the right-hand side is the original product of A. Subalgebras of $(A,\circ)$ are called *special* Jordan algebras. For example, the associative algebras $M_n(\mathbb{R})$, $M_n(\mathbb{C})$ and $M_n(\mathbb{H})$ of $n\times n$ matrices over the reals $\mathbb{R}$, complexes $\mathbb{C}$ and quaternions $\mathbb{H}$, respectively, are special Jordan algebras in the product $\circ$. However, the Jordan algebra $(H_3(\mathbb{O}),\circ)$ of 3×3 Hermitian matrices over the Cayley algebra $\mathbb{O}$ is not special.

Although the concept of a Jordan algebra was first introduced by Jordan, von Neumann and Wigner [5], under the name of an r*-number system*, to formulate an algebraic model for quantum mechanics, unexpected connections with Lie algebras and geometry were soon discovered.

How are Jordan algebras related to Lie algebras? A Lie algebra L is also a vector space equipped with an *anti-symmetric* bilinear product, usually denoted by the brackets $[a,b]$, not assumed to be associative, which satisfies the Jacobi identity

$$[[a,b],c] + [b,c],a] + [[c,a],b] = 0.$$

Comparing definitions, one sees no obvious relationship between the two, e.g. one is commutative but the other anti-commutative.

In fact, it is well-known to algebraists that several exceptional Lie algebras can be constructed from $H_3(\mathbb{O})$. Apart from this and, what is more relevant to our discussion, is the fact that there is a 1-1 correspondence between a class of Lie algebras and Jordan algebras (actually, Jordan triples, which are slightly more general than Jordan algebras).

A *Jordan triple* is a vector space V equipped with a *Jordan triple product*

$$(a,b,c) \in V \times V \times V \mapsto \{a,b,c\} \in V$$

which is linear and symmetric in the outer variables, but conjugate linear in the middle variable, and satisfies the *Jordan triple identity*

$$\{a,b,\{x,y,z\}\} = \{\{a,b,x\},y,z\} - \{x,\{b,a,y\},z\} + \{x,y,\{a,b,z\}\}.$$

Example. A real Jordan algebra $(A,\circ)$ is a Jordan triple in the canonical triple product

$$\{a,b,c\} = (a\circ b)\circ c + a\circ(b\circ c) - b\circ(a\circ c). \qquad (1)$$

If $(A,\circ)$ is a complex Jordan algebra equipped with an involution *, then it is also a Jordan triple in the canonical triple product

$$\{a,b,c\} = (a\circ b^*)\circ c + a\circ(b^*\circ c) - b^*\circ(a\circ c).$$

Jordan algebras are just Jordan triples $(V,\{\cdot,\cdot,\cdot\}_V)$ containing a *unit element* e, the latter means $\{e,a,e\}_V = a$ for all $a \in V$. Indeed, for such V, we can define a product

$$a\circ b := \{a,e,b\}_V \qquad (a,b \in V)$$

with which V becomes a Jordan algebra, and $\{\cdot,\cdot,\cdot\}_V$ is exactly the canonical triple product defined by the Jordan product $\circ$ as in (1).

Example. According to the preceding example, the real Jordan algebra $(M_n(\mathbb{R}),\circ)$ is a Jordan triple in the canonical triple product. The complex Jordan algebra $(M_n(\mathbb{C}),\circ)$ has an involution defined by conjugate transpose : $(a_{ij})^* := (\overline{a}_{ji})$, it is therefore a complex Jordan triple in the canonical triple product, which can be rewritten as

$$\{a,b,c\} = \frac{1}{2}(ab^*c + cb^*a) \quad (a,b,c \in M_n(\mathbb{C})).$$

The subspaces $Sk_n(\mathbb{C})$ and $S_n(\mathbb{C})$ of $M_n(\mathbb{C})$, consisting of skew-symmetric and symmetric matrices respectively, are also complex Jordan triples in the above triple product.

Example. The complex vector space $M_{mn}(\mathbb{C})$ of $m\times n$ complex matrices has no natural Jordan product if $m \neq n$. However, it is a complex Jordan triple in the triple product

$$\{a,b,c\} := \frac{1}{2}(ab^*c + cb^*c) \quad (a,b,c \in M_{mn}(\mathbb{C})).$$

A Lie algebra L is called 3*-graded* if there is a 3-grading $L = L_{-1} \oplus L_0 \oplus L_1$, where the summands are subspaces of L satisfying $[L_i,L_j] \subset L_{i+j}$ or $\{0\}$ if $i+j = \pm 2$.

One can find in [2] some references for the following correspondence between Jordan triples and Lie algebras, which can be infinite dimensional.

Theorem. *(Tits–Kantor–Koecher) There is a* 1-1 *correspondence between Jordan triples V and 3-graded Lie algebras $L = L_{-1} \oplus L_0 \oplus L_1$ with $L_0 = [L_{-1},L_1]$ and an involution $\theta : L \to L$ satisfying $\theta(L_j) = L_{-j}$.*

In this correspondence, we have $V = L_{\pm 1}$ and

$$\{a,b,c\} = [[a,\theta(b)],c] \qquad (a,b,c \in V)$$

which relates the Jordan triple identity in V and the Jacobi identity in L.

Symmetric manifolds

Our ensuing discussion of Jordan algebras and geometry can be summarised briefly by saying that Jordan algebras appear as tangent spaces of symmetric manifolds (Figure 2).

That Jordan algebras have something to do with symmetry may not be a surprise, given that they correspond to a class of Lie algebras and Lie theory describes fundamental laws of symmetry.

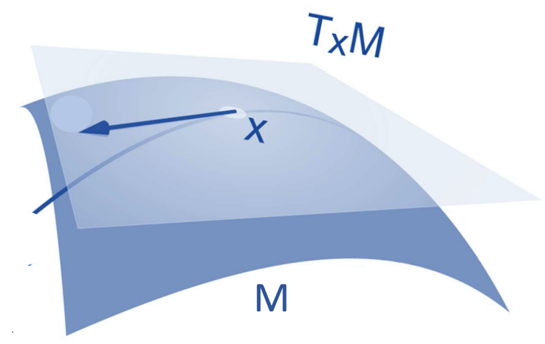

Figure 2. The tangent space T_xM at a point x in a symmetric manifold M carries a Jordan algebraic structure

A tangent space T_xM (Figure 2) of a manifold M at a point $x \in M$ is the vector space of all tangent vectors at x, it is a higher dimensional generalisation of a tangent line to a curve and a tangent plane to a surface. A tangent vector field X on M is a selection of a tangent vector $X(p) \in T_pM$ at each point $p \in M$.

A connected manifold M with a Riemannian metric g is called *symmetric* if it has a (unique) symmetry s_p at (about) each point $p \in M$, where a *symmetry* is an isometry $s_p : M \to M$ with respect to g satisfying two conditions below:

(i) $s_p \circ s_p$ is the identity map;

(ii) p is the only fixed-point of s_p in a neighbourhood of p.

Such a manifold is called a (Riemannian) *symmetric space*. Symmetric spaces are an important class of Riemannian manifolds and appear in many fields.

Example. The Euclidean space $\mathbb{R}^d$ is a symmetric space. The symmetry s_p about $p \in \mathbb{R}^d$ is given by $s_p(x) = 2p - x$.

A *Hermitian symmetric space* is a Riemannian symmetric space, which carries the structure of a complex manifold such that the given Riemannian metric is Hermitian, and the symmetries are holomorphic isometries. How are they related to Jordan algebras?

Let M be a Hermitian symmetric space. Fix a symmetry $s_p : M \to M$, which is an element in the automorphism group $\mathrm{Aut}\, M$, consisting of biholomorphic isometries of M. Here a *biholomorphism* $f : M \to M$ is a bijective holomorphic map whose inverse f^{-1} is also holomorphic. $\mathrm{Aut}\, M$ is a real Lie group with real Lie algebra $\mathfrak{L}$. Each element in $\mathfrak{L}$ is a (holomorphic) tangent vector field X on M, with $X(p) \in T_pM$. The symmetry s_p induces an involution $\sigma : \mathfrak{L} \to \mathfrak{L}$ for which $\mathfrak{L}$ has an eigenspace decomposition

$$\mathfrak{L} = \mathfrak{k} \oplus \mathfrak{p}$$

where

$$\mathfrak{k} = \{X \in \mathfrak{L} : \sigma(X) = X\},$$
$$\mathfrak{p} = \{X \in \mathfrak{L} : \sigma(X) = -X\}$$

and the map $X \in \mathfrak{p} \mapsto X(p) \in T_pM$ is a real linear isomorphism.

The complexification $\mathfrak{L}_c$ of $\mathfrak{L}$ is a 3-graded Lie algebra

$$\mathfrak{L}_c = \mathfrak{p}_+ \oplus \mathfrak{k}_c \oplus \mathfrak{p}_-$$

with an involution θ satisfying $\theta(\mathfrak{p}_\pm) = \mathfrak{p}_\mp$ and $\theta(\mathfrak{k}_c) = \mathfrak{k}_c$, where $\mathfrak{p}_+$ is complex linear isomorphic to T_pM.

By the Tits-Kantor-Koecher theorem above, $\mathfrak{p}_+$ is a Jordan triple and therefore the tangent space T_pM inherits the Jordan triple structure from $\mathfrak{p}_+$ via the linear isomorphism between them.

Further, T_pM is a so-called *Hermitian* Jordan triple, meaning that $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ in the eigenspace decomposition $\mathfrak{L} = \mathfrak{k} \oplus \mathfrak{p}$.

Conversely, given a complex Jordan triple V, Kaup [6] has shown that one can construct a real Lie algebra L with decomposition $L = k \oplus p$, and if V is Hermitian, meaning $[p, p] \subset k$, then there is a Hermitian symmetric space D such that V identifies in a natural way with a tangent space T_aD of D. We have therefore establish the following.

Theorem. *There is a* 1-1 *correspondence between Hermitian Jordan triples and Hermitian symmetric spaces.*

This theorem offers us an extra tool - to wit, Jordan triples, to study symmetric spaces. Indeed, it can even be extended to infinite dimension, which will be discussed briefly later. Thus we have a unified approach, using Jordan triples, to both finite and infinite dimensional symmetric spaces.

Symmetric spaces have been classified by É. Cartan using Lie theory. Let us consider the example of *nonpositively curved* Hermitian symmetric spaces and offer a Jordan perspective. The Hermitian Jordan triples corresponding to this class can be classified. They are finite direct sums

$$V_1 \oplus \cdots \oplus V_n$$

of six basic types of Jordan triples V_j $(j = 1, \dots, n)$. Each V_j is one of the following:

$$(1)\ M_{mn}(\mathbb{C}), \quad (2)\ Sk_n(\mathbb{C}), \quad (3)\ S_n(\mathbb{C}),$$
$$(4)\ Spin, \quad (5)\ M_{12}(\mathbb{O}), \quad (6)\ H_3(\mathbb{O})$$

where $H_3(\mathbb{O})$ consists of 3×3 Hermitian matrices over the complex Cayley algebra $\mathbb{O}$ and $M_{12}(\mathbb{O})$ consists of 1×2 matrices over $\mathbb{O}$. Each irreducible nonpositively curved Hermitian symmetric space in É. Cartan's classification list is biholomorphic to the open unit ball of one of these six types of Jordan triples of matrices.

Jordan algebras and symmetric cones

The real Jordan algebras classified in [5] are finite dimensional and assumed to be *formally real*, that is, they satisfy the condition

$$a_1^2 + \cdots + a_n^2 = 0 \Rightarrow a_1 = \cdots = a_n = 0.$$

We now reveal their relationship with Riemannian symmetric spaces. In short, they are in 1-1 correspondence with a class of cones which are Riemannian symmetric spaces.

Since all finite dimensional (Hausdorff) topological vector spaces are linearly homeomophic to a Euclidean space of the same dimension, there is only one (Hausdorff) topology on a finite dimensional vector space V making addition and scalar multiplication continuous. Therefore, there is no ambiguity to say that a set is *open* in V without referring to this topology.

A nonempty set Ω in a vector space V is called a *cone* if $\Omega + \Omega \subset \Omega$ and $\alpha\Omega \subset \Omega$ for all $\alpha > 0$.

Let V be finite dimensional. An open cone Ω in V is called *proper* if

$$\overline{\Omega} \cap -\overline{\Omega} = \{0\}$$

where $\overline{\Omega}$ is the closure of Ω. The cone $\overline{\Omega}$ induces a partial ordering $\leq$ in V so that $x \leq y \Leftrightarrow y - x \in \overline{\Omega}$.

An open cone Ω in V is called *linearly homogeneous* if for any $a, b \in \Omega$, there is a linear automorphism $\varphi : \Omega \to \Omega$ such that $\varphi(a) = b$. Here, a *linear automorphism* is a (continuous) linear isomorphism $\varphi : V \to V$ such that $\varphi(\Omega) = \Omega$.

If we consider an open cone Ω in V as a smooth manifold, then the tangent space $T_e\Omega$ at each point $e \in \Omega$ can be identified with V.

Theorem. *Let Ω be a proper linearly homogeneous open cone in a finite dimensional vector space V. Then Ω carries the structure of a Riemannian symmetric space whose linear automorphisms are isometries if and only if V admits the structure of a formally real Jordan algebra and $\overline{\Omega} = \{a^2 : a \in V\}$.*

Example. Let $V = \mathbb{R}^3$. The *light cone* (Figure 3)

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0, x_3^2 > x_1^2 + x_2^2\}$$

is a linearly homogeneous proper open cone in $\mathbb{R}^3$ and a Riemannian symmetric space. The corresponding Jordan algebraic structure of $\mathbb{R}^3$ is given by the Jordan product

$$(x_1, x_2, x_3) \circ (y_1, y_2, y_3) = (x_1y_3 + x_3y_1, \\ x_2y_3 + x_3y_2, x_1y_1 + x_2y_2 + x_3y_3).$$

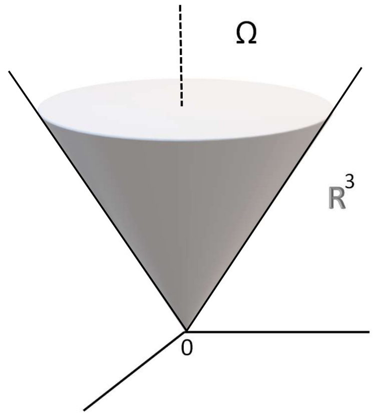

Figure 3. The tangent space R^3 of the symmetric light cone Ω is a formally real Jordan algebra

Infinite dimension

Now a few words about the infinite dimensional case. Essentially, assertions made in the previous discussion, particulary the last two theorems, can be extended to infinite dimension.

First, the toplogical vector spaces we need to consider are the ones equipped with a complete norm, namely, the *Banach spaces*. A norm $\|\cdot\|$ on a vector space V is said to be *complete* if V is a complete metric space in the metric $d(x, y) = \|x - y\|$ defined by the norm.

An infinite dimensional generalisation of a formally real Jordan algebra in finite dimension is the concept of a unital *JB-algebra*, which is a real unital Jordan algebra as well as a Banach space A satisfying

$$\|a^2\| = \|a\|^2, \quad \|a \circ b\| \leq \|a\|\|b\|, \\ \|a^2\| \leq \|a^2 + b^2\| \quad (a, b \in A).$$

A finite dimensional formally real Jordan algebra is a unital JB-algebra in the trace norm.

A finite dimensional manifold is modelled locally on $\mathbb{R}^d$ or $\mathbb{C}^d$. Analogously, a connected manifold can be modelled locally on a real or complex Banach space E. Such a manifold is called a real or complex connected *Banach manifold*, respectively. The manifold is called infinite dimensional if E is infinite dimensional.

In contrast to the finite dimensional manifolds, it is not meaningful to define a Riemannian metric on an infinite dimensional Banach manifold. Instead, one can define the notion of a *Finsler metric* on a Banach manifold, generalising that of a Riemannian metric (cf [1]). With this metric, we can extend the concept of a symmetric manifold to infinite dimension.

A Banach manifold equipped with a Finsler metric ν is called a *Finsler manifold*. A connected Finsler manifold M is called *symmetric* if there is a (unique) symmetry $s_p : M \to M$ at each $p \in M$, which is a ν-isometry satisfying the same conditions for a symmetry given before.

An open cone Ω in a Banach space V is called *normal* if there is a constant $\gamma > 0$ such that $0 \leq x \leq y \Rightarrow \|x\| \leq \gamma\|y\|$. A finite dimensional proper open cone is normal.

An infinite dimensional extension of the last theorem reads as follows (see [4]).

Theorem. *Let Ω be a normal linearly homogeneous open cone in a real Banach space V. Then Ω carries the structure of a Finsler symmetric manifold whose linear automorphisms are isometries if and only if V admits the structure of a unital JB-algebra and $\overline{\Omega} = \{a^2 : a \in V\}$.*

A final remark. Kaup [6] actually proved his aforementioned theorem for Hermitian Jordan triples in infinite dimension (see also [3]).

Theorem. (Kaup) *There is a 1-1 correspondence between complex symmetric Finsler manifolds and Hermitian Jordan triples that are Banach spaces with a continuous Jordan triple product.*

FURTHER READING

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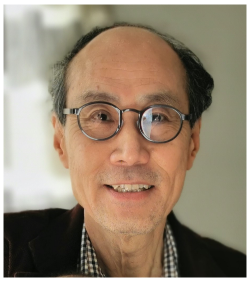

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