Centroids of Quadratic Jordan Superalgebras

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August 5, 2006

Abstract

The centroid of a Jordan superalgebra consists of the natural "superscalar multiplications" on the superalgebra. A philosophical question is whether the natural concept of "scalar" in the category of superalgebras should be that of superscalars or ordinary scalars. Basic examples of Jordan superalgebras are the simple Jordan superalgebras with semisimple even part, which were classified over an algebraically closed field of characteristic $\neq 2$ by M. Racine and E. Zelmanov. Here, we determine the centroids of the analogues of these superalgebras over general rings of scalars and show that they have no odd centroid, suggesting that ordinary scalars are the proper concept.

1 Jordan Superalgebras

Throughout, Φ will be an arbitrary ring of scalars; i.e. a commutative, associative, unital ring. In particular, we do not assume $\frac{1}{2} \in \Phi$, and so we work with quadratic Jordan algebras and superalgebras.

Definition 1. *A* **Φ-superalgebra** *is a $\mathbb{Z}_2$-graded algebra $B = B_0 \oplus B_1$, where $B_iB_j \subseteq B_{i+j}$. Here, B_0 is called the even part of B, and B_1 is called the odd part of B. Then $End_\Phi(B)$ is graded by $E_0 = E_{00} + E_{11} = End(B_0) \oplus End(B_1)$ and $E_1 = E_{01} + E_{10} = Hom(B_0, B_1) \oplus Hom(B_1, B_0)$. A homogeneous transformation $T_l \in E_l$ satisfies $T_l(B_i) \subset B_{i+l}$.*

An **associative superalgebra** *is just a $\mathbb{Z}_2$-graded associative algebra $B = B_0 \oplus B_1$, where $B_iB_j \subseteq B_{i+j}$.*

For example, if D is any associative Φ-algebra, then $B = M_{n+m}(D)$ is an associative superalgebra graded by $B_0 = \begin{pmatrix} M_n(D) & 0 \\ 0 & M_m(D) \end{pmatrix}$ and $B_1 = \begin{pmatrix} 0 & M_{n\times m}(D) \\ M_{m\times n}(D) & 0 \end{pmatrix}$. More generally, any idempotent e in a unital associative algebra B gives rise to a graded algebra B_e with $B_0 = B_{11} \oplus B_{00}$ and $B_1 = B_{10} \oplus B_{01}$, where $B_{ij} = e_i B e_j$ for $e_1 = e$ and $e_0 = 1 - e$.

A very important class of superalgebras is that of *superscalars*, the super-analogue of a scalar ring.

Definition 2. *A* **Φ-algebra of superscalars** *is a unital associative Φ-superalgebra $S = S_0 \oplus S_1$ which is supercommutative, so for all $x_i \in S_i$, $y_j \in S_j$,*

$$x_i \cdot y_j = (-1)^{ij} y_j \cdot x_i \qquad \textit{and} \qquad x_1^2 = 0.$$

The quintessential example of superscalars is the *Grassmann algebra* $G = G_0 \oplus G_1$, the exterior algebra $\Lambda(V)$ on a free module V of countably-infinite dimension. The requirement that odd superscalars square to zero is not universally accepted. Notice that if $\frac{1}{2} \in \Phi$, then $x_1 \cdot y_1 = -y_1 \cdot x_1$ implies $x_1^2 = 0$, but over a general scalar ring, we demand that $x_1^2 = 0$ (just as the true analogue of a skew-symmetric bilinear form for general scalars is an *alternating* bilinear form.

Definition 3. *A* **linear Jordan superalgebra** *is a superalgebra $J = A \oplus M$, where A is an ordinary Jordan algebra and M is an A-bimodule, such that J has a bilinear product $\langle \cdot, \cdot \rangle : J \times J \to J$ that satisfies the following identities:*

$$\langle x_i, y_j \rangle = (-1)^{ij} \langle y_j, x_i \rangle \qquad \textit{and} \qquad \sum_{cyclic\ (x_i, y_j, z_k)} (-1)^{(i+s)k} [\langle x_i, y_j \rangle, a_s, z_k] = 0,$$

where $[a, b, c] = \langle \langle a, b \rangle, c \rangle - \langle a, \langle b, c \rangle \rangle$ is the usual associator on J.

A **quadratic Jordan superalgebra** *is a Jordan superalgebra equipped with a bilinear product $\langle J_i, J_j \rangle \subseteq J_{i+j}$, a trilinear product $\langle J_i, J_j, J_k \rangle \subseteq J_{i+j+k}$, a squaring operator $^2 : J_0 \to J_0$, and quadratic operators $U_{x_0} : J_i \to J_i$ such that the Grassmann envelope $G(J) = J_0 \otimes G_0 + J_1 \otimes G_1$ is a quadratic Jordan algebra under the following products:*

$$U_{x_0 \otimes \gamma_0}(y_j \otimes \eta_j) = U_{x_0}(y_j) \otimes \gamma_0^2 \eta_j \qquad (x_0 \otimes \gamma_0)^2 = x_0^2 \otimes \gamma_0^2$$

$$U_{x_1 \otimes \gamma_1}(y_j \otimes \eta_j) = 0 \qquad (x_1 \otimes \gamma_1)^2 = 0$$

$$U_{x_i \otimes \gamma_i, z_k \otimes \mu_k}(y_j \otimes \eta_j) = \langle x_i, y_j, z_k \rangle \otimes \gamma_i \eta_j \mu_k$$

$$\langle x_i \otimes \gamma_i, y_j \otimes \eta_j \rangle = \langle x_i, y_j \rangle \otimes \gamma_i \eta_j.$$

We demand that the even bilinear and trilinear products $\langle x_0, y_0 \rangle = (x_0 + y_0)^2 - x_0^2 - y_0^2$ and $\langle x_0, y_0, z_0 \rangle = (U_{x_0 + z_0} - U_{x_0} - U_{z_0})(y_j)$ result from linearization of the quadratic products x_0^2 and U_{x_0}.

We demand explicitly that $\langle x_i, y_j, z_k \rangle$ be alternating *in the outer odd variables, not merely skew-symmetric, so $\langle x_1, y_j, x_1 \rangle = 0$. See (McCrimmon, 1992) for further details.*

We say a quadratic Jordan superalgebra is **unital** *if there exists an even element* $1_0 \in A$ *such that* $U_{1_0} = Id$, $x_0^2 = U_{x_0}(1_0)$, *and* $\langle 1_0, x_i \rangle = 2x_i$ *for all* $x_i \in J$.

A quadratic Jordan superalgebra A^+ can be created out of an associative superalgebra $A = A_0 \oplus A_1$ by defining the products as follows:

$$\langle x_i, y_j \rangle = x_i y_j + (-1)^{ij} y_j x_i \qquad \langle x_i, y_j, z_k \rangle = x_i y_j z_k + (-1)^{ij+jk+ki} z_k y_j x_i$$

$$U_{x_0}(y_j) = x_0 y_j x_0 \qquad x_0^2 = x_0 x_0.$$

Such a superalgebra is called **special**. If A is unital, so is A^+.

The simple Jordan superalgebras with semisimple even part were classified over an algebraically closed field K of characteristic not 2 by M. L. Racine and E. I. Zelmanov in (Racine and Zelmanov, 2003). If the characteristic of the field is not 3, then there are eight classes of superalgebras:

1. $K_3^{\lambda}(\Phi)$, the Kaplansky superalgebra
2. $D_4^{\lambda,\mu}(\Phi)$, the twisted quaternion superalgebra
3. K_{10}, the Kac superalgebra
4. $M_{n,m}(D)$, $n, m \geq 1$, the rectangular matrix superalgebra
5. $Q_n(D)$, $n \geq 2$, the square matrix superalgebra
6. $P_n(D, D_0)$, $n \geq 2$, the orthogonal superalgebra
7. $OSp_{n,2m}(\Phi)$, $n, m \geq 1$, the orthosymplectic superalgebra
8. the superalgebra of a nondegenerate supersymmetric bilinear form.

In characteristic 3, there are two additional "sporadic" superalgebras:

S1. $HS_3(\Phi_3)$, the sporadic conjugate superalgebra

S2. $H_3(C)$, the sporadic symplectic superalgebra.

Detailed descriptions of these classes will be provided in the subsequent sections. These superalgebras remain Jordan superalgebras when considered over arbitrary *rings* of scalars.

2 The Centroid

Definition 4. *Let J be a Jordan superalgebra. The* **linear** *or* **outer centroid** *of J, $L\Gamma(J) = L\Gamma_0(J) \oplus L\Gamma_1(J)$, is all $T = T_0 + T_1$ where $T_l \in End_{l\Phi}(J)$ satisfies*

$$T_l(\langle x_i, z_k \rangle) = \langle T_l(x_i), z_k \rangle = (-1)^{li} \langle x_i, T_l(z_k) \rangle. \qquad (L\Gamma 1)$$

If J is a quadratic superalgebra, we also demand that

$$
\begin{aligned}
T_l(\langle x_i, y_j, z_k\rangle) &= \langle T_l(x_i), y_j, z_k\rangle = (-1)^{li}\langle x_i, T_l(y_j), z_k\rangle \qquad (L\Gamma 2)\\
&= (-1)^{l(i+j)}\langle x_i, y_j, T_l(z_k)\rangle,
\end{aligned}
$$

$$
T_l(U_{x_0}(y_j)) = U_{x_0}(T_l(y_j)). \qquad (L\Gamma 3)
$$

The supermultiplication operators are $\langle x_i, y_j\rangle = L_{x_i}(u_j) = R_{y_j}(x_i)$ and $\langle x_i, y_j, z_k\rangle = L_{x_i,y_j}(z_K) = M_{x_i,z_k}(y_j) = R_{y_j,z_k}(x_i)$. We can write the linear centroid conditions as

$$
\begin{aligned}
T_l R_{y_j} &= R_{y_j} T_l & T_l L_{x_i} &= (-1)^{il} L_{x_i} T_l,\\
T_l R_{y_j,z_k} &= R_{y_j,z_k} T_l, & T_l M_{x_i,z_k} &= (-1)^{li} M_{x_i,z_k} T_l,\\
T_l L_{x_i,y_j} &= (-1)^{l(i+j)} L_{x_i,y_j} T_l, & T_l U_{x_0} &= U_{x_0} T_l.
\end{aligned}
$$

Thus we consider that R_{y_j} and R_{y_j,z_k} have degree 0 as *multiplication operators*, while L_{x_i} and M_{x_i,z_k} have degree i, and L_{x_i,y_j} has degree $i+j$.

Notice that the asymmetry of the interaction of the centroid with respect to left and right multiplication is due to the fact that we write our multiplication operators on the left (so they are tacitly left multiplication), which therefore *commute* with all right multiplications but only *supercommute* with left and middle multiplications.

Lemma 1. *$L\Gamma(J)$ is a subsuperalgebra of $End_\Phi(J)$.*

Proof. Let $\mathcal{X} = \{L_{x_i}, R_{y_j}, L_{x_i,y_j}, M_{x_i,z_k}, R_{y_j,z_k}, U_{x_0} : x_i, y_j, z_k \in J\}$. First, it must be shown that $L\Gamma(J)$ is closed under scalar multiplication. Let $\alpha \in \Phi$, $T_i \in L\Gamma_i(J)$, and $X_j \in \mathcal{X}$ have multiplication degree j. Then

$$
(\alpha T_i)X_j = \alpha((-1)^{ij} X_j T_i) = (-1)^{ij}\alpha X_j T_i = (-1)^{ij} X_j(\alpha T_i).
$$

Thus, $\alpha T_i \in L\Gamma_i(J)$.

Next, it must be shown that $L\Gamma(J)$ is closed under addition. This is true by definition for the sum of an even element and an odd element of $L\Gamma(J)$. Suppose $S_i, T_i \in L\Gamma_i(J)$ and $X_j \in \mathcal{X}$. Then $(S_i + T_i)$ is of degree i, and

$$
(S_i + T_i)X_j = S_iX_j + T_iX_j = (-1)^{ij} X_j S_i + (-1)^{ij} X_j T_i = (-1)^{ij} X_j(S_i + T_i),
$$

whence $S_i + T_i \in L\Gamma_i(J)$, and $L\Gamma(J)$ is closed under addition.

It must also be shown that $L\Gamma(J)$ is closed under composition. Let $S_i, T_j \in L\Gamma(J)$ and $X_k \in \mathcal{X}$. Then S_iT_j is of degree $i+j$, and

$$
\begin{aligned}
(S_iT_j)X_k &= S_i((-1)^{jk} X_k T_j) = (-1)^{jk} S_i X_k T_j\\
&= (-1)^{jk}(-1)^{ik} X_k S_i T_j = (-1)^{(j+i)k} X_k(S_iT_j).
\end{aligned}
$$

Thus, $S_iT_j \in L\Gamma(J)$, and $L\Gamma(J)$ is a Φ-subalgebra of $End_\Phi(J)$. $\square$

We remark that we cannot reduce centroid questions to the unital case, since $T \in L\Gamma(J)$ does *not* extend to the unital hull $\widetilde{J} = \Phi 1 \oplus J$ unless $T = \alpha Id$ is already a scalar in Φ and $T(\hat{1}) = \alpha\hat{1}$.

Definition 5. *The* **super-centroid** *of a quadratic Jordan superalgebra J, $\Gamma(J)$, has the additional properties*

$$
\begin{array}{llll}
U_{T_0(x_0)}(y_j) = T_0^2(U_{x_0}(y_j)), & (Q\Gamma 1) & (T_0(x_0))^2 = T_0^2(x_0^2), & (Q\Gamma 2)\\
U_{T_1(x_1)}(y_j) = 0, & (Q\Gamma 3) & (T_1(x_1))^2 = 0, & (Q\Gamma 4)\\
T_1^2 = 0. & (Q\Gamma 5) & &
\end{array}
$$

In other words, these endomorphisms T_l act like superscalar multiplications.

Note that $Q\Gamma 2, 4$ follow from $Q\Gamma 1, 3$ if J is unital. The conditions $Q\Gamma 3-5$ are somewhat controversial. One could define a weaker version of the centroid without making these demands. However, since we desire our superscalars to be alternating and we think of our centroid as superscalar multiplications, such a definition causes a few dilemmas. In particular, since $T_1(x_1)$ is even, its U-operator must be defined. We expect that $U_{T_1(x_1)} = T_1^2 U_{x_1}$, yet the U-operators are not defined for odd elements. This moral quandary is eliminated if we adopt the conditions above. In fact, these assumptions are rarely used, only appearing in the cases of the square matrix superalgebra $Q_n(D)$ and the Jordan superalgebra of a quadratic form. Furthermore, the scalar multiplications ΦId certainly belong to $\Gamma_0(J)$, as

$$\Phi Id \subseteq \Gamma_0(J) \subseteq L\Gamma(J).$$

We will show that in most cases $L\Gamma(J) = \Phi Id$, whence $\Gamma(J) = \Phi Id$ as well. Thus, our arguments seldom involve the quadratic conditions.

Note that there is no quandary if $\frac{1}{2} \in \Phi$, as $\Gamma(J) = L\Gamma(J)$ if the latter is supercommutative. In that case, we automatically have, for $T_i \in L\Gamma_i(J)$, $U_{T_0(x_0)} = \frac{1}{2}U_{T_0(x_0),T_0(x_0)} = \frac{1}{2}T_0^2 U_{x_0,x_0} = T_0^2 U_{x_0}$, $(T_0(x_0))^2 = \frac{1}{2}T_0(x_0)\circ T_0(x_0) = \frac{1}{2}T_0^2(x_0 \circ x_0) = T_0^2(x_0^2)$, $U_{T_1(x_1)} = \frac{1}{2}U_{T_1(x_1),T_1(x_1)} = \frac{1}{2}T_1^2 M_{x_1,x_1} = 0$ (by our hypothesis that $M_{x_1,x_1} = 0$), and $T_1^2 = \frac{1}{2}(T_1T_1 + T_1T_1) = \frac{1}{2}T_1T_1 - T_1T_1 = 0$ if $L\Gamma(J)$ is supercommutative.

The outer centroid is a unital associative superalgebra which is not, a priori, supercommutative. However, under mild assumptions about the superalgebra J, the centroid will be supercommutative.

Lemma 2 (Hiding Lemma)**.** *Suppose $B_k(x_i, y_j) : J \times J \to J$ is a bilinear map and S_s, T_t are B_k-linear maps (so $S_sB_k(x_i, y_j) = (-1)^{sk}B_k(S_s(x_i), y_j) = (-1)^{s(k+i)}B_k(x_i, S_s(y_j))$ and similarly for T_t). Then $(S_sT_t - (-1)^{st}T_tS_s)B_k = 0$.*

Proof.

$$
\begin{aligned}
(-1)^{st} T_t S_s B_k(x_i, y_j) &= (-1)^{st+sk} T_t B_k(S_s(x_i), y_j) \\
&= (-1)^{st+sk}(-1)^{t(k+s+i)} B_k(S_s(x_i), T_t(y_j)) \\
&= (-1)^{st+sk+tk+ts+ti} B_k(S_s(x_i), T_t(y_j)) \\
&= (-1)^{sk+tk+ti}(-1)^{sk} S_s B_k(x_i, T_t(y_j)) \\
&= (-1)^{tk+ti}(-1)^{t(k+i)} S_s T_t B_k(x_i, y_j) \\
&= S_s T_t B_k(x_i, y_j).
\end{aligned}
$$

□

Corollary 1. *Suppose J is a Jordan superalgebra such that either*

1. *the linear annihilator of J is trivial; i.e.,*

$$\langle z, J\rangle = \langle z, J, J\rangle = \langle J, z, J\rangle = 0 \text{ implies } z = 0 \text{ or}$$

2. *J is linearly idempotent; i.e.,*

$$J = \langle J, J\rangle + \langle J, J, J\rangle.$$

Then $L\Gamma(J)$ is supercommutative.

Proof. Define the following degree zero bilinear maps on $J \times J$: $B(x_i, y_j) := \langle x_i, y_j\rangle$, $B_{z_k}(x_i, y_j) := \langle x_i, y_j, z_k\rangle$, and $B'_{z_k}(x_i, y_j) := \langle x_i, z_k, y_j\rangle$. If $S_s, T_t \in L\Gamma(J)$, then by definition S_s and T_t are $B-$, $B_{z_k}-$, and $B'_{z_k}-$linear, so by the Hiding Lemma, $\Delta = S_s T_t - (-1)^{st} T_t S_s$ has

$$
\begin{aligned}
\Delta B(x_i, y_j) &= \Delta(\langle x_i, y_j\rangle) = \langle \Delta(x_i), y_j\rangle = 0 \\
\Delta B_{z_k}(x_i, y_j) &= \Delta(\langle x_i, y_j, z_k\rangle) = \langle \Delta(x_i), y_j, z_k\rangle = 0 \\
\Delta B'_{z_k}(x_i, y_j) &= \langle \Delta(x_i), z_k, y_j\rangle = 0.
\end{aligned}
$$

Thus, Δ maps J into the linear annihilator and kills all bilinear and trilinear products. If J has trivial linear annihilator or is linearly idempotent, then $\Delta = 0$ on J, and $L\Gamma(J)$ is supercommutative. □

Lemma 3. *If the linear centroid $L\Gamma(J)$ is supercommutative, then the supercentroid $\Gamma(J)$ is a subsuperalgebra of $End_\Phi(J)$.*

Proof. Since $\Gamma(J) \subseteq L\Gamma(J)$, we need to show that αT_0, $S_0 + T_0$, S_0T_0, $S_1T_1 \in L\Gamma_0(J)$ and αS_1, $S_1 + T_1$, S_0T_1, $T_1S_0 \in L\Gamma_1(J)$ satisfy the new quadratic conditions for $\alpha \in \Phi$, $S_0, T_0 \in \Gamma_0(J)$, and $S_1, T_1 \in \Gamma_1(J)$.

First,

$$\begin{aligned} U_{(\alpha T_0)(x_0)} &= \alpha^2 U_{T_0(x_0)} = \alpha^2 T_0^2 U_{x_0} = (\alpha T_0)^2 U_{x_0} \\ ((\alpha T_0)(x_0))^2 &= \alpha^2 (T_0(x_0))^2 = \alpha^2 T_0^2(x_0^2) = (\alpha T_0)^2(x_0^2), \end{aligned}$$

so $\alpha T_0 \in \Gamma_0(J)$.

For the sum $S_0 + T_0$, note that the quadratic product on J has the property that $U_{v_0,w_0}(z_i) = \langle v_0, z_i, w_0\rangle$, so $U_{v_0,v_0} = 2U_{v_0}$. Now

$$\begin{aligned} U_{(S_0+T_0)(x_0)} &= U_{S_0(x_0),T(x_0)} + U_{S_0(x_0)} + U_{T_0(x_0)} \\ &= \langle S_0(x_0), \cdot, T_0(x_0)\rangle + U_{S_0(x_0)} + U_{T_0(x_0)} \\ &= S_0T_0(\langle x_0, \cdot, x_0\rangle) + U_{S_0(x_0)} + U_{T_0(x_0)} \\ &= S_0T_0(2U_{x_0}) + U_{S_0(x_0)} + U_{T_0(x_0)} \\ &= 2S_0T_0(U_{x_0}) + S_0^2(U_{x_0}) + T_0^2(U_{x_0}) \\ &= (S_0+T_0)^2 U_{x_0} \text{ since } \Gamma(J) \text{ is supercommutative,} \quad \text{and} \end{aligned}$$

$$\begin{aligned} ((S_0+T_0)(x_0))^2 &= (S_0(x_0))^2 + (T_0(x_0))^2 + \langle S_0(x_0), T_0(x_0)\rangle \\ &= S_0^2(x_0^2) + T_0^2(x_0^2) + S_0T_0\langle x_0, x_0\rangle \\ &= \left(S_0^2 + T_0^2 + 2S_0T_0\right)(x_0^2) \\ &= (S_0+T_0)^2(x_0^2) \text{ since } \Gamma(J) \text{ is supercommutative,} \end{aligned}$$

so $S_0 + T_0 \in \Gamma_0(J)$.

For the composites S_0T_0 and S_1T_1,

$$\begin{aligned} U_{S_0(T_0(x_0))} &= S_0^2(U_{T_0(x_0)}) = S_0^2T_0^2(U_{x_0}) \\ &= (S_0T_0)^2 U_{x_0} \text{ since } \Gamma(J) \text{ is supercommutative.} \end{aligned}$$

$$\begin{aligned} ((S_0T_0)(x_0))^2 &= (S_0(T_0(x_0)))^2 = S_0^2(T_0(x_0))^2 = S_0^2T_0^2(x_0^2) \\ &= (S_0T_0)^2(x_0^2) \text{ since } \Gamma(J) \text{ is supercommutative,} \end{aligned}$$

so $S_0T_0 \in \Gamma_0(J)$, and

$$\begin{aligned} U_{(S_1T_1)(x_0)} &= U_{S_1(T_1(x_1))} = 0 = -S_1^2T_1^2U_{x_0} \\ &= (S_1T_1)^2U_{x_0} \text{ since } \Gamma(J) \text{ is supercommutative, and} \end{aligned}$$

$$\begin{aligned} ((S_1T_1)(x_0))^2 &= (S_1(T_1(x_0)))^2 = 0 = -S_1^2T_1^2(x_0^2) \\ &= (S_1T_1)^2(x_0^2) \text{ since } \Gamma(J) \text{ is supercommutative,} \end{aligned}$$

so $S_1T_1 \in \Gamma_0(J)$.

Next,

$$\begin{aligned} U_{(\alpha T_1)(x_1)} &= \alpha^2U_{T_1(x_1)} = 0, \\ ((\alpha T_1)(x_1))^2 &= \alpha^2(T_1(x_1))^2 = 0, \text{ and} \\ (\alpha T_1)^2 &= \alpha^2T_1^2 = 0, \qquad\qquad \text{so } \alpha T_1 \in \Gamma_1(J). \end{aligned}$$

For the sum $S_1 + T_1$,

$$\begin{aligned} U_{(S_1+T_1)(x_1)} &= U_{S_1(x_1)+T_1(x_1)} = U_{S_1(x_1)} + U_{T_1(x_1)} + U_{S_1(x_1),T_1(x_1)} \\ &= 0 + 0 + S_1U_{x_1,T_1(x_1)} \\ &= \pm S_1T_1M_{x_1,x_1} = 0 \text{ since } M_{x_1,x_1} = 0 \text{ by hypothesis,} \\ ((S_1+T_1)(x_1))^2 &= (S_1(x_1) + T_1(x_1))^2 \\ &= (S_1(x_1))^2 + (T_1(x_1))^2 + \langle S_1(x_1), T_1(x_1)\rangle \\ &= 0 + 0 + S_1\langle x_1, T_1(x_1)\rangle = -S_1T_1\langle x_1, x_1\rangle = 0, \text{ and} \end{aligned}$$

$$\begin{aligned} (S_1+T_1)^2 &= S_1^2 + T_1^2 + S_1T_1 + T_1S_1 \\ &= 0 + 0 + S_1T_1 - S_1T_1 = 0 \text{ since } \Gamma(J) \text{ is supercommutative,} \end{aligned}$$

so $S_1 + T_1 \in \Gamma_1(J)$.

Finally, for the composites S_0T_1 and T_1S_0,

$$\begin{aligned} U_{(S_0T_1)(x_1)} &= U_{S_0(T_1(x_1))} = S_0^2U_{T_1(x_1)} = 0, \\ U_{(T_1S_0)(x_1)} &= U_{T_1(S_0(x_1))} = 0, \\ ((S_0T_1)(x_1))^2 &= (S_0(T_1(x_1)))^2 = S_0^2(T_1(x_1))^2 = 0, \\ ((T_1S_0)(x_1))^2 &= (T_1(S_0(x_1)))^2 = 0, \text{ and} \\ (S_0T_1)^2 &= S_0^2T_1^2 = 0 = T_1^2S_0^2 = (T_1S_0)^2 \text{ since } \Gamma(J) \text{ is supercommutative,} \end{aligned}$$

so $S_0T_1, T_1S_0 \in \Gamma_1(J)$. Thus, $\Gamma(J)$ is a subsuperalgebra of $End_\Phi(J)$. □

The preceding lemma leads directly to the following result.

Theorem 1. *If $L\Gamma(J)$ is supercommutative, then $\Gamma_0(J)$ is a commutative, associative ring of scalars (called simply the* **centroid***), and J is a superalgebra over $\Gamma_0(J)$.*

When $L\Gamma(J)$ is supercommutative, the superalgebra J can be considered as a superalgebra over its supercentroid. Thus, the supercentroid provides a natural algebra of superscalars for J. This begs the following question: is the supercentroid *really* a set of superscalars, or is it just an ordinary set of scalars in disguise? In other words, are there any odd elements of the supercentroid? Since an odd supercentroidal element is a rather strange creature, one expects that a "nice" Jordan superalgebra would have no odd supercentroid. The classification in (Racine and Zelmanov, 2003) includes many of the "nice" examples of Jordan superalgebras over a field, so to provide support to this conjecture, we determine the supercentroids of these superalgebras over a general ring Φ. We will show that, under mild assumptions about the ring Φ and the superalgebra J, there is no odd supercentroid, and in most cases, the algebra is already centroidal: the supercentroid only consists of the scalar multiplications by Φ.

3 General Behavior of Centroid

Lemma 4. *Let $J = A \oplus M$ be a Jordan superalgebra.*

1. *The kernel of a centroidal $T \in L\Gamma(J)$ os an outer ideal in J (homogeneous if $T \in L\Gamma_i(J)$), so $T = \alpha Id$ as soon as $T = \alpha Id$ on an outer generating set.*

2. *In particular, if $T(A) = 0$ and $M = \langle A, M\rangle + U_A\langle A, M\rangle$, then $T = 0$.*

Proof. Note that (2) follows from (1) by setting $\alpha = 0$. If $T \in L\Gamma(J)$, then $T' = T - \alpha Id \in L\Gamma(J)$, and $T = \alpha Id$ if and only if $T' = 0$, so it suffices to show that the kernel of T is an outer ideal. The kernel of T_i is homogeneous, since $T_i(x_0 + x_1) = T_i(x_0) + T_i(x_1) \in J_i \oplus J_{i+1}$ vanishes if and only if $T_i(x_0) = T_i(x_1) = 0$. Let $\mathcal{M}$ denote the outer multiplication algebra generated by the set

$$\{Id, L_x, L_{x,y}, M_{x,y}, U_z : x, y \in A \cup M \text{ and } z \in A\}.$$

For a set of homogeneous elements $S \subset A \cup M$ (e.g., for $S = Ker(T)$), let $\mathcal{M}(S) = \{m(z) : m \in \mathcal{M}, z \in S\}$ be the outer ideal generated by S. Each element of $\mathcal{M}(S)$ can be written as linear combinations of elements of the form $(m_1(m_2(\cdots(m_n(z)))))$ for some $z \in S$ and some homogeneous $m_i \in \mathcal{M}_{\epsilon(i)}$, where $\epsilon(i) \in \{0, 1\}$ and M_j denotes the operators of multiplication degree j as described in Section 2. By definition, T supercommutes with all homogeneous outer multiplications, so

$$\begin{aligned} T(m_1(m_2(\cdots(m_n(z))))) &= \pm(m_1(m_2(\cdots(m_n(T(z)))))) \\ &\quad (= 0 \text{ if } z \in Ker(T)). \end{aligned}$$

If $S = Ker(T)$, this gives us that $T(\mathcal{M}(S)) = 0$, whence $\mathcal{M}(S) = Ker(T)$ is an outer ideal of J. □

Lemma 5. *Suppose $\mathcal{J}$ is a Jordan algebra or superalgebra with idempotent e. Let $\mathcal{J} = \mathcal{J}_2 \oplus \mathcal{J}_1 \oplus \mathcal{J}_0$ be the Peirce decomposition of $\mathcal{J}$ with respect to e. Then*

1. *$T(\mathcal{J}_i(e)) \subseteq \mathcal{J}_i(e)$ for $T \in L\Gamma_s(\mathcal{J})$ and $i = 0, 1, 2$.*
2. *If $T(e) = \alpha e$ for $\alpha \in \Phi$ (for example, if e is a* **reduced idempotent** *$J_2(e) = \Phi e$), then $T = \alpha Id$ on $\mathcal{J}_1$.*
3. *In particular, if $T(e) = 0$, $T(\mathcal{J}_1) = 0$.*
4. *Suppose $N \subseteq J_1$ is J_i-invariant for $i = 0$ or 2, so $\langle J_i, N\rangle \subseteq N$. If $Z_i = \{a \in J_i : \langle a, N\rangle = 0\}$ is 0, then $T(e) = 0$ implies $T(J_i) = 0$.*

Proof. Let E_i denote the even Peirce projection on $\mathcal{J}_i(e)$, so $E_2 = U_e$, $E_0 = U_{\hat{1}-e}$, and $E_1 = U_{e,\hat{1}-e}$. Then if $x_i \in \mathcal{J}_i(e)$, $T_0(x_i) = T(E_i(x_i)) = E_i(T_0(x_i)) \in \mathcal{J}_i(e)$.

Suppose $T(e) = \alpha e$. Then for any $x_1 \in \mathcal{J}_1$,

$$T(x_1) = T(\langle e, x_1\rangle) = \langle T(e), x_1\rangle = \langle \alpha e, x_1\rangle = \alpha\langle e, x_1\rangle = \alpha x_1.$$

To prove (4), J_i is T-invariant when $T(e) = 0$, so

$$\langle T(J_i), N\rangle = \langle J_i, T(N)\rangle \subseteq \langle J_i, T(J_1)\rangle = 0 \qquad \text{by (3).}$$

Thus, $T(J_i) \subseteq Z_i = 0$. □

Lemma 6. 1. *Suppose $J = A \oplus M$ is a Jordan superalgebra with idempotent $e \in A$ such that $\langle e, m\rangle = m$ for all $m \in M$. Then $L\Gamma_1(J) = 0$.*

2. *Suppose J has idempotents e_i and unit $1 = \sum e_i$, where all $M_2(e_i) = 0$. If $\frac{1}{2} \in \Phi$, then $L\Gamma_1(J) = 0$.*

Proof. 1. First, for $T_1 \in L\Gamma_1(J)$, $T_1(e) = T_1(U_e(e)) = U_e(T_1(e)) \in U_e(M) = 0$ since $M \subseteq J_1(e)$ by hypothesis. Then $T_1(M) \subseteq T_1(J_1(e)) = 0$ and $T_1(A_1(e)) = 0$ by Lemma 5. Also by Lemma 5, $T_1(A_2 \oplus A_0) \subset M \cap (J_2 \oplus J_0) = 0$. Thus, $T_1(A) = 0$ as well.

2. For each i, $T_1(e_i) = T(U_{e_i}(e_i)) = U_{e_i}(T_1(e_i)) \in U_{e_i}(M) = M_2(e_i) = 0$ by hypothesis, so $T_1(1) = \sum T_1(e_i) = 0$. Then $0 = \langle T_1(1), x_i\rangle = \langle 1, T_1(x_i)\rangle = 2T_1(x_i)$ for all x_i, so if $\frac{1}{2} \in \Phi$, then all $T_1(x_i) = 0$, and $T_1 = 0$.

□

Lemma 7. *Let $\mathcal{J}$ be a Jordan algebra or superalgebra, and let $T_s \in L\Gamma_s(\mathcal{J})$ or $T_s \in \Gamma_s(\mathcal{J})$. If $\mathcal{B}$ is a T_s-invariant subalgebra or subsuperalgebra of $\mathcal{J}$, then $T_s|_\mathcal{B} \in L\Gamma_s(\mathcal{B})$ or $T_s|_\mathcal{B} \in \Gamma_s(\mathcal{B})$.*

Proof. Let $x_i \in \mathcal{B}_i$ and $y_j \in \mathcal{B}_j$. Now $T_s|_\mathcal{B} = T_s$ on $\mathcal{B}$, so for $P = L_{x_i}, L_{x_i,y_j}, M_{x_i,y_j}, U_{x_0}$ of degree $k = i$, $i + j$, $i + j$, and 0, respectively,

$$T_s|_\mathcal{B} P = T_s P = (-1)^{ks} P T_s = (-1)^{ks} P T_s|_\mathcal{B}$$

as endomorphisms of $\mathcal{B}$. Thus, $T_s|_\mathcal{B} \in L\Gamma_s(\mathcal{B})$.

If $T_s \in \Gamma_s(\mathcal{J})$ (i.e., $U_{T_0(b_0)} = T_0^2 U_{b_0}$ if $s = 0$ or $U_{T_1(b_1)} = 0$ if $s = 1$),

$$U^\mathcal{B}_{T_0|_\mathcal{B}(b_0)} = U_{T_0(b_0)} = T_0^2(U_{b_0}) = T_0|^2_\mathcal{B}(U^\mathcal{B}_{b_0}), \qquad U^\mathcal{B}_{T_1|_\mathcal{B}(b_1)} = U_{T_1(b_1)} = 0$$

on $\mathcal{B}$. Thus, $T|_\mathcal{B} \in \Gamma_s(\mathcal{B})$. □

Lemma 8. *Suppose J is a unital quadratic Jordan superalgebra over a ring Φ such that $\frac{1}{2} \in \Phi$.*

1. *If $T \in L\Gamma(J)$ has $T(1_0) = \alpha 1_0$ for 1_0 the unit of J and $\alpha \in \Phi$, then $T = \alpha Id$ on all of J. In particular, if $T(1_0) = 0$, then $T = 0$.*

2. *If A is linearly centroidal, so $L\Gamma(A) = \Phi Id$, then $L\Gamma_0(J) = \Phi Id$.*

Proof.

1. Let $T \in L\Gamma(J)$, and suppose that $T(1_0) = \alpha 1_0$ for some $\alpha \in \Phi$. Since 1_0 is the unit of the quadratic Jordan superalgebra J, $\langle 1_0, x_i\rangle = 2x_i$ for all $x_i \in J_i$. Thus,

$$2T(x_i) = T(2x_i) = T(\langle 1_0, x_i\rangle) = \langle T(1_0), x_i\rangle = \langle \alpha 1_0, x_i\rangle = 2\alpha x_i.$$

Since $\frac{1}{2} \in \Phi$, this implies that $T(x_i) = \alpha x_i$.

2. If $T_0 \in L\Gamma_0(J)$, then $T_0|_A \in L\Gamma(A) = \Phi Id$ by Lemma 7, so $T_0(1_0) = \alpha 1_0$, and $T_0 = \alpha Id$ by (1).

□

Lemma 9. *Let D be a unital associative ring with involution $- : D \to D$ over its $*$-centroid $\Phi = \Gamma(D)$ (so $\bar{\alpha} = \alpha$ for $\alpha \in \Phi$). Let $1 \subseteq D_0 \subseteq H(D)$ be an ample subspace; i.e., $dD_0\bar{d} \subseteq D_0$ for all $d \in D$, so $d\bar{d} \in D_0$ and $d + \bar{d} \in D_0$. Let $J = H_n(D, D_0)$ be an ample outer ideal in the hermitian $n \times n$ matrices with respect to the $-$ transpose involution with diagonal entries in D_0. Then if $n \geq 2$, $L\Gamma(J) = \Phi Id$.*

Proof. Let $T \in L\Gamma(J))$. Note that J has n supplementary orthogonal idempotents $e_i = e_{ii}$. Let J_i denote the i^{th} Peirce space of J with respect to these idempotents. Then Lemma 5 gives us that $T(J_i) \subseteq J_i$, so $T(e_i) = \alpha_i e_i$ for some $\alpha_i \in D_0$. Now if $n \geq 2$ and $i \neq j$,

$$\begin{aligned} \alpha_j e_j &= T(e_j) = T(U_{e_{ij}+e_{ji}}(e_i)) \\ &= U_{e_{ij}+e_{ji}}(T(e_i)) = U_{e_{ij}+e_{ji}}(\alpha_i e_i) = \alpha_i e_j. \end{aligned}$$

Therefore, $\alpha_i = \alpha_j$ for all $1 \leq i, j \leq n$, so $T(e_i) = \alpha e_i$ for some $\alpha \in D_0$.

Now for any $d \in D$,

$$\begin{aligned} \alpha d e_{ij} + \overline{d} \alpha e_{ji} &= \{\alpha e_i, d e_{ij} + \overline{d} e_{ji}, e_j\} = \{T(e_i), d e_{ij} + \overline{d} e_{ji}, e_j\} \\ &= \{e_i, d e_{ij} + \overline{d} e_{ji}, T(e_j)\} = \{e_i, d e_{ij} + \overline{d} e_{ji}, \alpha e_j\} \\ &= d\alpha e_{ij} + \alpha \overline{d} e_{ji}. \end{aligned}$$

Thus, $\alpha d = d\alpha$ for all $d \in D$, whence $\alpha \in \Gamma(D) = \Phi$.

Let $T' = T - \alpha Id \in L\Gamma(J)$. Then $T'(e_i) = 0$ for all i. Now by Lemma 5, $T'(J_1)(e_i) = 0$, so if $n \geq 2$ and $i \neq j$, then $T'(De_{ij}) = 0$. Thus, $T = \alpha Id$. □

4 The Sporadic Superalgebras

When the base ring Φ has the property that $3\Phi = 0$, two additional "sporadic" algebras appear in the classification. Both arise as hermitian 3×3 matrices over an alternative superalgebra. Note that since Φ has "characteristic" 3, $\frac{1}{2} \in \Phi$.

4.1 The Sporadic Conjugate Superalgebra $\mathbf{HS_3}(\mathbf{\Phi_3})$

Suppose $\Phi = \Phi_3$ is a ring of scalars with $3\Phi = 0$, and let $A = H_3(\Phi)$ and $M = \overline{S_3(\Phi)} \oplus \overline{\overline{S_3(\Phi)}}$, where $S_3(\Phi)$ denotes the space of all skew 3×3 matrices $s^t = -s$. Then $J = A \oplus M$ with bilinear product defined for $h_i \in A = H_3(\Phi)$ and $s_i \in S_3(\Phi)$ by

$$\begin{aligned} \langle h_1, h_2 \rangle &= h_1 h_2 + h_2 h_1 \\ \langle h, \overline{s_1} \oplus \overline{\overline{s_2}} \rangle &= (\overline{h s_1 + s_1 h}) \oplus (\overline{\overline{h s_2 + s_2 h}}) \\ \langle \overline{S_3(\Phi)} \oplus \overline{\overline{0}}, \overline{S_3(\Phi)} \oplus \overline{\overline{0}} \rangle &= 0 = \langle \overline{0} \oplus \overline{\overline{S_3(\Phi)}}, \overline{0} \oplus \overline{\overline{S_3(\Phi)}} \rangle \\ \langle \overline{s_1} \oplus \overline{\overline{0}}, \overline{0} \oplus \overline{\overline{s_2}} \rangle &= s_1 s_2 + s_2 s_1 \end{aligned}$$

is a linear Jordan superalgebra, and since $\frac{1}{2} \in \Phi$, it is automatically a quadratic Jordan superalgebra with quadratic product defined by

$$U_h(h') = hh'h \qquad\qquad h^2 = hh$$

$$U_h(\overline{s_1} \oplus \overline{\overline{s_2}}) = (\overline{hs_1h} \oplus \overline{\overline{hs_2h}}).$$

Note that J is a unital quadratic Jordan superalgebra with unit I_3, as $U_{I_3} = Id$, $h^2 = U_h I)3$, and $\langle I_3, x_i \rangle = 2x_i$ for each $x_i \in J$.

This superalgebra can be viewed as the hermitian 3×3 matrices over the alternative superalgebra $B(1,2) = B_0 \oplus B_1 = \Phi 1 \oplus (\Phi x \oplus \Phi y)$, where $xy = -yx = 1$, under the conjugate superinvolution $(b_0 \oplus b_1)^* = b_0 \oplus -b_1$. See (Shestakov, 1999) for more details.

Theorem 2. *$L\Gamma(HS_3(\Phi_3)) = \Phi_3 Id$.*

Proof. Let $s_i = (\sigma_{jk})$ and $s_i' = (\sigma_{jk}')$ be skew matrices. Then since $\frac{1}{2} \in \Phi$, $\sigma_{ii} = 0 = \sigma_{ii}'$ for $i = 1,2,3$. Now $M_2(e_{ii}) = 0$, since

$$U_{e_{ii}}\left(\overline{s_i} \oplus \overline{\overline{s_i'}}\right) = \overline{\sigma_{ii}e_{ii}} \oplus \overline{\overline{\sigma_{ii}'e_{ii}}} = 0.$$

By Lemma 6 (2), $L\Gamma_1(J) = 0$.

By Lemma 9, $H_3(\Phi_3)$ is centroidal, so Lemma 8 (2) implies $L\Gamma_0(J) = \Phi Id$. □

4.2 The Sporadic Symplectic Superalgebra $\mathbf{H_3(C)}$

Let $\Phi = \Phi_3$ be a ring of scalars with $3\Phi = 0$, so $\frac{1}{2} \in \Phi$. Let $C = C_0 \oplus C_1$, where $C_0 = M_2(\Phi)$ and $C_1 = \Phi m_1 \oplus \Phi m_2$ is the free module over Φ with basis $\{m_1, m_2\}$. Now C has basis $\{e_{11}, e_{12}, e_{21}, e_{22}, m_1, m_2\}$, where the product on these elements is given by the following table.

	e_{11}	e_{12}	e_{21}	e_{22}	m_1	m_2
e_{11}	e_{11}	e_{12}	0	0	m_1	0
e_{12}	0	0	e_{11}	e_{12}	m_2	0
e_{21}	e_{21}	e_{22}	0	0	0	m_1
e_{22}	0	0	e_{21}	e_{22}	0	m_2
m_1	0	$-m_2$	0	m_1	$-e_{21}$	e_{11}
m_2	m_2	0	$-m_1$	0	$-e_{22}$	e_{12}

Then C is an alternative superalgebra with superinvolution given by $(a \oplus m)^* = \overline{a} \oplus (-m)$, where $- : M_2(\Phi) \to \Phi$ is the usual symplectic involution

$$\overline{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

Let $J = H_3(C)$, the hermitian 3×3 matrices with respect to the $*$-transpose superinvolution. Then J is a linear (and hence quadratic since $\frac{1}{2} \in \Phi$) Jordan superalgebra with even part $A = H_3(C_0)$ and odd part $M = H_3(C_1)$. The bilinear product is given by $\langle x_i, y_j \rangle = x_i y_j + (-1)^{ij} y_j x_i$, and the quadratic product is given by $U_{x_0}(y_j) = x_0 y_j x_0$. Note that since $m^* = -m$, the elements of $M = H_3(C_1)$ are skew matrices, and the superalgebra J is unital with unit $I = \begin{pmatrix} I_2 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{pmatrix}$.

Theorem 3. *$L\Gamma(H_3(B)) = \Phi Id$.*

Proof. First, note that $J = H_3(B)$ has three orthogonal idempotents,

$$E_1 = \begin{pmatrix} I_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and } E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_2 \end{pmatrix}.$$

Let $T_1 \in L\Gamma_1(J)$, and consider $T_1(E_i)$. Note that $M_2(E_i) = 0$, since M is skew and $\frac{1}{2} \in \Phi$ implies that the diagonal entries of skew elements are zero. By Lemma 6 (2), $L\Gamma_1(J) = 0$.

Now $A = H_3(C_0)$ is centroidal, since $L\Gamma(H_3(C_0)) = L_{\Gamma(C_0)} = L_{\Gamma(M_2(\Phi))} = L_{\Phi I_2} = \Phi Id$ by (McCrimmon, 2004). Thus, Lemma 8 implies $L\Gamma_0(J) = \Phi Id$.

□

5 The Small-dimensional Superalgebras

5.1 The Kaplansky Superalgebra $\mathbf{K_3^{(\lambda)}(\Phi)}$

Let Φ be a ring of scalars, let $A = \Phi e$ be the free module over Φ with basis e, and let $M = \Phi x \oplus \Phi y$ be the free module over Φ with basis $\{x, y\}$. Then for $\lambda \in \Phi$, $J = K_3^{(\lambda)}(\Phi) = A \oplus M$ with bilinear product given by

$$e^2 = e, \quad \langle e, x \rangle = x, \quad \langle e, y \rangle = y, \quad \langle x, y \rangle = \lambda e$$

$$\langle e, e \rangle = 2e, \quad \langle x, x \rangle = 0, \quad \langle y, y \rangle = 0$$

and quadratic product with $U_e = Id$ on A and $U_e = 0$ on M is a quadratic Jordan superalgebra over Φ. Details on the quadratic product and trilinear product can be found in (King, 2001). Note that e is an idempotent, and J has Peirce decomposition $J = J_2 \oplus J_1 \oplus J_0$ with respect to e, where $J_2 = \Phi e = A$, $J_1 = \Phi x \oplus \Phi y = M$, and $J_0 = 0$.

Theorem 4. *$L\Gamma\left(K_3^{(\lambda)}\right) = \Phi Id$.*

Proof. First, e is an idempotent that satisfies the hypotheses of Lemma 6, so K_3 has no odd centroid. Let $T_0 \in L\Gamma_0(K_3)$. Since e is a reduced idempotent and $x, y \in J_1(e)$, by Lemma 5 (2), $T_0 = \alpha Id$. $\square$

5.2 The Twisted Quaternion Superalgebra $\mathbf{D}_4^{(\lambda,\mu)}(\mathbf{\Phi})$

Let Φ be a ring of scalars with $\lambda, \mu \in \Phi$ and $\mu \neq 0$. Let $A = \Phi e_1 \oplus \Phi e_2$ be the free module over Φ with basis $\{e_1, e_2\}$, and let $M = \Phi x \oplus \Phi y$ be the free module over Φ with basis $\{x, y\}$. Then $J = D_4^{(\lambda,\mu)}(\Phi) = A \oplus M$ with bilinear product given by

$$e_i^2 = e_i, \qquad \langle e_i, e_i \rangle = 2e_i, \qquad \langle e_1, e_2 \rangle = 0,$$
$$\langle e_i, x \rangle = x, \quad \langle e_i, y \rangle = y, \quad \langle x, y \rangle = \lambda e_1 + \mu e_2 \qquad \langle x, x \rangle = 0 \qquad \langle y, y \rangle = 0$$

and quadratic product with

$$U_{e_i}(e_j) = \delta_{ij} e_i, \qquad U_{e_i} = 0 \text{ on } M,$$
$$U_{e_1,e_2} = Id \text{ on } M, \quad U_{e_1,e_2} = 0 \text{ on } A$$

is a quadratic Jordan superalgebra over Φ. Details on the quadratic product and trilinear product can be found in (King, 2001). Notice that as a linear space, J is isomorphic to the split quaternion superalgebra $M_2(\Phi)^+$ graded by e_{ii} via $x \to \lambda e_{12}$, $y \to e_{21}$, but the odd product $\langle x, y \rangle$ is a twisted $\lambda e_{11} + \mu e_{22}$ imbedding of $[\lambda e_{12}, e_{21}] = \lambda e_{11} - \lambda e_{22}$.

Theorem 5. $L\Gamma\left(D_4^{(\lambda,\mu)}(\Phi)\right) = \Phi Id.$

Proof. Note that e_1 is an idempotent with Peirce decomposition $J_2 = \Phi e_1$, $J_1 = M$, and $J_0 = \Phi e_2$. Now e_1 satisfies the conditions of Lemma 6, so $L\Gamma_1(J) = 0$. Let $T_0 \in L\Gamma_0(J)$. Since e_1 is reduced, $T_0(e_1) = \alpha e_1$ for some $\alpha \in \Phi$. Let $T_0' = T_0 - \alpha Id \in L\Gamma_0(J)$. Then $T_0'(e_1) = 0$, and by Lemma 5 (3), $T_0'(J_1) = 0$. Now by Lemma 5 (4), since $Z_2(e_1) = Z_0(e_1) = 0$, $T_0' = 0$ on $J_2(e_1)$ and $J_0(e_1)$. Hence, $T_0'(J) = 0$, and $T_0 = \alpha Id$ on J. $\square$

5.3 The Kac Superalgebra $\mathbf{K_{10}}$

Let Φ be a ring of scalars, let $A = \Phi e \oplus V \oplus \Phi f$ be the free module over Φ, where V is free with basis $B_0 = \{v_1, v_2, v_3, v_4,\}$, and let $M = \Phi x_1 \oplus \Phi y_1 \oplus \Phi x_2 \oplus \Phi y_2$ be the free module over Φ with basis $\{x_1, y_1, x_1, y_2\}$. Then $J = K_{10} = A \oplus M$ with bilinear product given by

$$e^2 = e, \ f^2 = f, \qquad \langle e, e \rangle = 2e, \qquad \langle f, f \rangle = 2f$$
$$\langle e, f \rangle = 0, \qquad \langle e, v_i \rangle = 2v_i \qquad \langle f, v_i \rangle = 0$$
$$\langle e, m \rangle = m = \langle f, m \rangle, \ m \in M$$

$$\langle v_1, y_1 \rangle = y_2, \quad \langle v_1, x_2 \rangle = x_1, \quad \langle v_2, x_1 \rangle = x_2, \quad \langle v_2, y_2 \rangle = y_1,$$
$$\langle v_3, y_1 \rangle = -x_2, \quad \langle v_3, y_2 \rangle = x_1, \quad \langle v_4, x_1 \rangle = y_2, \quad \langle v_4, x_2 \rangle = -y_1,$$

$$\langle x_i, y_i\rangle = e - 3f, \quad \langle x_1, x_2\rangle = -2v_3, \quad \langle x_1, y_2\rangle = 2v_1,$$
$$\langle y_1, x_2\rangle = -2v_2, \quad \langle y_1, y_2\rangle = -2v_4$$

and all other products of basis elements $\langle u, w\rangle = 0$ is a quadratic Jordan superalgebra.

The quadratic and trilinear products on J are rather awkward to describe, and thus we will limit our discussion to the cases needed in the following arguments. See (McCrimmon, 2006) for full details on these products. In particular,

$$\langle x_1, y_1, e\rangle = e, \qquad \langle x_1, y_1, v_1\rangle = 2v_1,$$
$$\langle x_1, y_1, v_4\rangle = 0, \qquad \text{and} \qquad \langle x_1, y_1, v_3\rangle = 2v_3.$$

Theorem 6. *$L\Gamma(K_{10}) = \Phi Id$.*

Proof. First, note that the element e is an idempotent, and the Peirce decomposition with respect to e for $J = K_{10}$ is given by $J_2 = \Phi e \oplus \Phi v_1 \oplus \Phi v_2 \oplus \Phi v_3 \oplus \Phi v_4$, $J_1 = M$, and $J_0 = \Phi f$. Now e satisfies the conditions of Lemma 6, so $L\Gamma_1(J) = 0$.

Let $T_0 \in L\Gamma_0(J)$. Let $T_0(e) = \alpha e + \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 + \beta_4 v_4$. Now, from the trilinear products described above, we have

$$\begin{aligned} T_0(e) &= T_0(\langle x_1, y_1, e\rangle) = \langle x_1, y_1, T_0(e)\rangle \\ &= \langle x_1, y_1, \alpha e + \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 + \beta_4 v_4\rangle \\ &= \alpha e + \sum_{i=1}^{4} \beta_i \langle x_1, y_1, v_i\rangle = \alpha e + 2\beta_1 v_1 + 2\beta_3 v_3. \end{aligned}$$

Thus, $\alpha e + \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 + \beta_4 v_4 = \alpha e + 2\beta_1 v_1 + 2\beta_3 v_3$, and then $0 = \beta_1 v_1 - \beta_2 v_2 + \beta_3 v_3 - \beta_4 v_4$. Since the v_i's are independent, we have $\beta_i = 0$ for all i, whence $T_0(e) = \alpha e$.

Let $T_0' = T_0 - \alpha Id \in L\Gamma_0(J)$. Then $T_0'(e) = 0$, and by Lemma 5 (3), $T_0'(J_1(e)) = 0$. By Lemma 5 (4), $T_0' = 0$ on $J_2(e)$ and $J_0(e)$, since $Z_2(e) = \{a_2 \in J_2(e) : \langle a_2, M\rangle = 0\}$ and $Z_0(e) = \{\gamma f : \langle \alpha f, M\rangle = 0\}$ are both zero (see (McCrimmon, 2006) for products). This gives us that $T_0' = 0$ on J, whence $T_0 = \alpha Id$. □

6 The Matrix Superalgebras

6.1 The Rectangular Matrix Superalgebra $\mathbf{M_{n,m}(D)}$, $(\mathbf{n, m \geq 1})$

Let D be a unital associative algebra (not necessarily commutative) over its (associative) centroid $\Gamma(D) = \Phi$. For $n, m \geq 1$, the matrix algebra $B = M_{n,m}(D) := M_{n+m}(D)$ can be viewed as an associative superalgebra with even part consisting of the diagonal matrices $B_0 = M_n(D) \oplus M_m(D)$

and odd part consisting of the off-diagonal matrices $B_1 = \overline{M_{n\times m}(D)} \oplus \overline{\overline{M_{m\times n}(D)}}$. We have $1 = E_{11} + E_{00}$, where $E_{11} = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$ and $E_{00} = \begin{pmatrix} 0 & 0 \\ 0 & I_m \end{pmatrix}$. Then $B_0 = U_{E_{11}}(B) \oplus U_{E_{00}}(B) = B_{11} \oplus B_{00}$, and $B_1 = E_{11}BE_{00} \oplus E_{00}BE_{11} = B_{10} \oplus B_{01}$.

View $M_{n,m}(D)$ as an algebra over $\Phi = \Gamma(D)$. Note that Φ acts faithfully on both A and M. Indeed, if $\gamma e_{ij} = 0$ or $\gamma \overline{e_{ij}} = 0$ for a single matrix unit e_{ij}, then $\gamma = 0$.

Theorem 7. *1. If B is a semiprime unital associative algebra over $\Phi = \Gamma(B)$, and $e \neq 0, 1$ is an idempotent of B, then $\Gamma(B_e^+) = \Phi Id$.*

2. If D is semiprime over $\Phi = \Gamma(D)$, then $L\Gamma(M_{n,m}(D)) = \Phi Id$.

Proof. First, note that (2) is a special case of (1) since D semiprime implies $B = M_{n,m}(D)$ is semiprime. To prove (1), note that e is an idempotent that satisfies the conditions of Lemma 6, so $J = B_e^+$ has $L\Gamma_1(J) = 0$.

Next, we will show that the even elements of the centroid are just the Φ-multiplications. Let $T_0 \in L\Gamma_0(J)$. The Peirce decompositions relative to $e = e_1$ and $e_0 = 1 - e$ are $J_2(e_1) = B_{11} = J_0(e_0)$, $J_0(e_1) = B_{00} = J_2(e_0)$, and $J_1(e_1) = B_{10} \oplus B_{01} = J_1(e_0)$. Also, $A = B_{11} \oplus B_{00}$, and $M = B_{10} \oplus B_{01}$. By Lemma 5, T_0 can be written as $T_0 = T_{11} \oplus T_{00}$ for $T_{ii} \in End_\Phi(B_{ii})$, and by Lemma 7, $T_{ii} \in L\Gamma(B_{ii}^+)$.

Lemma 10. *If B is a semiprime over its centroid Φ, then $T_{ii} = \alpha_i Id$ for some $\alpha_i \in \Phi$ $(i = 1, 0)$.*

Proof. If B is semiprime, then the $B_{ii} = e_i B e_i$ are semiprime as associative algebras. By (McCrimmon, 1999) (Corollary 3.4), since B is a unital semiprime associative algebra, $L\Gamma(B_{ii}^+) = L\Gamma(B_{ii}) = L_\Phi$, so $T_{ii} = \alpha_i E_{ii}$ for some $\alpha_i \in \Phi$. □

Now $M \subseteq J_1(e_i)$, so Lemma 5 implies that $T_0 = \alpha_1 Id = \alpha_0 Id$ on M. Since Φ acts faithfully on M, $\alpha_1 = \alpha_0$, which gives us that $T_0 = \alpha Id$ on both A and M. □

6.2 The Square Matrix Superalgebra $\mathbf{Q_n(D)}$, $(\mathbf{n \geq 2})$

Let D be a unital associative algebra (not necessarily commutative) over its centroid $\Gamma(D) = \Phi$. For $n \geq 2$, let

$$Q_n(D) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in M_n(D) \right\} \cong M_n(D) \otimes \Omega_0,$$

where $\Omega_0 = \Phi 1_0 \oplus \Phi\overline{\omega_1}$ and $\overline{\omega_1}^2 = 1_0$ for $1_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\omega_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $Q_n(D)$ is a subsuperalgebra of $M_{n,n}(D)$ with even part

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in M_n(D) \right\} \cong M_n(D)^+ \cong M_n(D)^+ \otimes \Phi 1_0$$

and odd part

$$M = \left\{ \overline{b} = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} = b\overline{\omega_1} : b \in M_n(D) \right\} \cong M_n(D)^- \cong M_n(D)^+ \otimes \overline{\omega_1}.$$

We will identify both A and M with $M_n(D)$. Then A is spanned by $\{\alpha e_{ij} : 1 \leq i,j \leq n,\ \ \alpha \in D\}$ and M is spanned by $\{\overline{\beta e_{ij}} : 1 \leq i,j \leq n,\ \ \beta \in D\}$.

The quadratic product on $Q_n(D)$ is given by

$$U_a(b) = aba \qquad\qquad U_a(\overline{b}) = \overline{aba},$$

the bilinear product on $Q_n(D)$ is given by

$$\langle a,b\rangle = a \circ b = ab + ba, \qquad \langle a,\overline{b}\rangle = \overline{a \circ b} = \overline{ab+ba}, \qquad \langle \overline{a},\overline{b}\rangle = [a,b] = ab - ba,$$

and the trilinear product is given by

$$\begin{aligned}
\langle a,b,c\rangle &= abc + cba \\
\langle a,b,\overline{c}\rangle &= \langle a,\overline{b},c\rangle = \langle \overline{a},b,c\rangle = \overline{abc + cba} \\
\langle a,\overline{b},\overline{c}\rangle &= \langle \overline{a},b,\overline{c}\rangle = \langle \overline{a},\overline{b},c\rangle = abc - cba \\
\langle \overline{a},\overline{b},\overline{c}\rangle &= \overline{abc - cba}.
\end{aligned}$$

View $Q_n(D)$ as an algebra over $\Phi = \Gamma(D)$. As in $M_{n,m}(D)$, Φ acts faithfully on both A and M, and if $\gamma e_{ij} = 0$ or $\gamma\overline{e_{ij}} = 0$ for a single matrix unit e_{ij}, then $\gamma = 0$.

Theorem 8. *If $\Phi = \Gamma(D)$ is semiprime (e.g., if D is semiprime), then for $n \geq 2$, $\Gamma(Q_n(D)) = \Phi Id$.*

Proof. First, we will show that there is no odd centroid for $J = Q_n(D)$. Let $T_1 \in \Gamma_1(\mathcal{J})$. Then $M_2(e_{ii}) = \overline{De_{ii}}$, $A_2(e_{ii}) = De_{ii}$, $T_1(e_{ii}) = T_1(U_{e_{ii}}(e_{ii})) = U_{e_{ii}}(T_1(e_{ii})) = \overline{\gamma_{ii}e_{ii}}$ for some $\gamma_{ii} \in D$, and $T_1(\overline{e_{ii}}) = U_{e_{ii}}T_1(\overline{e_{ii}}) = \delta_{ii}$ for some $\delta_{ii} \in D$. Now for $i \neq j$ and any $\alpha \in D$,

$$\alpha e_{ij} = \langle \alpha e_{ij}, e_{jj}\rangle = \langle e_{ii}, \alpha e_{ij}\rangle = \langle \overline{e_{ii}}, \alpha\overline{e_{ij}}\rangle$$

implies

$$\begin{aligned}
T_1(\alpha e_{ij}) &= \langle \alpha e_{ij}, \gamma_{jj}\overline{e_{jj}}\rangle = \langle \gamma_{ii}\overline{e_{ii}}, \alpha e_{ij}\rangle = \langle \delta_{ii}e_{ii}, \alpha\overline{e_{ij}}\rangle \\
&= \alpha\gamma_{jj}\overline{e_{ij}} = \gamma_{ii}\alpha\overline{e_{ij}} = \delta_{ii}\alpha\overline{e_{ij}}.
\end{aligned}$$

Thus, $\alpha\gamma_{jj} = \gamma_{ii}\alpha = \delta_{ii}\alpha$, and in particular, if $\alpha = 1$, then $\gamma_{jj} = \gamma_{ii} = \delta_{ii}$, and all have a common value $\gamma = \gamma_{jj} = \gamma_{ii} = \delta_{ii}$. Now $\alpha\gamma = \gamma\alpha$ for all $\alpha \in D$ implies that $\gamma \in \Gamma(D)$.

By $(Q\Gamma 3)$, $0 = U_{T_1(\overline{e_{ii}})}(e_{ii}) = U_{\gamma e_{ii}}(e_{ii}) = \gamma^2 e_{ii}$, so $\gamma^2 = 0$. Since $\gamma \in \Phi = \Gamma(D)$ and Φ is semiprime, $\gamma = 0$. Now $T_1(\alpha_{ij}e_{ij}) = T_1(\langle \alpha_{ij}e_{ij}, e_{jj}\rangle) = 0$, and $T_1(\alpha_{ij}\overline{e_{ij}}) = T_1(\langle \alpha_{ij}e_{ij}, \overline{e_{jj}}\rangle) = 0$, so $T_1 = 0$. This shows that $T_1(J) = 0$.

Let $T_0 \in \Gamma_0(\mathcal{J})$. Recall that $A \cong M_n(D)^+$. By Lemma 7, $T_0|_A \in \Gamma(M_n(D)^+)$. Then by (McCrimmon, 1999) (Corollary 3.4), since D is semiprime over Φ, $T_0|_A = \sigma Id$ for some $\sigma \in \Phi$. Let $T_0' = T_0 - \sigma Id \in \Gamma_0(\mathcal{J})$, so $T_0'(A) = 0$. Since M is spanned by all $\alpha\overline{e_{ij}} = \langle e_{ii}, \alpha\overline{e_{ij}}\rangle$ and $\alpha\overline{e_{ii}} = U_{e_{ii}}(\langle e_{ij}, \alpha e_{ji}\rangle)$, Lemma 5 (2) implies that $T_0' = 0$, whence $T_0 = \sigma Id$. □

Remark 1. *If we do not demand that $T_1^2 = 0$ or $U_{T_1(\bar{b})} = 0$ for $T_1 \in \Gamma_1(\mathcal{J})$, then $T_1 = L_{\gamma\omega_1}$ with $2\gamma = 0$ would be an odd element of the centroid.*

6.3 The Orthogonal Matrix Superalgebra $\mathbf{P_n(D, D_0)}$, $\mathbf{n \geq 2}$

Let D be a unital associative ring with involution $*$ and D_0 an ample subspace. Then $\star : M_n(D) \to M_n(D)$ given by $a^\star = (a^*)^t$ is an involution. For $n \geq 2$, let $P_n(D, D_0)$ denote the following subsuperalgebra of $M_{n,n}(D)$:

$$P_n(D, D_0) = \left\{ \begin{pmatrix} a & s \\ h & a^\star \end{pmatrix} : a \in M_n(D),\ s \in S_n(D) \text{ is alternating},\ h \in H_n(D, D_0) \right\}.$$

Here, s alternating means $s = sk(u) = u - u^\star$ for some $u \in M_n(D)$. Now $S_n(D)$ is spanned by all $\overline{\alpha s_{ij}} := \alpha e_{ij} - \alpha^\star e_{ji}$ for $\alpha \in D$ and $i \neq j$ and all $\overline{\alpha s_{ii}} := (\alpha - \alpha^\star)e_{ii}$ for $\alpha \in D$, and $H_n(D, D_0)$ is spanned by all $\overline{\overline{\alpha h_{ij}}} := \alpha e_{ij} + \alpha^\star e_{ji}$ for $\alpha \in D$ and $i \neq j$ and all $\overline{\overline{\gamma h_{ii}}} := \gamma e_{ii}$ for $\gamma \in D_0$.

Define $\sharp : M_{n,n}(D) \to M_{n,n}(D)$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\sharp = \begin{pmatrix} d^\star & -b^\star \\ c^\star & a^\star \end{pmatrix}.$$

Proposition 1. *$\sharp$ is a superinvolution on $M_{n,n}(D)$.*

Proof. Clearly, $B^{\sharp\sharp} = B$ for all $B \in M_{n,n}(D)$, so $\sharp$ is of period 2. It remains to show that $(B_iC_j)^\sharp = (-1)^{ij}C_j^\sharp B_i^\sharp$ for all $B_i \in M_{n,n}(D)_i$ and $C_j \in M_{n,n}(D)_j$. First,

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right]^\sharp = \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix}^\sharp$$
$$= \begin{pmatrix} b'^\star c^\star + d'^\star d^\star & -b'^\star a^\star - d'^\star b^\star \\ a'^\star c^\star + c'^\star d^\star & a'^\star a^\star + c'^\star b^\star \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}^{\sharp}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\sharp} = \begin{pmatrix} d'^{\star} & -b'^{\star} \\ c'^{\star} & a'^{\star} \end{pmatrix}\begin{pmatrix} d^{\star} & -b^{\star} \\ c^{\star} & a^{\star} \end{pmatrix} = \begin{pmatrix} d'^{\star}d^{\star} - b'^{\star}c^{\star} & -d'^{\star}b^{\star} - b'^{\star}a^{\star} \\ c'^{\star}d^{\star} + a'^{\star}c^{\star} & -c'^{\star}b^{\star} + a'^{\star}a^{\star} \end{pmatrix}.$$

If either element is even, then $b^{\star}$ and $c^{\star}$ or $b'^{\star}$ and $c'^{\star}$ are zero, whence the expressions above are equal. On the other hand, if both elements are odd, then $a^{\star}$, $d^{\star}$, $a'^{\star}$, and $d'^{\star}$ are all zero, and

$$\left[\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\begin{pmatrix} 0 & b' \\ c' & 0 \end{pmatrix}\right]^{\sharp} = -\begin{pmatrix} 0 & b' \\ c' & 0 \end{pmatrix}^{\sharp}\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}^{\sharp} = \begin{pmatrix} -b'^{\star}c^{\star} & 0 \\ 0 & -c'^{\star}b^{\star} \end{pmatrix}.$$

Thus, $\sharp$ is a superinvolution. □

Then $P_n(D, D_0)$ is an ample outer ideal in the superalgebra of hermitian elements with respect to the superinvolution $\sharp$; it is a superalgebra with even part $A = \{(a, a^{\star t}) : a \in M_n(D)\} \cong M_n(D)^{+}$ and odd part $M = \overline{S_n(D)} \oplus \overline{\overline{H_n(D, D_0)}}$. Since $A \cong M_n(D)^{+}$, we will identify $(a, a^{\star t})$ with a. For $a, b, c \in A$, $s, s', s'' \in S_n(D)$, and $h, h', h'' \in H_n(D, D_0)$, the bilinear product is given by

$$\begin{aligned} \langle a, b\rangle &= ab + ba \\ \langle a, \overline{s} \oplus \overline{\overline{h}}\rangle &= \overline{as + sa^{\star}} \oplus \overline{\overline{a^{\star}h + ha}} \\ \langle \overline{s} \oplus \overline{\overline{h}}, \overline{s'} \oplus \overline{\overline{h'}}\rangle &= sh' - s'h. \end{aligned}$$

The triple product on $P_n(D)$ is given by

$$\begin{aligned} \langle a, b, c\rangle &= abc + cba \\ \langle a, b, \overline{s} \oplus \overline{\overline{h}}\rangle &= \overline{abs + sa'^{\star}a^{\star}} \oplus \overline{\overline{a^{\star}a'^{\star}h + hba}} \\ \langle a, \overline{s} \oplus \overline{\overline{h}}, b\rangle &= \overline{asa'^{\star} + bsa^{\star}} \oplus \overline{\overline{a^{\star}hb + a'^{\star}ha}} \\ \langle a, \overline{s} \oplus \overline{\overline{h}}, \overline{s'} \oplus \overline{\overline{h'}}\rangle &= ash' - s'ha \\ \langle \overline{s} \oplus \overline{\overline{h}}, a, \overline{s'} \oplus \overline{\overline{h'}}\rangle &= sah' - s'a^{\star}h \\ \langle \overline{s} \oplus \overline{\overline{h}}, \overline{s'} \oplus \overline{\overline{h'}}, \overline{s''} \oplus \overline{\overline{h''}}\rangle &= \overline{sh's'' - s''h's} \oplus \overline{\overline{hs'h'' - h''s'h}}. \end{aligned}$$

Finally, the quadratic product is given by

$$U_a\left(b \oplus (\overline{s} \oplus \overline{\overline{h}})\right) = aba \oplus (\overline{asa^{\star}} \oplus \overline{\overline{a^{\star}ha}}).$$

Note that for any skew s, t, the following are still alternating:

$$\begin{aligned} as + sa^{\star} &= sk(as) \qquad asb^{\star} + bsa^{\star} = sk(asb^{\star}) \\ sht - ths &= sk(sht) \quad\; a(sk(u))a^{\star} = sk(aua^{\star}). \end{aligned}$$

Also, the following remain in D_0 by ampleness:

$$
\begin{aligned}
a^\star b^\star h + hba &= tr(hba), & a^\star ha,\\
a^\star hb + b^\star ha &= tr(a^\star hb), & a^\star h + ha,\\
hsh' - h'sh &= tr(hsh'). &
\end{aligned}
$$

Thus, the products above produce the appropriate homogeneous elements.

Theorem 9. *If $n \geq 2$ and D is semiprime, then $L\Gamma(P_n(D, D_0)) = \Phi Id$ for $\Phi = \Gamma(D)$.*

Proof. Let $J = P_n(D, D_0)$ and $T_1 \in L\Gamma_1(J)$. First, we will show that $T_1(I_n) = 0$.

Lemma 11. $T_1(I_n) = 0$.

Proof. Note that, since $T_1(a)$ is an odd element for any $a \in A$, there exist functions $S : A \to S_n(D)$ and $H : A \to H_n(D, D_0)$ such that $T_1(a) = \overline{S(a)} \oplus \overline{\overline{H(a)}}$ for all $a \in A$. Let $S(e_{ii}) = (\sigma_{kl})$, and let $H(e_{ii}) = (\alpha_{kl})$ with $\alpha_{ii} \in D_0$.

$$
\begin{aligned}
\text{Then } T_1(e_{ii}) &= T_1(U_{e_{ii}}(e_{ii})) = U_{e_{ii}}(T_1(e_{ii})) = U_{e_{ii}}\left(\overline{S(e_{ii})} \oplus \overline{\overline{H(e_{ii})}}\right)\\
&= \overline{\sigma_{ii}e_{ii}} \oplus \overline{\overline{\alpha_{ii}e_{ii}}}.
\end{aligned}
$$

Since $n \geq 2$, there exists $j \neq i$. Then

$$
\begin{aligned}
0 &= T_1(\langle e_{jj}, e_{ji}\rangle) - T_1(\langle e_{ji}, e_{ii}\rangle) = \langle T_1(e_{jj}), e_{ji}\rangle - \langle e_{ji}, T(e_{ii})\rangle\\
&= \langle \overline{\sigma_{jj}e_{jj}} \oplus \overline{\overline{\alpha_{jj}e_{jj}}}, e_{ji}\rangle - \langle e_{ji}, \overline{\sigma_{ii}e_{ii}} \oplus \overline{\overline{\alpha_{ii}e_{ii}}}\rangle\\
&= \overline{\sigma_{jj}(e_{ji}e_{jj} + e_{jj}e_{ij})} \oplus \overline{\overline{\alpha_{jj}(e_{ij}e_{jj} + e_{jj}e_{ji})}} - \overline{\sigma_{ii}(e_{ji}e_{ii} + e_{ii}e_{ij})} \oplus \overline{\overline{\alpha_{ii}(e_{ij}e_{ii} + e_{ii}e_{ji})}}\\
&= \overline{\overline{\alpha_{jj}(e_{ij} + e_{ji})}} - \overline{\sigma_{ii}(e_{ji} + e_{ij})}.
\end{aligned}
$$

Hence, $\overline{\sigma_{ii}(e_j i + e_{ij})} = \overline{\overline{0}, \overline{\alpha_{jj}(e_{ij} + e_{ji})}}$, so $\sigma_{ii} = 0 = \alpha_{jj}$ for all i, j, and $T_1(e_{ii}) = T_1(e_{ji}) = 0$, so $T_1(I_n) = 0$. □

Lemma 12. $T_1(M) = 0$.

Proof. We will show that for any $\overline{m} = \overline{s_n} \oplus \overline{\overline{h_n}}$, $T_1(\overline{m}) = 0$. Let $a \in A$ and $\overline{m} \in M$, and let $T_1(\overline{m}) = (\tau_{ij})$ (recall that this is really $(\tau_{ij}) \oplus (\tau^*_{ji})$). Since $T_1(I_n) = 0$,

$$0 = \langle a, T_1(I_n), \overline{m}\rangle = \langle a, I_n, T_1(\overline{m})\rangle = aT_1(\overline{m}) + T_1(\overline{m})a.$$

This says that for any $a \in A$, $-aT_1(\overline{m}) = T_1(\overline{m})a$. Then for $i \neq j$, $\tau_{ij}e_{ij} = e_{ii}T_1(\overline{m})e_{jj} = -T_1(\overline{m})e_{ii}e_{jj} = 0$. Thus, $\tau_{ij} = 0$ for $i \neq j$, whence

$$T_1(\overline{m}) = diag(\tau_{11}, \tau_{22}, \ldots \tau_{nn})$$

is a diagonal matrix.

The above argument says that for any odd $\overline{m} \in M$, $T_1(\overline{m})$ is a diagonal matrix. Then for $i \neq j$ and $\overline{m}$ as above, $\langle \overline{m}, e_{ij} \rangle$ is odd, so $T_1(\langle \overline{m}, e_{ij} \rangle)$ must be diagonal. However,

$$T_1(\langle \overline{m}, e_{ij} \rangle) \quad = \quad \langle T_1(\overline{m}), e_{ij\rangle} \langle diag(\tau_{11}, \tau_{22}, \ldots \tau_{nn}), e_{ij} \rangle = (\tau_{ii} + \tau_{jj}) e_{ij},$$

and these are *not* all diagonal unless $\tau_{ii} = -\tau_{jj}$ for all $i \neq j$.

For $\alpha \in D_0$, $\overline{\alpha h_{ii}} \in H_n(D, D_0)$. Let $T_1(\overline{\alpha h_{ii}}) = diag(\rho_1, \rho_2, \ldots \rho_n)$. Note that $U_{e_{ii}}(\overline{\alpha h_{ii}}) = \overline{\alpha h_{ii}}$, so by Lemma 5, $T_1(\overline{\alpha h_{ii}}) = \rho_i e_{ii}$, which gives us that $\rho_j = 0$ for $j \neq i$. However, we have already shown above that $\rho_i = -\rho_j$ for $i \neq j$ and $n > 1$, whence $\rho_i = 0$ and $T_1(\overline{\alpha h_{ii}}) = 0$. Also note that $U_{e_{ii}}(\overline{\beta s_{ii}}) = \overline{\beta s_{ii}}$, so a similar argument yields that $T_1(\overline{\beta s_{ii}}) = 0$.

Now for $\overline{m} = \overline{s_n} \oplus \overline{\overline{h_n}}$ as above, let $s_n = (\beta_{kl})$ and $h_n = (\alpha_{kl})$. Then

$$\begin{aligned} \tau_{ii} e_{ii} &= U_{e_{ii}} T_1(\overline{m}) = T_1(U_{e_{ii}}(\overline{m})) \\ &= T_1(U_{e_{ii}}((\overline{s_n}, \overline{h_n}))) = T_1(\overline{\beta_{ii} s_{ii}}) + T_1(\overline{\alpha_{ii} h_{ii}}) = 0. \end{aligned}$$

Thus, $0 = \tau_{ii} = -\tau_{jj}$ for each i, and then $T_1(\overline{m}) = diag(\tau_{11}, \tau_{22}, \ldots, \tau_{nn}) = 0$. □

Lemma 13. $A \subset \mathcal{M}(\overline{\overline{h_{11}}})$.

Proof. Recall that $\overline{\overline{h_{ii}}} = e_{ii} \in H_n(D, D_0)$ and $\overline{s_i j} = e_{ij} - e_{ji} \in S_n(D)$. We will show that the spanning set for A is contained in $\mathcal{M}(\overline{\overline{h_{11}}})$. First, for $j \neq 1$, $\langle \overline{\overline{h_{11}}}, \overline{\alpha s_{1j}} \rangle = \alpha e_{j1}$, so $\alpha e_{j1} \in \mathcal{M}(\overline{\overline{h_{11}}})$. Now $\langle \alpha e_{j1}, e_{1j} \rangle = \alpha e_{11} + \alpha e_{jj}$, and applying $U_{e_{11}}$ and $U_{e_{jj}}$, we get αe_{11} and αe_{jj}, respectively, in $M(\overline{\overline{h_{11}}})$. Then $\langle \alpha e_{11}, e_{1j} \rangle = \alpha e_{1j} \in \mathcal{M}(\overline{\overline{h_{11}}})$, and if $j \neq k$ and $j, k \neq 1$, by the above, $\langle \alpha e_{j1}, e_{1k} \rangle = \alpha e_{jk} \in \mathcal{M}(\overline{\overline{h_{11}}})$. Thus, the spanning set for A is contained in $\mathcal{M}(\overline{\overline{h_{11}}})$, whence $A \subset \mathcal{M}(\overline{\overline{h_{11}}})$. □

Since $T_1(\overline{\overline{h_{11}}}) = 0$, Lemma 4 gives us that $T_1(A) = 0$ as well. Hence, in view of Lemma 12, $T_1(J) = 0$.

We now know that $L\Gamma_1(J) = 0$, so consider $T_0 \in L\Gamma_0(J)$. Note that $T_0|_A \in L\Gamma(M_n(D)^+) = L\Gamma(M_n(D)) = \{L_{\alpha I_n} : \alpha \in \Phi\} = \Phi Id$ by Lemma 10. Therefore, there exists $\alpha \in \Gamma(D)$ such that $T_0|_A = \alpha Id$. Let $T_0' = T_0 - \alpha Id$. Then $T_0' \in L\Gamma_0(J)$ has $T_0'|_A = 0$.

We will show that $T_0' = 0$ on the spanning set for M. Note that $e_{ii} \in A$ is an idempotent such that for $i \neq j$, $\{\overline{\beta s_{ij}} : \beta \in D\} \cup \{\overline{\overline{\gamma h_{ij}}} : \gamma \in D\} \subset J_1(e_{ii})$. Then by Lemma 5, since $T_0'(e_{ii}) = 0$, $T_0'(\overline{\beta s_{ij}}) = 0 = T_0'(\overline{\overline{\gamma h_{ij}}})$ for all $\beta, \gamma \in D$ and $i \neq j$. It remains to show that $T_0'(\overline{\beta s_{ii}}) = 0 = T_0'(\overline{\overline{\gamma h_{ii}}})$ for $\beta \in D$ and $\gamma \in D_0$.

Since the kernel of T_0' is an outer ideal of J which contains A and $\overline{Ds_{ij}}$, $\overline{\overline{Dh_{ij}}}$ for $i \neq j$, it also contains $\langle e_{ji}, \overline{\beta s_{ij}} \rangle = \langle e_{ji}, \overline{\beta e_{ij} - \beta^* e_{ji}} \rangle = \overline{(\beta - \beta^*) e_{jj}} = \overline{\beta s_{jj}}$.

Note that $U_{e_{ii}}(\overline{\overline{\gamma h_{ii}}}) = \overline{\overline{\gamma h_{ii}}}$. Then

$$\begin{aligned} T_0'(\overline{\overline{\gamma h_{ii}}}) &= T_0'(U_{e_{ii}}(\overline{\overline{\gamma h_{ii}}})) = U_{e_{ii}}(T_0'(\overline{\overline{\gamma h_{ii}}})) = \overline{\beta' s_{ii}} \oplus \overline{\overline{\gamma' h_{ii}}}, \quad \text{and} \\ 0 &= \langle T_0'(e_{ij}), \overline{\overline{\gamma h_{ij}}} \rangle = \langle e_{ij}, T_0'(\overline{\overline{\gamma h_{ij}}}) \rangle = \langle e_{ij}, \overline{(\beta' - \beta'^*) e_{ii}} \oplus \overline{\overline{\gamma' e_{ii}}} \rangle \\ &= \overline{\overline{\gamma'(e_{ji} + e_{ij})}}. \end{aligned}$$

Hence, $\gamma' = 0$. Similarly,

$$\begin{aligned} 0 &= \langle T_0'(e_{ji}), \overline{\overline{\gamma h_{ij}}} \rangle = \langle e_{ji}, T_0'(\overline{\overline{\gamma h_{ij}}}) \rangle = \langle e_{ji}, \overline{(\beta' - \beta'^*) e_{ii}} \oplus \overline{\overline{\gamma' e_{ii}}} \rangle \\ &= \overline{(\beta' - \beta'^*)(e_{ji} + e_{ij})}. \end{aligned}$$

Hence, $\beta' = \beta'^*$. Then $\overline{\beta' s_{ii}} = 0$, whence $T_0'(\overline{\overline{\gamma h_{ii}}}) = 0$ for all i. This gives us that $T_0'|_M = 0$, so $T_0|_M = \alpha Id$ as well. □

6.4 The Orthosymplectic Superalgebra $\mathbf{OSp_{n,2m}}(\boldsymbol{\Phi})$

Let Φ be a (commutative) ring of scalars. For $n, m > 0$, let $* : M_{n,2m}(\Phi) \to M_{n,2m}(\Phi)$ be given by

$$\begin{aligned} \begin{pmatrix} a_n & b_{n\times 2m} \\ c_{2m\times n} & d_{2m} \end{pmatrix}^* &= \begin{pmatrix} I_n & 0 \\ 0 & S_{2m} \end{pmatrix} \begin{pmatrix} a_n^t & -c_{2m\times n}^t \\ b_{n\times 2m}^t & d_{2m}^t \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & S_{2m} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} a_n^t & -c_{2m\times n}^t S_{2m}^{-1} \\ S_{2m} b_{n\times 2m}^t & S_{2m} d_{2m}^t S_{2m}^{-1} \end{pmatrix} \end{aligned}$$

where $S_{2m} = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$.

Lemma 14. *$*$ is a superinvolution.*

Proof. Throughout, $b, b' \in M_{n\times 2m}(\Phi)$, $c, c' \in M_{2m\times n}(\Phi)$, $a, a' \in M_n(\Phi)$, $d, d' \in M_{2m}(\Phi)$, and $S = S_{2m}$. First, $*$ is of period 2, since

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{**} &= \begin{pmatrix} a^t & -c^t S^{-1} \\ S b^t & S d^t S^{-1} \end{pmatrix}^* \\ &= \begin{pmatrix} a & -(S b^t)^t S^{-1} \\ S(-c^t S)^t & S(S d^t S^{-1})^t S^{-1} \end{pmatrix} \\ &= \begin{pmatrix} a & b I_{2m} \\ I_{2m} c & -I_{2m} d(-I_{2m}) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \end{aligned}$$

Now

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right]^* = \begin{pmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb'+dd' \end{pmatrix}^*$$
$$= \begin{pmatrix} a'^t a^t + c'^t b' & -(a'^t c^t + c'^t d^t)S^{-1} \\ S(b'^t a^t + d'^t b^t) & S(b'^t c^t + d'^t d^t)S^{-1} \end{pmatrix},$$

and

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}^* \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} a'^t & -c'^t S^{-1} \\ Sb'^t & Sd'^t S^{-1} \end{pmatrix}\begin{pmatrix} a^t & -c^t S^{-1} \\ Sb^t & Sd^t S^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} a'^t a^t - c'^t S^{-1} S b^t & -a'^t c^t S^{-1} - c'^t S^{-1} S d^t S^{-1} \\ Sb'^t a^t + Sd'^t S^{-1} S b^t & -Sb'^t c^t S^{-1} + Sd'^t S^{-1} S d^t S^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} a'^t a^t - c'^t b^t & -(a'^t c^t + c'^t d^t)S^{-1} \\ S(b'^t a^t + d'^t b^t) & S(-b'^t c^t + d'^t d^t)S^{-1} \end{pmatrix}.$$

If at least one element is even, $b = c = 0$ or $b' = c' = 0$, so the expressions above are equal. If both are odd, then $a = d = 0$ and $a' = d' = 0$, and

$$\left[\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\begin{pmatrix} 0 & b' \\ c' & 0 \end{pmatrix}\right]^* = -\begin{pmatrix} 0 & b' \\ c' & 0 \end{pmatrix}^*\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}^*.$$

Thus, $(x_i y_j)^* = (-1)^{ij} y_j^* x_i^*$, whence $*$ is a superinvolution. □

Let $J = OSp_{n,2m}(\Phi)$ denote the hermitian elements with respect to this superinvolution. Then J is a subsuperalgebra of $M_{n,2m}(\Phi)$ with even part $A \cong H_n(\Phi) \oplus \mathcal{R}$, where $\mathcal{R} = \{d \in M_{2m}(\Phi) : Sd^t S^{-1} = d\}$. Throughout, for $1 \le k, l \le m$, let $k' = k + m$ and $l' = l + m$. Here, $\mathcal{R}$ is spanned by the following elements:

$$r_{kl} = r_{k'l'} = 0 \oplus (e_{kl} + e_{l'k'}), \quad r_{kl'} = 0 \oplus (e_{kl'} - e_{lk'}), \quad r_{k'l} = 0 \oplus (e_{k'l} - e_{l'k}).$$

The odd part is

$$\left\{ \overline{b_{n\times 2m}} = \begin{pmatrix} 0 & b_{n\times 2m} \\ S_{2m} b^t_{n\times 2m} & 0 \end{pmatrix} : b_{n\times 2m} \in M_{n\times 2m}(\Phi) \right\} \cong \overline{M_{n\times 2m}(\Phi)}.$$

The quadratic product on J is given by

$$U_{h\oplus r}(h' \oplus r') = hh'h \oplus rr'r$$
$$U_{h\oplus r}(\bar{b}) = \overline{hbr},$$

the bilinear product is given by

$$\begin{aligned}\langle h \oplus r, h' \oplus r' \rangle &= (hh' + h'h) \oplus (rr' + r'r) \\ \langle \overline{b}, \overline{c} \rangle &= (bSc^t - cSb^t) \oplus (Sb^t c - Sc^t b) \\ \langle h \oplus r, \overline{b} \rangle &= \overline{hb + br},\end{aligned}$$

and the trilinear product is given by

$$\begin{aligned}\langle h \oplus r, h' \oplus r', h'' \oplus r'' \rangle &= (hh'h'' + h''h'h) \oplus (rr'r'' + r''r'r) \\ \langle h \oplus r, h' \oplus r', \overline{b} \rangle &= \overline{hh'b + br'r} \\ \langle h \oplus r, \overline{b}, h' \oplus r' \rangle &= \overline{hbr' + h'br} \\ \langle h \oplus r, \overline{b}, \overline{c} \rangle &= (hbSc^t - cSb^t h) \oplus (rSb^t c - Sc^t br) \\ \langle \overline{b}, h \oplus r, \overline{c} \rangle &= (bhSc^t - crSb^t) \oplus (Sb^t hc - Sc^t hb) \\ \langle \overline{b}, \overline{c}, \overline{d} \rangle &= \overline{bSc^t d - dSc^t b}\end{aligned}$$

Theorem 10. $L\Gamma(OSp_{n,2m}(\Phi)) = \Phi\ Id$.

Proof. Let $J = OSp_{2,2m}(\Phi)$ and $T_1 \in L\Gamma_1(J)$. First, note that as in $M_{n,2m}(\Phi)$, J contains the idempotent $e = I_n \oplus 0$, which satisfies the conditions of Lemma 6, so $T_1 = 0$.

Let $T_0 \in L\Gamma_0(J)$. By Lemma 5, $T_0 : H_n(\Phi) \to H_n(\Phi)$, and by Lemma 7, $T_0|_{H_n(\Phi)} \in L\Gamma(H_n(\Phi))$. By Lemma 9, T_0 is multiplication by some $\alpha \in \Phi$ on $H_n(\Phi)$. Let $T_0' = T_0 - \alpha Id \in L\Gamma_0(J)$. Now $M = J_1(e)$ for $e = I_n \oplus 0$, so $T_0' = 0$ on M by Lemma 5. Finally, note that $J_0 = (R)$, and suppose that $r \in Z_0 = \{r \in \mathcal{R} : \langle r, J_1 \rangle = 0\}$. Then $\langle r, \overline{B} \rangle = \overline{Br} = 0$ for any $B \in M_{n,2m}(\Phi)$. Let $r = \sum_{i,j=1}^{2m} \rho_{ij} E_{ij}$. Then

$$0 = E_{1k} r E_{1l}^t = E_{1k} \left(\sum_{i,j=1}^{2m} \rho_{ij} E_{ij} \right) E_{1l}^t = \rho_{kl} E_{11} = 0,$$

whence $\rho_{kl} = 0$ for all $k, l = 1 \ldots 2m$. Hence, $r = 0$, so $Z_0 = 0$. By Lemma 5, $T_0'(\mathcal{R}) = 0$ as well, whence $T_0 = \alpha Id$ on all of J. □

7 Quadratic Form Superalgebras

Suppose $J = A \oplus M$ is a superspace over a ring of scalars Φ equipped with a quadratic form $Q : A \to \Phi$ and an alternating bilinear form $B_1 : M \times M \to \Phi$. Throughout, assume Φ acts faithfully on J. Let B_0 be the bilinear form associated with Q, given by

$$B_0(x_0, y_0) = Q(x_0 + y_0) - Q(x_0) - Q(y_0).$$

Then the bilinear form $B = B_0 \oplus B_1$, where $B(A, M) = 0$, is a supersymmetric bilinear form on J.

Suppose J has a basepoint $1_0 \in A$ such that $Q(1_0) = 1$. Define an involution $- : J \to J$ by $\overline{y_j} = tr(y_j)1_0 - y_j$, where $tr(y_j) := B(y_j, 1_0)$. Note that since $A \perp M$, $tr(M) = 0$, whence $\overline{y_1} = -y_1$ for all $y_1 \in M$. The superspace $J = J(Q, B_1, 1_0)$ can be turned into a unital quadratic Jordan superalgebra, called the superalgebra of the superform, by defining the following products:

$$\begin{aligned} U_{x_0}(y_i) &= B(x_0, \overline{y_i})x_0 - Q(x_0)\overline{y_j}, \qquad U_{x_0}(y_1) = Q(x_0)y_1 \\ \langle x_i, y_j \rangle &= tr(x_i)y_j + (-1)^{ij} tr(y_j)x_i - B(x_i, y_j)1_0 \\ \langle x_i, y_j, z_k \rangle &= B(x_i, \overline{y_j})z_k + (-1)^{ij+jk+ki} B(z_k, \overline{y_j})x_i - (-1)^{jk} B(x_i, z_k)\overline{y_j}. \end{aligned}$$

Since $A \perp M$ and $tr(M) = 0$, these products often become much simpler, depending on the combination of even and odd elements. In particular,

$$\begin{aligned} \langle x_0, y_0 \rangle &= tr(x_0)y_0 + tr(y_0)x_0 - B_0(x_0, y_0)1_0 \\ \langle x_0, y_1 \rangle &= tr(x_0)y_1 \\ \langle y_1, x_0 \rangle &= tr(x_0)y_1 \\ \langle x_1, y_1 \rangle &= -B_1(x_1, y_1)1_0. \end{aligned}$$

Note that the bilinear product $\langle M, M \rangle$ is alternating.

$$\begin{aligned} \langle x_0, y_0, z_0 \rangle &= B_0(x_0, \overline{y_0})z_0 + B_0(z_0, \overline{y_0})x_0 - B_0(x_0, z_0)\overline{y_0} \\ \langle x_1, y_0, z_0 \rangle &= B_0(z_0, \overline{y_0})x_1 \\ \langle z_0, y_0, x_1 \rangle &= B_0(z_0, \overline{y_0})x_1 \\ \langle x_0, y_1, z_0 \rangle &= B_0(x_0, z_0)y_1 \\ \langle x_1, y_1, z_0 \rangle &= -B_1(x_1, y_1)z_0 \\ \langle z_0, y_1, x_1 \rangle &= B_1(x_1, y_1)z_0 \\ \langle x_1, y_0, z_1 \rangle &= -B_1(x_1, z_1)\overline{y_0} \\ \langle x_1, y_1, z_1 \rangle &= -B_1(x_1, y_1)z_1 + B_1(z_1, y_1)x_1 - B_1(x_1, z_1)y_1. \end{aligned}$$

Again, note that the trilinear products $\langle M, M, J \rangle$, $\langle M, J, M \rangle$, $\langle J, M, M \rangle$, and $\langle M, M, M \rangle$ are alternating in the variables from M.

What is the centroid $\Gamma(J)$? Since we think of the elements of the centroid as "superscalars," it is reasonable to conjecture that they interact like superscalars with the bilinear form B and the quadratic form Q. The following results will show that this is true under mild assumptions about the superalgebra.

Definition 6. *For* $B = B_0 \oplus B_1$*, the* **centroid of** (Q, B)*,* $\Gamma(Q, B)$*, is the set of homogeneous degree-zero transformations* $T = T_A \oplus T_M \in End_\Phi(J)$ *such that the following hold for all* $x, y, z \in J$ *and* $w \in A$*:*

$$\begin{aligned} B(x,y)T(z) &= B(T(x),y)z = B(x,T(y))z \\ Q(T(w))y &= Q(w)T^2(y). \end{aligned}$$

Definition 7. *Let* $\beta : V \times V \to \Phi$ *be a bilinear form and* W *a* Φ*-module. We say* β *is*

(i) **full-valued**[1] *if* $1 \in \Phi$ *is contained in the ideal generated by* $\beta(V, V)$*;*

(ii) **cancellable-valued** *on* W *if* $\beta(u, v)$ *is cancellable on* W *for some* u, v*, so*

$$\beta(u,v)w = 0 \ \textit{ for } w \in W \quad \textit{implies} \quad w=0;$$

(iii) **alternatingly cancellable** *on* W *if* $\bigcup_{v\in V} \beta(V,v)\beta(V,v)$ *is cancellable on* W*, so*

$$\beta(u,v)\beta(u',v)w = 0 \ \textit{ for all } u, e', v \in V \ \textit{ implies} \quad w = 0;$$

(iv) **cancellable** *on* W *if* $\beta(V, V)$ *is cancellable on* W*, so*

$$\beta(u,v)w = 0 \ \textit{ for all } u, v \in V \ \textit{ implies} \quad w = 0.$$

Lemma 15. *For a bilinear form* $\beta : V \times V \to \Phi$*,*

1. *If* β *is (i) full-valued or (ii) cancellable-valued on* W*, then (iii)* β *is alternatingly cancellable on* W*, which implies that (iv)* β *is cancellable on* W*.*

2. *If* W *is a faithful* Φ*-module, so* $\alpha W = 0$ *implies* $\alpha = 0$*, then (iii)* β *is alternatingly cancellable or (iv)* β *is cancellable on* W *implies (v)* β *is alternatingly cancellable or cancellable on* Φ*, which implies (vi)* V *is torsion-free, so* $\alpha V = 0$ *for* $\alpha \in \Phi$ *implies* $\alpha = 0$*.*

Proof. 1. Suppose (i) holds, so $1 = \sum_{i=1}^{n} \beta(u_i, v_i)$ for $u_i, v_i \in V$. For any k,

$$1 = 1^{n(k-1)+1} = \left(\sum_{i=1}^{n} \beta(u_i, v_i) \right)^{n(k-1)+1} \in \sum_{i=1}^{n} \Phi \beta(u_i, v_i)^k$$

by the Pigeonhole Principle, since in each monomial of length $n(k-1)+1$, the n different $\beta(u_i, v_i)$ can't all appear less than $k-1$ times, so one must appear k times. Suppose $w \in V$ has $\beta(V, v)\beta(V, v)w = 0$ for all $v \in V$. Then $\beta(u_i, v_i)^2 w = 0$ for all i, and

$$w = 1w \in \sum_{i=1}^{n} \Phi \beta(u_i, v_i)^2 w = 0.$$

[1] Thanks to Ottmar Loos for suggesting the full-valued case.

Hence, β is alternatingly cancellable on W.

Suppose (ii) holds, so there exist $u, v \in V$ such that $\beta(u, v)$ is cancellable on W. Then $\beta(V, v)\beta(V, v)w = 0$ implies $\beta(u, v)\beta(u, v)w = 0$, which implies $w = 0$ since $\beta(u, v)$ is cancellable. Hence, β is alternatingly cancellable on W.

Suppose (iii) holds, and suppose $w \in V$ has $\beta(V, V)w = 0$. Then
$\beta(V, v)\beta(V, v)w = 0$ for any v, which implies $w = 0$ since β is alternatingly cancellable on W. Hence, β is also cancellable on W.

2. Suppose (iii) or (iv) holds, and let $S(V, V) = \bigcup_{v \in V} \beta(V, v)\beta(V, v)$ or $S(V, V) = \beta(V, V)$, respectively. If $S\alpha = 0$, then $S(\alpha w) = 0$. Thus, αw is killed by all $\beta(V, v)\beta(V, v)$ or $\beta(V, V)$, whence $\alpha W = 0$ by (iii) or (iv) on W. Since W is a faithful Φ-module, $\alpha = 0$, and hence (v) β is alternatingly faithful on Φ.

   Suppose (v) holds, and suppose $\alpha V = 0$. Then $0 = S(V, \alpha V) = S(V, V)\alpha$, so $\alpha = 0$ by (v). Hence, (vi) V is torsion-free.

□

Remark 2. *If B_0 is cancellable on $A = J_0$ or B_1 is cancellable on $M = J_1$, then $\Gamma(Q, B)$ is a commutative ring of scalars by the usual hiding trick:*

$$\begin{aligned} B_i(u_i, v_i)TS(z) &= B_i(T(u_i), v_i)S(z) = B_i(T(u_i), S(v_i))z \\ &= B_i(u_i, S(v_i))T(z) = B_i(u_i, v_i)ST(z), \end{aligned}$$

so $B_i(J_i, J_i)[TS(z) - ST(z)] = 0$, whence $TS = ST$ since B_i is cancellable on J_i.

Inspecting the product rules, it is clear that we can view the superalgebra $J(Q, B_1, 1_0)$ as an algebra over $\Phi' = \Gamma(Q, B)$.

Theorem 11. *If $J = J(Q, B_1, 1_0)$, where B_0 and B_1 are cancellable on J and B_1 is alternatingly cancellable on M, then the centroid of J is just the centroid of the quadratic form (Q, B), so $\Gamma(J(Q, B_1, 1_0)) = \Gamma(Q, B) = \Phi'$.*

Proof. Since B_0 and B_1 are faithful, we know J is an algebra over Φ'. We will show $\Gamma_1(J) = 0$ and $\Gamma_0(J) = \Phi'$.

Lemma 16. *If B_0 is cancellable on A and B_1 is alternatingly cancellable on M, then J has no odd centroid, so $\Gamma_1(J) = 0$.*

Proof. Let $T_1 \in \Gamma_1(J)$. Then for all $v_j \in J_j$, we have

1. $B_1(z_1, y_1)T_1(z_1) = 0$;

2. $B_0(z_0, y_0)T_1(x_1) = B_1(x_1, T_1(\overline{y_0}))z_0$;

3. $B_1(z_1, y_1)T_1(x_0) = -B_0(x_0, T_1(z_1))y_1$;

4. $B_1(z_1, w_1)B_1(z_1, y_1)B_1(x_1, T_1(x_0)) = 0$.

First, (1) follows since $0 = T_1(\langle z_1, y_1, z_1\rangle) = \langle T_1(z_1), y_1, z_1\rangle = B_1(z_1, y_1)T_1(z_1)$. Next, $T_1(\langle z_0, \overline{y_0}, x_1\rangle) = \langle z_0, T_1(\overline{y_0}), x_1$, so (2) holds, and (3) follows from
$T_1(\langle x_0, y_1, z_1\rangle) = -\langle x_0, y_1, T_1(z_1)\rangle$. Applying $B_1(z_1, w_1)B_1(x_1, \cdot)$ to (3), we obtain

$$\begin{aligned}
B_1(z_1, w_1)B_1(z_1, y_1)B_1(x_1, T_1(x_0)) &= B_1(z_1, w_1)B_1(x_1, [B_1(z_1, y_1)T_1(x_0)]) \\
&= B_1(z_1, w_1)B_1(x_1, [-B_0(x_0, T_1(z_1))y_1]) \\
&= -B_1(x_1, y_1)B_1(z_1, w_1)B_0(x_0, T_1(z_1)) \\
&= -B_1(x_1, y_1)B_0(x_0, [B_1(z_1, w_1)T_1(z_1)]) \\
&= 0 \text{ by (1).}
\end{aligned}$$

Since B_1 is alternatingly cancellable on Φ, (4) implies that $B_1(x_1, T_1(x_0)) = 0$ for all x_1, x_0, so by (2), $B_0(z_0, y_0)T_1(x_1) = 0$ for all z_0, y_0, x_1. Now $B_0(A, A)T_1(x_1) = 0$, so $T_1(x_1) = 0$ since B_0 is cancellable on A. Thus, $T_1(M) = 0$, and by (3), $B_1(M, M)T_1(A) = 0$. Since B_1 is cancellable on M, $T_1(A) = 0$. Hence, $T_1 = 0$ on $J = A \oplus M$.

□

Lemma 17. *Suppose B_0 and B_1 are cancellable on J. Then if $T_0 \in \Gamma_0(J)$,*

$$B_i(T_0(x_i), y_i) = B_i(x_i, T_0(y_i)),$$

so T_0 "hops" inside B_i. Additionally, $B_i(T_0(x_i), y_i)z_k = B_i(x_i, y_i)T_0(z_k)$ for $k \neq i$, so T_0 hops out of B_i onto z_{1-i}.

Proof. If $i = j = 0$,

$$\begin{aligned}
B_0(T_0(x_0), y_0)z_1 &= \langle T_0(x_0), z_1, y_0\rangle \\
&= \langle x_0, z_1, T_0(y_0)\rangle = B_0(x_0, T_0(y_0))z_1 \\
&= \langle x_0, T_0(z_1), y_0\rangle = B_0(x_0, y_0)T_0(z_1).
\end{aligned}$$

Then $B_0(T_0(x_0), y_0)z_1 - B_0(x_0, y_0)T_0(z_1) = 0$, so T_0 always hops out of B_0 onto any z_1. We also have $\big[B_0(T_0(x_0), y_0) - B_0(x_0, T_0(y_0))\big]z_1 = 0$ for any $z_1 \in M$. Since B_0 is cancellable on J, Lemma 15 (vi) gives us that $B_0(T_0(x_0), y_0) - B_0(x_0, T_0(y_0)) = 0$.

If $i = j = 1$,

$$\begin{aligned} B_1(T_0(x_1), y_1)z_0 &= \langle z_0, y_1, T_0(x_1)\rangle \\ &= \langle z_0, T_0(y_1), x_1\rangle = B_1(x_1, T_0(y_1))z_0 \\ &= \langle T_0(z_0), y_1, x_1\rangle = B_1(x_1, y_1)T_0(z_0). \end{aligned}$$

Then $B_1(T_0(x_1), y_1)z_0 - B_1(x_1, y_1)T_0(z_0) = 0$, so T_0 hops out of B_1 onto any z_0. Also, $\big[B_1(T_0(x_1), y_1) - B_1(x_1, T_0(y_1))\big]z_0 = 0$ for any $z_0 \in A$. Since B_1 is cancellable on J, Lemma 15 (vi) gives us that $B_1(T_0(x_1), y_1) - B_1(x_1, T_0(y_1)) = 0$. □

Note that Lemma 17 states that the elements of the centroid not only act as superscalars within *products*, but also within the bilinear form.

We would also expect a superscalar to be able to move from inside the bilinear form to outside the bilinear form. In other words, we would like T to "hop" into the bilinear form. Under what conditions is $B(x_i, y_j)T_l(z_k) = B(T_l(x_i), y_j)z_k$? It is clearly true for $l = 1$ since all odd elements of the centroid are identically zero. Thus, we will restrict our study of this concept to the more interesting case of even elements T_0.

First, note that since $A \perp M$, if $i \neq j$, then $B(x_i, y_j)T_0(z_k) = 0 = B(T_0(x_i), y_j)z_k$. By Lemma 17, all that remains is the case when $i = j = k$.

Lemma 18. *If B_0 and B_1 are cancellable on J, then $T_0 \in \Gamma_0(J)$ hops out of B onto like elements; i.e.,*

$$B_i(T_0(x_i), y_i)z_i = B_i(x_i, y_i)T_0(z_i) \text{ for } i = 0, 1.$$

Proof. If $i \neq j$, Lemma 17 gives us that

$$\beta_j T_0(x_i) := B_j(y_j, z_j)T_0(x_i) = B_j(T_0(y_j), z_j)x_i =: \beta'_j x_i.$$

Set $\Delta_i := B_i(x_i, y_i)(\beta'_j) - B_i(T_0(x_i), y_i)z_i \in J_i$. Then

$$\begin{aligned} \beta_j \Delta_i &= B_i(x_i, y_i)(\beta'_j z_i) - B_i(\beta'_j x_i, y_i)z_i \\ &= \beta'_j \left[B_i(x_i, y_i)z_i - B_i(x_i, y_i)z_i\right] = 0. \end{aligned}$$

Thus, $B_j(J_i, J_i)\Delta_i = 0$, and since B_j is cancellable on J_i, we have $\Delta_i = 0$ for $i = 0, 1$, whence $B_i(T_0(x_i), y_i)z_i = B_i(x_i, y_i)T_0(z_i)$. □

Lemma 19. *If B_1 is cancellable on J, $Q(T_0(x_0))y_0 = Q(x_0)T_0^2(y_0)$.*

Proof. Since $T_0 \in \Gamma(J)$, $U_{T_0(x_0)}(\overline{y_0}) = T_0^2(U_{x_0}(\overline{y_0}))$. Expanding these products, we get

$$B_0(T_0(x_0), y_0)T_0(x_0) - Q(T_0(x_0))y_0 = B_0(x_0, y_0)T_0^2(x_0) - Q(x_0)T_0^2(y_0).$$

Then

$$\begin{aligned} Q(T_0(x_0))y_0 - Q(x_0)T_0^2(y_0) &= B_0(T_0(x_0), y_0)T_0(x_0) - B_0(x_0, y_0)T_0(T_0(x_0)) \\ &= B_0(T_0(T_0(x_0)), y_0)x_0 - B_0(T_0(T_0(x_0)), y_0)x_0 \text{ by Lemma 18} \\ &= 0. \end{aligned}$$

Hence, $Q(T_0(x_0))y_0 = Q(x_0)T_0^2(y_0)$. □

Lemma 20. *For any $x_0 \in A$ and any $y_1 \in M$,*

$$Q(T_0(x_0))y_1 = Q(x_0)T_0^2(y_1).$$

Proof. Recall that $U_{T_0(x_0)}(y_1) = T_0^2(U_{x_0}(y_1))$. Thus, $Q(T_0(x_0))y_1 = Q(x_0)T_0^2(y_1)$. □

Hence, we have shown that $\Gamma(J(Q, B_1, 1_0)) = \Gamma(Q, B) = \Phi'$. □

Remark 3. *If B_1 is cancellable on J, the above results state that $T_0 \in \Gamma_0(J)$ is* almost *a scalar. If $\beta = B_1(u_1, v_1)$ is a cancellable scalar, let*

$$\widetilde{\Phi} = \Phi < \beta^{-1} >= \left\{ \frac{\varphi}{\beta^n} : \varphi \in \Phi,\ \ n \in \mathbb{N} \right\},$$

so $\widetilde{\Phi}$ is the usual "localization of Φ at β." Then $B_0(T_0(u_1), v_1)Id = B_1(u_1, v_1)T_0$ by Lemma **??**, *so*

$$T_0 = \beta^{-1}B_1(T_0(u_1), v_1)Id \in \widetilde{\Phi}Id.$$

In other words, $\beta T_0 \in \Phi Id$ for all $T_0 \in \Gamma_0(J)$.

8 Conclusion

This evidence leads to the conclusion that the natural concept of scalars in a Jordan superalgebra should be the usual notion of scalars rather than superscalars. We adopt as the correct notion of centroid for a Jordan superalgebra the set $\Gamma_0(J)$, when $\Gamma_0(J)$ is supercommutative. In this case, the centroid is a commutative, associative ring of scalars, and J is a $\Gamma_0(J)$-superalgebra via Definitions 4 and 5.

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