A NON-ORTHOGONAL CAYLEY-DICKSON DOUBLING

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ABSTRACT. Let R be an integral domain in which 2 is not an invertible element, with quotient field K of characteristic not 2. A construction method for octonion algebras over R is presented for which the resulting algebra does not necessarily contain a composition subalgebra.

INTRODUCTION

Composition algebras over rings have been studied repeatedly over the last few years. Those containing a composition subalgebra of half their rank can be constructed by a Cayley-Dickson doubling process which, in its most general version, is due to Petersson [P]. For rings where 2 is invertible, all composition algebras of rank 8 (called *octonion algebras*) which possess a quadratic étale subalgebra S, can be built with the help of hermitian spaces over S of rank 3 and trivial determinant, as described by Knus, Parimala and Sridharan [KPS] . Moreover, for rings where 2 is an invertible element, there exists a general construction which yields all composition algebras of rank 4 (i.e., all quaternion algebras) [Pu]. So far the question whether it is possible to explicitly construct octonion algebras without composition subalgebras of rank 2 or 4, or even without any subalgebra, seems to have been treated in the literature only in [KPS]. There, examples of octonion division algebras over the polynomial ring $k[x, y]$, k a suitable field of characteristic not 2, are given which are obtained by an elaborate gluing process. These algebras have a norm which is an indecomposable rank 7 quadratic space when it is restriced to the trace zero elements (thus they do not contain any subalgebra other than R itself).

The method presented in this paper yields octonion algebras over rings where 2 is not invertible, which do not necessarily contain any composition subalgebra. Coxeter's order of integral octaves is one of them: it is an

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octonion algebra over $\mathbb{Z}$ which does not contain any composition subalgebra and hence cannot be obtained by one of the known construction methods for octonion algebras.

1. Preliminaries

Let R be a unital commutative associative ring. The term "R-algebra" refers to unital nonassociative algebras which are finitely generated projective as R-modules.

An R-algebra A is called *quadratic* if there exists a quadratic form $n\colon A \to R$ such that $n(1_A) = 1$ and $x^2 - n(1_A, x)x + n(x)1_A = 0$ for all $x \in A$. The form n is uniquely determined and called the *norm* of the quadratic algebra A. An R-algebra C is called a *composition algebra*, if it carries a quadratic form $n\colon C \to R$ satisfying the following two conditions: (i) Its induced symmetric bilinear form $n(x, y)\colon = n(x+y) - n(x) - n(y)$ is nondegenerate, i.e. determines an R-module isomorphism $C \xrightarrow{\sim} C^\vee = \mathrm{Hom}_R(C, R)$. (ii) n permits composition, that is $n(xy) = n(x)n(y)$ for all $x, y \in C$.

An algebra is called *alternative* if its associator $[x, y, z] = (xy)z - x(yz)$ is alternating. Composition algebras are quadratic alternative algebras. A quadratic form n on the composition algebra satisfying (i) and (ii) above agrees with its norm as a quadratic algebra and thus is unique. It is called the *norm* of the composition algebra C and is also denoted by n_C. A quadratic alternative algebra is a composition algebra if and only if its norm is nondegenerate [M, 4.6]. Composition algebras only exist in ranks 1, 2, 4 or 8. Those of rank 2 are exactly the quadratic étale R-algebras, those of rank 4 are called *quaternion algebras*, those of rank 8 *octonion algebras*. A composition algebra C has a *canonical involution* $\sigma = \bar{\ }$ given by $\overline{x} = t(x)1_C - x$, where $t\colon C \to R$ is the *trace* given by $t(x)\colon = n(1_C, x)$. This involution satisfies $\bar{x}x = n(x)1 \in R1$. Moreover, $t(x, y)\colon = t(xy)$ is a nondegenerate symmetric bilinear form on C, also called the *trace form* of C.

Given a composition algebra D over R of rank at most 4 and an element $\mu \in R^\times$, the R-module $D \oplus D$ becomes a composition algebra via the multiplication

$$(u, v)(u', v') = (uu' + \mu\overline{v'}v, v'u + v\overline{u'})$$

for $u, u', v, v' \in D$, with norm $n = n_D(u) - \mu n_D(v)$. This is the *(classical) Cayley-Dickson doubling* of D. The resulting algebra is called $\mathrm{Cay}(D, \mu)$. The iterated Cayley-Dickson doubling $\mathrm{Cay}(\mathrm{Cay}(D, \mu), \nu)$ is also denoted by

$\mathrm{Cay}(D, \mu, \nu)$. The Cayley-Dickson doubling depends on the scalar μ only up to an invertible square.

Over fields, the classical Cayley-Dickson process generates all possible composition algebras. Over rings, a more general version is required, which yields all composition algebras containing a composition subalgebra of half their rank [P].

2. The Construction method

Let R be an integral domain with quotient field $K = \mathrm{Quot}(R)$ of characteristic not 2. Assume that 2 is not an invertible element in R. Let $C = \mathrm{Cay}(K, a, b, c)$ be an octonion algebra over K such that $a, b, c \in R$ with standard basis $1, i, j, k, e$, that is $i^2 = a, j^2 = b, k^2 = -ab, e^2 = c, ij = k, ji = -ij, ej = -je, ek = -ke, ei = -ie, jk = -bi = -kj, ik = aj = -ki$. Define $h = \frac{1}{2}(i + j + k + e)$. The following generalizes [C, 5.1].

1.1 Proposition *The R-submodule $\Lambda(a, b, c)$ in C generated by*

$$1, i, j, k, h, ih, jh, kh$$

is multiplicatively closed, i.e. an R-subalgebra in $C = \mathrm{Cay}(K, a, b, c)$, provided that

$$(a + b - ab + c) \equiv 0 \bmod 4.$$

Proof A straightforward computation yields that

$$\begin{aligned}
&h^2 = \tfrac{1}{4}(a + b - ab + c),\\
&(ih)h = \tfrac{1}{4}i(a + b - ab + c),\\
&(jh)h = \tfrac{1}{4}j(a + b - ab + c),\\
&(kh)h = \tfrac{1}{4}k(a + b - ab + c),\\
&(ih)i = \tfrac{1}{2}(ai - aj - ak - ae) = a(i - h),\\
&(jh)j = \tfrac{1}{2}b(-i + j - k - e) = b(j - h),\\
&(jh)i = kh + aj,\\
&(kh)k = \tfrac{1}{2}ab(i + j - k + e) = ab(h - k),\\
&h(ih) = ah - \tfrac{1}{4}i(a + b - ab + c),\\
&h(jh) = bh - \tfrac{1}{4}j(a + b - ab + c),\\
&h(kh) = -\tfrac{1}{4}(a + b - ab + c)k - abh = -h(hk) - abh,\\
&(ih)^2 = -\tfrac{1}{4}a(a + b - ab + c) + aih,\\
&(jh)^2 = -\tfrac{1}{4}b(a + b - ab + c) + bjh,\\
&(kh)^2 = \tfrac{1}{4}ab(a + b - ab + c) - abkh,\\
&hi = \tfrac{1}{2}(a - k - aj + ei) = a - ih,\\
&hj = \tfrac{1}{2}(k + b + bi + ej) = b - jh,
\end{aligned}$$

$hk = \frac{1}{2}(aj - bi - ab + ek) = -ab - kh,$
$(jh)k = bih - abj,$
$(kh)i = ajh + ak,$
$(ih)j = bi - kh,$
$(kh)j = -bih + bk,$
$(ih)k = -ajh - abi,$
$(jh)(ih) = -\frac{1}{4}(a + b - ab + c)k - abh + bih,$
$(kh)(jh) = \frac{1}{4}b(a + b - ab + c)i - abjh - abh,$
$(ih)(kh) = -\frac{1}{4}a(a + b - ab + c)j - abh + akh,$
$k(jh) = -j(kh),$
$j(kh) = -bih + bei = bih - ab - kb - abj,$
$i(kh) = -k(ih),$
$k(ih) = -ajh + aje = -ajh + 2ajh + ak - ab + abi,$
$j(ih) = -ajh + aje = -i(jh),$
$i(jh) = kh - ke = kh - 2kh + ki + kj - ab = -kh - aj + bi - ab$, and so on. □

Obviously, $e = 2h - i - j - k \in \Lambda(a, b, c)$. If $\Lambda(a, b, c)$ is multiplicatively closed it is a quadratic alternative R-algebra of rank 8 with a scalar involution. This scalar involution is also denoted by σ, since it is given by $\sigma|_{\Lambda(a,b,c)}$. If $\Lambda(a, b, c)$ is multiplicatively closed, $\Lambda(c, b, a)$ is not necessarily multiplicatively closed as well, since this would require that also $(c + b - bc + a) \equiv 0 \bmod 4$, which is apparently not always the case.

1.2 Lemma *With respect to the standard basis of C, an element $u \in \Lambda(a, b, c)$ can be written as*

$$u = b_0 1 + b_1 i + b_2 j + b_3 k + b_4 e + b_5 ie + b_6 je + b_7 ke$$

with $b_i \in R$ or $b_i \in \frac{1}{2}R$. In particular, if a_5 is "odd", this influences b_0, b_2 and b_3, if a_6 is "odd", this influences b_0, b_1 and b_3, etc.

Proof Write

$$u = a_0 1 + a_1 i + a_2 j + a_3 k + a_4 h + a_5 ih + a_6 jh + a_7 kh,$$

then

$$u = b_0 1 + b_1 i + b_2 j + b_3 k + b_4 e + b_5 ie + b_6 je + b_7 ke$$

with $b_0 = \frac{1}{2}(2a_0 + a_5 a + a_6 b - a_7 ab)$, $b_1 = \frac{1}{2}(2a_1 + a_4 - a_6 b + a_7 b)$, $b_2 = \frac{1}{2}(2a_2 + a_4 + a_5 a - a_7 a)$, $b_3 = \frac{1}{2}(2a_3 + a_4 + a_5 - a_6)$, $b_4 = \frac{1}{2}a_4$, $b_5 = \frac{1}{2}a_5$, $b_6 = \frac{1}{2}a_6$, $b_7 = \frac{1}{2}a_7$. □

Since $ih = \frac{1}{2}(a + k + aj + ie)$, $jh = \frac{1}{2}(-k + b - bi + je)$, and

$kh = \frac{1}{2}(-aj + bi - ab + ke)$, $hi = \frac{1}{2}(a - k - aj - ie) = -ih + a$,
we obtain

$\sigma(ih) = \frac{1}{2}(a - k - aj - ie) = -ih + a = hi$,
$\sigma(jh) = \frac{1}{2}(k + b + bi - je) = -jh + b = hj$,
$\sigma(kh) = \frac{1}{2}(aj - bi - ab - ke) = -kh + ab = hk$,
$\sigma(h) = -h$, $\sigma(ie) = ei$, $\sigma(je) = ej$, $\sigma(ke) = ek$, $\sigma(e) = -e$

for the canonical involution σ on C. View $C = \mathrm{Cay}(K, a, b, c)$ as $\mathrm{Cay}(D, c)$ with $D = (a, b)_K$. Define $D_0 = R1 \oplus Ri \oplus Rj \oplus Rk$ then $\Lambda(a, b, c) = D_0 \oplus D_0 h$. Now write $u \in \Lambda(a, b, c)$ as $u = r + qh$ with $r = a_0 1 + a_1 i + a_2 j + a_3 k$, and $q = a_4 1 + a_5 i + a_6 j + a_7 k$, then

$$\sigma(u) = \sigma(r + qh) = \sigma(r) - qh + (a_5 a + a_6 b - a_7 ab)1,$$

or equivalently, using the above identities, $\sigma(u) = \sigma(r) - a_4 h + hq_0$, where $q_0 = a_5 i + a_6 j + a_7 k$ is the pure part of q. Thus the norm of u can be written as

$$n(u) = u\sigma(u) = (r + qh)\sigma(r + qh)$$
$$= n_D(r) + (a_5 a + a_6 b - a_7 ab)u - r(qh) + (qh)\sigma(r) - (qh)^2.$$

The same computation but using $n(u) = \sigma(u)u$ instead implies the identity

$$-r(qh) + (qh)\sigma(r) = -(qh)r + \sigma(r)(qh).$$

If $r, q \in R \oplus Ri \oplus Rj \oplus Rk$, as above, then

(i) $n(r) = n_D(r)$,
(ii) $h\sigma(q) = qh - (a_5 a + a_6 b - a_7 ab)1$,
(iii) $(qh)(h\sigma(q)) = (h\sigma(q))(qh)$,
(iv) $(qh)^2 = \frac{1}{4}(a + b - ab + c)n_D(q) + (a_5 a + a_6 b - a_7 ab)qh$,
(v) $t(u) = t(r + qh) = t_D(r) + (a_5 a + a_6 b - a_7 ab)1$.

We summarize this in the following

1.3 Lemma *Let $u = r + qh \in \Lambda(a, b, c)$.*

(a) The involution on $\Lambda(a, b, c)$ is given by

$$\sigma(u) = \sigma(r + qh) = \sigma(r) - qh + (a_5 a + a_6 b - a_7 ab)1.$$

(b) The norm on $\Lambda(a, b, c)$ is given by

$$n(u) = n_D(r) + (a_5 a + a_6 b - a_7 ab)u - r(qh) + (qh)\sigma(r) - (qh)^2$$
$$= n_D(r) + (a_5 a + a_6 b - a_7 ab)r - r(qh) + (qh)\sigma(r) - \frac{1}{4}(a + b - ab + c)n_D(q).$$

(c) The trace on $\Lambda(a, b, c)$ is given by

$$t(u) = t(r + qh) = t_D(r) + (a_5 a + a_6 b - a_7 ab)1.$$

In particular, $t(u) \in R$.

This implies $n(u) \in R$ for all $a, b, c \in R$ satisfying $a+b-ab+c \equiv 0 \bmod 4$.

1.4 Theorem $\Lambda(a,b,c)$ *is an octonion algebra over* R*, provided that* a, b, c *are invertible elements in* R *satisfying* $a+b-ab+c \equiv 0 \bmod 4$.

Proof Using the identities from above the discriminant of $\Lambda(a,b,c)$ with respect to the trace is the determinant of the matrix

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & a & b & -ab \\ 0 & 2\,a & 0 & 0 & a & 0 & -ab & ab \\ 0 & 0 & 2\,b & 0 & b & ab & 0 & -ab \\ 0 & 0 & 0 & -2\,ab & -ab & -ab & ab & 0 \\ 0 & a & b & -ab & K\,c & 0 & 0 & 0 \\ a & 0 & ab & -ab & 0 & -Ka+a^2 & ab & -a^2b \\ b & -ab & 0 & ab & 0 & ab & -Kb+b^2 & -ab^2 \\ -ab & ab & -ab & 0 & 0 & -a^2b & -ab^2 & Kab+a^2b^2 \end{bmatrix}$$

where $K = (a+b-ab+c)/2$. The determinant can be computed to be $-a^4b^4c^4$. Hence n_C restricted to $\Lambda(a,b,c)$ is nondegenerate and the assertion is proved.

□

For $K = \mathbb{Q}$, $C = \mathrm{Cay}(K,-1,-1,-1)$ is up to isomorphism the only octonion division algebra. The Coxeter lattice $\Lambda(-1,-1,-1)$ is up to isomorphism the only maximal $\mathbb{Z}$-order in C [C]. It is an octonion algebra over $\mathbb{Z}$ which does not contain any composition subalgebras. The construction presented here can be used to find other such algebras over suitable rings R. The questions whether or when the octonion algebra $\Lambda(a,b,c)$ contains composition subalgebras will not be answered in general here. We are only going to exclude some obvious cases for now.

1.5 Lemma *Let* $\Lambda(a,b,c)$ *be an octonion algebra over* R*. Let* $K = (a+b-ab+c)/2 \neq 0$.

(a) If $b-ab+c$ *is invertible in* R *then* $S = R \oplus Rih$ *is a composition subalgebra of* $\Lambda(a,b,c)$.

(b) If $a+b+c$ *is invertible in* R *then* $S = R \oplus Rjh$ *is a composition subalgebra of* $\Lambda(a,b,c)$.

(c) If $b-ab+c$ *is invertible in* R *then* $S = R \oplus Rkh$ *is a composition subalgebra of* $\Lambda(a,b,c)$.

(d) If $-4cK^2+2aK+2acK-a^2$ *is invertible in* R *then* $D = R\oplus Ri\oplus Rh\oplus Rih$ *is a quaternion subalgebra of* $\Lambda(a,b,c)$.
(e) If $-4cK^2-2abK-2abcK+a^2b^2$ *is invertible in* R *then* $D = R1\oplus Rj\oplus Rh\oplus Rjh$ *is a quaternion subalgebra of* $\Lambda(a,b,c)$.
(f) If $-4cK^2+2aK+2acK-a^2$ *is invertible in* R *then* $D = R\oplus Rk\oplus Rh\oplus Rkh$ *is a quaternion subalgebra of* $\Lambda(a,b,c)$.

Proof In all cases considered we easily compute the determinant of the trace restricted to the algebra, which is the determinant of a submatrix of the big one from above to be
(a): $-a(b-ab+c)$ (and resp. a^2 if we assumed that $K=0$),
(b): $-b(a-ab+c)$ (and resp. b^2 if we assumed that $K=0$),
(c): $ab(a+b+c)$ (and resp. a^2b^2 if we assumed that $K=0$),
(d): $a^2(-4cK^2+2aK+2acK-a^2)$ (and resp. $-a^4$ if we assumed that $K=0$),
(e): $b^2(-4cK^2+2bK+2bcK-b^2)$ (and resp. $-b^4$ if we assumed that $K=0$),
(f): $a^2b^2(-4cK^2-2abK-2abcK+a^2b^2)$ (and resp. a^4b^4 if we assumed that $K=0$). □

The algebra $S = R1\oplus Rh$ is never a composition subalgebra of $\Lambda(a,b,c)$, since its determinant is $2Kc$ and thus not invertible in R.

1.6 Remark An analogous construction is of course possible for quaternion algebras. Let $D=(a,b)_K$ be a quaternion algebra over K such that $a,b\in R$, with standard basis $1,i,j,k$, that is $i^2=a, j^2=b, k^2=-ab, ij=k, ji=-ij, jk=-bi=-kj, ik=aj=-ki$. Define $h=\frac{1}{2}(i+j)$. The full R-lattice $\Lambda(a,b)$ in D generated by

$$1, i, h, ih$$

is multiplicatively closed, i.e. an R-order in $(a,b)_K$, provided that

$$(a+b)\equiv 0 \bmod 4.$$

(An easy computation yields $h^2=\frac{1}{4}(a+b)$, $(ih)h=\frac{1}{4}i(a+b)$, $(ih)i=a(i-h)=a(h-j)$, $i(ih)=ah$, $h(ih)=ah-\frac{1}{4}i(a+b)$, $(ih)^2=-\frac{1}{4}a(a+b)+aih$, $hi=\frac{1}{2}(a-k)=a-ih$, $j=2h-i\in\Lambda(a,b)$.) If $\Lambda(a,b)$ is multiplicatively closed it is a quadratic alternative R-algebra of rank 4 with a scalar involution.

With respect to the standard basis of D, again an element $u\in\Lambda(a,b)$ can be written as $u=b_01+b_1i+b_2j+b_3k$ with $b_i\in R$ or $b_i\in\frac{1}{2}R$. (If $u=a_01+a_1i+a_2h+a_3ih$, then $b_0=\frac{1}{2}(2a_0+a_3a)$, $b_1=\frac{1}{2}(2a_1+a_2)$, $b_2=\frac{1}{2}a_2$, $b_3=\frac{1}{2}a_3$.) Since $ih=\frac{1}{2}(a+k)$ and $hi=-ih+a$, we obtain

$\sigma(ih) = -ih + a = hi$ and $\sigma(h) = -h$, for the canonical involution σ on D. View D as $\mathrm{Cay}(S,b)$ where $S = K(i) = K \oplus Ki$ is a quadratic étale algebra over K. Define $S_0 = R1 \oplus Ri$ then $\Lambda(a,b) = S_0 \oplus S_0 h$. Now write $u \in \Lambda(a,b)$ as $u = r + qh$ with $r = a_0 1 + a_1 i \in S_0$, and $q = a_2 1 + a_3 i \in S_0$, then $\sigma(u) = \sigma(r+qh) = \sigma(r) - qh + a_3 a$, or equivalently, using the above identities, $\sigma(u) = \sigma(r) - a_2 h + h a_3 i$. Thus the norm of u can be written as $n(u) = u\sigma(u)$ $= n_S(r) + a_3 a u - rqh + qh\sigma(r) - (qh)^2$. The same computation but using $n(u) = \sigma(u)u$ instead implies the identity $-rqh + qh\sigma(r) = -qhr + \sigma(r)qh$. Furthermore, then $n(r) = n_S(r)$, $h\sigma(q) = qh - a_3 a$, $qh^2\sigma(q) = n_S(q)h^2$, $(qh)^2 = \frac{1}{4}(a+b)n_S(q) + a_3 a q h$, $t(u) = t(r+qh) = t_S(r) + a_3 a$.

Thus the involution is given by $\sigma(u) = \sigma(r+qh) = \sigma(r) - qh + a_3 a$, the norm is given by $n(u) = n_S(r) + a_3 a r - rqh + qh\sigma(r) - \frac{1}{4}(a+b)n_S(q)$, and the trace is given by $t(u) = t(r+qh) = t_S(r) + a_3 a$. In particular, $t(u) \in R$. Again $n(u) \in R$ for all $a, b \in R$ satisfying $a + b \equiv 0 \bmod 4$.

$\Lambda(a,b)$ is an quaternion algebra over R, provided that a, b are invertible elements in R satisfying $a + b \equiv 0 \bmod 4$: Using the identities from above the discriminant of $\Lambda(a,b)$ with respect to the trace is the determinant of the matrix

$$\begin{bmatrix} 2 & 0 & 0 & a \\ 0 & 2a & a & 0 \\ 0 & a & (a+b)/2 & 0 \\ a & 0 & 0 & a(a-b)/2 \end{bmatrix}$$

This determinant is $-a^2b^2$. Hence n_C restricted to $\Lambda(a,b)$ is nondegenerate.

However, a straightforward computation shows that $S = R1 \oplus Rih$ is a quadratic étale subalgebra of $\Lambda(a,b)$, since its determinant with respect to the trace is easily calculated to be $-ab$, and hence is an invertible element in R. Thus $\Lambda(a,b)$ can always be obtained by a Cayley-Dickson doubling out of this algebra S, and our construction does not yield anything new in this context.

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