Nonassociative Algebras*

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One of the earliest surveys on nonassociative algebras is the article by Shirshov [Shi58] which introduced the phrase "rings that are nearly associative". The first book in the English language devoted to a systematic study of nonassociative algebras is Schafer [Sch66]. A comprehensive exposition of the work of the Russian school is Zhevlakov, Slinko, Shestakov and Shirshov [ZSS82]. A collection of open research problems in algebra, including many problems on nonassociative algebra, is the *Dniester Notebook* [FKS93]; the survey article by Kuzmin and Shestakov [KS95] is from the same period. Three books on Jordan algebras which contain substantial material on general nonassociative algebras are Braun and Koecher [BK66], Jacobson [Jac68] and McCrimmon [McC04]. Recent research appears in the Proceedings of the International Conferences on Nonassociative Algebra and its Applications [Gon94], [CGG00], [SSS05]. The present section provides very limited information on Lie algebras, since they are the subject of Section §16.4. The last part (§9) of the present section presents three applications of computational linear algebra to the study of polynomial identities for nonassociative algebras: pseudorandom vectors in a nonassociative algebra, the expansion matrix for a nonassociative operation, and the representation theory of the symmetric group.

1 Introduction

Definitions:

An **algebra** is a vector space A over a field F together with a bilinear **multiplication** $(x, y) \mapsto xy$ from

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$A \times A$ to A; that is, **distributivity** holds for all $a, b \in F$ and all $x, y, z \in A$:

$$(ax + by)z = a(xz) + b(yz), \qquad x(ay + bz) = a(xy) + b(xz).$$

The **dimension** of an algebra A is its dimension as a vector space.

An algebra A is **finite dimensional** if A is a finite dimensional vector space.

The **structure constants** of a finite dimensional algebra A over F with basis $\{x_1, \ldots, x_n\}$ are the scalars $c_{ij}^k \in F$ $(i, j, k = 1, \ldots, n)$ defined by:

$$x_i x_j = \sum_{k=1}^{n} c_{ij}^k x_k.$$

An algebra A is **unital** if there exists an element $1 \in A$ for which

$$1x = x1 = x \quad \text{for all } x \in A.$$

An **involution** of the algebra A is a linear mapping $j\colon A \to A$ satisfying

$$j(j(x)) = x \quad \text{and} \quad j(xy) = j(y)j(x) \quad \text{for all } x, y \in A.$$

An algebra A is a **division algebra** if for every $x, y \in A$ with $x \neq 0$ the equations $xv = y$ and $wx = y$ are solvable in A.

The **associator** in an algebra is the trilinear function

$$(x, y, z) = (xy)z - x(yz).$$

An algebra A is **associative** if the associator vanishes identically:

$$(x, y, z) = 0 \quad \text{for all } x, y, z \in A.$$

An algebra is **nonassociative** if the above identity is not necessarily satisfied.

An algebra A is **alternative** if it satisfies the **right and left alternative identities**

$$(y, x, x) = 0 \quad \text{and} \quad (x, x, y) = 0 \quad \text{for all } x, y \in A.$$

An algebra A is **anticommutative** if it satisfies the identity

$$x^2 = 0 \quad \text{for all } x \in A.$$

(This implies that $xy = -yx$, and the converse holds in characteristic $\neq 2$.)

The **Jacobian** in an anticommutative algebra is defined by

$$J(x, y, z) = (xy)z + (yz)x + (zx)y.$$

A **Lie algebra** is an anticommutative algebra satisfying the **Jacobi identity**

$$J(x, y, z) = 0 \quad \text{for all } x, y, z \in A.$$

A **Malcev algebra** is an anticommutative algebra satisfying the identity

$$J(x, y, xz) = J(x, y, z)x \quad \text{for all } x, y, z \in A.$$

The **commutator** in an algebra A is the bilinear function

$$[x, y] = xy - yx.$$

The **minus algebra** A^- of an algebra A is the algebra with the same underlying vector space as A but with $[x, y]$ as the multiplication.

An algebra A is **commutative** if it satisfies the identity

$$xy = yx \quad \text{for all } x, y \in A.$$

A **Jordan algebra** is a commutative algebra satisfying the **Jordan identity**

$$(x^2, y, x) = 0 \quad \text{for all } x, y \in A.$$

The **Jordan product** (or **anticommutator**) in an algebra A is the bilinear function

$$x * y = xy + yx.$$

(The notation $x \circ y$ is also common.)

The **plus algebra** A^+ of an algebra A over a field F of characteristic $\neq 2$ is the algebra with the same underlying vector space as A but with $x \cdot y = \frac{1}{2}(x * y)$ as the multiplication.

A Jordan algebra is called **special** if it is isomorphic to a subalgebra of A^+ for some associative algebra A; otherwise it is called **exceptional**.

Given two algebras A and B over a field F, a **homomorphism** from A to B is a linear mapping $f\colon A \to B$ which satisfies $f(xy) = f(x)f(y)$ for all $x, y \in A$.

An **isomorphism** is a homomorphism which is a linear isomorphism of vector spaces.

Let A be an algebra. Given two subsets $B, C \subseteq A$ we write BC for the **subspace spanned by the products** yz where $y \in B$, $z \in C$.

A **subalgebra** of A is a subspace B satisfying $BB \subseteq B$.

The **subalgebra generated by a set** $S \subseteq A$ is the smallest subalgebra of A containing S.

A (two-sided) **ideal** of an algebra A is a subalgebra B satisfying $AB + BA \subseteq B$.

Given two algebras A and B over the field F, the **(external) direct sum** of A and B is the vector space direct sum $A \oplus B$ with the multiplication

$$(w, x)(y, z) = (wy, xz) \quad \text{for all } w, y \in A \text{ and all } x, z \in B.$$

Given an algebra A with two ideals B and C, we say that A is the **(internal) direct sum** of B and C if $A = B \oplus C$ (direct sum of subspaces).

Facts ([Shi58], [Sch66], [ZSS82], [KS95]):

1. Every finite dimensional associative algebra over a field F is isomorphic to a subalgebra of a matrix algebra $F^{n\times n}$ for some n.
2. The algebra A^- is always anticommutative. If A is associative then A^- is a Lie algebra.
3. (Poincaré-Birkhoff-Witt Theorem or PBW Theorem) Every Lie algebra is isomorphic to a subalgebra of A^- for some associative algebra A.
4. The algebra A^+ is always commutative. If A is associative then A^+ is a Jordan algebra. (See Example 2 in §9.) If A is alternative then A^+ is a Jordan algebra.
5. The analogue of the PBW theorem for Jordan algebras is false: not every Jordan algebra is special. (See Example 4 below.)
6. Every associative algebra is alternative.
7. (Artin's Theorem) An algebra is alternative if and only if every subalgebra generated by two elements is associative.
8. Every Lie algebra is a Malcev algebra.
9. Every Malcev algebra generated by two elements is a Lie algebra.
10. If A is an alternative algebra then A^- is a Malcev algebra. (See Example 3 in §9.)
11. In an external direct sum of algebras, the summands are ideals.

Examples:

1. Associativity is satisfied when the elements of the algebra are mappings of a set into itself with the composition of mappings taken as multiplication. Such is the multiplication in the algebra $\operatorname{End} V$, the algebra of linear operators on the vector space V. One can show that every associative algebra is isomorphic to a subalgebra of the algebra $\operatorname{End} V$, for some V. Thus, the condition of associativity of multiplication characterizes the **algebras of linear operators**. (Note that $\operatorname{End} V$ is also denoted $L(V, V)$ elsewhere in this book, but $\operatorname{End} V$ is the standard notation in the study of algebras.)

2. **Cayley-Dickson doubling process.** Let A be a unital algebra over F with an involution $x \mapsto \overline{x}$ satisfying

$$x + \overline{x},\ x\,\overline{x} \in F \text{ for all } x \in A. \tag{1}$$

Let $a \in F$, $a \neq 0$. The algebra (A, a) is defined as follows: the underlying vector space is $A \oplus A$, addition and scalar multiplication are defined by the vector space formulas

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2), \qquad c(x_1, x_2) = (cx_1, cx_2) \text{ for all } c \in F, \tag{2}$$

and multiplication is defined by the formula

$$(x_1, x_2)(y_1, y_2) = (\, x_1 y_1 + a y_2 \overline{x_2},\ \overline{x_1} y_2 + y_1 x_2 \,). \tag{3}$$

This algebra has an involution defined by

$$\overline{(x_1, x_2)} = (\overline{x_1}, -x_2). \tag{4}$$

In particular, starting with a field F of characteristic $\neq 2$, we obtain the following examples:

(a) The algebra $\mathbb{C}(a) = (F, a)$ is commutative and associative. If the polynomial $x^2 + a$ is irreducible over F then $\mathbb{C}(a)$ is a field; otherwise $\mathbb{C}(a) \cong F \oplus F$ (algebra direct sum).

(b) The algebra $\mathbb{H}(a, b) = (\mathbb{C}(a), b)$ is an algebra of **generalized quaternions**, which is associative but not commutative.

(c) The algebra $\mathbb{O}(a, b, c) = (\mathbb{H}(a, b), c)$ is an algebra of **generalized octonions** or a **Cayley-Dickson algebra**, which is alternative but not associative. (See Example 1 in §9.)

In fact, the algebras of generalized quaternions and octonions may also be defined over a field of characteristic 2 (see [Sch66], [ZSS82]).

3. **Real division algebras** [EHH91, Part B]: In the previous example, taking F to be the field $\mathbb{R}$ of real numbers and $a = b = c = -1$, we obtain the field $\mathbb{C}$ of **complex numbers**, the associative division algebra $\mathbb{H}$ of **quaternions**, and the alternative division algebra $\mathbb{O}$ of **octonions** (also known as the **Cayley numbers**). Real division algebras exist only in dimensions 1, 2, 4 and 8, but there are many other examples: the algebras $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ with the multiplication $x \cdot y = \overline{x}\,\overline{y}$ are still division algebras but they are not alternative and they are not unital.

4. **The Albert algebra.** Let $\mathbb{O}$ be the octonions and let $M_3(\mathbb{O})$ be the algebra of 3×3 matrices over $\mathbb{O}$ with involution induced by the involution of $\mathbb{O}$, that is $(a_{ij}) \mapsto (\overline{a_{ji}})$. The subalgebra $H_3(\mathbb{O})$ of Hermitian matrices in $M_3(\mathbb{O})^+$ is an exceptional Jordan algebra, the Albert algebra: there is no associative algebra A such that $H_3(\mathbb{O})$ is isomorphic to a subalgebra of A^+.

2 General properties

Definitions:

Given an algebra A and an ideal I, the **quotient algebra** A/I is the quotient space A/I with multiplication defined by $(x+I)(y+I) = xy + I$ for all $x, y \in A$.

The algebra A is **simple** if $AA \neq \{0\}$ and A has no ideals apart from $\{0\}$ and A.

The algebra A is **semisimple** if it is the direct sum of simple algebras. (The definition of semisimple that is used in the theory of Lie algebras is different; see Section §16.4.)

Set $A^1 = A^{(1)} = A$, and then by induction define

$$A^{n+1} = \sum_{i+j=n+1} A^i A^j \quad \text{and} \quad A^{(n+1)} = A^{(n)} A^{(n)} \quad \text{for } n \geq 1.$$

The algebra A is **nilpotent** if $A^n = \{0\}$ for some n and **solvable** if $A^{(s)} = \{0\}$ for some s. The smallest natural number n (respectively s) with this property is the **nilpotency index** (respectively **solvability index**) of A.

An element $x \in A$ is **nilpotent** if the subalgebra it generates is nilpotent.

A **nil algebra** (respectively **nil ideal**) is an algebra (respectively ideal) in which every element is nilpotent.

An algebra is **power associative** if every element generates an associative subalgebra.

An **idempotent** is an element $e \neq 0$ of an algebra A satisfying $e^2 = e$. Two idempotents e, f are **orthogonal** if $ef = fe = 0$.

For an algebra A over a field F, the **degree** of A is defined to be the maximal number of mutually orthogonal idempotents in the scalar extension $\overline{F} \otimes_F A$ where $\overline{F}$ is the algebraic closure of F.

The **associator ideal** $D(A)$ of the algebra A is the ideal generated by all the associators. The **associative center** or **nucleus** $N(A)$ of A is defined by

$$N(A) = \{\, x \in A \,|\, (x, A, A) = (A, x, A) = (A, A, x) = \{0\} \,\}.$$

The **center** $Z(A)$ of the algebra A is defined by

$$Z(A) = \{\, x \in N(A) \,|\, [x, A] = \{0\} \,\}.$$

The **right and left multiplication operators** by an element $x \in A$ are defined by

$$R_x \colon y \mapsto yx, \qquad L_x \colon y \mapsto xy.$$

The **multiplication algebra** of the algebra A is the subalgebra $M(A)$ of the associative algebra $\operatorname{End} A$ (of endomorphisms of the vector space A) generated by all R_x and L_x for $x \in A$.

The **right multiplication algebra** of the algebra A is the subalgebra $R(A)$ of $\operatorname{End} A$ generated by all R_x for $x \in A$.

The **centroid** $C(A)$ of the algebra A is the centralizer of the multiplication algebra $M(A)$ in the algebra $\operatorname{End} A$; that is,

$$C(A) = \{\, T \in \operatorname{End} A \,|\, TR_x = R_xT = R_{Tx},\ TL_x = L_xT = L_{Tx},\ \text{for any } x \in A \,\}.$$

An algebra A over a field F is **central** if $C(A) = F$.

The **unital hull** $A^\sharp$ of an algebra A over a field F is defined as follows: If A is unital, then $A^\sharp = A$; and when A has no unit, we set $A^\sharp = A \oplus F$ (vector space direct sum) and define multiplication by assuming that A is a subalgebra and the unit of F is the unit of A.

Let $\mathcal{M}$ be a class of algebras closed under homomorphic images. A subclass $\mathcal{R}$ of $\mathcal{M}$ is said to be **radical** if

1. $\mathcal{R}$ is closed under homomorphic images,
2. for each $A \in \mathcal{M}$ there is an ideal $\mathcal{R}(A)$ of A such that $\mathcal{R}(A) \in \mathcal{R}$ and $\mathcal{R}(A)$ contains every ideal of A contained in $\mathcal{R}$,
3. $\mathcal{R}(A/\mathcal{R}(A)) = \{0\}$.

In this case we call the ideal $\mathcal{R}(A)$ the **$\mathcal{R}$-radical** of A. The algebra A is said to be **$\mathcal{R}$-semisimple** if $\mathcal{R}(A) = \{0\}$.

If the subclass Nil of nil-algebras is radical in the class $\mathcal{M}$, then the corresponding ideal $\operatorname{Nil} A$, for $A \in \mathcal{M}$, is called the **nil radical** of A. In this case, the algebra A is called **nil-semisimple** if $\operatorname{Nil} A = \{0\}$. By definition, $\operatorname{Nil} A$ contains all two-sided nil-ideals of A, and the quotient algebra $A/\operatorname{Nil} A$ is nil-semisimple; that is, $\operatorname{Nil}(A/\operatorname{Nil} A) = \{0\}$.

If the subclass $Nilp$ of nilpotent algebras (or the subclass $Solv$ of solvable algebras) is radical in the class $\mathcal{M}$, then the corresponding ideal $\operatorname{Nilp} A$ (respectively $\operatorname{Solv} A$), for $A \in \mathcal{M}$, is called the **nilpotent radical** (respectively the **solvable radical**) of A.

For an algebra A over F, an A-**bimodule** is a vector space M over F with bilinear mappings

$$A \times M \to M,\ (x,m) \mapsto xm \quad \text{and} \quad M \times A \to M,\ (m,x) \mapsto mx.$$

The **split null extension** $E(A,M)$ of A by M is the algebra over F with underlying vector space $A \oplus M$ and multiplication

$$(x+m)(y+n) = xy + (xn + my) \quad \text{for all } x,y \in A,\ m,n \in M.$$

For an algebra A, the **regular bimodule** $\operatorname{Reg}(A)$ is the underlying vector space of A considered as an A-bimodule, interpreting mx and xm as multiplication in A.

If M is an A-bimodule, then the mappings

$$\rho(x)\colon m \mapsto mx, \qquad \lambda(x)\colon m \mapsto xm,$$

are linear operators on M, and the mappings

$$x \mapsto \rho(x), \qquad x \mapsto \lambda(x),$$

are linear mappings from A to the algebra $\operatorname{End}_F M$. The pair (λ, ρ) is called the **birepresentation** of A associated with the bimodule M.

The notions of **sub-bimodule, homomorphism of bimodules**, **irreducible bimodule** and **faithful birepresentation** are defined in the natural way. The sub-bimodules of a regular A-bimodule are exactly the two-sided ideals of A.

Facts ([Sch66], [Jac68], [ZSS82]):

1. If A is a simple algebra, then $AA = A$.

2. (Isomorphism theorems) (1) If $f\colon A \to B$ is a homomorphism of algebras over the field F, then $A/\ker(F) \cong \operatorname{im}(f) \subseteq B$. (2) If B_1 and B_2 are ideals of the algebra A with $B_2 \subseteq B_1$, then $(A/B_2)/(B_1/B_2) \cong A/B_1$. (3) If S is a subalgebra of A and B is an ideal of A, then $B \cap S$ is an ideal of S and $(B+S)/B \cong S/(B \cap S)$.

3. The algebra A is nilpotent of index n if and only if any product of n elements (with any arrangement of parentheses) equals zero, and if there exists a nonzero product of $n-1$ elements.

4. Every nilpotent algebra is solvable; the converse is not generally true. (See Example 1 below.)

5. In any algebra A, the sum of two solvable ideals is again a solvable ideal. If A is finite-dimensional, then A contains a unique maximal solvable ideal $\operatorname{Solv} A$, and the quotient algebra $A/\operatorname{Solv} A$ does not contain nonzero solvable ideals. In other words, the subclass $Solv$ of solvable algebras is radical in the class of all finite dimensional algebras.

6. An algebra A is associative if and only if $D(A) = \{0\}$, if and only if $N(A) = A$.

7. Every solvable associative algebra is nilpotent.

8. The subclass $Nilp$ of nilpotent algebras is radical in the class of all finite dimensional associative algebras.

9. A finite dimensional associative algebra A is semisimple if and only if $\operatorname{Nilp} A = \{0\}$.

10. The previous two facts imply that every finite dimensional associative algebra A contains a unique maximal nilpotent ideal N such that the quotient algebra A/N is isomorphic to a direct sum of simple algebras.

11. Over an algebraically closed field F, every finite dimensional simple associative algebra is isomorphic to the algebra $F^{n \times n}$ of $n \times n$ matrices over F, for some $n \geq 1$.

12. The subclass $Nilp$ is not radical in the class of finite dimensional Lie algebras. (See Example 1 below.)

13. Over a field of characteristic zero, an algebra is power associative if and only if

$$x^2 x = x x^2 \quad \text{and} \quad (x^2 x)x = x^2 x^2 \quad \text{for all } x.$$

14. Every power associative algebra A contains a unique maximal nil ideal $\mathrm{Nil}\,A$, and the quotient algebra $A/\mathrm{Nil}\,A$ is nil-semisimple; that is, it does not contain nonzero nil ideals. In other words, the subclass Nil of nil-algebras is radical in the class of all power associative algebras.

15. For a finite dimensional alternative or Jordan algebra A we have $\mathrm{Nil}\,A = \mathrm{Solv}\,A$.

16. For finite dimensional commutative power associative algebras, the question of the equality of the nil and solvable radicals is still open, and is known as *Albert's problem*. An equivalent question is: Are there any simple finite dimensional commutative power associative nil algebras?

17. Every nil-semisimple finite dimensional commutative power associative algebra over a field of characteristic $\neq 2, 3, 5$ has a unit element and decomposes into a direct sum of simple algebras. Every such simple algebra is either a Jordan algebra or a certain algebra of degree 2 over a field of positive characteristic.

18. Direct expansion shows that these two identities are valid in every algebra:

$$x(y,z,w) + (x,y,z)w = (xy,z,w) - (x,yz,w) + (x,y,zw),$$
$$[xy,z] - x[y,z] - [x,z]y = (x,y,z) - (x,z,y) + (z,x,y).$$

From these it follows that the associative center and the center are subalgebras, and

$$D(A) = (A,A,A) + (A,A,A)A = (A,A,A) + A(A,A,A).$$

19. If $z \in Z(A)$ then for any $x \in A$ we have

$$R_z R_x = R_x R_z = R_{zx} = R_{xz}.$$

20. If A is unital then its centroid $C(A)$ is isomorphic to its center $Z(A)$. If A is simple then $C(A)$ is a field which contains the base field F.

21. Let A be a finite dimensional algebra with multiplication algebra $M(A)$. Then

(a) A is nilpotent if and only if $M(A)$ is nilpotent.

(b) If A is semisimple then so is $M(A)$.

(c) If A is simple then so is $M(A)$, and $M(A) \cong \operatorname{End}_{C(A)} A$.

22. An algebra A is simple if and only if the bimodule $\operatorname{Reg}(A)$ is irreducible.

23. If A is an alternative algebra (respectively a Jordan algebra) then its unital hull $A^\sharp$ is also alternative (respectively Jordan).

Examples:

1. Let A be algebra with basis x, y, and multiplication given by $x^2 = y^2 = 0$ and $xy = -yx = y$. Then A is a Lie algebra and $A^{(2)} = \{0\}$ but $A^n \neq \{0\}$ for any $n \geq 1$. Thus A is solvable but not nilpotent.

2. Let A be an algebra over a field F with basis x_1, x_2, y, z and the following nonzero products of basis elements:

$$yx_1 = ax_1y = x_2, \qquad zx_2 = ax_2z = x_1,$$

where $0 \neq a \in F$. Then $I_1 = Fx_1 + Fx_2 + Fy$ and $I_2 = Fx_1 + Fx_2 + Fz$ are different maximal nilpotent ideals in A. By choosing $a = 1$ or $a = -1$ we obtain a commutative or anticommutative algebra A.

3. In general, in a nonassociative algebra, a power of an element is not uniquely determined. In the previous example, for the element $w = x_1 + x_2 + y + z$ we have

$$w^2w^2 = 0 \quad \text{but} \quad w(ww^2) = (1 + a)(x_1 + x_2).$$

4. Let $A_1, \ldots, A_n$ be simple algebras over a field F with bases

$$\{v_i^1 \mid i \in I_1\}, \; \ldots, \; \{v_i^n \mid i \in I_n\}.$$

Consider the algebra $A = Fe \oplus A_1 \oplus \cdots \oplus A_n$ (vector space direct sum) with multiplication defined by the following conditions:

(a) the A_i are subalgebras of A;

(b) $A_iA_j = \{0\}$ for $i \neq j$;

(c) $ev_i^j = v_i^je = e$ for all i, j;

(d) $e^2 = e$.

Then $I = Fe$ is the unique minimal ideal in A, and $I^2 = I$. In particular, $\operatorname{Solv} A = \{0\}$, but A does not decompose into an algebra direct sum.

5. **Suttles' example.** (Notices AMS 19 (1972) A-566) Let A be a commutative algebra over a field F of characteristic $\neq 2$, with basis x_i $(1 \leq i \leq 5)$ and the following multiplication table (all other products are zero):

$$x_1x_2 = x_2x_4 = -x_1x_5 = x_3, \qquad x_1x_3 = x_4, \qquad x_2x_3 = x_5.$$

Then A is a solvable power associative nil algebra that is not nilpotent.

3 Composition algebras

Definitions:

A **composition algebra** is an algebra A with unit 1 over a field F of characteristic $\neq 2$ together with a **norm** $n(x)$ (a non-degenerate quadratic form on the vector space A) which **admits composition** in the sense that

$$n(xy) = n(x)n(y) \text{ for all } x \in A.$$

A **quadratic algebra** A over a field F is a unital algebra in which every $x \in A$ satisfies the condition $x^2 \in \mathrm{Span}(x, 1)$. In other words, every subalgebra of A generated by a single element has dimension ≤ 2.

A composition algebra A is **split** if it contains zero-divisors; that is, if $xy = 0$ for some nonzero $x, y \in A$.

Facts ([Sch66], [Jac68], [ZSS82], [Bae02]):

1. Every composition algebra A is alternative and quadratic. Moreover, every element $x \in A$ satisfies the equation

$$x^2 - t(x)x + n(x) = 0,$$

where $t(x)$ is a linear form on A (the **trace**) and $n(x)$ is the original quadratic form on A (the norm).

2. For a composition algebra A the following conditions are equivalent: (1) A is split; (2) $n(x) = 0$ for some nonzero $x \in A$; (3) A contains an idempotent $e \neq 1$.

3. Let A be a unital algebra over a field F with an involution $x \mapsto \overline{x}$ satisfying conditions (1) from Example 2 of Section 1. The Cayley-Dickson doubling process gives the algebra (A, a) defined by equations (2)–(4). It is clear that A is isomorphically embedded into (A, a) and that $\dim(A, a) = 2\dim A$. For $v = (0, 1)$ we have $v^2 = a$ and $(A, a) = A \oplus Av$. For any $y = y_1 + y_2v \in (A, a)$ we have $\overline{y} = \overline{y_1} - y_2v$.

4. In a composition algebra A, the mapping $x \mapsto \overline{x} = t(x) - x$ is an involution of A fixing the elements of the field $F = F1$. Conversely, if A is an alternative algebra with unit 1 and involution $x \mapsto \overline{x}$ satisfying conditions (1) from Example 2 of Section 1, then $x\overline{x} \in F$ and the quadratic form $n(x) = x\overline{x}$ satisfies $n(xy) = n(x)n(y)$.

5. The mapping $y \mapsto \overline{y}$ is an involution of (A,a) extending the involution $x \mapsto \overline{x}$ of A. Moreover, $y+\overline{y}$ and $y\,\overline{y}$ are in F for every $y \in (A,a)$. If the quadratic form $n(x) = x\,\overline{x}$ is non-degenerate on A then the quadratic form $n(y) = y\,\overline{y}$ is non-degenerate on (A,a), and the form $n(y)$ admits composition on (A,a) if and only if A is associative.

6. Every composition algebra over a field F of characteristic $\neq 2$ is isomorphic to F or to one of the algebras of types 2a–2c obtained from F by the Cayley-Dickson process as in Example 2 of Section 1.

7. Every split composition algebra over a field F is isomorphic to one of the algebras $F \oplus F$, $M_2(F)$, $\mathrm{Zorn}(F)$ described in Examples 2 to 4 below.

8. Every finite dimensional composition algebra without zero divisors is a division algebra, and so every composition algebra is either split or a division algebra.

9. Every composition algebra of dimension > 1 over an algebraically closed field is split, and so every composition algebra over an algebraically closed field F is isomorphic to one of the algebras F, $F \oplus F$, $M_2(F)$, $\mathrm{Zorn}(F)$.

Examples:

1. The fields of real numbers $\mathbb{R}$ and complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$, are real composition algebras with the Euclidean norm $n(x) = x\overline{x}$. The first three are associative; the algebra $\mathbb{O}$ provides us with the first and most important example of a nonassociative alternative algebra.

2. Let F be a field and let $A = F \oplus F$ be the direct sum of two copies of the field with the exchange involution $\overline{(a,b)} = (b,a)$, the trace $t((a,b)) = a+b$, and the norm $n((a,b)) = ab$. Then A is a two-dimensional split composition algebra.

3. Let $A = M_2(F)$ be the algebra of 2×2 matrices over F with the symplectic involution:
$$x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \overline{x} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$
the matrix trace $t(x) = a+d$, and the determinant norm $n(x) = ad-bc$. Then A is a 4-dimensional split composition algebra.

4. An 8-dimensional split composition algebra is the **Zorn vector-matrix algebra** (or the **Cayley-Dickson matrix algebra**), obtained by taking $A = \mathrm{Zorn}(F)$, which consists of all 2×2 block matrices with scalars on the diagonal and 3×1 column vectors off the diagonal:
$$\mathrm{Zorn}(F) = \left\{ x = \begin{bmatrix} a & \mathbf{u} \\ \mathbf{v} & b \end{bmatrix} \;\Big|\; a,b \in F,\ \mathbf{u},\mathbf{v} \in F^3 \right\},$$

with norm, involution, and product

$$n(x) = ab - (\mathbf{u}, \mathbf{v}), \qquad \overline{x} = \begin{bmatrix} b & -\mathbf{u} \\ -\mathbf{v} & a \end{bmatrix},$$

$$x_1 x_2 = \begin{bmatrix} a_1 a_2 + (\mathbf{u}_1, \mathbf{v}_2) & a_1 \mathbf{u}_2 + \mathbf{u}_1 b_2 - \mathbf{v}_1 \times \mathbf{v}_2 \\ \mathbf{v}_1 a_2 + b_1 \mathbf{v}_2 + \mathbf{u}_1 \times \mathbf{u}_2 & b_1 b_2 + (\mathbf{v}_1, \mathbf{u}_2) \end{bmatrix};$$

the scalar and vector products are defined for $\mathbf{u} = [u_1, u_2, u_3]^T$ and $\mathbf{v} = [v_1, v_2, v_3]^T$ by

$$(\mathbf{u}, \mathbf{v}) = u_1 v_1 + u_2 v_2 + u_3 v_3, \qquad \mathbf{u} \times \mathbf{v} = [u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1].$$

Application:

1. If we write the equation $n(x)n(y) = n(xy)$ in terms of the coefficients of the algebra elements x, y with respect to an orthogonal basis for each of the composition algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ given in Example 3 of Section 1, then we obtain an identity expressing the multiplicativity of a quadratic form:

$$\left(x_1^2 + \cdots + x_k^2\right)\left(y_1^2 + \cdots + y_k^2\right) = z_1^2 + \cdots + z_k^2.$$

   Here the z_i are bilinear functions in the x_i and y_i: to be precise, z_i is the coefficient of the i-th basis vector in the product of the elements $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$. By Hurwitz' theorem such a *k-square identity* exists only for $k = 1, 2, 4, 8$.

4 Alternative algebras

Definitions:

A **left alternative algebra** is one satisfying the identity $(x, x, y) = 0$.

A **right alternative algebra** is one satisfying the identity $(y, x, x) = 0$.

A **flexible algebra** is one satisfying the identity $(x, y, x) = 0$.

An **alternative algebra** is one satisfying all three identities (any two imply the third).

The **Moufang identities** play an important role in the theory of alternative algebras:

$$(xy \cdot z)y = x(yzy) \qquad \textbf{right Moufang identity}$$

$$(yzy)x = y(z \cdot yx) \qquad \textbf{left Moufang identity}$$

$$(xy)(zx) = x(yz)x \qquad \textbf{central Moufang identity}$$

(The terms yzy and $x(yz)x$ are well-defined by the flexible identity.)

An **alternative bimodule** over an alternative algebra A is an A-bimodule M for which the split null extension $E(A, M)$ is alternative.

Let A be an alternative algebra, let M be an alternative A-bimodule, and let (λ, ρ) be the associated birepresentation of A. The algebra A **acts nilpotently** on M if the subalgebra of $\operatorname{End} M$, which is generated by the elements $\lambda(x)$, $\rho(x)$ for all $x \in A$, is nilpotent. If $y \in A$ then y **acts nilpotently** on M if the elements $\lambda(y)$, $\rho(y)$ generate a nilpotent subalgebra of $\operatorname{End} M$.

A finite dimensional alternative algebra A over a field F is **separable** if the algebra $A_K = K \otimes_F A$ is nil-semisimple for any extension K of the field F.

Facts ([Sch66], [ZSS82]); additional facts about alternative algebras are given in Section 1, and facts about right alternative algebras are given in Section 6:

1. Every commutative or anticommutative algebra is flexible.

2. Substituting $x + z$ for x in the left alternative identity, and using distributivity, we obtain

   $$(x, z, y) + (z, x, y) = 0.$$

   This is the **linearization on** x of the left alternative identity. Linearizing the right alternative identity in the same way, we get

   $$(y, x, z) + (y, z, x) = 0.$$

   From the last two identities, it follows that in any alternative algebra A the associator (x, y, z) is a skew-symmetric (alternating) function of the arguments x, y, z.

3. Every alternative algebra is power associative (Corollary of Artin's Theorem). In particular, the nil radical $\operatorname{Nil} A$ exists in the class of alternative algebras.

4. Every alternative algebra satisfies the three Moufang identities and the identities

   $$(x, y, yz) = (x, y, z)y, \qquad (x, y, zy) = y(x, y, z).$$

5. A bimodule M over an alternative algebra A is alternative if and only if the following relations hold in the split null extension $E(A, M)$:

   $$(x, m, x) = 0 \quad \text{and} \quad (x, m, y) = (m, y, x) = (y, x, m) \quad \text{for all } x, y \in A \text{ and } m \in M.$$

6. It follows from the definition of alternative bimodule and the Moufang identities that

   $$[\rho(x), \lambda(x)] = 0, \qquad \rho(x^k) = (\rho(x))^k \text{ for } k \geq 1, \qquad \lambda(x^k) = (\lambda(x))^k \text{ for } k \geq 1.$$

   This implies that any nilpotent element of an alternative algebra acts nilpotently on any bimodule.

7. If every element of an alternative algebra A acts nilpotently on a finite dimensional alternative A-bimodule M, then A acts nilpotently on M.

8. A nilpotent algebra A acts nilpotently on the A-bimodule M if and only if the algebra $E(A, M)$ is nilpotent.

9. In a finite dimensional alternative algebra A, every nil subalgebra is nilpotent. In particular, the nil radical $\operatorname{Nil} A$ is nilpotent.

10. The subclass $Nilp$ of nilpotent algebras is radical in the class of all finite dimensional alternative algebras. For any finite dimensional alternative algebra A we have

$$\operatorname{Nil} A = \operatorname{Solv} A = \operatorname{Nilp} A.$$

11. Let A be a finite dimensional alternative algebra. The quotient algebra $A/\operatorname{Nil} A$ is semisimple; that is, it decomposes into a direct sum of simple algebras. Every finite dimensional nil-semisimple alternative algebra is isomorphic to a direct sum of simple algebras, where every simple algebra is either a matrix algebra over a skew-field or a Cayley-Dickson algebra over its center.

12. Let A be a finite dimensional alternative algebra over a field F. If the quotient algebra $A/\operatorname{Nil} A$ is separable over F, then there exists a subalgebra B of A such that B is isomorphic to $A/\operatorname{Nil} A$ and $A = B \oplus \operatorname{Nil} A$ (vector space direct sum).

13. Every alternative bimodule over a separable alternative algebra is completely reducible (as in the case of associative algebras).

14. Let A be a finite dimensional alternative algebra, let M be a faithful irreducible A-bimodule, and let (λ, ρ) be the associated birepresentation of A. Either M is an associative bimodule over A (which must then be associative), or one of the following holds:

    (a) The algebra A is an algebra of generalized quaternions, λ is a (right) associative irreducible representation of A, and $\rho(x) = \lambda(\overline{x})$ for every $x \in A$.

    (b) The algebra $A = \mathbb{O}$ is a Cayley-Dickson algebra and M is isomorphic to $\operatorname{Reg}(\mathbb{O})$.

15. Every simple alternative algebra (of any dimension) is either associative or is isomorphic to a Cayley-Dickson algebra over its center.

5 Jordan algebras

In this section we assume that the base field F has characteristic $\neq 2$.

Definitions:

A **Jordan algebra** is a commutative nonassociative algebra satisfying the **Jordan identity**

$$(x^2y)x = x^2(yx).$$

The **linearization on** x of the Jordan identity is

$$2((xz)y)x + (x^2y)z = 2(xz)(xy) + x^2(yz).$$

A Jordan algebra J is **special** if it is isomorphic to a subalgebra of the algebra A^+ for some associative algebra A; otherwise, it is **exceptional**.

Facts ([BK66], [Jac68], [ZSS82], [McC04]); additional facts about Jordan algebras are given in Section 1, and facts about noncommutative Jordan algebras are given in Section 6:

1. (Zelmanov's Simple Theorem) Every simple Jordan algebra (of any dimension) is isomorphic to one of the following: (a) an algebra of a bilinear form, (b) an algebra of Hermitian type, (c) an Albert algebra; for definitions see Examples 3, 4, 5 below.

2. Let J be a Jordan algebra. Consider the regular birepresentation $x \mapsto L_x$, $x \mapsto R_x$ of the algebra J. Commutativity and the Jordan identity imply that for all $x, y \in J$ we have

$$L_x = R_x, \qquad [R_x, R_{x^2}] = 0, \qquad R_{x^2y} - R_y R_{x^2} + 2R_x R_y R_x - 2R_x R_{yx} = 0.$$

   Linearizing the last equation on x we see that for all $x, y, z \in J$ we have

$$R_{(xz)y} - R_y R_{xz} + R_x R_y R_z + R_z R_y R_x - R_x R_{yz} - R_z R_{yx} = 0.$$

3. For every $k \geq 1$, the operator R_{x^k} belongs to the subalgebra $A \subseteq \operatorname{End} J$ generated by R_x and R_{x^2}. Since A is commutative we have $[R_{x^k}, R_{x^\ell}] = 0$ for all $k, \ell \geq 1$, which can be written as $(x^k, J, x^\ell) = \{0\}$.

4. It follows from the previous fact that every Jordan algebra is power associative and the radical $\operatorname{Nil} J$ is defined.

5. Let J be a finite dimensional Jordan algebra. As for alternative algebras, we have

$$\operatorname{Nil} J = \operatorname{Solv} J = \operatorname{Nilp} J;$$

   that is, the radical $\operatorname{Nil} J$ is nilpotent. The quotient algebra $J/\operatorname{Nil} J$ is semisimple; that is, isomorphic to a direct sum of simple algebras. If the quotient algebra $J/\operatorname{Nil} J$ is separable over F, then there exists a subalgebra B of J such that B is isomorphic to $J/\operatorname{Nil} J$ and $J = B \oplus \operatorname{Nil} J$ (vector space direct sum).

6. If a Jordan algebra J contains an idempotent e, the operator R_e satisfies the equation $R_e(2R_e-1)(R_e-1)=0$, and the algebra J has the following analogue of the Pierce decomposition from the theory of associative algebras:

$$J = J_1 \oplus J_{1/2} \oplus J_0 \quad \text{where} \quad J_i = J_i(e) = \{x \in J \,|\, xe = ix\}.$$

For $i,j=0,1$ $(i \neq j)$ we have the inclusions

$$J_i^2 \subseteq J_i, \quad J_iJ_{1/2} \subseteq J_{1/2}, \quad J_iJ_j = \{0\}, \quad J_{1/2}^2 \subseteq J_1 + J_2.$$

More generally, if J has unit $1 = \sum_{i=1}^n e_i$ where e_i are orthogonal idempotents, then

$$J = \bigoplus_{i \leq j} J_{ij} \quad \text{where} \quad J_{ii} = J_1(e_i), \quad J_{ij} = J_{1/2}(e_i) \cap J_{1/2}(e_j) \text{ for } i \neq j,$$

and the components J_{ij} are multiplied according to the rules

$$J_{ii}^2 \subseteq J_{ii}, \quad J_{ij}J_{ii} \subseteq J_{ij}, \quad J_{ij}^2 \subseteq J_{ii} + J_{jj}, \quad J_{ii}J_{jj} = \{0\} \quad \text{for distinct } i,j,$$
$$J_{ij}J_{jk} \subseteq J_{ik}, \quad J_{ij}J_{kk} = \{0\}, \quad J_{ij}J_{k\ell} = \{0\} \quad \text{for distinct } i,j,k,\ell.$$

7. Every Jordan algebra which contains > 3 strongly connected orthogonal idempotents is special. (Orthogonal idempotents e_1, e_2 are **strongly connected** if there exists an element $u_{12} \in J_{12}$ for which $u_{12}^2 = e_1 + e_2$.)

8. (Coordinatization Theorem) Let J be a Jordan algebra with unit $1 = \sum_{i=1}^n e_i$ $(n \geq 3)$ where the e_i are mutually strongly connected orthogonal idempotents. Then J is isomorphic to the Jordan algebra $H_n(D)$ of Hermitian $n \times n$ matrices over an alternative algebra D (which is associative for $n > 3$) with involution $*$ such that $H(D,*) \subseteq N(D)$, where $N(D)$ is the associative center of D.

9. Every Jordan bimodule over a separable Jordan algebra is completely reducible, and the structure of irreducible bimodules is known.

Examples:

1. **The algebra** A^+. If A is an associative algebra, then the algebra A^+ is a Jordan algebra. Every subspace J of A closed with respect to the operation $x \cdot y = \frac{1}{2}(xy+yx)$ is a subalgebra of the algebra A^+ and is therefore a special Jordan algebra. The subalgebra of A generated by J is called the **associative enveloping algebra** of J. Properties of the algebras A and A^+ are closely related: A is simple (respectively nilpotent) if and only if A^+ is simple (respectively nilpotent).

2. The algebra A^+ may be a Jordan algebra for nonassociative A: for instance, if A is a right alternative (in particular, alternative) algebra, then A^+ is a special Jordan algebra.

3. **The algebra of a bilinear form**. Let X be a vector space of dimension > 1 over F, with a symmetric nondegenerate bilinear form $f(x, y)$. Consider the vector space direct sum $J(X, f) = F \oplus X$, and define on it a multiplication by assuming that the unit element $1 \in F$ is the unit element of $J(X, f)$ and by setting $xy = f(x, y)1$ for any $x, y \in X$. Then $J(X, f)$ is a simple special Jordan algebra; its associative enveloping algebra is the Clifford algebra $C(X, f)$ of the bilinear form f. When $F = \mathbb{R}$ and $f(x, y)$ is the ordinary dot product on X, the algebra $J(X, f)$ is called a **spin-factor**.

4. **Algebras of Hermitian type**. Let A be an associative algebra with involution $*$. The subspace $H(A, *) = \{x \in A \,|\, x^* = x\}$ of $*$-symmetric elements is closed with respect to the Jordan multiplication $x \cdot y$ and therefore is a special Jordan algebra. For example, let D be an associative composition algebra with involution $x \mapsto \overline{x}$ and let $D^{n\times n}$ be the algebra of $n \times n$ matrices over D. Then the mapping $S\colon (x_{ij}) \mapsto (\overline{x_{ji}})$ is an involution of $D^{n\times n}$ and the set of D-Hermitian matrices $H_n(D) = H(D^{n\times n}, S)$ is a special Jordan algebra. If A is $*$-simple (if it contains no proper ideal I with $I^* \subseteq I$) then $H(A, *)$ is simple. In particular, all the algebras $H_n(D)$ are simple. Every algebra A^+ is isomorphic to the algebra $H(B, *)$ where $B = A \oplus A^{\rm opp}$ (algebra direct sum) and $(x_1, x_2)^* = (x_2, x_1)$.

5. **Albert algebras**. If $D = \mathbb{O}$ is a Cayley-Dickson algebra, then the algebra $H_n(\mathbb{O})$ of Hermitian matrices over D is a Jordan algebra only for $n \leq 3$. For $n = 1, 2$ the algebras are isomorphic to algebras of bilinear forms and are therefore special. The algebra $H_3(\mathbb{O})$ is exceptional (not special). An algebra J is called an Albert algebra if $K \otimes_F J \cong H_3(\mathbb{O})$ for some extension K of the field F. Every Albert algebra is simple, exceptional, and has dimension 27 over its center.

6 Power associative algebras, noncommutative Jordan algebras, and right alternative algebras

A natural generalization of Jordan algebras is the class of algebras which satisfy the Jordan identity but which are not necessarily commutative. If the algebra has a unit element, then the Jordan identity easily implies the flexible identity. The right alternative algebras have been the most studied among the power associative algebras that do not satisfy the flexible identity.

As in the previous section, we assume that F is a field of characteristic $\neq 2$.

Definitions:

A **noncommutative Jordan algebra** is an algebra satisfying the flexible and Jordan identities. In this definition the Jordan identity may be replaced by any of the identities

$$x^2(xy) = x(x^2y), \qquad (yx)x^2 = (yx^2)x, \qquad (xy)x^2 = (x^2y)x.$$

A subspace V of an algebra A is **right nilpotent** if $V^{\langle n\rangle} = \{0\}$ for some $n \geq 1$, where $V^{\langle 1\rangle} = V$ and $V^{\langle n+1\rangle} = V^{\langle n\rangle}V$.

Facts ([Sch66], [ZSS82], [KS95]):

1. Let A be a finite dimensional power associative algebra with a bilinear symmetric form (x, y) satisfying the following conditions:

   (a) $(xy, z) = (x, yz)$ for all $x, y, z \in A$.

   (b) $(e, e) \neq 0$ for every idempotent $e \in A$.

   (c) $(x, y) = 0$ if the product xy is nilpotent.

   Then $\operatorname{Nil} A = \operatorname{Nil} A^+ = \{x \in A \,|\, (x, A) = \{0\}\}$, and if F has characteristic $\neq 2, 3, 5$ then the quotient algebra $A/\operatorname{Nil} A$ is a noncommutative Jordan algebra.

2. Let A be a finite dimensional nil-semisimple flexible power associative algebra over an infinite field of characteristic $\neq 2, 3$. Then A has a unit element and is a direct sum of simple algebras, each of which is either a noncommutative Jordan algebra or (in the case of positive characteristic) an algebra of degree 2.

3. The structure of arbitrary finite dimensional nil-semisimple power associative algebras is still unclear. In particular, it is not known whether they are semisimple. It is known that in this case new simple algebras arise even in characteristic zero.

4. An algebra A is a noncommutative Jordan algebra if and only if it is flexible and the corresponding plus-algebra A^+ is a Jordan algebra.

5. Let A be a noncommutative Jordan algebra. For $x \in A$, the operators R_x, L_x, L_{x^2} generate a commutative subalgebra in the multiplication algebra $M(A)$, containing all the operators R_{x^k} and L_{x^k} for $k \geq 1$.

6. Every noncommutative Jordan algebra is power associative.

7. Let A be a finite dimensional nil-semisimple noncommutative Jordan algebra over F. Then A has a unit element and is a direct sum of simple algebras. If F has characteristic 0 then every simple summand is either a (commutative) Jordan algebra, a quasi-associative algebra (see Example 3 below), or a quadratic flexible algebra. In the case of positive characteristic, there are more examples of simple noncommutative Jordan algebras.

8. Unlike alternative and Jordan algebras, an analogue of the Wedderburn principal theorem on splitting of the nil radical does not hold in general for noncommutative Jordan algebras.

9. Every quasi-associative algebra (see Example 3 below) is a noncommutative Jordan algebra.

10. Every flexible quadratic algebra is a noncommutative Jordan algebra.

11. The right multiplication operators in every right alternative algebra A satisfy

$$R_{x^2} = R_x^2, \qquad R_{x \cdot y} = R_x \cdot R_y,$$

where $x \cdot y = \frac{1}{2}(xy + yx)$ is the multiplication in the algebra A^+. (Recall that in this section we assume that the characteristic of the base field is $\neq 2$.)

12. If A is a right alternative algebra (respectively noncommutative Jordan algebra) then its unital hull $A^\sharp$ is also right alternative (respectively noncommutative Jordan).

13. If A is a right alternative algebra then the mapping $x \mapsto R_x$ is a homomorphism of the algebra A^+ into the special Jordan algebra $R(A)^+$. If A has a unit element, then this mapping is injective. For every right alternative algebra A, the algebra A^+ is embedded into the algebra $R(A^\sharp)^+$ and hence is a special Jordan algebra.

14. Every right alternative algebra A satisfies the identity

$$R_{x^k} R_{x^\ell} = R_{x^{k+\ell}} \quad \text{for any } x \in A \text{ and } k, \ell \geq 1.$$

Therefore A is power associative and the nil radical $\operatorname{Nil} A$ is defined.

15. Let A be an arbitrary right alternative algebra. Then the quotient algebra $A/\operatorname{Nil} A$ is alternative. In particular, every right alternative algebra without nilpotent elements is alternative.

16. Every simple right alternative algebra that is not a nil algebra is alternative (and hence either associative or a Cayley-Dickson algebra). The non-nil restriction is essential: there exists a non-alternative simple right alternative nil algebra.

17. A finite dimensional right alternative nil algebra is right nilpotent and therefore solvable, but such an algebra can be non-nilpotent. In particular, the subclass $Nilp$ is not radical in the class of finite dimensional right alternative algebras.

Examples:

1. The Suttles algebra (Example 5 in Section 2) is a power associative algebra which is not a noncommutative Jordan algebra. For another example see Example 5 below.

2. The class of noncommutative Jordan algebras contains, apart from Jordan algebras, all alternative algebras (and thus all associative algebras) and all anticommutative algebras.

3. **Quasi-associative algebras**. Let A be an algebra over a field F and let $a \in F$, $a \neq \frac{1}{2}$. Define a new multiplication on A as follows:

$$x \cdot_a y = axy + (1-a)yx,$$

and denote the resulting algebra by A^a. The passage from A to A^a is reversible: $A = (A^a)^b$ for $b = a/(2a-1)$. Properties of A and A^a are closely related: the ideals (respectively subalgebras) of A are those of A^a; the algebra A^a is nilpotent (respectively solvable, simple) if and only if the same holds for A. If A is associative, then A^a is a noncommutative Jordan algebra; furthermore, if the identity $[[x,y],z] = 0$ does not hold in A then A^a is not associative. In particular, if A is a simple noncommutative associative algebra, then A^a is an example of a simple nonassociative noncommutative Jordan algebra. The algebras of the form A^a for an associative algebra A are **split** quasi-associative algebras. More generally, an algebra A is quasi-associative if it has a scalar extension which is a split quasi-associative algebra.

4. **Generalized Cayley-Dickson algebras.** For $a_1, \ldots, a_n \in F - \{0\}$ let

$$A(a_1) = (F, a_1), \quad \ldots, \quad A(a_1, \ldots, a_n) = \big(A(a_1, \ldots, a_{n-1}), a_n\big),$$

be the algebras obtained from F by successive application of the Cayley-Dickson process (Example 2 of Section 1). Then $A(a_1, \ldots, a_n)$ is a central simple quadratic noncommutative Jordan algebra of dimension 2^n.

5. Let V be a vector space of dimension $2n$ over a field F with a nondegenerate skew-symmetric bilinear form (x,y). On the vector space direct sum $A = F \oplus V$ define a multiplication (as for the Jordan algebra of bilinear form) by letting the unit element 1 of F be the unit of A and by setting $xy = (x,y)1$ for any $x, y \in V$. Then A is a simple quadratic algebra (and hence it is power associative) but A is not flexible and thus is not a noncommutative Jordan algebra.

7 Malcev algebras

Some of the theory of Malcev algebras generalizes the theory of Lie algebras. For information about Lie algebras, the reader is advised to consult Section §16.4.

Definitions:

A **Malcev algebra** is an anticommutative algebra satisfying the identity

$$J(x,y,xz) = J(x,y,z)x \quad \text{where} \quad J(x,y,z) = (xy)z + (yz)x + (zx)y.$$

In a **left-normalized product** we omit the parenthesis: for example,

$$xyzx = ((xy)z)x, \qquad yzxx = ((yz)x)x.$$

A **representation** of a Malcev algebra A is a linear mapping $\rho\colon A \to \operatorname{End} V$ satisfying the following identity for all $x, y, z \in A$:

$$\rho(xy \cdot z) = \rho(x)\rho(y)\rho(z) - \rho(z)\rho(x)\rho(y) + \rho(y)\rho(zx) - \rho(yz)\rho(x).$$

We call V a **Malcev module** for A. The anticommutativity of A implies that the notion of a Malcev module is equivalent to that of Malcev bimodule: we set $xv = -vx$ for all $x \in A$, $v \in V$.

The **Killing form** $K(x, y)$ on a Malcev algebra A is defined (as for a Lie algebra) by

$$K(x, y) = \operatorname{trace}(R_x R_y).$$

Facts ([KS95], [She00], [PS04], [SZ05]):

1. After expanding the Jacobians, the Malcev identity takes the form

$$xyzx + yzxx + zxxy = xy \cdot xz$$

(using our convention on left-normalized products). If F has characteristic $\neq 2$ the Malcev identity is equivalent to the more symmetric identity

$$xyzt + yztx + ztxy + txyz = xz \cdot yt.$$

2. Any two elements in a Malcev algebra generate a Lie subalgebra.

3. The structure theory of finite dimensional Malcev algebras repeats the main features of the corresponding theory for Lie algebras. For any alternative algebra A, the minus algebra A^- is a Malcev algebra. Let $\mathbb{O} = \mathbb{O}(a, b, c)$ be a Cayley-Dickson algebra over a field F of characteristic $\neq 2$. Then $\mathbb{O} = F \oplus M$ (vector space direct sum), where $M = \{x \in \mathbb{O} \mid t(x) = 0\}$. The subspace M is a subalgebra (in fact an ideal) of the Malcev algebra $\mathbb{O}^-$, and $M \cong \mathbb{O}^-/F$ (in fact $\mathbb{O}^-$ is the Malcev algebra direct sum of the ideals F and M). The Malcev algebra $M = M(a, b, c)$ is central simple and has dimension 7 over F; if F has characteristic $\neq 3$ then M is not a Lie algebra.

4. Every central simple Malcev algebra of characteristic $\neq 2$ is either a Lie algebra or an algebra $M(a, b, c)$. There are no non-Lie simple Malcev algebras in characteristic 3.

5. Two Malcev algebras $M(a, b, c)$, $M(a', b', c')$ are isomorphic if and only if the corresponding Cayley-Dickson algebras $\mathbb{O}(a, b, c)$, $\mathbb{O}(a', b', c')$ are isomorphic.

6. Let A be a finite dimensional Malcev algebra over a field F of characteristic 0 and let $\operatorname{Solv} A$ be the solvable radical of A. The algebra A is semisimple (it decomposes into a direct sum of simple algebras) if and only if $\operatorname{Solv} A = \{0\}$ (in fact this is often used as the definition of "semisimple" for Malcev algebras, following the terminology for Lie algebras). If the quotient algebra $A/\operatorname{Solv} A$ is separable then A contains a subalgebra $B \cong A/\operatorname{Solv} A$ and $A = B \oplus \operatorname{Solv} A$ (vector space direct sum).

7. The Killing form $K(x,y)$ is symmetric and associative:

$$K(x,y) = K(y,x), \qquad K(xy,z) = K(x,yz).$$

The algebra A is semisimple if and only if the form $K(x,y)$ is nondegenerate. For the solvable radical we have $\operatorname{Solv} A = \{ x \in A \mid K(x,A^2) = \{0\} \}$. In particular, A is solvable if and only if $K(A,A^2) = \{0\}$.

8. If all the operators $\rho(x)$ for $x \in A$ are nilpotent, then they generate a nilpotent subalgebra in $\operatorname{End} V$ (A acts nilpotently on V). If the representation ρ is almost faithful (that is, $\ker \rho$ does not contain nonzero ideals of A), then A is nilpotent.

9. Every representation of a semisimple Malcev algebra A is completely reducible.

10. If A is a Malcev algebra and V is an A-bimodule, then V is a Malcev module for A if and only if the split null extension $E(A,V)$ is a Malcev algebra.

11. Let A be a Malcev algebra, and let V be a faithful irreducible A-module. Then the algebra A is simple, and either V is a Lie module over A (which must then be a Lie algebra) or one of the following holds:

    (a) $A \cong M(a,b,c)$ and V is a regular A-module.

    (b) A is isomorphic to the Lie algebra $sl(2,F)$ with $\dim V = 2$ and $\rho(x) = x^*$ where x^* is the matrix adjoint to $x \in A \subseteq M_2(F)$. (Here the matrix adjoint is defined by the equation $x\,x^* = x^*x = \det(x)I$ where I is the identity matrix.)

12. The **speciality problem** for Malcev algebras is still open: Is every Malcev algebra embeddable into the algebra A^- for some alternative algebra A? This is the generalization of the Poincaré-Birkhoff-Witt theorem for Malcev algebras.

8 Akivis and Sabinin algebras

The theory of Akivis and Sabinin algebras generalizes the theory of Lie algebras and their universal enveloping algebras. For information about Lie algebras, the reader is advised to consult Section §16.4.

Definitions:

An **Akivis algebra** is a vector space A over a field F, together with an anticommutative bilinear operation $A \times A \to A$ denoted $[x,y]$, and a trilinear operation $A \times A \times A \to A$ denoted (x,y,z), satisfying the **Akivis identity** for all $x,y,z \in A$:

$$[[x,y],z] + [[y,z],x] + [[z,x],y] =$$
$$(x,y,z) + (y,z,x) + (z,x,y) - (x,z,y) - (y,x,z) - (z,y,x).$$

A **Sabinin algebra** is a vector space A over a field F, together with multilinear operations

$$\langle x_1, \ldots, x_m; y, z \rangle \qquad (m \geq 0),$$

satisfying these identities:

$$\langle x_1, \ldots, x_m; y, y \rangle = 0, \tag{5}$$

$$\begin{aligned} &\langle x_1, \ldots, x_r, u, v, x_{r+1}, \ldots, x_m; y, z \rangle - \langle x_1, \ldots, x_r, v, u, x_{r+1}, \ldots, x_m; y, z \rangle \\ &\quad + \sum_{k=0}^{r} \sum_{s} \langle x_{s(1)}, \ldots, x_{s(k)}, \langle x_{s(k+1)}, \ldots, x_{s(r)}; u, v \rangle, x_{r+1}, \ldots, x_m; y, z \rangle = 0, \end{aligned} \tag{6}$$

$$K_{u,v,w} \Big[\langle x_1, \ldots, x_r, u; v, w \rangle + \sum_{k=0}^{r} \sum_{s} \langle x_{s(1)}, \ldots, x_{s(k)}; \langle x_{s(k+1)}, \ldots, x_{s(r)}; v, w \rangle, u \rangle \Big] = 0, \tag{7}$$

where s is a $(k, r-k)$-shuffle (a permutation of $1, \ldots, r$ satisfying $s(1) < \cdots < s(k)$ and $s(k+1) < \cdots < s(r)$) and the operator $K_{u,v,w}$ denotes the sum over all cyclic permutations.

An algebra $\mathcal{F}_{\mathcal{M}}[X]$ from a class $\mathcal{M}$, with a set of generators X, is called the **free algebra** in $\mathcal{M}$ with the set X of free generators, if any mapping of X into an arbitrary algebra $A \in \mathcal{M}$ extends uniquely to a homomorphism of $\mathcal{F}_{\mathcal{M}}[X]$ to A.

Let I be any subset of $\mathcal{F}_{\mathcal{M}}[X]$. The T-**ideal** in $\mathcal{F}_{\mathcal{M}}[X]$ determined by I, denoted by $T = T(I, X)$, is the smallest ideal of $\mathcal{F}_{\mathcal{M}}[X]$ containing all elements of the form $f(x_1, \ldots, x_n)$ for all $f \in I$ and all $x_1, \ldots, x_n \in \mathcal{F}_{\mathcal{M}}[X]$.

Facts ([HS90], [She99], [SU02], [GH03], [Per05], [BHP05], [BDE05]):

1. Free algebras may be constructed as follows. Let S be a set of generating elements and let Ω be a set of operation symbols. Let $r\colon \Omega \to \mathbb{N}$ (the nonnegative integers) be the arity function; that is, $\omega \in \Omega$ will represent an n-ary operation for $n = r(\omega)$. The set $W(S, \Omega)$ of nonassociative Ω-words on the set S is defined inductively as follows:

   (a) $S \subseteq W(S, \Omega)$.

   (b) If $\omega \in \Omega$ and $x_1, \ldots, x_n \in W(S, \Omega)$ where $n = r(\omega)$, then $\omega(x_1, \ldots, x_n) \in W(S, \Omega)$.

   Let F be a field and let $F(S, \Omega)$ be the vector space over F with basis $W(S, \Omega)$. For each $\omega \in \Omega$ we define an n-ary operation with $n = r(\omega)$ on $F(S, \Omega)$, denoted by the same symbol ω, as follows: given any basis elements $x_1, \ldots, x_n \in W(S, \Omega)$ we set the value of ω on the arguments $x_1, \ldots, x_n$ equal to the nonassociative word $\omega(x_1, \ldots, x_n)$, and extend linearly to all of $F(S, \Omega)$. The algebra $F(S, \Omega)$ is the free Ω-algebra on the generating set S over the field F with the operations $\omega \in \Omega$.

2. The quotient algebra $F(S, \Omega)/T(I, S, \Omega)$ is the free $\mathcal{M}$-algebra for the class $\mathcal{M} = \mathcal{M}(I)$ of Ω-algebras defined by the set of identities I.

3. Every subalgebra of a free Akivis algebra is again free.

4. Every Akivis algebra is isomorphic to a subalgebra of *Akivis*(A) for some nonassociative algebra A. This generalizes the Poincaré-Birkhoff-Witt theorem for Lie algebras. The free nonassociative algebra is the universal enveloping algebra of the free Akivis algebra. (See Example 1 below.)

5. The free nonassociative algebra with generating set X has a natural structure of a (nonassociative) Hopf algebra, generalizing the Hopf algebra structure on the free associative algebra. The **Akivis elements** (the elements of the subalgebra generated by X using the commutator and associator) are properly contained in the **primitive elements** (the elements satisfying $\Delta(x) = x \otimes 1 + 1 \otimes x$ where Δ is the comultiplication). The Akivis elements and the primitive elements have a natural structure of an Akivis algebra. The primitive elements have the additional structure of a Sabinin algebra.

6. The Witt dimension formula for free Lie algebras (the primitive elements in the free associative algebra) has a generalization to the primitive elements of the free nonassociative algebra.

7. Sabinin algebras are a nonassociative generalization of Lie algebras in the following sense: the tangent space at the identity of any local analytic loop (without associativity assumptions) has a natural structure of a Sabinin algebra, and the classical correspondence between Lie groups and Lie algebras generalizes to this case.

8. Every Sabinin algebra arises as the subalgebra of primitive elements in some nonassociative Hopf algebra.

9. Another (equivalent) way to define Sabinin algebras, which exploits the Hopf algebra structure, is as follows. Let A be a vector space and let $T(A)$ be the tensor algebra of A. We write $\Delta\colon T(A) \to T(A) \otimes T(A)$ for the comultiplication on $T(A)$: the algebra homomorphism which extends the diagonal mapping $\Delta\colon u \mapsto 1 \otimes u + u \otimes 1$ for $u \in A$. We will use the Sweedler notation and write $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$ for any $x \in T(A)$. Then A is a Sabinin algebra if it is equipped with a trilinear mapping

$$T(A) \otimes A \otimes A \to A, \qquad x \otimes y \otimes z \mapsto \langle x; y, z \rangle, \qquad \text{for } x \in T(A) \text{ and } y, z \in A,$$

satisfying the identities

$$\langle x; y, y \rangle = 0, \tag{8}$$

$$\langle x \otimes u \otimes v \otimes x'; y, z \rangle - \langle x \otimes v \otimes u \otimes x'; y, z \rangle + \sum \langle x_{(1)} \otimes \langle x_{(2)}; u, v \rangle \otimes x'; y, z \rangle$$
$$= 0, \tag{9}$$

$$K_{u,v,w} \Big[\langle x \otimes u; v, w \rangle + \sum \langle x_{(1)}; \langle x_{(2)}; v, w \rangle, u \rangle \Big] = 0, \tag{10}$$

where $x, x' \in T(A)$ and $u, v, w, y, z \in A$. Identities (8–10) exploit the Sweedler notation to express identities (5–7) in a more compact form.

Examples:

1. Any nonassociative algebra A becomes an Akivis algebra $\mathit{Akivis}(A)$ if we define $[x, y]$ and (x, y, z) to be the commutator $xy - yx$ and the associator $(xy)z - x(yz)$. If A is an associative algebra then the trilinear operation of $\mathit{Akivis}(A)$ is identically zero; in this case the Akivis identity reduces to the Jacobi identity, and so $\mathit{Akivis}(A)$ is a Lie algebra. If A is an alternative algebra then the alternating property of the associator shows that the right side of the Akivis identity reduces to $6(x, y, z)$.

2. Every Lie algebra is an Akivis algebra with the identically zero trilinear operation. Every Malcev algebra (over a field of characteristic $\neq 2, 3$) is an Akivis algebra with the trilinear operation equal to $\frac{1}{6}J(x, y, z)$.

3. Every Akivis algebra A is a Sabinin algebra if we define

$$\langle a, b \rangle = -[a, b], \qquad \langle x; a, b \rangle = (x, b, a) - (x, a, b), \qquad \langle x_1, \ldots, x_m; a, b \rangle = 0 \ (m > 1),$$

for all $a, b, x, x_i \in A$.

4. Let L be a Lie algebra with a subalgebra $H \subseteq L$ and a subspace $V \subseteq L$ for which $L = H \oplus V$. We write $P_V \colon L \to V$ for the projection onto V with respect to this decomposition of L. We define an operation

$$\langle -, -; - \rangle \colon T(V) \otimes V \otimes V \to V,$$

by (using the Sweedler notation again)

$$\big\{ x \otimes a \otimes b \big\} + \sum \big\{ x_{(1)} \otimes \langle x_{(2)}; a, b \rangle \big\} = 0,$$

where for $x = x_1 \otimes \cdots \otimes x_n \in T(V)$ we write

$$\big\{ x \big\} = P_V\big([x_1, [\ldots, [x_{n-1}, x_n]] \cdots] \big).$$

Then the vector space V together with the operation $\langle -, -; - \rangle$ is a Sabinin algebra, and every Sabinin algebra can be obtained in this way.

9 Computational methods

For homogeneous multilinear polynomial identities of degree n, the number of associative monomials is $n!$ and the number of association types is C_n (the Catalan number); hence the number of nonassociative monomials grows superexponentially:

$$n! \cdot \frac{1}{n} \binom{2n-2}{n-1} = \frac{(2n-2)!}{(n-1)!} > n^{n-1}.$$

One way to reduce the size of the computations is to apply the theory of superalgebras [Vau98]. Another technique is to decompose the space of multilinear identities into irreducible representations of the symmetric group S_n. The application of the representation theory of the symmetric group to the theory of polynomial identities for algebras was initiated in 1950 by Malcev and Specht. The computational implementation of these techniques was pioneered by Hentzel in the 1970s [Hen77]; for detailed discussions of recent applications see [HP97], [BH04]. Another approach has been implemented in the Albert system [Jac03]. In this section we present three small examples ($n \leq 4$) of computational techniques in nonassociative algebra.

Examples

1. *The identities of degree 3 satisfied by the division algebra of real octonions.*

$$\begin{bmatrix}
-3 & -9 & -9 & -3 & -3 & -9 & -3 & -9 & -9 & -3 & -3 & -9 \\
1 & -5 & -3 & 5 & -1 & -1 & -3 & -1 & 1 & 1 & -5 & 3 \\
-1 & 1 & 3 & -5 & 1 & -3 & 1 & -1 & 1 & -3 & 3 & -5 \\
2 & 2 & -4 & -2 & 0 & -2 & 2 & 2 & -4 & -2 & 0 & -2 \\
1 & 1 & 1 & 3 & -9 & 3 & 5 & -3 & -3 & 7 & -5 & -1 \\
-3 & -3 & 3 & -1 & 5 & -1 & -3 & -3 & 3 & -1 & 5 & -1 \\
-10 & -2 & 0 & 4 & 2 & 4 & -8 & -4 & -2 & 6 & 4 & 2 \\
0 & 0 & 0 & 6 & -2 & -2 & -2 & 2 & 2 & 4 & -4 & 0
\end{bmatrix}$$

Table 1: The octonion evaluation matrix

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 1 & -1
\end{bmatrix}$$

Table 2: The reduced row echelon form of the octonion evaluation matrix

$$\begin{bmatrix}
-1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

Table 3: A basis for the nullspace of the octonion evaluation matrix

There are 12 distinct multilinear monomials of degree 3 for a nonassociative algebra:

$$(xy)z,\ (xz)y,\ (yx)z,\ (yz)x,\ (zx)y,\ (zy)x,\ x(yz),\ x(zy),\ y(xz),\ y(zx),\ z(xy),\ z(yx).$$

We create a matrix of size 8×12 and initialize it to zero; the columns correspond to the nonassociative monomials. We use a pseudorandom number generator to produce three octonions x, y, z represented as vectors with respect to the standard basis $1,\ i,\ j,\ k,\ \ell,\ m,\ n,\ p$. We store the evaluation of monomial

j in column j of the matrix. For example, generating random integers from the set $\{-1, 0, 1\}$ using the base 3 expansion of $1/\sqrt{2}$ gives

$$x = [1, -1, 0, -1, -1, 1, 0, 0], \quad y = [-1, 1, 1, 1, 0, 0, 1, 0], \quad z = [-1, 1, 1, 1, 0, 0, 0, -1].$$

Evaluation of the monomials gives the matrix in Table 1; its reduced row echelon form appears in Table 2. The nullspace contains the identities satisfied by the octonion algebra: the span of the rows of the matrix in Table 3. These rows represent the linearizations of the right alternative identity (row 1), the left alternative identity (row 2), and the flexible identity (row 5), together with the assocyclic identities $(x, y, z) = (y, z, x)$ and $(x, y, z) = (z, x, y)$ (rows 3 and 4).

2. *The identities of degree 4 satisfied by the Jordan product $x * y = xy + yx$ in every associative algebra over a field of characteristic 0.*

$$\begin{bmatrix}
1&0&0&0&0&0&1&1&0&0&0&1&1&0&0\\
0&1&0&0&0&0&0&0&1&1&0&1&1&0&0\\
0&0&1&0&0&0&1&1&0&1&0&0&0&1&0\\
0&0&0&1&0&0&0&0&0&1&1&1&0&1&0\\
0&0&0&0&1&0&0&1&1&1&0&0&0&0&1\\
0&0&0&0&0&1&0&1&0&0&1&1&0&0&1\\
1&0&1&1&0&0&0&0&0&0&1&0&1&0&0\\
0&1&0&0&1&1&0&0&0&0&1&0&1&0&0\\
0&0&1&1&0&1&1&0&0&0&0&0&0&0&1\\
0&0&0&0&0&1&0&1&0&0&1&1&0&0&1\\
0&0&0&1&1&1&0&0&1&0&0&0&0&1&0\\
0&0&0&1&0&0&0&0&0&1&1&1&0&1&0\\
1&1&1&0&0&0&0&0&1&0&0&0&0&1&0\\
0&0&0&1&1&1&0&0&1&0&0&0&0&1&0\\
1&1&0&0&1&0&1&0&0&0&0&0&0&0&1\\
0&0&0&0&1&0&0&1&1&1&0&0&0&0&1\\
0&1&0&0&1&1&0&0&0&0&1&0&1&0&0\\
0&1&0&0&0&0&0&0&1&1&0&1&1&0&0\\
1&1&0&0&1&0&1&0&0&0&0&0&0&0&1\\
0&0&1&1&0&1&1&0&0&0&0&0&0&0&1\\
1&1&1&0&0&0&0&0&1&0&0&0&0&1&0\\
0&0&1&0&0&0&1&1&0&1&0&0&0&1&0\\
1&0&1&1&0&0&0&0&0&0&1&0&1&0&0\\
1&0&0&0&0&0&1&1&0&0&0&1&1&0&0
\end{bmatrix}$$

Table 4: The Jordan expansion matrix in degree 4

The operation $x * y$ satisfies commutativity in degree 2, and there are no new identities of degree 3, so we consider degree 4. There are 15 distinct multilinear monomials for a commutative nonassociative operation, 12 for association type $((--)-)-$ and 3 for association type $(--)(--)$:

$$((w{*}x){*}y){*}z, \ ((w{*}x){*}z){*}y, \ ((w{*}y){*}x){*}z, \ ((w{*}y){*}z){*}x, \ ((w{*}z){*}x){*}y,$$

$$((w{*}z){*}y){*}x, \ ((x{*}y){*}w){*}z, \ ((x{*}y){*}z){*}w, \ ((x{*}z){*}w){*}y, \ ((x{*}z){*}y){*}w,$$

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}$$

Table 5: The reduced row echelon form of the Jordan expansion matrix

$$\begin{bmatrix}
0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & -1 & 0 & -1 & 0 & 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Table 6: A basis for the nullspace of the Jordan expansion matrix

$$((y{*}z){*}x){*}w,\ ((y{*}z){*}w){*}x,\ (w{*}x){*}(y{*}z),\ (w{*}y){*}(x{*}z),\ (w{*}z){*}(x{*}y).$$

When each of these monomials is expanded in terms of the associative product, there are 24 possible terms, namely the permutations of w, x, y, z in lexicographical order: $wxyz, \dots, zyxw$. We construct a 24×15 matrix in which the i, j entry is the coefficient of the i-th associative monomial in the expansion of the j-th commutative monomial; see Table 4. The non-trivial identities of degree 4 satisfied by $x * y$ correspond to the nonzero vectors in the nullspace. The reduced row echelon form appears in Table 5. The rank is 11, and so the nullspace has dimension 4. A basis for the nullspace consists of the rows of Table 6. The first row represents the linearization of the Jordan identity; this is the only identity which involves monomials of both association types. (This proves that the plus algebra A^+ of any associative algebra A is a Jordan algebra.) The Jordan identity implies the identities in the other three rows, which are permuted forms of the identity

$$w{*}(x{*}(yz)) - x{*}(w{*}(yz)) = (w{*}(x{*}y)){*}z - (x{*}(w{*}y)){*}z + y{*}(w{*}(x{*}z)) - y{*}(x{*}(w{*}z));$$

that is, the commutator of multiplication operators is a derivation.

3. *The identities of degree 4 satisfied by the commutator* $[x, y] = xy - yx$ *in every alternative algebra over a field of characteristic zero.*

λ	4	31	22	211	1111
d_λ	1	3	2	3	1

Table 7: Partitions of 4 and irreducible representations of S_4

$$
A_\lambda = \left[\begin{array}{c|cc}
 & 0 & 0 \\
\text{lifted alternative identities} & \vdots & \vdots \\
 & 0 & 0 \\
\hline
\text{expansion of } [[[x,y],z],w] & -I & 0 \\
\text{expansion of } [[x,y],[z,w]] & 0 & -I
\end{array} \right]
$$

Table 8: The matrix of Malcev identities for partition λ

$$
\left[\begin{array}{cccccccccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & 2 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 2 & -1 & 1 & 0 & 0 & 1 & -1 & -1 & -2 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{array} \right]
$$

Table 9: The lifted and expansion identities for partition $\lambda = 22$

The group algebra $\mathbb{Q}S_n$ decomposes as a direct sum of full matrix algebras of size $d_\lambda \times d_\lambda$ where the index λ runs over all partitions λ of the integer n. We choose the "natural representation" to fix a particular decomposition. For each λ there is a projection p_λ from $\mathbb{Q}S_n$ onto the matrix algebra of size $d_\lambda \times d_\lambda$. In the case $n = 4$ the partitions and the dimensions of the corresponding irreducible representation S_4 are given in Table 7. For a nonassociative operation in degree 4 there are 5 association types:

$$
((--)-)-, \quad (-(--))-, \quad (--)(--), \quad -((--)-), \quad -(-(--)),
$$

and so any nonassociative identity can be represented as an element of the direct sum of 5 copies of $\mathbb{Q}S_n$: given a partition λ, the nonassociative identity projects via p_λ to a matrix of size $d_\lambda \times 5d_\lambda$. For

$$
\left[\begin{array}{cccccccccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
$$

Table 10: The reduced row echelon form of the lifted and expansion identities

$$
\left[\begin{array}{cccccccccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{array}\right]
$$

Table 11: The anticommutative identities for partition $\lambda = 22$

$$
\left[\begin{array}{cccccccccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
$$

Table 12: The reduced row echelon form of the anticommutative identities

an anticommutative operation in degree 4 there are 2 association types:

$$
[[[-,-],-],-], \quad [[-,-],[-,-]],
$$

and so any anticommutative identity projects via p_λ to a matrix of size $d_\lambda \times 2d_\lambda$. The linearizations of the left and right alternative identities are:

$$
L(x,y,z) = (xy)z - x(yz) + (yx)z - y(xz), \qquad R(x,y,z) = (xy)z - x(yz) + (xz)y - x(zy).
$$

Each of these can be "lifted" to degree 4 in five ways; for $L(a,b,c)$ we have

$$
wL(x,y,z), \quad L(xw,y,z), \quad L(x,yw,z), \quad L(x,y,zw), \quad L(x,y,z)w;
$$

and similarly for $R(x,y,z)$. Altogether we have 10 lifted alternative identities which project via p_λ to a matrix of size $10d_\lambda \times 5d_\lambda$. Using the commutator to expand the two anticommutative association types gives

$$
\begin{aligned}
[[[x,y],z],w] = {} & ((xy)z)w - ((yx)z)w - (z(xy))w + (z(yx))w \\
& - w((xy)z) + w((yx)z) + w(z(xy)) - w(z(yx)),
\end{aligned}
$$

$$
\begin{aligned}
[[x,y],[z,w]] = (xy)(zw) - (yx)(zw) - (xy)(wz) + (yx)(wz) \\
- (zw)(xy) + (zw)(yx) + (wz)(xy) - (wz)(yx).
\end{aligned}
$$

Given a partition λ we can store these two relations in a matrix of size $2d_\lambda \times 7d_\lambda$: we use all 7 association types, store the right sides of the relations in the first 5 types, and $-I$ (I is the identity matrix) in type 6 (respectively 7) for the first (respectively second) expansion. For each partition λ, all of this data can be stored in a matrix A_λ of size $12d_\lambda \times 7d_\lambda$ which is schematically displayed in Table 8. We compute the reduced row echelon form of A_λ: let i be the largest number for which rows 1–i of $\mathrm{RREF}(A_\lambda)$ have a nonzero entry in the first 5 association types. Then the remaining rows of $\mathrm{RREF}(A_\lambda)$ have only zero entries in the first 5 types; if one of these rows contains nonzero entries in the last 2 types, this row represents an identity which is satisfied by the commutator in every alternative algebra. However, such an identity may be a consequence of the obvious anticommutative identities:

$$
\begin{aligned}
&[[[x,y],z],w] + [[[y,x],z],w] = 0, \\
&[[x,y],[z,w]] + [[y,x],[z,w]] = 0, \qquad [[x,y],[z,w]] + [[z,w],[x,y]] = 0.
\end{aligned}
$$

These identities are represented by a matrix of size $3d_\lambda \times 7d_\lambda$ in which the first $5d_\lambda$ columns are zero. We need to determine if any of the rows $i+1$ to $12d_\lambda$ of $\mathrm{RREF}(A_\lambda)$ do not lie in the row space of the matrix of anticommutative identities. If such a row exists, it represents a non-trivial identity satisfied by the commutator in every alternative algebra. For example, consider the partition $\lambda = 22$ ($d_\lambda = 2$). The 24×14 matrix A_λ for this partition appears in Table 9, and its reduced row echelon form appears in Table 10. The 6×14 matrix representing the anticommutative identities for this partition appears in Table 11, and its reduced row echelon form appears in Table 12. Comparing the last four rows of Table 10 with Table 12 we see that there is one new identity for $\lambda = 22$ represented by the third-last row of Table 10. Similar computations for the other partitions show that there is one non-trivial identity for partition $\lambda = 211$ and no non-trivial identities for the other partitions. The two identities from partitions 22 and 211 are the irreducible components of the Malcev identity: the submodule generated by the linearization of the Malcev identity (in the S_4-module of all multilinear anticommutative polynomials of degree 4) is the direct sum of two irreducible submodules corresponding to these two partitions.

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