

NEW CLASSES OF DOMAINS WITH EXPLICIT BERGMAN KERNEL

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ABSTRACT. We introduce two classes of egg type domains, built on general bounded symmetric domains, for which we obtain the Bergman kernel in explicit formulas.

In 1921, S. Bergman introduced a kernel function, which is now known as the Bergman kernel function. It is well known that there exists a unique Bergman kernel function for each bounded domain in \mathbb{C}^n . For which domains can the Bergman kernel function be computed by explicit formulas? This is an important problem. Explicit formulas of the Bergman kernel function can help to solve important conjectures. We illustrate this point by two cases. Mostow and Siu have given a counterexample to the important conjecture that the universal covering of a compact Kähler manifold of negative sectional curvature should be biholomorphic to the ball. In their counterexample the explicit calculation of the Bergman kernel function and metric of the egg domain

$$\{z \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{14} < 1\}$$

plays an essential role [19]. Another example is Lu Qikeng conjecture: in order to give a counterexample to the Lu Qikeng conjecture, an explicit formula for the Bergman kernel function is used in [3]. Therefore, computation of the Bergman kernel function by explicit formulas is an important research direction in several complex variables. Up to now, there are still many mathematicians working in that direction.

If the group of holomorphic automorphisms of a bounded homogeneous domain is known, then one can get its Bergman kernel function in explicit form. L.K. Hua [14] has obtained the Bergman kernel functions in explicit form for the four types of Cartan domains by using that method (it is called Hua's method). It is well known that there are two exceptional Cartan domains, of dimension 16 and 27. Weiping Yin [25] has obtained the Bergman kernel functions in explicit form for these two exceptional Cartan domains by using the Hua's method. A general expression of the Bergman kernel for all symmetric bounded homogeneous domains can also be given using the theory of Jordan triple systems; the Bergman kernel is then, up to a constant, a (negative) power of the "generic minimal polynomial" (cf. Loos [18]).

For a non-symmetric homogeneous domain, we know that it is holomorphically equivalent to a Siegel domain (or N-Siegel domain in the sense of Yichao Xu);

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S.G. Gindikin [11] has computed the Bergman kernel functions in explicit form for homogeneous Siegel domains by using generalized power functions. Yichao Xu [24] has also obtained the Bergman kernel functions in explicit form for N-Siegel domains. In the middle of sixties, Jiaqing Zhong and Weiping Yin [31],[32] have constructed some new types of non-symmetric homogeneous domains and their extension spaces; Weiping Yin [26],[27],[28] has computed their Bergman kernel functions in explicit form by using Hua's method. But their papers were published only at the beginning of the eighties, after the end of Cultural Revolution in China.

Besides homogeneous domains, domains for which the Bergman kernel function can be computed in explicit form are the egg domains (also called complex ellipsoids or complex ovals). In general, an egg domain has the following form:

$$E(p_1, \dots, p_n) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_1|^{2p_1} + \dots + |z_n|^{2p_n} < 1\},$$

where p_1, \dots, p_n are positive real numbers. Here the z_j ($j = 1, \dots, n$) are complex numbers. A more general case is obtained when the z_j are complex vectors, that is $z_j = (z_{j1}, \dots, z_{jm_j})$ and $|z_j|^{2p_j} = [\sum_{k=1}^{m_j} |z_{jk}|^2]^{p_j}$; the corresponding domain is then denoted by $E(p_1, \dots, p_n; m_1, \dots, m_n)$ or simply by $E_{p;m}$. S. Bergman ([2], p.82) has computed the Bergman kernel function in explicit form on $E(p_1, 1)$ by summing a series; although he states that $1/p_1$ is a positive integer, his computation is valid for arbitrary p_1 . The explicit form of the Bergman kernel functions on $E(1, \dots, 1, p_n)$ and on $E(1, \dots, 1, p_n; 1, \dots, 1, m)$ were obtained by D'Angelo in [5] and [6] respectively. In the case $1/p_1, \dots, 1/p_n$ are positive integers, Zinov'ev [33] obtained the explicit form of the Bergman kernel function on $E(p_1, \dots, p_n)$. If $1/p_1, \dots, 1/p_{n-1}$ are positive integers and $1/p_n = p$ is a positive real number, then the Bergman kernel function on $E(p_1, \dots, p_n)$ can also be computed explicitly. If p_1, \dots, p_n are positive integers, then the Bergman kernel function on $E(p_1, \dots, p_n)$ has an explicit expression in terms of multivariate hypergeometric functions ([9, Theorem 1]).

We are not able to compute an explicit formula for the Bergman kernel function on any egg domain. If there are two (or more) numbers among the positive real numbers $1/p_1, \dots, 1/p_n$ which are not integers (nor inverse of integers), then one cannot get an explicit formula for the Bergman kernel function on $E(p_1, \dots, p_n)$ or on $E_{p;m}$. So, one needs to estimate the Bergman kernel function on egg domains. This work has been done by Sheng Gong and Xuean Zheng [12],[13]. In his thesis of doctoral degree (Purdue Univ 1997-1998) titled "The Bergman kernel on Reinhardt domains", Chieh-hsien Tiao has obtained an estimation for the Bergman kernel function on general Reinhardt domains [23].

During the period of the second author's stay at the Institut des Hautes Études Scientifiques (IHES) in February 1998, he and G. Roos introduced the following four types of domains, called *super-Cartan domains* or *Cartan-Hartogs domains* (N, m, n are positive integers and $K > 0$ is real):

$$\begin{aligned} Y_I(N, m, n; K) &:= \{W \in \mathbb{C}^N, Z \in R_I(m, n) : |W|^{2K} < \det(I - ZZ^T)\}, \\ Y_{II}(N, p; K) &:= \{W \in \mathbb{C}^N, Z \in R_{II}(p) : |W|^{2K} < \det(I - ZZ^T)\}, \\ Y_{III}(N, q; K) &:= \{W \in \mathbb{C}^N, Z \in R_{III}(q) : |W|^{2K} < \det(I - ZZ^T)\}, \\ Y_{IV}(N, n; K) &:= \{W \in \mathbb{C}^N, Z \in R_{IV}(n) : \\ &\quad |W|^{2K} < 1 - ZZ^T - [(ZZ^T)^2 - |ZZ'|^2]^{1/2}\}. \end{aligned}$$

where $R_I(m, n)$, $R_{II}(p)$, $R_{III}(q)$ and $R_{IV}(n)$ denote respectively the Cartan domains of first, second, third and fourth type in the sense of L.K. Hua. Here Z^T denotes the conjugate and transpose of Z , \det denotes the determinant of a square matrix.

Weiping Yin has got the Bergman kernel function in explicit form for these four Cartan-Hartogs domains [29],[30]. We would like to point out that if we can get the Bergman kernel in explicit form for a domain, then this domain is a good domain for research.

Quite recently, Weiping Yin has introduced the following four types of domains, which may be called *Cartan egg domains*:

$$\begin{aligned} CE_I(M, N, m, n; K) &:= \{W_1 \in \mathbb{C}^M, W_2 \in \mathbb{C}^N, Z \in R_I(m, n) : \\ &\quad |W_1|^2 + |W_2|^{2K} < \det(I - ZZ^T)\}, \\ CE_{II}(M, N, p; K) &:= \{W_1 \in \mathbb{C}^M, W_2 \in \mathbb{C}^N, Z \in R_{II}(p) : \\ &\quad |W_1|^2 + |W_2|^{2K} < \det(I - ZZ^T)\}, \\ CE_{III}(M, N, q; K) &:= \{W_1 \in \mathbb{C}^M, W_2 \in \mathbb{C}^N, Z \in R_{III}(q) : \\ &\quad |W_1|^2 + |W_2|^{2K} < \det(I - ZZ^T)\}, \\ CE_{IV}(M, N, n; K) &:= \{W_1 \in \mathbb{C}^M, W_2 \in \mathbb{C}^N, Z \in R_{IV}(n) : \\ &\quad |W_1|^2 + |W_2|^{2K} < 1 - ZZ^T - [(ZZ^T)^2 - |ZZ'|^2]^{1/2}\}. \end{aligned}$$

More generally, we will consider in this paper the two following classes of domains, built on an arbitrary irreducible bounded circled homogeneous domain Ω :

$$\begin{aligned} Y(q, \Omega; k) &:= \{(W, Z) \in \mathbb{C}^q \times \Omega; |W|^{2k} < N(Z, Z)\}, \\ E(p, q, \Omega; k) &:= \{(W_1, W_2, Z) \in \mathbb{C}^p \times \mathbb{C}^q \times \Omega; |W_1|^2 + |W_2|^{2k} < N(Z, Z)\}, \end{aligned}$$

where $N(Z, Z)$ is the *generic norm* of Ω , which is the proper generalization of $\det(I - ZZ^T)$.

If we are able to compute the Bergman kernel when $p = q = 1$, then, by using twice the *Principle of inflation* (which will be stated below), one can get the Bergman kernel functions of Cartan egg domains for general p, q . Therefore, we consider first $Y(1, \Omega, k)$ and $E(1, 1, \Omega; k)$. The domain $E(1, 1, \Omega; k)$ is not homogeneous, so we cannot use Hua's method to get the Bergman kernel function in explicit form. Also the domain $E(1, 1, \Omega; k)$ is not Reinhardt, so we cannot get its Bergman kernel function by summing an infinite series. Our new method is a combination of these two methods.

In Section 1, we review the properties of Jordan triple systems, which are necessary for the general definition of $Y(q, \Omega; k)$ and $E(p, q, \Omega; k)$. Section 2 is devoted to auxiliary results: the computation of $\int_{\Omega} N(x, x)^s \alpha^n$, which appears to be a simple consequence of an integral computed by Selberg, the notion of "semi-Reinhardt domains" and description of complete orthonormal systems for these domains, and the "Principle of inflation", which allows to derive the Bergman kernel for $Y(q, \Omega; k)$ and $E(p, q, \Omega; k)$ from the special case $p = q = 1$. In Section 3, we study automorphisms of the domains $Y(q, \Omega; k)$ and $E(p, q, \Omega; k)$ and compute their Bergman kernels. The last section gives tables stating specific results for all types (classical and exceptional) of bounded symmetric domains.

1. BOUNDED SYMMETRIC DOMAINS AND JORDAN TRIPLE SYSTEMS

Hereunder we give a review of properties of the Jordan triple structure associated to a complex bounded symmetric domain (see [18], [21]).

1.1. Jordan triple system associated to a bounded symmetric domain.

Let Ω be an irreducible bounded circled homogeneous domain in a complex vector space V . Let K be the identity component of the (compact) Lie group of (linear) automorphisms of Ω leaving 0 fixed. Let ω be a volume form on V , invariant by K and by translations. Let \mathcal{K} be the Bergman kernel of Ω with respect to ω , that is, the reproducing kernel of the Hilbert space $H^2(\Omega, \omega) = \text{Hol}(\Omega) \cap L^2(\Omega, \omega)$. The Bergman metric at $z \in \Omega$ is defined by

$$h_z(u, v) = \partial_u \bar{\partial}_v \log \mathcal{K}(z).$$

The *Jordan triple product* on V is defined by

$$h_0(\{uvw\}, t) = \partial_u \bar{\partial}_v \partial_w \bar{\partial}_t \log \mathcal{K}(z) |_{z=0}.$$

The triple product $(x, y, z) \mapsto \{xyz\}$ is complex bilinear and symmetric with respect to (x, z) , complex antilinear with respect to y . It satisfies the *Jordan identity*

$$\{xy\{uvw\}\} - \{uv\{xyw\}\} = \{\{xyu\}vw\} - \{u\{vxy\}w\}.$$

The space V endowed with the triple product $\{xyz\}$ is called a (*Hermitian*) *Jordan triple system*. For $x, y, z \in V$, denote by $D(x, y)$ and $Q(x, z)$ the operators defined by

$$\{xyz\} = D(x, y)z = Q(x, z)y.$$

The Bergman metric at 0 is related to D by

$$h_0(u, v) = \text{tr } D(u, v).$$

A Jordan triple system is called *Hermitian positive* if $(u|v) = \text{tr } D(u, v)$ is positive definite. As the Bergman metric of a bounded domain is always definite positive, the Jordan triple system associated to a bounded symmetric domain is Hermitian positive.

The *quadratic representation*

$$Q : V \longrightarrow \text{End}_{\mathbb{R}}(V)$$

is defined by $Q(x)y = \frac{1}{2}\{xyx\}$. The following fundamental identity for the quadratic representation is a consequence of the Jordan identity:

$$Q(Q(x)y) = Q(x)Q(y)Q(x).$$

The *Bergman operator* B is defined by

$$B(x, y) = I - D(x, y) + Q(x)Q(y),$$

where I denotes the identity operator in V . It is also a consequence of the Jordan identity that the following fundamental identity holds for the Bergman operator:

$$Q(B(x, y)z) = B(x, y)Q(z)B(y, x).$$

The Bergman operator gets its name from the following property:

$$h_z(B(z, z)u, v) = h_0(u, v) \quad (z \in \Omega; u, v \in V).$$

If $\Phi \in (\text{Aut } \Omega)_0$, the identity component of the automorphism group of Ω , the relation

$$(1.1) \quad B(\Phi x, \Phi y) = d\Phi(x) \circ B(x, y) \circ d\Phi(y)^*$$

holds for $x, y \in \Omega$, where $*$ denotes the adjoint with respect to the hermitian metric h_0 . As a consequence, the Bergman kernel of Ω is given by

$$(1.2) \quad \mathcal{K}(z) = \frac{1}{\text{vol } \Omega} \frac{1}{\det B(z, z)}.$$

1.2. Spectral theory. An Hermitian positive Jordan triple system is always *semi-simple*, that is the direct sum of a finite family of simple subsystems with component-wise triple product.

As the domain Ω is assumed to be irreducible, the associated Jordan triple system V is *simple*, that is V is not the direct sum of two non trivial subsystems.

An *automorphism* $f : V \rightarrow V$ of the Jordan triple system V is a complex linear isomorphism preserving the triple product : $f\{u, v, w\} = \{fu, fv, fw\}$. The automorphisms of V form a group, denoted $\text{Aut } V$, which is a compact Lie group; we will denote by K its identity component.

An element $c \in V$ is called *tripotent* if $\{ccc\} = 2c$. If c is a tripotent, the operator $D(c, c)$ annihilates the polynomial $T(T-1)(T-2)$.

Let c be a tripotent. The decomposition $V = V_0(c) \oplus V_1(c) \oplus V_2(c)$, where $V_j(c)$ is the eigenspace $V_j(c) = \{x \in V ; D(c, c)x = jx\}$, is called the *Peirce decomposition* of V (with respect to the tripotent c).

Two tripotents c_1 and c_2 are called *orthogonal* if $D(c_1, c_2) = 0$. If c_1 and c_2 are orthogonal tripotents, then $D(c_1, c_1)$ and $D(c_2, c_2)$ commute and $c_1 + c_2$ is also a tripotent.

A non zero tripotent c is called *primitive* if it is not the sum of non zero orthogonal tripotents. A tripotent c is *maximal* if there is no non zero tripotent orthogonal to c . The set of maximal tripotents is equal to the *Shilov boundary* of the domain Ω .

A *frame* of V is a maximal sequence (c_1, \dots, c_r) of pairwise orthogonal primitive tripotents. The frames of V form a manifold \mathcal{F} , which is called the *Satake-Furstenberg boundary* of Ω .

Let $\mathbf{c} = (c_1, \dots, c_r)$ be a frame. For $0 \leq i \leq j \leq r$, let

$$V_{ij}(\mathbf{c}) = \{x \in V \mid D(c_k, c_k)x = (\delta_i^k + \delta_j^k)x, 1 \leq k \leq r\} :$$

the decomposition $V = \bigoplus_{0 \leq i \leq j \leq r} V_{ij}(\mathbf{c})$ is called the *simultaneous Peirce decomposition* with respect to the frame \mathbf{c} .

Let V be a simple Hermitian positive Jordan triple system. Then there exist frames for V . All frames have the same number of elements, which is the *rank* r of V . The subspaces $V_{ij} = V_{ij}(\mathbf{c})$ of the simultaneous Peirce decomposition have the following properties: $V_{00} = 0$; $V_{ii} = \mathbb{C}e_i$ ($0 < i$); all V_{ij} 's ($0 < i < j$) have the same dimension a ; all V_{0i} 's ($0 < i$) have the same dimension b .

The *numerical invariants* of V (or of Ω) are the rank r and the two integers

$$\begin{aligned} a &= \dim V_{ij} \quad (0 < i < j), \\ b &= \dim V_{0i} \quad (0 < i). \end{aligned}$$

The *genus* of V is the number g defined by

$$g = 2 + a(r-1) + b.$$

The HPJTS V and the domain Ω are said to be of *tube type* if $b = 0$.

Let V be a simple Hermitian positive Jordan triple system. Then any $x \in V$ can be written in a unique way

$$x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_p c_p,$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_p > 0$ and c_1, c_2, \dots, c_p are pairwise orthogonal tripotents. The element x is *regular* iff $p = r$; then (c_1, c_2, \dots, c_r) is a frame of V . The decomposition $x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_p c_p$ is called the *spectral decomposition* of x .

1.3. The generic minimal polynomial. Let V be a Jordan triple system of rank r . There exist polynomials m_1, \dots, m_r on $V \times \bar{V}$, homogeneous of respective bidegrees $(1, 1), \dots, (r, r)$, such that for each regular $x \in V$, the polynomial

$$m(T, x, y) = T^r - m_1(x, y)T^{r-1} + \cdots + (-1)^r m_r(x, y)$$

satisfies

$$m(T, x, x) = \prod_{i=1}^r (T - \lambda_i^2),$$

where $x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_r c_r$ is the spectral decomposition of x . Here \bar{V} denotes the space V with the conjugate complex structure. The polynomial

$$m(T, x, y) = T^r - m_1(x, y)T^{r-1} + \cdots + (-1)^r m_r(x, y)$$

is called the *generic minimal polynomial* of V (at (x, y)). The (inhomogeneous) polynomial $N : V \times \bar{V} \rightarrow \mathbb{C}$ defined by

$$N(x, y) = m(1, x, y)$$

is called the *generic norm*. The following identities hold:

$$(1.3) \quad \begin{aligned} \det B(x, y) &= N(x, y)^g, \\ \operatorname{tr} D(x, y) &= g m_1(x, y). \end{aligned}$$

1.4. The spectral norm. Let V be an HPJTS. The map $x \mapsto \lambda_1$, where $x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_p c_p$ is the spectral decomposition of x ($\lambda_1 > \lambda_2 > \cdots > \lambda_p > 0$) is a norm on V , called the *spectral norm*. The *bounded symmetric domain* Ω is the unit ball of V for the *spectral norm*. It is also characterized by the set of polynomial inequalities

$$\left. \frac{\partial^j}{\partial T^j} m(T, x, x) \right|_{T=1} > 0, \quad 0 \leq j \leq r-1.$$

Proposition 1.1. *Let*

$$\Phi : \mathcal{F} \times \{\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0\} \longrightarrow V_{\text{reg}}$$

be defined by

$$\Phi((c_1, \dots, c_r), (\lambda_1, \dots, \lambda_r)) = \sum_{j=1}^r \lambda_j c_j ;$$

here V_{reg} is the open dense subset of regular elements of V . Then Φ is a diffeomorphism; its restriction

$$\Phi_0 : \mathcal{F} \times \{1 > \lambda_1 > \lambda_2 > \cdots > \lambda_r > 0\} \longrightarrow \Omega_{\text{reg}} = \Omega \cap V_{\text{reg}}$$

is a diffeomorphism onto the open dense subset Ω_{reg} of regular elements of Ω .

Let α be the Kähler form on V associated to the Hermitian inner product m_1 :

$$(1.4) \quad \alpha = \frac{i}{2\pi} \partial \bar{\partial} m_1.$$

In the following, we endow V with the volume form $\omega = \alpha^n$ ($n = \dim_{\mathbb{C}} V$), so that the volume of the unit ball associated to m_1 is equal to 1. The following property is well-known (see e.g., [16, (5.1.1)]):

Proposition 1.2. *Let V be an irreducible HPJTS of dimension n , rank r and invariants a, b . The pull-back of the volume element α^n by Φ is*

$$\Phi^* \alpha^n = \Theta \wedge \prod_{j=1}^r \lambda_j^{2b+1} \prod_{1 \leq j < k \leq r} (\lambda_j^2 - \lambda_k^2)^a d\lambda_1 \wedge \dots \wedge d\lambda_r,$$

where a and b are the numerical invariants of V and Θ is a K -invariant volume form on \mathcal{F} .

2. AUXILIARY RESULTS

2.1. The Selberg integral. Using an integral of Selberg, we compute in this subsection the integral $\int_{\Omega} N(x, x)^s \alpha^n$, where $N(x, x)$ is the generic norm of the irreducible bounded symmetric domain Ω . The integral of Selberg was used by Korányi [15] for computing the volume of a general bounded symmetric domain. The integral $\int_{\Omega} N(x, x)^s \alpha^n$ has been calculated by Hua [14] for the four series of classical domains.

Proposition 2.1. *Let $\text{vol } \Omega$ and $\text{vol } \mathcal{F}$ be the volumes of Ω and \mathcal{F} w.r. to α^n and Θ :*

$$\text{vol } \Omega = \int_{\Omega} \alpha^n, \quad \text{vol } \mathcal{F} = \int_{\mathcal{F}} \Theta.$$

Then, for $s \in \mathbb{C}$, $\text{Re } s > -1$

$$(2.1) \quad \int_{\Omega} N(x, x)^s \alpha^n = F(s) \text{vol } \mathcal{F} = \frac{F(s)}{F(0)} \text{vol } \Omega,$$

with

$$(2.2) \quad F(s) = \frac{1}{2^r r!} \prod_{j=1}^r \frac{\Gamma(b+1+(j-1)\frac{a}{2}) \Gamma(s+1+(j-1)\frac{a}{2}) \Gamma(j\frac{a}{2}+1)}{\Gamma(s+b+2+(r+j-2)\frac{a}{2}) \Gamma(\frac{a}{2}+1)},$$

$$\frac{F(s)}{F(0)} = \prod_{j=1}^r \frac{\Gamma(s+1+(j-1)\frac{a}{2}) \Gamma(b+2+(r+j-2)\frac{a}{2})}{\Gamma(1+(j-1)\frac{a}{2}) \Gamma(s+b+2+(r+j-2)\frac{a}{2})}.$$

Proof. According to Propositions 1.1 and 1.2, we have

$$F(s) = \int_{1 > \lambda_1 > \lambda_2 > \dots > \lambda_r > 0} \dots \int \prod_{j=1}^r (1 - \lambda_j^2)^s \prod_{j=1}^r \lambda_j^{2b+1} \prod_{1 \leq j < k \leq r} (\lambda_j^2 - \lambda_k^2)^a d\lambda_1 \wedge \dots \wedge d\lambda_r;$$

by the change of variables $t_j = \lambda_j^2$, we have

$$F(s) = \frac{1}{2^r} \int_{1 > \lambda_1 > \lambda_2 > \dots > \lambda_r > 0} \dots \int \prod_{j=1}^r (1 - t_j)^s \prod_{j=1}^r t_j^b \prod_{1 \leq j < k \leq r} (t_j - t_k)^a dt_1 \wedge \dots \wedge dt_r.$$

Extending to integration over the cube $[0, 1]^r$, we also have

$$F(s) = \frac{1}{2^r r!} \int_0^1 \cdots \int_0^1 \prod_{j=1}^r (1-t_j)^s \prod_{j=1}^r t_j^b \prod_{1 \leq j < k \leq r} |t_j - t_k|^a dt_1 \wedge \cdots \wedge dt_r.$$

The above integral has been evaluated by Selberg [22]:

Theorem 2.2 (Selberg). *For $\operatorname{Re} x > 0$, $\operatorname{Re} y > 0$, $\operatorname{Re} z > -\min\left(\frac{1}{n}, \frac{\operatorname{Re} x}{n-1}, \frac{\operatorname{Re} y}{n-1}\right)$, one has*

$$\begin{aligned} \int_0^1 \cdots \int_0^1 \prod_{j=1}^n t_j^{x-1} (1-t_j)^{y-1} \prod_{1 \leq j < k \leq n} |t_j - t_k|^{2z} dt_1 \cdots dt_n \\ = \prod_{j=1}^n \frac{\Gamma(x + (j-1)z) \Gamma(y + (j-1)z) \Gamma(jz + 1)}{\Gamma(x + y + (n+j-2)z) \Gamma(z + 1)}. \end{aligned}$$

See [1] for a simple proof.

Applying this result for $n \leftarrow r$, $x \leftarrow b+1$, $y \leftarrow s+1$, $z \leftarrow \frac{a}{2}$, we obtain the expression of $F(s)$:

$$F(s) = \frac{1}{2^r r!} \prod_{j=1}^r \frac{\Gamma(b+1 + (j-1)\frac{a}{2}) \Gamma(s+1 + (j-1)\frac{a}{2}) \Gamma(j\frac{a}{2} + 1)}{\Gamma(s+b+2 + (r+j-2)\frac{a}{2}) \Gamma(\frac{a}{2} + 1)},$$

valid for $\operatorname{Re} s > -1$. It follows that

$$\frac{F(s)}{F(0)} = \prod_{j=1}^r \frac{\Gamma(s+1 + (j-1)\frac{a}{2}) \Gamma(b+2 + (r+j-2)\frac{a}{2})}{\Gamma(1 + (j-1)\frac{a}{2}) \Gamma(s+b+2 + (r+j-2)\frac{a}{2})}.$$

This proves Proposition 2.1.

More precisely, $\int_{\Omega} N(x, x)^s \alpha^n$ is a very simple rational function of s :

Theorem 2.3. *Let V be an irreducible HPJTS of dimension n . There exists a polynomial χ of degree n , the zeroes of which are all negative integers or half integers, such that*

$$\chi(s) \int_{\Omega} N(x, x)^s \alpha^n$$

is independent of s . The polynomial χ is equal to

$$(2.3) \quad \chi(s) = \prod_{j=1}^r \left(s + 1 + (j-1)\frac{a}{2} \right)_{1+b+(r-j)a}.$$

Here we use the classical notation $(s)_k$ for the polynomial of degree k (“Pochhammer polynomial”):

$$(s)_k = \prod_{j=0}^{k-1} (s+j) = s(s+1) \cdots (s+k-1).$$

Proof. It follows from (2.1) and (2.2) that $\int_{\Omega} N(x, x)^s \alpha^n$ is, up to a constant, equal to

$$(2.4) \quad \frac{1}{\chi(s)} = \prod_{j=1}^r \frac{\Gamma(s+1 + (j-1)\frac{a}{2})}{\Gamma(s+b+2 + (r+j-2)\frac{a}{2})}.$$

This may also be written (changing j into $r + 1 - j$ in the denominator)

$$\begin{aligned}\chi(s) &= \frac{\prod_{j=1}^r \Gamma(s + b + 2 + (2r - j - 1)\frac{a}{2})}{\prod_{j=1}^r \Gamma(s + 1 + (j - 1)\frac{a}{2})} \\ &= \prod_{j=1}^r \left(s + 1 + (j - 1)\frac{a}{2} \right)_{1+b+(r-j)a}.\end{aligned}$$

□

As a consequence of Theorem 2.3, we have for $s \in \mathbb{C}$, $\operatorname{Re} s > -1$

$$(2.5) \quad \int_{\Omega} N(x, x)^s \alpha^n = \frac{\chi(0)}{\chi(s)} \operatorname{vol} \Omega.$$

Note also that the degree of χ :

$$\sum_{j=1}^r (1 + b + (r - j)a) = r + rb + \frac{r(r-1)}{2}a$$

is equal to the dimension of V .

2.2. Semi-Reinhardt domains.

Definition 2.1. A bounded domain D in \mathbb{C}^{m+n} is called a semi-Reinhardt domain if $0 \in D$ and if

$$(e^{i\theta_1} w_1, \dots, e^{i\theta_m} w_m, e^{i\theta} z_1, \dots, e^{i\theta} z_n) \in D$$

for all $(w, z) \in D$ and $\theta_1, \dots, \theta_m, \theta$ real.

That means, D is ‘‘Reinhardt’’ w.r. to w and is circular w.r. to z . Obviously, a semi-Reinhardt domain is a circular domain, but the converse is not true.

It is well-known that: if D_1 is a Reinhardt domain containing the origin in \mathbb{C}^m , then $\{w_1^{j_1} \dots w_m^{j_m}\}$ is a complete orthogonal system for D_1 ; if D_2 is a circular domain containing the origin in \mathbb{C}^n , then a complete orthonormal system for D_2 is given by

$$\left\{ P_{ki}; k \in \mathbb{N}, 1 \leq i \leq m_k = \binom{n+k-1}{k} \right\},$$

where, for any fixed k , $\{P_{k1}, P_{k2}, \dots, P_{km_k}\}$ is an orthonormal basis of the space of homogeneous polynomials of degree k in z_1, \dots, z_n . We give hereunder a generalization of these facts for semi-Reinhardt domains.

Let D be a semi-Reinhardt domain in \mathbb{C}^{m+n} . For each multi-index $j \in \mathbb{N}^m$, consider the weighted scalar product on polynomials on \mathbb{C}^n defined by

$$(f | g)^{(j)} = \int_D |w^j|^2 f(z) \overline{g(z)}.$$

As D is circular w.r. to z , polynomials of different degrees are orthogonal for this scalar product. For each $k \in \mathbb{N}$, choose a basis of the space \mathcal{P}_k of homogeneous polynomials of degree k in z_1, \dots, z_n , which is orthonormal w.r. to $(|)^{(j)}$:

$$\left\{ P_{ki}^{(j)} \right\}_{1 \leq i \leq m_k},$$

where $m_k = \dim \mathcal{P}_k = \binom{n+k-1}{k}$. Then $\{P_{ki}^{(j)}; k \in \mathbb{N}, 1 \leq i \leq m_k\}$ is a complete orthonormal system for the space of holomorphic functions such that

$$\int_D |w^j|^2 |f(z)|^2 < \infty.$$

Theorem 2.4. *Let D be a semi-Reinhardt domain in \mathbb{C}^{m+n} . Then a complete orthonormal system for D is given by*

$$\left\{ w_1^{j_1} \dots w_m^{j_m} P_{ki}^{(j)}(z); j = (j_1, \dots, j_m) \in \mathbb{N}^m, k \in \mathbb{N}, 1 \leq i \leq m_k \right\}.$$

Proof. Let $\Phi_{jki}(w, z) = w^j P_{ki}^{(j)}(z)$. As D is semi-Reinhardt, it is clear that the system $\{\Phi_{jki}\}$ is orthonormal.

Let $f(z, w)$ be a holomorphic, square-integrable function on D . Let $b_{jki} = \int_D f(w, z) \overline{\Phi_{jki}(w, z)}$. It suffices to show that

$$(2.6) \quad \int_D |f(w, z)|^2 = \sum_{j, k, 1 \leq i \leq m_k} |b_{jki}|^2.$$

Let $D_z = \{w; (z, w) \in D\}$. Then D_z is ‘‘Reinhardt’’, $f(w, z) = \sum b_j(z) w^j$,

$$b_j(z) = \frac{\int_{w \in D_z} f(w, z) \overline{w^j}}{\int_{w \in D_z} |w^j|^2},$$

and $\int_{w \in D_z} |f(w, z)|^2 = \sum_j |b_j(z)|^2 \int_{w \in D_z} |w^j|^2$. The functions $b_j(z)$ are holomorphic, for they may also be obtained locally as Cauchy integrals of $f(z, w)$ on the Shilov boundary of a polydisc in \mathbb{C}_w^m . Let Ω be the projection of D on the second factor \mathbb{C}^n ; then Ω is circular, b_j is square-integrable on Ω , $b_j = \sum_{k, i} c_{jki} P_{ki}^{(j)}$, with

$$c_{jki} = \int_{\Omega} |w^j|^2 b_j(z) \overline{P_{ki}^{(j)}(z)} = \int_{\Omega} f(w, z) \overline{w^j} \overline{P_{ki}^{(j)}} b_{jki} = b_{jki}$$

and $\int_{\Omega} |w^j|^2 |b_j(z)|^2 = \sum_{k \in \mathbb{N}, 1 \leq i \leq m_k} |b_{jki}|^2$. The equality (2.6) then easily follows. \square

2.3. The principle of inflation. Let Ω be a bounded complete Hartogs domain in \mathbb{C}^{n+1} :

$$\Omega = \{(z, \zeta); z \in D, \zeta \in \mathbb{C}, |\zeta|^2 < \phi(z)\},$$

where ϕ is a bounded, positive, continuous function on some bounded domain D in \mathbb{C}^n . Due to the circular symmetry w.r. to the one-dimensional variable, the Bergman kernel of Ω can be written

$$\mathcal{K}_{\Omega}(z, \zeta) = L(z, |\zeta|^2).$$

The ‘‘Principle of inflation’’, given in [4], allows to compute the Bergman kernel of the ‘‘inflated’’ domain G , obtained from Ω when replacing ζ by an m -dimensional variable Z and $|\zeta|^2$ by $\|Z\|^2 = |Z_1|^2 + \dots + |Z_m|^2$.

Theorem 2.5. ([4]) *Let G be defined by*

$$G = \{(z, Z); z \in D, Z \in \mathbb{C}^m, \|Z\|^2 < \phi(z)\}.$$

The spaces \mathbb{C}^{n+1} and \mathbb{C}^{n+m} being respectively endowed with the translation invariant volume forms $\beta(z) \wedge \frac{i}{2\pi} \partial\bar{\partial}|\zeta|^2$ and $\beta(z) \wedge \left(\frac{i}{2\pi} \partial\bar{\partial} \|Z\|^2\right)^m$, the Bergman kernel function \mathcal{K}_G of G is

$$K_G(z, Z) = \frac{1}{m!} \frac{\partial^{m-1}}{\partial r^{m-1}} L(z, r) \Big|_{r=\|Z\|^2}.$$

Trivial, but basic example: $n = 0$, $\phi = 1$; Ω is the unit disc, G the unit hermitian ball,

$$\mathcal{K}_\Omega(\zeta) = \frac{1}{(1 - |\zeta|^2)^2}, \quad \mathcal{K}_G(Z) = \frac{1}{(1 - \|Z\|^2)^{m+1}}.$$

Proof. See [4, Subsection 2.2].

3. EGG DOMAINS BUILT ON BOUNDED SYMMETRIC DOMAINS

3.1. The domains $Y(q, \Omega; k)$ and $E(p, q, \Omega; k)$. Let Ω be any bounded irreducible circled homogeneous domain. For $q \in \mathbb{N}$ and $k \in \mathbb{R}$, $k > 0$, define the domain $Y(q, \Omega; k)$ by

$$Y(q, \Omega; k) := \left\{ (W, Z) \in \mathbb{C}^q \times \Omega; \|W\|^{2k} < N(Z, Z) \right\};$$

for $p, q \in \mathbb{N}$ and $k \in \mathbb{R}$, $k > 0$, define the domain $E(p, q, \Omega; k)$ by

$$E(p, q, \Omega; k) := \left\{ (W_1, W_2, Z) \in \mathbb{C}^p \times \mathbb{C}^q \times \Omega; \|W_1\|^2 + \|W_2\|^{2k} < N(Z, Z) \right\}.$$

Note that, as it is the case for Ω , the boundary of $Y(q, \Omega; k)$ and $E(p, q, \Omega; k)$ is not smooth when the rank of the symmetric domain Ω is greater than 1.

For Ω belonging to one of the four classical series, an explicit form of the Bergman kernel for $Y(q, \Omega; k)$ has been obtained by Weiping Yin ([29], [30]).

The spaces \mathbb{C}^p and \mathbb{C}^q are equipped with their canonical Hermitian structure and with the volume forms

$$\begin{aligned} \omega_1(W_1) &= \left(\frac{i}{2\pi} \partial\bar{\partial} \|W_1\|^2 \right)^p, \\ \omega_2(W_2) &= \left(\frac{i}{2\pi} \partial\bar{\partial} \|W_2\|^2 \right)^q, \end{aligned}$$

which give volume 1 to the Hermitian unit balls. The Bergman kernel of $Y(q, \Omega; k)$ (resp. $E(p, q, \Omega; k)$) is considered with respect to the volume form $\omega_2(W) \wedge \omega(Z)$ (resp. $\omega_1(W_1) \wedge \omega_2(W_2) \wedge \omega(Z)$), where $\omega = \alpha^n$ is defined by (1.4).

3.2. Automorphisms. We consider here the case where $p = q = 1$ and we write for short

$$\begin{aligned} Y(\Omega; k) &= Y(1, \Omega; k), \\ E(\Omega; k) &= E(1, 1, \Omega; k). \end{aligned}$$

Let $\Phi \in (\text{Aut } \Omega)_0$, the identity component of the automorphism group of Ω , and let $J\Phi(Z) = \det D\Phi(Z)$ be the complex Jacobian of Φ at $Z \in \Omega$. Let Z_0 be the inverse image of 0 under Φ : $\Phi(Z_0) = 0$. From the relations $B(\Phi(Z), \Phi(T)) = d\Phi(Z) \circ B(Z, T) \circ d\Phi(T)^*$, $B(Z, 0) = I$ and $\det B(Z, T) = N(Z, T)^g$, we obtain

$$(3.1) \quad 1 = J\Phi(Z) N(Z, Z_0)^g \overline{J\Phi(Z_0)},$$

which shows that $N(Z, Z_0)$ never vanishes when $Z \in \Omega$. In particular, we have

$$(3.2) \quad J\Phi(Z) = \frac{1}{J\Phi(Z_0)N(Z, Z_0)^g}$$

and

$$(3.3) \quad |J\Phi(Z_0)|^2 = \frac{1}{N(Z_0, Z_0)^g}$$

Consider the holomorphic function $A(\cdot, Z_0)$ defined by

$$(3.4) \quad A(Z, Z_0) = \frac{N(Z_0, Z_0)^{\frac{1}{2}}}{N(Z, Z_0)};$$

up to multiplication by a complex number of modulus 1, $J\Phi(Z)$ is then the g -th power of $A(Z, Z_0)$. As a bounded circled symmetric domain Ω is always convex and $Z \mapsto A(Z, Z_0)$ is a non-vanishing holomorphic function on Ω , the holomorphic function $Z \mapsto \ln A(Z, Z_0)$ is well-defined on Ω (say, with the initial condition that $\ln A(Z_0, Z_0)$ is real); for each $\lambda \in \mathbb{C}$, we then define the holomorphic function $Z \mapsto A(Z, Z_0)^\lambda$ by $A(Z, Z_0)^\lambda = \exp \lambda \ln A(Z, Z_0)$.

Lemma 3.1. *Let Ω be a bounded irreducible circled homogeneous domain of genus g . Then for each $\Phi \in (\text{Aut } \Omega)_0$ and for complex numbers α_1, α_2 of modulus 1, the map $\Psi : E(\Omega; k) \rightarrow E(\Omega; k)$ defined by*

$$(3.5) \quad \Psi(W_1, W_2, Z) = (\alpha_1 A(Z, Z_0) W_1, \alpha_2 A(Z, Z_0)^{\frac{1}{k}} W_2, \Phi(Z)),$$

where $Z_0 = \Phi^{-1}(0)$, is a holomorphic automorphism of $E(\Omega; k)$. Moreover, we have

$$(3.6) \quad |J\Psi(W_1, W_2, Z)| = |A(Z, Z_0)|^{1 + \frac{1}{k} + g}.$$

In particular,

$$(3.7) \quad |J\Psi(W_1, W_2, Z_0)|^2 = N(Z_0, Z_0)^{-1 - \frac{1}{k} - g}.$$

Proof. From (3.2), (3.3), (3.4) we deduce that

$$(3.8) \quad |J\Phi(Z)| = |A(Z, Z_0)|^g.$$

From this, it is easily deduced that Ψ is a holomorphic automorphism of $E(\Omega; k)$. The differential of Ψ has the form

$$d\Psi(W_1, W_2, Z) = \begin{pmatrix} \alpha_1 A(Z, Z_0) & 0 & 0 \\ 0 & \alpha_2 (A(Z, Z_0))^{\frac{1}{k}} & 0 \\ * & * & d\Phi(Z) \end{pmatrix},$$

which implies

$$|J\Psi(W_1, W_2, Z)| = |A(Z, Z_0)|^{1 + \frac{1}{k} + g},$$

that is (3.6); (3.7) then follows from $A(Z_0, Z_0) = N(Z_0, Z_0)^{-\frac{1}{2}}$. \square

As a corollary (or directly along the same lines), we have

Corollary 3.2. *Let Ω be a bounded irreducible circled homogeneous domain of genus g . Then for each $\Phi \in (\text{Aut } \Omega)_0$ and for a complex number α of modulus 1, the map $\Psi_0 : Y(\Omega; k) \rightarrow Y(\Omega; k)$ defined by*

$$\Psi_0(W, Z) = (\alpha A(Z, Z_0)^{\frac{1}{k}} W, \Phi(Z)),$$

where $\Phi(Z_0) = 0$, is a holomorphic automorphism of $Y(\Omega; k)$. We have

$$|J\Psi_0(W, Z)| = |A(Z, Z_0)|^{\frac{1}{k}+g}$$

and

$$(3.9) \quad |J\Psi_0(W, Z_0)|^2 = N(Z_0, Z_0)^{-\frac{1}{k}-g}.$$

3.3. The Bergman kernel of $Y(q, \Omega; k)$. In view of the Principle of inflation, we consider first $Y(\Omega; k) = Y(1, \Omega; k)$. Let \mathcal{K}_Y denote the Bergman kernel of $Y(\Omega; k)$. Assume $\Phi \in (\text{Aut } \Omega)_0$ is an automorphism of Ω which maps $Z \in D$ to 0. Let $W^* = A(Z, Z)^{\frac{1}{k}}W$; then by the transformation law of the Bergman kernel and (3.9), we have

$$\mathcal{K}_Y(W, Z) = \mathcal{K}_Y(W^*, 0) N(Z, Z)^{-\frac{1}{k}-g}.$$

So the determination of \mathcal{K}_Y reduces to computing $\mathcal{K}_Y(W^*, 0)$.

The domain $Y(\Omega; k)$ is clearly semi-Reinhardt in \mathbb{C}^{1+n} ; applying Theorem 2.4, it has a complete orthonormal system of the form

$$\left\{ W^j P_{\ell i}^{(j)}(Z); j \in \mathbb{N}, k \in \mathbb{N}, 1 \leq i \leq m_\ell = \binom{n+\ell-1}{\ell} \right\},$$

where the $P_{\ell i}^{(j)}$'s are polynomials of degree ℓ on V ; in particular, $P_{01}^{(j)}$ is the constant polynomial a_j defined by

$$|a_j|^2 \int_{Y(\Omega; k)} |W^j|^2 \omega_2(W) \wedge \omega(Z) = 1.$$

The Bergman kernel of $Y(\Omega; k)$ at $(W, 0)$ is then

$$\mathcal{K}_Y(W, 0) = \sum \left| W^j P_{\ell i}^{(j)}(0) \right|^2 = \sum |a_j|^2 |W^j|^2.$$

Lemma 3.3.

$$(3.10) \quad |a_j|^2 = \frac{1}{\chi(0) \text{vol } \Omega} (j+1) \chi\left(\frac{j+1}{k}\right).$$

Proof. We have

$$\begin{aligned} |a_j|^{-2} &= \int_{Y(\Omega; k)} |W^j|^2 \omega_2(W) \wedge \omega(Z) \\ &= \int_{Z \in \Omega} \left(\int_{|W|^{2k} < N(Z, Z)} |W|^{2j} \omega_2(W) \right) \omega(Z); \end{aligned}$$

as

$$\int_{|W|^{2k} < R^2} |W|^{2j} \omega_2(W) = 2 \int_0^{R^{1/k}} r^{2j+1} dr = \frac{R^{2(j+1)/k}}{j+1},$$

then, by Theorem 2.3,

$$|a_j|^{-2} = \frac{1}{j+1} \int_{\Omega} N(Z, Z)^{(j+1)/k} \omega(Z) = \frac{1}{j+1} \frac{\chi(0)}{\chi((j+1)/k)} \text{vol } \Omega.$$

This proves (3.10). \square

So we have

$$\mathcal{K}_Y(W, 0) = \sum |a_j|^2 |W^j|^2 = \frac{1}{\chi(0) \operatorname{vol} \Omega} \sum_{j=0}^{\infty} (j+1) \chi \left(\frac{j+1}{k} \right) |W|^{2j}.$$

Let F be defined by

$$F(X) = \sum_{j=0}^{\infty} \left(\frac{j+1}{k} \right) \chi \left(\frac{j+1}{k} \right) X^j;$$

then

$$\mathcal{K}_Y(W, 0) = \frac{k}{\chi(0) \operatorname{vol} \Omega} F(|W|^2)$$

and

$$\begin{aligned} \mathcal{K}_Y(W, Z) &= \mathcal{K}_Y(W^*, 0) N(Z, Z)^{-\frac{1}{k}-g} \\ &= \frac{k}{\chi(0) \operatorname{vol} \Omega} F \left(\frac{|W|^2}{N(Z, Z)^{1/k}} \right) N(Z, Z)^{-\frac{1}{k}-g}. \end{aligned}$$

So we have proved

Theorem 3.4. *The Bergman kernel of $Y(\Omega; k) = Y(1, \Omega; k)$ is*

$$(3.11) \quad \mathcal{K}_Y(W, Z) = \frac{k}{\chi(0) \operatorname{vol} \Omega} F \left(\frac{|W|^2}{N(Z, Z)^{1/k}} \right) N(Z, Z)^{-\frac{1}{k}-g},$$

where

$$(3.12) \quad F(X) = \sum_{j=0}^{\infty} \left(\frac{j+1}{k} \right) \chi \left(\frac{j+1}{k} \right) X^j.$$

As $\left(\frac{j+1}{k} \right) \chi \left(\frac{j+1}{k} \right)$ is a polynomial function of j , F is a rational function of X (a finite linear combination of derivatives of $(1-X)^{-1}$). So $\mathcal{K}_Y(W, Z)$ is the product of the Bergman kernel of Ω :

$$\mathcal{K}_\Omega(Z) = \frac{1}{\operatorname{vol} \Omega} \frac{1}{N(Z, Z)^g}$$

by

$$\frac{k}{\chi(0)} F \left(\frac{|W|^2}{N(Z, Z)^{1/k}} \right) N(Z, Z)^{-\frac{1}{k}} = G(|W|^2, N(Z, Z)^{1/k}),$$

where $G(X_1, X_2)$ is a rational function of X_1, X_2 .

The Bergman kernel of $Y(q, \Omega; k)$ is then obtained from \mathcal{K}_Y by using the Principle of inflation (Theorem 2.5):

Corollary 3.5. *The Bergman kernel of $Y(q, \Omega; k)$ is*

$$(3.13) \quad \mathcal{K}_{Y(q, \Omega; k)}(W, Z) = \frac{1}{q!} \frac{k}{\chi(0) \operatorname{vol} \Omega} F^{(q-1)} \left(\frac{\|W\|^2}{N(Z, Z)^{1/k}} \right) N(Z, Z)^{-\frac{q}{k}-g},$$

where F is the rational function defined by (3.12).

3.4. The Bergman kernel of $E(p, q, \Omega; k)$. Let \mathcal{K}_E denote the Bergman kernel of $E(\Omega; k) = E(1, 1, \Omega; k)$. Assume $\Phi \in (\text{Aut } \Omega)_0$ is an automorphism of Ω which maps $Z \in D$ to 0. Let $W_1^* = A(Z, Z)W_1$, $W_2^* = A(Z, Z)^{\frac{1}{k}}W_2$; then by the transformation law of the Bergman kernel and (3.7), we have

$$\mathcal{K}_E(W_1, W_2, Z) = \mathcal{K}_E(W_1^*, W_2^*, 0) N(Z, Z)^{-1 - \frac{1}{k} - g}.$$

So the determination of \mathcal{K}_E reduces to computing $\mathcal{K}_E(W_1^*, W_2^*, 0)$.

The domain $E(\Omega; k)$ is semi-Reinhardt in \mathbb{C}^{2+n} ; applying Theorem 2.4, it has a complete orthonormal system of the form

$$\left\{ W_1^{j_1} W_2^{j_2} P_{\ell i}^{(j_1, j_2)}(Z); j_1, j_2, \ell \in \mathbb{N}, 1 \leq i \leq m_\ell = \binom{n + \ell - 1}{\ell} \right\},$$

where the $P_{\ell i}^{(j_1, j_2)}$'s are polynomials of degree ℓ ; in particular, $P_{01}^{(j_1, j_2)}$ is the constant polynomial $a_{j_1 j_2}$ defined by

$$|a_{j_1 j_2}|^2 \int_{E(\Omega; k)} \left| W_1^{j_1} W_2^{j_2} \right|^2 \omega_1(W_1) \wedge \omega_2(W_2) \wedge \omega(Z) = 1.$$

The Bergman kernel of $E(\Omega; k)$ at $(W_1, W_2, 0)$ is then

$$(3.14) \quad \begin{aligned} \mathcal{K}_E(W_1, W_2, 0) &= \sum \left| W_1^{j_1} W_2^{j_2} P_{\ell i}^{(j_1, j_2)}(0) \right|^2 \\ &= \sum |a_{j_1 j_2}|^2 \left| W_1^{j_1} \right|^2 \left| W_2^{j_2} \right|^2. \end{aligned}$$

Lemma 3.6. *Let $h = j_1 + 1 + \frac{j_2 + 1}{k}$; then*

$$(3.15) \quad |a_{j_1 j_2}|^2 = \frac{k}{\chi(0) \text{vol } \Omega} \frac{\Gamma(h+1) \chi(h)}{\Gamma(j_1+1) \Gamma\left(\frac{j_2+1}{k}\right)}.$$

Proof. We have

$$\begin{aligned} |a_{j_1 j_2}|^{-2} &= \int_{E(\Omega; k)} \left| W_1^{j_1} W_2^{j_2} \right|^2 \omega_1(W_1) \wedge \omega_2(W_2) \wedge \omega(Z) \\ &= \int_{Z \in \Omega} \left(\int_{|W_1|^2 + |W_2|^{2k} < N(Z, Z)} \left| W_1^{j_1} W_2^{j_2} \right|^2 \omega_1(W_1) \wedge \omega_2(W_2) \right) \omega(Z) \end{aligned}$$

and, by standard computations,

$$\begin{aligned} &\int_{|W_1|^2 + |W_2|^{2k} < R^2} \left| W_1^{j_1} W_2^{j_2} \right|^2 \omega_1(W_1) \wedge \omega_2(W_2) \\ &= 4 \int_{r_1^2 + r_2^{2k} < R^2, r_j \geq 0} r_1^{2j_1+1} r_2^{2j_2+1} dr_1 dr_2 \\ &= \frac{4}{k} \int_{s_1^2 + s_2^2 < R^2, s_j \geq 0} s_1^{2j_1+1} s_2^{\frac{2(j_2+1)}{k}-1} ds_1 ds_2 \\ &= \frac{4}{k} \int_0^R \rho^{2j_1 + \frac{2(j_2+1)}{k} + 1} d\rho \int_0^{\pi/2} \cos^{2j_1+1} \theta \sin^{\frac{2(j_2+1)}{k}-1} \theta d\theta \\ &= \frac{4}{k} \frac{R^{2(j_1+1 + \frac{j_2+1}{k})}}{2 \Gamma(j_1+1) \Gamma\left(\frac{j_2+1}{k}\right)} \\ &= \frac{1}{k} \frac{\Gamma(j_1+1) \Gamma\left(\frac{j_2+1}{k}\right)}{\Gamma\left(j_1+1 + \frac{j_2+1}{k} + 1\right)} R^{2(j_1+1 + \frac{j_2+1}{k})}. \end{aligned}$$

Let $h = j_1 + 1 + \frac{j_2+1}{k}$; then, by Theorem 2.3,

$$\begin{aligned} |a_{j_1 j_2}|^{-2} &= \frac{1}{k} \frac{\Gamma(j_1 + 1) \Gamma\left(\frac{j_2+1}{k}\right)}{\Gamma(h + 1)} \int_{\Omega} N(Z, Z)^h \omega(Z) \\ &= \frac{1}{k} \frac{\Gamma(j_1 + 1) \Gamma\left(\frac{j_2+1}{k}\right)}{\Gamma(h + 1)} \frac{\chi(0)}{\chi(h)} \text{vol } \Omega, \end{aligned}$$

which proves (3.15). \square

From (3.15) and (3.14), we obtain

$$\mathcal{K}_E(W_1, W_2, 0) = \frac{1}{\text{vol } \Omega} \frac{k}{\chi(0)} \sum_{j_1, j_2=0}^{\infty} \frac{\Gamma(h + 1) \chi(h)}{\Gamma(j_1 + 1) \Gamma\left(\frac{j_2+1}{k}\right)} |W_1^{j_1}|^2 |W_2^{j_2}|^2.$$

Let b_j ($1 \leq j \leq n + 2$) be the constants defined by

$$h(h - 1) \chi(h) = \sum_{j=1}^{n+2} b_j (h + 1)_j = \sum_{j=1}^{n+2} b_j \frac{\Gamma(h + j + 1)}{\Gamma(h + 1)};$$

so

$$\begin{aligned} |a_{j_1 j_2}|^2 &= \frac{1}{\text{vol } \Omega} \frac{k}{\chi(0)} \frac{\Gamma(h + 1) \chi(h)}{\Gamma(j_1 + 1) \Gamma\left(\frac{j_2+1}{k}\right)} \\ &= \frac{k}{\chi(0) \text{vol } \Omega} \sum_{j=1}^{n+2} b_j \frac{\Gamma(h + j + 1) \Gamma(h - 1)}{\Gamma(h + 1) \Gamma(j_1 + 1) \Gamma\left(\frac{j_2+1}{k}\right)} \end{aligned}$$

and

$$\begin{aligned} \mathcal{K}_E(W_1, W_2, 0) &= \frac{1}{\text{vol } \Omega} \frac{k}{\chi(0)} \\ &\quad \sum_{j_1, j_2=0}^{\infty} \sum_{j=1}^{n+2} b_j \frac{\Gamma(h + j + 1) \Gamma(h - 1)}{\Gamma(h + 1) \Gamma(j_1 + 1) \Gamma\left(\frac{j_2+1}{k}\right)} |W_1^{j_1}|^2 |W_2^{j_2}|^2. \end{aligned}$$

Let $s = (j_2 + 1)/k$, $t_1 = |W_1|^2$, $t_2 = |W_2|^2$, then we have

$$\mathcal{K}_E(W_1, W_2, 0) = \Lambda(t_1, t_2)$$

where

$$(3.16) \quad \Lambda(t_1, t_2) = \frac{1}{\text{vol } \Omega} \frac{k}{\chi(0)} \sum_{j=1}^{n+2} b_j \sum_{j_2=0}^{\infty} \left(\sum_{j_1=0}^{\infty} \frac{\Gamma(h + j + 1) \Gamma(j_1 + s)}{\Gamma(h + 1) \Gamma(j_1 + 1) \Gamma(s)} t_1^{j_1} \right) t_2^{j_2}.$$

But

$$\begin{aligned} \sum_{j_1=0}^{\infty} \frac{\Gamma(h + j + 1) \Gamma(j_1 + s)}{\Gamma(h + 1) \Gamma(j_1 + 1) \Gamma(s)} t_1^{j_1} &= \sum_{j_1=0}^{\infty} \frac{\Gamma(j_1 + j + 2 + s) \Gamma(j_1 + s)}{\Gamma(j_1 + 2 + s) \Gamma(j_1 + 1) \Gamma(s)} t_1^{j_1} \\ &= \frac{\Gamma(j + 2 + s)}{\Gamma(s + 2)} {}_2F_1(s + 2 + j, s; s + 2; t_1) \end{aligned}$$

where ${}_2F_1(a, b; c; z)$ denotes the Gauss hypergeometric function

$${}_2F_1(a, b; c; z) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{z^m}{\Gamma(m + 1)}.$$

By Euler's transformation formula (see [10, (1.2.10)], or [7]):

$${}_2F_1(a, b; c; z) = (1 - z)^{c-a-b} {}_2F_1(c - a, c - b; c; z),$$

we get

$${}_2F_1(s + 2 + j, s; s + 2; t_1) = (1 - t_1)^{-s-j} {}_2F_1(-j, 2; s + 2; t_1),$$

where ${}_2F_1(-j, 2; s + 2; t_1)$ is in fact a polynomial of degree j :

$${}_2F_1(-j, 2; s + 2; t_1) = \sum_{m=0}^j \frac{(-j)_m (2)_m}{(s + 2)_m} \frac{t_1^m}{\Gamma(m + 1)}.$$

(It can be expressed with help of a Jacobi polynomial, but we will not use this fact here.) This proves

$$(3.17) \quad \sum_{j_1=0}^{\infty} \frac{\Gamma(h + j + 1) \Gamma(j_1 + s)}{\Gamma(h + 1) \Gamma(j_1 + 1) \Gamma(s)} t_1^{j_1} \\ = \Gamma(j + 2 + s) (1 - t_1)^{-s-j} \sum_{m=0}^j \frac{(-j)_m (2)_m}{\Gamma(s + 2 + m)} \frac{t_1^m}{\Gamma(m + 1)}.$$

From (3.16) and (3.17), we deduce

$$\Lambda(t_1, t_2) = \frac{1}{\text{vol } \Omega} \frac{k}{\chi(0)} (1 - t_1)^{-1/k} \sum_{j=1}^{n+2} b_j (1 - t_1)^{-j} \\ \sum_{m=0}^j (-j)_m (2)_m \frac{t_1^m}{\Gamma(m + 1)} \sum_{j_2=0}^{\infty} \frac{\Gamma(j + 2 + \frac{j_2+1}{k})}{\Gamma(\frac{j_2+1}{k} + 2 + m)} \left(\frac{t_2}{(1 - t_1)^{1/k}} \right)^{j_2}.$$

Consider now, for $0 \leq m \leq j$, the function H_{jm} defined by

$$H_{jm}(\lambda) = \sum_{p=0}^{\infty} \frac{\Gamma(j + 2 + \frac{p+1}{k})}{\Gamma(\frac{p+1}{k} + 2 + m)} \lambda^p = \sum_{p=0}^{\infty} \left(\frac{p+1}{k} + 2 + m \right)_{j-m} \lambda^p;$$

as $\left(\frac{p+1}{k} + 2 + m \right)_{j-m}$ is a polynomial of p , H_{jm} is a rational function of λ (more precisely, a finite linear combination of derivatives of the function $(1 - \lambda)^{-1}$). So we finally obtain:

Theorem 3.7. *The Bergman kernel \mathcal{K}_E of $E(\Omega; k)$ is*

$$(3.18) \quad \mathcal{K}_E(W_1, W_2, Z) = \Lambda \left(\frac{|W_1|^2}{N(Z, Z)}, \frac{|W_2|^2}{N(Z, Z)^{1/k}} \right) N(Z, Z)^{-1 - \frac{1}{k} - g},$$

where Λ is defined by

$$(3.19) \quad \Lambda(t_1, t_2) = \frac{1}{\text{vol } \Omega} \frac{k}{\chi(0)} (1 - t_1)^{-1/k} H \left(t_1, \frac{t_2}{(1 - t_1)^{1/k}} \right), \\ H(t_1, \lambda) = \sum_{j=1}^{n+2} b_j (1 - t_1)^{-j} \sum_{m=0}^j (-j)_m (2)_m \frac{t_1^m}{\Gamma(m + 1)} H_{jm}(\lambda), \\ H_{jm}(\lambda) = \sum_{p=0}^{\infty} \left(\frac{p+1}{k} + 2 + m \right)_{j-m} \lambda^p,$$

$$h(h-1)\chi(h) = \sum_{j=1}^{n+2} b_j (h+1)_j.$$

We see that $\mathcal{K}_E(W_1, W_2, Z)$ is the product of the Bergman kernel of Ω :

$$\mathcal{K}_\Omega(Z) = \frac{1}{\text{vol } \Omega} \frac{1}{N(Z, Z)^g}$$

by the function

$$\frac{k}{\chi(0)} H \left(\frac{|W_1|^2}{N(Z, Z)}, \frac{|W_2|^2}{(N(Z, Z) - |W_1|^2)^{1/k}} \right) N(Z, Z)^{-1 - \frac{1}{k}},$$

where $H(X_1, X_2)$ is a rational function of X_1, X_2 .

The Bergman kernel $\mathcal{K}_{E(p, q, \Omega; k)}$ of $E(p, q, \Omega; k)$ is then obtained from \mathcal{K}_E using twice the Principle of inflation (Theorem 2.5):

Corollary 3.8. *The Bergman kernel of $E(p, q, \Omega; k)$ is*

$$\begin{aligned} & \mathcal{K}_{E(p, q, \Omega; k)}(W_1, W_2, Z) \\ &= \frac{1}{p!q!} \Lambda^{(p-1, q-1)} \left(\frac{\|W_1\|^2}{N(Z, Z)}, \frac{\|W_2\|^2}{N(Z, Z)^{1/k}} \right) N(Z, Z)^{-p - \frac{q}{k} - g}, \end{aligned}$$

where

$$\Lambda^{(p-1, q-1)}(t_1, t_2) = \frac{\partial^{p+q-2} \Lambda}{\partial t_1^{p-1} \partial t_2^{q-1}}(t_1, t_2)$$

is the partial derivative of the function Λ defined by (3.19).

4. TABLES

The following examples exhaust the list of simple Hermitian positive Jordan triple systems (see [17]). The HPJTS occurring in the four infinite series $I_{p, q}$, II_n , III_n , IV_n are called *classical*; the two HPJTS of type V and VI are called *exceptional*. There is some overlapping between the classical series, due to a finite number of isomorphisms in low dimension. We give hereunder for each type:

- the definition of the space V , its Jordan triple product, and the corresponding bounded circled homogeneous domain;
- the generic norm;
- the numerical invariants $r, a, b, g = 2 + a(r-1) + b$;
- the polynomial

$$\chi(s) = \prod_{j=1}^r \left(s + 1 + (j-1) \frac{a}{2} \right)_{1+b+(r-j)a}.$$

from which one can deduce the behavior of the function

$$s \mapsto \int_{\Omega} N(x, x)^s \alpha^n$$

and the Bergman kernels of $Y(q, \Omega; k)$ and $E(p, q, \Omega; k)$.

4.1. **Type I** $_{m,n}$ ($1 \leq m \leq n$). $V = \mathcal{M}_{m,n}(\mathbb{C})$ (space of $m \times n$ matrices with complex entries), endowed with the triple product

$$\{xyz\} = x^t \bar{y}z + z^t \bar{y}x.$$

The domain Ω is the set of $m \times n$ matrices x such that $I_m - x^t \bar{x}$ is definite positive. The generic minimal polynomial is

$$m(T, x, y) = \text{Det}(TI_m - x^t \bar{y}),$$

where Det is the usual determinant of square matrices. The numerical invariants are $r = m$, $a = 2$, $b = n - m$, $g = m + n$. These HPJTS are of tube type only for $m = n$.

Applying (2.3), we obtain the polynomial χ :

$$\chi(s) = \prod_{j=1}^m (s + j)_{m+n+1-2j},$$

which may also be written

$$\chi(s) = \prod_{j=1}^m (s + j)_n.$$

4.2. **Type II** $_n$ ($n \geq 2$). $V = \mathcal{A}_n(\mathbb{C})$ (space of $n \times n$ alternating matrices) with the same triple product as for Type I. The domain Ω is the set of $n \times n$ alternating matrices x such that $I_n + x \bar{x}$ is definite positive.

4.2.1. *Type II* $_{2p}$ ($n = 2p$ even). The generic minimal polynomial is here given by

$$m(T, x, y)^2 = \text{Det}(TI_n + x \bar{y}).$$

The numerical invariants are $r = \frac{n}{2} = p$, $a = 4$, $b = 0$, $g = 2(n - 1)$; these HPJTS are of tube type.

The polynomial χ is

$$\chi(s) = \prod_{j=1}^p (s + 2j - 1)_{1+4(p-j)} = \prod_{j=1}^p (s + 2j - 1)_{2p-1}.$$

4.2.2. *Type II* $_{2p+1}$ ($n = 2p + 1$ odd). The generic minimal polynomial is given by

$$Tm(T, x, y)^2 = \text{Det}(TI_n + x \bar{y}).$$

The numerical invariants are $r = \lceil \frac{n}{2} \rceil = p$, $a = 4$, $b = 2$, $g = 2(n - 1)$; these HPJTS are not of tube type.

The polynomial χ is

$$\chi(s) = \prod_{j=1}^p (s + 2j - 1)_{3+4(p-j)} = \prod_{j=1}^p (s + 2j - 1)_{2p+1}.$$

4.3. **Type III_n** ($n \geq 1$). $V = \mathcal{S}_n(\mathbb{C})$ (space of $n \times n$ symmetric matrices) with the same triple product as for Type I. The domain Ω is the set of $n \times n$ symmetric matrices x such that $I_n - x\bar{x}$ is definite positive. The generic minimal polynomial is

$$m(T, x, y) = \text{Det}(TI_n - x\bar{y}).$$

The numerical invariants are $r = n$, $a = 1$, $b = 0$, $g = n + 1$. These HPJTS are of tube type.

The polynomial χ is

$$\chi(s) = \prod_{j=1}^n \left(s + \frac{j+1}{2} \right)_{1+n-j}.$$

4.4. **Type IV_n** ($n \neq 2$). $V = \mathbb{C}^n$ with the quadratic operator defined by

$$Q(x)y = q(x, \bar{y})x - q(x)\bar{y},$$

where $q(x) = \sum x_i^2$, $q(x, y) = 2 \sum x_i y_i$. The domain Ω is the set of points $x \in \mathbb{C}^n$ such that

$$1 - q(x, \bar{x}) + |q(x)|^2 > 0, \quad 2 - q(x, \bar{x}) > 0.$$

The generic minimal polynomial is

$$m(T, x, y) = T^2 - q(x, \bar{y}) + q(x)q(\bar{y}).$$

The numerical invariants are $r = 2$, $a = n - 2$, $b = 0$, $g = n$. These HPJTS are of tube type.

The polynomial χ is

$$\chi(s) = (s+1)_{n-1} \left(s + \frac{n}{2} \right).$$

4.5. **Type V**. $V = \mathcal{M}_{2,1}(\mathbb{O}_{\mathbb{C}})$, the subspace of $\mathcal{H}_3(\mathbb{O}_{\mathbb{C}})$ consisting in matrices of the form

$$\begin{pmatrix} 0 & a_3 & \tilde{a}_2 \\ \tilde{a}_3 & 0 & 0 \\ a_2 & 0 & 0 \end{pmatrix}$$

with the same quadratic operator as for type VI (see below). Here \tilde{a} denotes the Cayley conjugate of $a \in \mathbb{O}_{\mathbb{C}}$. The generic minimal polynomial is

$$m(T, x, y) = T^2 - (x|y)T + (x^\sharp|y^\sharp).$$

The domain Ω is the “exceptional domain of dimension 16” defined by

$$1 - (x|x) + (x|x^\sharp) > 0, \quad 2 - (x|x) > 0.$$

The numerical invariants are $r = 2$, $a = 6$, $b = 4$, $g = 12$. This HPJTS is not of tube type.

The polynomial χ is

$$\chi(s) = \prod_{j=1}^r \left(s + 1 + (j-1)\frac{a}{2} \right)_{1+b+(r-j)a}.$$

$$\chi(s) = (s+1)_{11} (s+4)_5;$$

it can also be written

$$\chi(s) = (s+1)_8 (s+4)_8.$$

4.6. **Type VI.** $V = \mathcal{H}_3(\mathbb{O}_{\mathbb{C}})$, the space of 3×3 matrices with entries in the space $\mathbb{O}_{\mathbb{C}}$ of octonions over \mathbb{C} , which are Hermitian with respect to the Cayley conjugation; the quadratic operator is defined by

$$Q(x)y = (x|y)x - x^{\sharp} \times \bar{y},$$

where \times denotes the Freudenthal product, x^{\sharp} the adjoint matrix in $\mathcal{H}_3(\mathbb{O}_{\mathbb{C}})$ and $(x|y)$ the standard Hermitian product in $\mathcal{H}_3(\mathbb{O}_{\mathbb{C}})$ (for details, see [20]). The domain Ω is the “exceptional domain of dimension 27” defined by

$$\begin{aligned} 1 - (x|x) + (x^{\sharp}|x^{\sharp}) - |\det x|^2 &> 0, \\ 3 - 2(x|x) + (x^{\sharp}|x^{\sharp}) &> 0, \\ 3 - (x|x) &> 0. \end{aligned}$$

The generic minimal polynomial is

$$m(T, x, y) = T^3 - (x|y)T^2 + (x^{\sharp}|y^{\sharp})T - \det x \det \bar{y},$$

where \det denotes the determinant in $\mathcal{H}_3(\mathbb{O}_{\mathbb{C}})$. The numerical invariants are $r = 3$, $a = 8$, $b = 0$, $g = 18$. This HPJTS is of tube type.

The polynomial χ is

$$\begin{aligned} \chi(s) &= (s+1)_{17}(s+5)_9(s+9) \\ &= (s+1)_9(s+5)_9(s+9)_9. \end{aligned}$$

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