LIE G-TORI OF SYMPLECTIC TYPE

GEORGIA BENKART[1,*] AND YOJI YOSHII[2]

Abstract. We classify centerless Lie G-tori of type C_r including the most difficult case $r = 2$ by applying techniques due to Seligman. In particular, we show that the coordinate algebra of a Lie G-torus of type C_2 is either an associative G-torus with involution or is a Clifford G-torus. Our results generalize the classification of the core of the extended affine Lie algebras of type C_r by Allison and Gao.

Dedicated to Professor George Seligman with admiration

1. Introduction

The extended affine Lie algebras of [AABGP] are natural generalizations of the affine and toroidal Lie algebras, which have played such a pivotal role in diverse areas of mathematics and physics. Their root systems (the so-called extended affine root systems) are essential in the work of Saito ([S1], [S2]) and Slodowy [Sl] on singularities. An extended affine Lie algebra $\mathcal{E}$ possesses a nondegenerate invariant symmetric bilinear form and a finite-dimensional self-centralizing ad-diagonalizable subalgebra $\mathcal{H}$. The root system R of $\mathcal{E}$ relative to $\mathcal{H}$ has a decomposition $R = R^0 \cup R^\times$ into isotropic roots R^0 and nonisotropic roots $R^\times$. The subalgebra $\mathcal{E}_c$ generated by the root spaces corresponding to the nonisotropic roots is an ideal of $\mathcal{E}$, called the *core* of $\mathcal{E}$. The algebra $\mathcal{E}$ is said to be *tame* if the centralizer of $\mathcal{E}_c$ in $\mathcal{E}$ is just the center $\mathcal{Z}(\mathcal{E}_c)$ of $\mathcal{E}_c$.

The core features prominently in the classification of the tame extended affine Lie algebras (see [AABGP], [BGK], [BGKN], [AG], [AY], [ABG3]). It is graded by a finite (possibly nonreduced) irreducible root system Δ and is root-graded in the sense of [BM] and [ABG2]. It also has a grading by the

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[*]Corresponding author
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free abelian group Λ generated by R^0. Moreover, $\mathcal{E}_c/\mathcal{Z}(\mathcal{E}_c)$ is what is now referred to as a centerless Lie torus (as in [N1] and [N2]), and every centerless Lie torus is the centerless core of an extended affine Lie algebra (see [Y6]). The family of centerless Lie tori consists of mostly infinite-dimensional Lie algebras, but these Lie algebras are well-behaved counterparts of the finite-dimensional split simple Lie algebras. Roughly speaking, the axioms that characterize tame extended affine Lie algebras and that get used heavily in the determination of their cores are replaced in the notion of a Lie torus by gradings by an irreducible root system Δ and by a free abelian group Λ of finite rank, by a certain property that is sometimes called the division property, and by the assumption that the homogeneous pieces have dimension at most 1. When the rank of Λ is ≥ 1, the Lie algebra is infinite-dimensional, but when it is 0, the algebra is just a finite-dimensional split simple Lie algebra.

In this paper, we study Lie G-tori, where the free abelian group Λ in the definition of a Lie torus is replaced by an arbitrary abelian group G. Lie G-tori were first introduced by the second author in [Y5] and [Y6] as a special class of root-graded Lie algebras. The classification of Lie G-tori of type A_r can be easily derived from results in [BGK], [BGKN], [Y1], [Y2], [Y4] or [AY]. They are coordinatized by G-tori (in the sense of Definition 4.1 below), which are associative when $r \geq 3$, alternative when $r = 2$, and Jordan when $r = 1$. The Lie G-tori of type B_r ($r \geq 3$) were determined in [Y5], where it was shown that their coordinate algebras are a special type of Jordan G-torus, called a Clifford G-torus. Our aim is to classify centerless Lie G-tori of type $\mathrm{B}_2 = \mathrm{C}_2$ together with those of type C_r for higher rank $r \geq 3$. Lie G-tori of type F_4, G_2, and BC_r have not yet been classified, although the centerless cores of extended affine Lie algebras (the centerless Lie tori) of types F_4, G_2, and BC_r have been determined in [AG], [AY], [ABG3], [AFY], and [F].

2. Preparation

Throughout we will assume that all algebras are over a field $\mathbb{F}$ of characteristic zero.

Let Δ be a finite irreducible root system (not necessarily reduced) as in [B, Chap. VI, §1.1]. For each root $\mu \in \Delta$, let $\mu^\vee$ be the corresponding coroot so that $\langle \nu, \mu^\vee \rangle = 2(\nu, \mu)/(\mu, \mu)$ is the Cartan integer for all $\nu \in \Delta$.

Set $\Delta_{\text{ind}} = \{\mu \in \Delta \mid \frac{1}{2}\mu \notin \Delta\}$. Let $G = (G, +, 0)$ be an additive abelian group. For any subset S of G, we denote the subgroup of G that S generates by $\langle S \rangle$.

Definition 2.1. A Lie algebra $\mathcal{L}$ is a *Lie G-torus of type* Δ if

(1) $\mathcal{L}$ has a decomposition into subspaces $\mathcal{L} = \bigoplus_{\mu \in \Delta \cup \{0\},\, g \in G} \mathcal{L}_\mu^g$ such that $[\mathcal{L}_\mu^g, \mathcal{L}_\nu^h] \subseteq \mathcal{L}_{\mu+\nu}^{g+h}$;

(2) For every $g \in G$, $\mathcal{L}_0^g = \sum_{\mu \in \Delta, h \in G} [\mathcal{L}_\mu^h, \mathcal{L}_{-\mu}^{g-h}]$;

(3) (a) For each nonzero $x \in \mathcal{L}_\mu^g$ ($\mu \in \Delta, g \in G$), there exists a $y \in \mathcal{L}_{-\mu}^{-g}$ so that $t := [x, y] \in \mathcal{L}_0^0$ satisfies $[t, z] = \langle \nu, \mu^\vee \rangle z$ for all $z \in \mathcal{L}_\nu^h$, ($\nu \in \Delta \cup \{0\}, h \in G$).
    (b) $\dim \mathcal{L}_\mu^g \leq 1$ and $\dim \mathcal{L}_\mu^0 = 1$ if $\mu \in \Delta_{\text{ind}}$;

(4) $G = \langle \operatorname{supp} \mathcal{L} \rangle$ where

$$\operatorname{supp} \mathcal{L} := \{g \in G \mid \mathcal{L}_\mu^g \neq 0 \text{ for some } \mu \in \Delta \cup \{0\}\}.$$

2.2. Remarks on Definition 2.1

Condition (4) is simply a convenience. If it fails to hold, we may replace G by the subgroup generated by $\operatorname{supp} \mathcal{L}$.

It follows from (1) that $\mathcal{L}$ is graded by the group G. Thus, if $\mathcal{L}^g := \bigoplus_{\mu \in \Delta \cup \{0\}} \mathcal{L}_\mu^g$, then $\mathcal{L} = \bigoplus_{g \in G} \mathcal{L}^g$ and $[\mathcal{L}^g, \mathcal{L}^h] \subseteq \mathcal{L}^{g+h}$.

The Lie algebra $\mathcal{L}$ also admits a grading by the root lattice $Q(\Delta)$: if $\mathcal{L}_\lambda := \bigoplus_{g \in G} \mathcal{L}_\lambda^g$ for $\lambda \in Q(\Delta)$, where $\mathcal{L}_\lambda^g = 0$ if $\lambda \notin \Delta \cup \{0\}$, then $\mathcal{L} = \bigoplus_{\lambda \in Q(\Delta)} \mathcal{L}_\lambda$ and $[\mathcal{L}_\lambda, \mathcal{L}_\mu] \subseteq \mathcal{L}_{\lambda+\mu}$.

From (3) we see for $\mu \in \Delta_{\text{ind}}$ that there exist elements $e_\mu \in \mathcal{L}_\mu^0$, $f_\mu \in \mathcal{L}_{-\mu}^0$, and $t_\mu := [e_\mu, f_\mu]$ so that $[t_\mu, z] = \langle \nu, \mu^\vee \rangle z$ for all $z \in \mathcal{L}_\nu^h$, ($\nu \in \Delta, h \in G$). Thus, the elements e_μ, f_μ, t_μ determine a canonical basis for a copy of the Lie algebra $\mathfrak{sl}_2$. In addition, the products $[t_\lambda, t_\mu] = 0$ for $\lambda, \mu \in \Delta_{\text{ind}}$. It follows that the subalgebra $\mathfrak{g}$ of $\mathcal{L}$ generated by the subspaces $\mathcal{L}_\mu^0$ for $\mu \in \Delta_{\text{ind}}$ is a split simple Lie algebra with split Cartan subalgebra $\mathfrak{h} := \sum_{\mu \in \Delta_{\text{ind}}} [\mathcal{L}_\mu^0, \mathcal{L}_{-\mu}^0]$. We may identify the coroot $\mu^\vee$ with an element of $\mathfrak{h}$. Then t_μ is equivalent to $\mu^\vee$ modulo the center $\mathcal{Z}(\mathcal{L})$ of $\mathcal{L}$. Condition (3a) (or equivalently, existence of canonical $\mathfrak{sl}_2$-basis elements $e_\mu \in \mathcal{L}_\mu^0, f_\mu \in \mathcal{L}_{-\mu}^0$, and $t_\mu = [e_\mu, f_\mu]$ such that $t_\mu \equiv \mu^\vee \mod \mathcal{Z}(\mathcal{L})$ for each $\mu \in \Delta_{\text{ind}}$) is often called the *division property*, and $\mathcal{L}$ is said to be *division graded.*

It follows then that a Lie G-torus $\mathcal{L}$ is a Lie algebra graded by the root system Δ in the following sense:

Definition 2.3. A Lie algebra $\mathcal{L}$ is said to be *graded by the root system* Δ (where Δ is finite and irreducible) or to be Δ-*graded* if

(Δ1) $\mathcal{L}$ contains as a subalgebra a finite-dimensional split "simple" Lie algebra $\mathfrak{g}$, called the *grading subalgebra*, with root system $\Delta_{\mathfrak{g}}$ relative to a split Cartan subalgebra $\mathfrak{h}$;
(Δ2) $\mathcal{L}$ has a decomposition into subspaces $\mathcal{L} = \bigoplus_{\mu\in\Delta\cup\{0\}} \mathcal{L}_\alpha$, where $\mathcal{L}_\mu = \{v \in \mathcal{L} \mid [t, v] = \mu(t)v \text{ for all } t \in \mathfrak{h}\}$.
(Δ3) $\mathcal{L}_0 = \sum_{\mu\in\Delta} [\mathcal{L}_\mu, \mathcal{L}_{-\mu}]$;
(Δ4) either Δ is reduced and equals the root system $\Delta_{\mathfrak{g}}$ of $(\mathfrak{g}, \mathfrak{h})$ or $\Delta = \mathrm{BC}_r$ and $\Delta_{\mathfrak{g}}$ is of type B_r, C_r, or D_r.

The word simple is in quotes above, because in all instances except two, $\mathfrak{g}$ is a simple Lie algebra. The sole exceptions are when Δ is of type BC_2, $\Delta_{\mathfrak{g}}$ is of type $\mathrm{D}_2 = \mathrm{A}_1\times\ \mathrm{A}_1$, and $\mathfrak{g}$ is the direct sum of two copies of $\mathfrak{sl}_2$; or when Δ is of type BC_1, $\Delta_{\mathfrak{g}}$ is of type D_1, and $\mathfrak{g} = \mathfrak{h}$ is one-dimensional. Neither of these exceptions will play a role in this work.

The definition above is due to Berman-Moody [BM] for the case $\Delta = \Delta_{\mathfrak{g}}$. The extension to the nonreduced root systems BC_r was developed by Allison-Benkart-Gao in [ABG2] for $r \geq 2$ and by Benkart-Smirnov in [BS] for $r = 1$.

A Lie G-torus $\mathcal{L}$ of type BC_r has grading subalgebra $\mathfrak{g}$ generated by the root spaces $\mathcal{L}_\mu^g$ for $\mu \in \Delta_{\mathrm{ind}}$, and so $\mathfrak{g}$ will be of type B_r, since those root spaces have dimension one by (3b) of Definition 2.1.

The original definition of a Lie G-torus in [Y5] required the Lie algebra $\mathcal{L}$ to be Δ-graded. As mentioned above, this holds automatically.

Sometimes in what follows, we stipulate that a Lie algebra is (Δ, G)-*graded*. By that we mean it is a Δ-graded Lie algebra $\mathcal{L}$ which is also G-graded, $\mathcal{L} = \bigoplus_{g\in G} \mathcal{L}^g$, such that the grading subalgebra $\mathfrak{g}$ of $\mathcal{L}$ is contained in $\mathcal{L}^0$ and the support $\{g \in G \mid \mathcal{L}^g \neq 0\}$ generates G. It follows that every (Δ, G)-graded Lie algebra has a decomposition $\mathcal{L} = \bigoplus_{\mu\in\Delta\cup\{0\},\, g\in G} \mathcal{L}_\mu^g$, where $\mathcal{L}_\mu^g = \mathcal{L}_\mu \cap \mathcal{L}^g$, and that condition (4) of Definition 2.1 holds.

In [BN, Defn. 3.12], a Lie G-torus is defined to be a (Δ, G)-graded Lie algebra $\mathcal{L}$ satisfying (3) of Definition 2.1. The definition in [BN] permits the grading subalgebra to be type C_r $(r \geq 1)$ or D_r $(r \geq 3)$ when Δ is of type BC_r.

2.4. Lie algebras graded by C_r, $r \geq 2$.

We specialize now to Lie algebras graded by the root systems C_r, $r \geq 2$, since ultimately we intend to classify the centerless Lie G-tori of type C_r. Towards this purpose, let V be a $2r$-dimensional vector space over $\mathbb{F}$ with a nondegenerate skew-symmetric bilinear form. Let $\{v_1, \ldots, v_{2r}\}$ be a basis of V, and let $\mathfrak{g}$ denote the symplectic Lie algebra $\mathfrak{sp}(V)$ of skew endomorphisms of V. Using the basis above, we identify $\mathfrak{g}$ with $\mathfrak{sp}_{2r}(\mathbb{F})$, the Lie algebra of $2r \times 2r$ matrices x over $\mathbb{F}$ that satisfy $x^t M = -Mx$, where M is the matrix whose (i,j)-entry is $\mathrm{sign}(i-j)\delta_{i+j,2r+1}$ for $1 \leq i,j \leq 2r$. We also identify a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ with the set of diagonal matrices in $\mathfrak{g}$. Thus, the elements $\{E_{1,1} - E_{2r,2r}, \ldots, E_{r,r} - E_{r+1,r+1}\}$ determine a basis for $\mathfrak{h}$, where the $E_{i,j}$ are the standard matrix units. Let $\{\varepsilon_1, \ldots, \varepsilon_r\}$ be the dual basis in $\mathfrak{h}^*$. Then $\mathfrak{g}$ has a decomposition into one-dimensional root spaces relative to $\mathfrak{h}$, and a basis for these root spaces may be taken as follows:

($\mathfrak{g}$1) $E_{i,j} - E_{2r+1-j,2r+1-i}$ for $\varepsilon_i - \varepsilon_j$,
($\mathfrak{g}$2) $E_{i,2r+1-j} + E_{j,2r+1-i}$ for $\varepsilon_i + \varepsilon_j$,
($\mathfrak{g}$3) $E_{2r+1-j,i} + E_{2r+1-i,j}$ for $-\varepsilon_i - \varepsilon_j$

where $1 \leq i,j \leq r$. The corresponding root system Δ decomposes into the set $\Delta_{sh} := \{\pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \leq i \neq j \leq r\}$ of short roots and the set $\Delta_{lg} = \{\pm 2\varepsilon_i \mid 1 \leq i \leq r\}$ of long roots.

Let $\mathfrak{s}$ denote the set of $2r \times 2r$ matrices s of trace zero satisfying $s^t M = Ms$. Then $\mathfrak{s}$ is a $\mathfrak{g}$-module under the action $x.s = [x,s] = xs - sx$ $(x \in \mathfrak{g}, s \in \mathfrak{s})$, and $\mathfrak{s}$ has a decomposition into one-dimensional weight spaces relative to $\mathfrak{h}$. A basis for these weight spaces may be chosen as follows:

($\mathfrak{s}$1) $E_{i,j} + E_{2r+1-j,2r+1-i}$ for $\begin{cases} \varepsilon_i + \varepsilon_j & \text{if } i < j \\ -\varepsilon_i - \varepsilon_j & \text{if } i > j \end{cases}$
($\mathfrak{s}$2) $E_{i,2r+1-j} - E_{j,2r+1-i}$ for $\varepsilon_i - \varepsilon_j$,
($\mathfrak{s}$3) $E_{2r+1-m,i} - E_{2r+1-i,j}$ for $-\varepsilon_i + \varepsilon_j$,

where $1 \leq i \neq j \leq r$.

A C_r-graded Lie algebra $\mathfrak{L}$ decomposes into copies of $\mathfrak{g}$, $\mathfrak{s}$, and the trivial one-dimensional $\mathfrak{g}$-module relative to the adjoint action of the grading subalgebra $\mathfrak{g}$. By collecting isomorphic summands, we may assume there are

$\mathbb{F}$-vector spaces A, B, D so that

$$\mathfrak{L} = (\mathfrak{g} \otimes A) \oplus (\mathfrak{s} \otimes B) \oplus D,$$

where D is the sum of the trivial modules. By [ABG1], there is a symmetric product $\circ$ and a skew-symmetric product $[\cdot,\cdot]$ on $\mathfrak{a} = A \oplus B$ so that $\mathfrak{a}$ with the multiplication

$$\alpha\alpha' = \frac{1}{2}(\alpha \circ \alpha') + \frac{1}{2}[\alpha, \alpha'] \tag{2.5}$$

for $\alpha, \alpha' \in \mathfrak{a}$ is the *coordinate algebra* of $\mathfrak{L}$. The space D is a Lie subalgebra of L, which acts as derivations on $\mathfrak{a}$. When $\mathfrak{L}$ is centerless, then D is spanned by the inner derivations $D_{\alpha,\alpha'}$ for $\alpha, \alpha' \in \mathfrak{a}$. The precise expression for $D_{\alpha,\alpha'}$ depends on the rank and is displayed in (2.7) below. Moreover by [ABG1], the multiplication in a centerless C_r-graded Lie algebra $\mathfrak{L}$ is given by

$$\begin{aligned}
&[x \otimes a, y \otimes a'] = [x, y] \otimes \frac{1}{2} a \circ a' + x \circ y \otimes \frac{1}{2}[a, a'] + \mathfrak{tr}(xy) D_{a,a'} \\
&[x \otimes a, s \otimes b] = x \circ s \otimes \frac{1}{2}[a, b] + [x, s] \otimes \frac{1}{2} a \circ b \\
&[s \otimes b, t \otimes b'] = [s, t] \otimes \frac{1}{2} b \circ b' + s \circ t \otimes \frac{1}{2}[b, b'] + \mathfrak{tr}(st) D_{b,b'} \\
&[d, x \otimes a + s \otimes b] = x \otimes d(a) + s \otimes d(b)
\end{aligned} \tag{2.6}$$

for $x, y \in \mathfrak{g}$, $s, t \in \mathfrak{s}$, $a, a' \in A$, $b, b' \in B$ and $d \in D$, where

$$\begin{aligned}
w \circ z &= wz + zw - \frac{1}{r}\mathfrak{tr}(wz)\,\mathrm{id} \\
[w, z] &= wz - zw
\end{aligned}$$

for all $w, z \in \mathfrak{gl}_{2r}(\mathbb{F})$. Here $\mathfrak{tr}$ denotes the usual matrix trace, and id is the identity matrix. Note that $x \circ y \in \mathfrak{s}$, $x \circ s \in \mathfrak{g}$, $[x, s] \in \mathfrak{s}$, $[s, t] \in \mathfrak{g}$ and $s \circ t \in \mathfrak{s}$ for $x, y \in \mathfrak{g}$, $s, t \in \mathfrak{s}$. There exists a distinguished element $1 \in A$ so that the grading subalgebra $\mathfrak{g}$ of $\mathfrak{L}$ is identified with $\mathfrak{g} \otimes 1$, and $1 \circ \alpha = 2\alpha$ and $[1, \alpha] = 0$ for all $\alpha \in \mathfrak{a}$.

An important remark for the $r = 2$ case is that $s \circ t = 0$ for all $s, t \in \mathfrak{s}$, and so the skew product on B can be defined arbitrarily in that case. This flexibility in defining the skew product is crucial in the determination of the coordinate algebra in Section 5.

By [ABG1] (see also [Se]) $\mathfrak{a}$ under the product (2.5) is an associative algebra for $r \geq 4$ and is an alternative algebra for $r = 3$ with A contained in the nucleus of $\mathfrak{a}$. The linear isomorphism σ, defined as $a^\sigma = a$ and $b^\sigma = -b$ for $a \in A$ and $b \in B$, is an involution of $\mathfrak{a}$ since

$$A \circ A \subseteq A, \quad [A, A] \subseteq B, \quad A \circ B \subseteq B,$$
$$[A, B] \subseteq A, \quad B \circ B \subseteq A, \quad [B, B] \subseteq B.$$

Finally,

$$D_{\alpha,\alpha'} = \begin{cases} \frac{1}{2r}\Big([L_\alpha, L_{\alpha'}] + [R_\alpha, R_{\alpha'}] - [L_\alpha, R_{\alpha'}] + [L_{\alpha^\sigma}, L_{\alpha'^\sigma}] \\ \qquad + [R_{\alpha^\sigma}, R_{\alpha'^\sigma}] + [L_{\alpha^\sigma}, R_{\alpha'^\sigma}]\Big) & \text{for } r \geq 3, \\ \frac{1}{2}\Big([L^+_\alpha, L^+_{\alpha'}] + [L^+_{\alpha^\sigma}, L^+_{\alpha'^\sigma}]\Big) & \text{for } r = 2, \end{cases} \tag{2.7}$$

where L (resp. R) is the left (resp. right) multiplication operator on $\mathfrak{a}$, and L^+ is the left multiplication operator on the plus algebra $\mathfrak{a}^+$, which is $\mathfrak{a}$ with the multiplication

$$\alpha \cdot \alpha' := \frac{1}{2}\alpha \circ \alpha' \tag{2.8}$$

for $\alpha, \alpha' \in \mathfrak{a}$.

2.9. Examples of C_r-graded Lie algebras

Suppose $\mathfrak{a}$ is an algebra with unit element 1 and with product denoted by juxtaposition. Set $\alpha \circ \alpha' = \alpha\alpha' + \alpha'\alpha$ and $[\alpha, \alpha'] = \alpha\alpha' - \alpha'\alpha$ for all $\alpha, \alpha' \in \mathfrak{a}$. Assume $\mathfrak{a}$ has an involution σ, and A (resp. B) is the set of symmetric (resp. skew-symmetric) elements of $\mathfrak{a}$ relative to σ. Let $\mathfrak{g}, \mathfrak{s}$ be as in the previous section, and define $D_{\alpha,\alpha'}$ as in (2.7). If $\mathfrak{L} = (\mathfrak{g} \otimes A) \oplus (\mathfrak{s} \otimes B) \oplus D_{\mathfrak{a},\mathfrak{a}}$ under the multiplication in (2.6) is a Lie algebra, then we denote it by $\mathfrak{sp}_{2r}(\mathfrak{a})$. In particular, if $\mathfrak{a}$ is any unital associative algebra with involution having symmetric elements A and skew elements B, then $\mathfrak{sp}_{2r}(\mathfrak{a})$ is a centerless C_r-graded Lie algebra for any $r \geq 2$, and any centerless C_r-graded Lie algebra for $r \geq 4$ is isomorphic to $\mathfrak{sp}_{2r}(\mathfrak{a})$ for some unital associative algebra $\mathfrak{a}$ with involution. The centerless C_3-graded Lie algebras are exactly the Lie algebras $\mathfrak{sp}_6(\mathfrak{a})$, where $\mathfrak{a}$ is a unital alternative algebra with involution whose symmetric elements A lie in the nucleus of $\mathfrak{a}$.

Now suppose A is a commutative, associative algebra with unit element, and let B be a left A-module. Assume there is an A-bilinear symmetric form

$\zeta : B \times B \to A$, and define a multiplication on $\mathfrak{a} = A \oplus B$ by $(a+b)(a'+b') = aa' + \zeta(b,b') + ab' + a'b$. Then $\mathfrak{a}$ with this product is a Jordan algebra (*of Clifford type*). For any Jordan algebra $\mathfrak{a}$ of Clifford type, $\mathfrak{sp}_4(\mathfrak{a})$ is a centerless C_2-graded Lie algebra. There is a construction described in [AG] or [Y3] which results in a B_r-graded Lie algebra $\mathfrak{o}_{2r+1}(\mathfrak{a})$, and when $\mathfrak{a}$ is a Jordan algebra of Clifford type, $\mathfrak{sp}_4(\mathfrak{a}) \cong \mathfrak{o}_5(\mathfrak{a})$.

2.10. $\mathfrak{a}^+$ is a Jordan algebra

The plus algebra $\mathfrak{a}^+ = (\mathfrak{a}, \cdot)$ with product $\alpha \cdot \alpha' = \frac{1}{2}\alpha \circ \alpha'$ for $\alpha, \alpha' \in \mathfrak{a}$ is a Jordan algebra for any alternative algebra $\mathfrak{a}$ (see for example, [M, III, Ex. 3.1]), hence for the coordinate algebra of any C_r-graded Lie algebra, $r \geq 3$ By [ABG1, Secs. 2.45 and 2.48] and [ABG2, Prop. 6.75] (see also [N3, Sec. 4.9]), the coordinate algebra $\mathfrak{a}$ of any C_2-graded Lie algebra can be identified with the half space J_{12} of a Jordan algebra J with a *triangle* (p_1, p_2, q). Here we show for any Jordan algebra J with a triangle that the half space J_{12} under a suitable product (see (2.14)) has the structure of a Jordan algebra. As a consequence, we obtain that $\mathfrak{a}^+$ is a Jordan algebra for the coordinate algebra $\mathfrak{a}$ of any C_2-graded Lie algebra.

Let J be a Jordan algebra with a *triangle* (p_1, p_2, q) and with product denoted by juxtaposition. (Facts about triangles quoted here can be found in [J, Chap. III] or [M, III.6-III.8].) Thus, the elements $p_1, p_2, q \in J$ satisfy

$$p_1^2 = p_1, \quad p_2^2 = p_2, \quad p_1 p_2 = 0, \tag{2.11}$$

$$p_1 q = \frac{1}{2} q, \quad p_2 q = \frac{1}{2} q, \quad \text{and} \quad q^2 = p_1 + p_2 = 1. \tag{2.12}$$

We have the Peirce decomposition $J = J_{11} \oplus J_{12} \oplus J_{22}$ of J, where $J_{11},\ J_{22}$, and J_{12} are the 1, 0, and $\frac{1}{2}$-eigenspaces, respectively, of the left multiplication operator L_{p_1} of the idempotent p_1. These spaces have the following multiplication properties:

$$J_{11}J_{11} \subseteq J_{11}, \quad J_{22}J_{22} \subseteq J_{22}, \quad J_{11}J_{22} = 0,$$
$$J_{11}J_{12} + J_{22}J_{12} \subseteq J_{12} \quad \text{and} \quad J_{12}J_{12} \subseteq J_{11} + J_{22}.$$

Also,

$$x_{11}(x_{22}x_{12}) = x_{22}(x_{11}x_{12}) \tag{2.13}$$

for $x_{11} \in J_{11}$, $x_{12} \in J_{12}$, $x_{22} \in J_{22}$. The *connection involution* determined by the triangle is the transformation $\sigma : J \to J$ defined by $\sigma(x) = 2(qx)q - x$. The mapping σ is an automorphism of J of order 2 which stabilizes J_{12}, interchanges J_{11} and J_{22}, and satisfies $\sigma L_q = L_q \sigma = L_q$. We denote its restriction to J_{12} simply by σ. Thus, one can write $J_{12} = J_{12}^{(+)} \oplus J_{12}^{(-)}$, where $J_{12}^{(\pm)}$ is the ± 1-eigenspace for σ, and

$$J_{12}^{(-)} = \{b \in J_{12} \mid qb = 0\}.$$

When J_{12} is the coordinate algebra $\mathfrak{a} = A \oplus B$ of a C_2-graded Lie algebra, then the connection involution is the involution on $\mathfrak{a}$ in the previous section, and $A = J_{12}^{(+)}$ and $B = J_{12}^{(-)}$. Moreover, using the fact that $x_{11} \mapsto x_{11}q$ is a linear isomorphism from J_{11} onto $J_{12}^{(+)}$, we have that the product $\cdot$ on $\mathfrak{a}^+$ is given by

$$(a+b)\cdot(a'+b') = \frac{1}{2}\Big(x_{11}a' + x'_{11}a + x_{11}b' + x_{11}^{\sigma}b' + x'_{11}b + (x'_{11})^{\sigma}b\Big) - (bb')q \tag{2.14}$$

for $a, a' \in A$, $b, b' \in B$, $x_{11}, x'_{11} \in J_{11}$, $a = x_{11}q$ and $a' = x'_{11}q$.

Note that the skew product on B is chosen to be 0 in [ABG1] i.e., $[B, B] = 0$, which is essential for determining the central extensions of a C_2-graded Lie algebra, but for any choice of a skew product on B, the plus product is given by the expression in (2.14).

Theorem 2.15. *Let $J = J_{11} \oplus J_{12} \oplus J_{22}$ be a Jordan algebra with a triangle (p_1, p_2, q). Then the product $\cdot$ on J_{12} defined by (2.14) coincides with the product of the q-isotope $J^{(q)}$ of J on J_{12}, i.e., $(J_{12}, \cdot) = J_{12}^{(q)}$. In particular, $(J_{12}, \cdot)$ is a Jordan algebra.*

Proof. For $u, v \in J$, the product $\cdot_q$ is defined by

$$u \cdot_q v = (uq)v + u(qv) - (uv)q.$$

Thus, if $a = x_{11}q$, $a' = x'_{11}q \in J_{12}^{(+)}$ and $b, b' \in J_{12}^{(-)}$, then

$$\begin{aligned}
&(a+b)\cdot_q(a'+b') \\
&\quad = (aq)(a'+b') + (a+b)(qa') - \big((a+b)(a'+b')\big)q \quad (\text{since } Bq = 0) \\
&\quad = (aq)a' + (aq)b' + a(qa') + b(qa') - (aa')q \\
&\qquad\qquad -(ab')q - (ba')q - (bb')q \\
&\quad = (aq)a' + (aq)b' + a(qa') + b(qa') - (aa')q - (bb')q,
\end{aligned}$$

since $(ab')q = (ab')^\sigma q = -(ab')q$ and $(ba')q = (ba')^\sigma q = -(ba')q$. Note that

$$\begin{aligned} (aq)a' &= \big((x_{11}q)q\big)a' = \frac{1}{2}(x_{11}^\sigma + x_{11})a' && (2.16)\\ a(qa') &= a\big(q(x'_{11}q)\big) = \frac{1}{2}a\big((x'_{11})^\sigma + x'_{11}\big), && (2.17)\end{aligned}$$

but $x_{11}^\sigma a' = x'_{11}a$ and $(x'_{11})^\sigma a = x_{11}a'$ by (2.13), and hence $(aq)a' + a(qa') = x_{11}a' + x'_{11}a$. Also, we have $(aq)b' = \big((x_{11}q)q\big)b' = \frac{1}{2}(x_{11}^\sigma + x_{11})b'$ and $b(qa') = b\big(q(x'_{11}q)\big) = \frac{1}{2}b\big((x'_{11})^\sigma + x'_{11}\big)$. Thus, it is enough to show that

$$(aa')q = \frac{1}{2}\big((aq)a' + a(qa')\big). \tag{2.18}$$

We use the following two identities to establish (2.18):

$$\begin{aligned} (qx_{ii})x_{12} &= (qx_{12})x_{ii} + (q(x_{12}x_{ii}))p_j && (2.19)\\ (x_{ii}y_{ii})x_{12} &= (x_{ii}x_{12})y_{ii} && (2.20)\end{aligned}$$

for $x_{ii}, y_{ii} \in J_{ii}$, $i, j \in \{1, 2\}$, $i \neq j$, and $x_{12} \in J_{12}$.

Now for (2.19), the formula [MN, (1.3.3)], which was stated for Jordan triple systems, can be adapted for Jordan algebras to say

$$\begin{aligned} &(mx_{12})x_{ii} + m(x_{12}x_{ii}) - (mx_{ii})x_{12}\\ &\qquad = (m(x_{ii}x_{12}))p_i + m((x_{ii}x_{12})p_i) - (mp_i)(x_{ii}x_{12})\end{aligned}$$

for $m \in J_{12}$. Let $m = q$. Then, since $p_i(x_{ii}x_{12}) = \frac{1}{2}x_{ii}x_{12}$ and $qp_i = \frac{1}{2}q$, we have

$$\begin{aligned} &(qx_{12})x_{ii} + q(x_{12}x_{ii}) - (qx_i)x_{12}\\ &\qquad = (q(x_{ii}x_{12}))p_i + \frac{1}{2}q(x_{ii}x_{12}) - \frac{1}{2}q(x_{ii}x_{12})\\ &\qquad = (q(x_{ii}x_{12}))p_i.\end{aligned}$$

Note that $q(x_{ii}x_{12}) \in J_{12}J_{12} \subset J_{11} \oplus J_{22}$, and so $(q(x_{ii}x_{12})) = (q(x_{ii}x_{12}))p_1 + (q(x_{ii}x_{12}))p_2$. Hence, $(qx_{12})x_{ii} + (q(x_{12}x_{ii}))p_j - (qx_{ii})x_{12} = 0$, which is (2.19). Also observe that $(qx_{12})x_{ii} = ((qx_{12})x_{ii})p_i$ since $qx_{12} \in J_1 \oplus J_2$ and $J_1J_2 = 0$. Thus, (2.20) follows from applying [MN, (1.3.5)] to Jordan algebras.

In demonstrating (2.18), we write $y = y_1 + y_2$ for $y \in J_{11} \oplus J_{22}$ $(y_i \in J_{ii})$ to simplify the notation. Then we have

$$\begin{aligned}
(aa')q &= ((qx_{11})a')q \\
&= ((qa')x_{11})q + (q(a'x_{11}))_2 q \quad \text{by (2.19)} \\
&= ((qa')_1 x_{11})q + (q(a'x_{11}))_2 q \quad \text{since } J_{22}J_{11} = 0 \\
&= (qa')_1(x_{11}q) + (q(a'x_{11}))_2 q \quad \text{by (2.20)} \\
&= (qa')_1 a + (q(a'x_{11}))_2 q.
\end{aligned}$$

Note that $a = qx_{11} = qx_{11}^{\sigma}$ and $x_{11}^{\sigma} \in J_{22}$, and so, by a similar argument, we also get $(aa')q = (qa')_2 a + (q(a'x_{11}^{\sigma}))_1 q$. Hence,

$$2(aa')q = (qa')a + (q(a'x_{11}))_2 q + (q(a'x_{11}^{\sigma}))_1 q.$$

Since $L_q = L_q\sigma$, we have $q(a'x_{11}^{\sigma}) = q(a'x_{11})$. Hence, by (2.16),

$$(q(a'x_{11}))_2 q + (q(a'x_{11}^{\sigma}))_1 q = (q(a'x_{11}))q = \frac{1}{2}(a'x_{11}^{\sigma} + a'x_{11}) = a'(qa).$$

Thus, (2.18) holds, and the proof is finished. □

Suppose that J is a Jordan algebra with a triangle (p_1, p_2, q) and connection involution σ. Let $\mathfrak{a} = (J_{12}, \cdot)$, $A = J_{12}^{(+)}$, and $B = J_{12}^{(-)}$, where the product $\cdot$ is as in (2.14). Define a new multiplication on $\mathfrak{a}$ by

$$\begin{aligned}
aa' &= x_{11} \cdot a', \\
ab &= x_{11} \cdot b, \\
ba &= x_{11}^{\sigma} \cdot b, \\
bb' &= -(b \cdot b') \cdot q
\end{aligned}$$

for $a = x_{11} \cdot q$, $a' \in A$, and $b, b' \in B$ as in [ABG1, Sec. 2.48] so that $\alpha \cdot \alpha' = \frac{1}{2}(\alpha\alpha' + \alpha'\alpha)$ for all $\alpha, \alpha' \in \mathfrak{a}$. Thus, $\mathfrak{a}$ is the coordinate algebra of a Lie algebra graded by C_2, and every coordinate algebra $\mathfrak{a}$ of a Lie algebra graded by C_2 has this form, (see the discussion in [ABG1, Sec. 2.51]). So in summary, we have

Corollary 2.21. *Let $\mathfrak{a} = A \oplus B$ be a coordinate algebra of a Lie algebra graded by C_2. Then $\mathfrak{a}^+ = (\mathfrak{a}, \cdot)$ is a Jordan algebra with involution. In particular, $\mathfrak{a}$ is a Jordan admissible algebra with involution (for any choice of skew product on B).*

Remark 2.22. In [Se], Seligman proved that $(A, \cdot)$ is a Jordan algebra by the following argument. (Actually Seligman was working with finite-dimensional simple Lie algebras graded by C_2, but the same proof applies in the general setting.) The universal relation $\mathfrak{tr}(x[y,z]) = \mathfrak{tr}(y[z,x]) = \mathfrak{tr}(z[x,y])$ implies that

$$D_{a,a'\circ a''} + D_{a',a''\circ a'} + D_{a'',a\circ a'} = 0$$

for $a, a', a'' \in A$. In particular, $D_{a,a^2} = 0$. Moreover,

$$D_{a,a'}a'' = \frac{2}{r}\Big(a \circ (a' \circ a'') - a' \circ (a \circ a'')\Big).$$

Combining those results gives $a \cdot (a^2 \cdot a') = a^2 \cdot (a \cdot a')$ for all $a, a' \in A$ so that $(A, \cdot)$ is a Jordan algebra. In our classification of Lie G-tori of type C_2 in Section 5, we only require the fact that $(A, \cdot)$ is a Jordan algebra. However, we have included the proof that $(J_{12}, \cdot)$ and $\mathfrak{a}^+ = (\mathfrak{a}, \cdot)$ are Jordan algebras, as those results may be of independent interest.

3. The coordinate algebra of a (C_r, G)-graded Lie algebra

In this section, we show that the coordinate algebra of a (C_r, G)-graded Lie algebra $\mathcal{L}$ is G-graded. More specifically, we prove the following theorem.

Theorem 3.1. (i) *Let $\mathcal{L}$ be a (C_r, G)-graded Lie algebra. Then for $r \geq 3$, the coordinate algebra $\mathfrak{a} = A \oplus B$ of $\mathcal{L}$ is G-graded and has a graded involution. Also, for $r \geq 2$, $(A, \cdot)$ is an $\langle L \rangle$-graded Jordan algebra with graded involution, where $\langle L \rangle$ is the subgroup of G generated by $L = \{g \in G \mid \mathcal{L}_\mu^g \neq 0, \mu \in \Delta_{lg}\}$.*

(ii) *$\mathfrak{sp}_{2r}(\mathfrak{a})$ is a centerless (C_r, G)-graded Lie algebra for any G-graded associative algebra $\mathfrak{a}$ with graded involution if $r \geq 2$, or for any G-graded alternative algebra $\mathfrak{a}$ with graded involution whose symmetric elements A are in the nucleus of $\mathfrak{a}$ if $r = 3$. In addition, $\mathfrak{sp}_4(\mathfrak{a})$ is a centerless (C_2, G)-graded Lie algebra for any G-graded Jordan algebra $\mathfrak{a}$ of Clifford type.*

Proof. (i) We suppose that $\mathcal{L}$ is a (C_r, G)-graded Lie algebra. Thus, we are assuming that $\mathcal{L}$ is Δ-graded, $\mathcal{L} = \bigoplus_{\mu \in \Delta \cup \{0\}} \mathcal{L}_\mu$ for $\Delta = \mathrm{C}_r$ $(r \geq 2)$ with grading subalgebra $\mathfrak{g}$; $\mathcal{L}$ is G-graded $\mathcal{L} = \bigoplus_{g \in G} \mathcal{L}^g$ and has a decomposition

$$\mathcal{L} = \bigoplus_{\mu \in \Delta \cup \{0\}} \bigoplus_{g \in G} \mathcal{L}^g_\mu,$$

where $\mathcal{L}^g_\mu = \mathcal{L}_\mu \cap \mathcal{L}^g$; $\mathfrak{g} \subseteq \mathcal{L}^0$; and $\operatorname{supp} \mathcal{L}$ generates G. Then we have

$$\mathcal{L}_\mu = \begin{cases} \mathfrak{g}_\mu \otimes A & \text{if } \mu \in \Delta_{lg} \\ (\mathfrak{g}_\mu \otimes A) \oplus (\mathfrak{s}_\mu \otimes B) & \text{if } \mu \in \Delta_{sh}. \end{cases} \tag{3.2}$$

Set

$$L = \{g \in G \mid \mathcal{L}^g_\mu \neq 0, \mu \in \Delta_{lg}\}$$

and let $\langle L \rangle$ be the subgroup of G generated by L. For all $\mu \in \Delta_{lg}$ and $g \in G$, we define A^g_μ using

$$\mathcal{L}^g_\mu = \mathfrak{g}_\mu \otimes A^g_\mu.$$

Then $A = \bigoplus_{g \in G} A^g_\mu$, and in particular, $A^g_\mu = 0$ if $g \notin L$. For any $\mu, \nu \in \Delta_{lg}$, there exist $\gamma_1, \gamma_2 \in \Delta_{sh}$ such that $\nu + \gamma_1 \in \Delta_{sh}$ and $\mu = \nu + \gamma_1 + \gamma_2$. Then,

$$\mathfrak{g}_\mu \otimes A^g_\mu = [[\mathfrak{g}_\nu \otimes A^g_\nu, \mathfrak{g}_{\gamma_1} \otimes 1], \mathfrak{g}_{\gamma_2} \otimes 1] = \mathfrak{g}_\mu \otimes \ A^g_\nu \quad \text{for all } g \in G.$$

Therefore, $A^g_\mu = A^g_\nu$ for all $\mu, \nu \in \Delta_{lg}$, and for $g \in G$, we specify that

$$A^g := A^g_\mu \quad \text{for any choice of } \mu \in \Delta_{lg}.$$

Then

$$A = \bigoplus_{g \in G} A^g = \bigoplus_{l \in L} A^l, \tag{3.3}$$

where $\mathcal{L}^l_\mu = \mathfrak{g}_\mu \otimes A^l$ for all $\mu \in \Delta_{lg}$ and $l \in L$. The algebra A is graded by the group $\langle L \rangle$, and $1 \in A^0$.

Let $\mu \in \Delta_{sh}$ and $g \in G$. We define A^g_μ and B^g_μ via the relations

$$\mathfrak{g}_\mu \otimes A^g_\mu = (\mathfrak{g}_\mu \otimes A) \cap \mathcal{L}^g_\mu \quad \text{and} \quad \mathfrak{s}_\mu \otimes B^g_\mu = (\mathfrak{s}_\mu \otimes B) \cap \mathcal{L}^g_\mu.$$

We claim that

$$\mathcal{L}^g_\mu = (\mathfrak{g}_\mu \otimes A^g_\mu) \oplus (\mathfrak{s}_\mu \otimes B^g_\mu) \quad \text{and} \quad A^g_\mu = A^g. \tag{3.4}$$

To see this, let $w \in \mathcal{L}^g_\mu$. Then by (3.2), $w = u + v$ for some $u \in \mathfrak{g}_\mu \otimes A$ and $v \in \mathfrak{s}_\mu \otimes B$. We need to show that $u, v \in \mathcal{L}^g_\mu$. Now by (3.3), we have $u = \sum_{l \in L} \ e_\mu \otimes a_l$ for some $0 \neq e_\mu \in \mathfrak{g}_\mu$ and $a_l \in A^l$. We can find some $\nu \in \Delta$ such that $\mu + \nu \in \Delta_{lg}$. Let $0 \neq e_\nu \in \mathfrak{g}_\nu = \mathfrak{g}_\nu \otimes 1 \subset \mathcal{L}^0_\nu$. Then

$$[e_\nu, w] \in \mathcal{L}^g_{\mu+\nu} = \begin{cases} 0 & \text{if } g \notin L \\ \mathfrak{g}_{\mu+\nu} \otimes A^g & \text{if } g \in L. \end{cases} \tag{3.5}$$

Let $0 \neq s_\mu \in \mathfrak{s}_\mu$. Since weight spaces of $\mathfrak{s}$ relative to $\mathfrak{h}$ are one-dimensional, $v = s_\mu \otimes b$ for some $b \in B$, and

$$[e_\nu, v] = [e_\nu, s_\mu \otimes b] = [e_\nu, s_\mu] \otimes b \ \Big(+ e_\nu \circ s_\mu \otimes \frac{1}{2}[1, b]\Big).$$

But $[e_\nu, s_\mu] \otimes b = 0$ since $\mu + \nu \in \Delta_{lg}$, and hence $[e_\nu, v] = 0$. Thus we obtain $[e_\nu, u] = [e_\nu, w]$.

If $g \notin L$, then, by (3.5), $0 = [e_\nu, w] = [e_\nu, u] = \sum_{l \in L} [e_\nu, e_\mu \otimes a_l] = 0$, and so $[e_\nu, e_\mu \otimes a_l] = 0$ for all $l \in L$. Since $[\mathfrak{g}_\nu, \mathfrak{g}_\mu] \neq 0$, we get $a_l = 0$ for all $l \in L$, i.e., $u = 0$. Therefore, $w = v \in \mathcal{L}_\mu^g$.

If $g \in L$, then $[e_\nu, u] = [e_\nu, w] \in \mathcal{L}^g_{\mu+\nu}$, and so $u = e_\mu \otimes a_g \in \mathfrak{g}_\mu \otimes A^g$. Note that there exists $\gamma \in \Delta_{lg}$ such that $\mu - \gamma \in \Delta$. So by (3.3), we have

$$\mathfrak{g}_\mu \otimes A^g = [\mathfrak{g}_\gamma \otimes A^g, \mathfrak{g}_{\mu-\gamma} \otimes 1] \subseteq [\mathcal{L}^g_\gamma, \mathcal{L}^0_{\mu-\gamma}] \subseteq \mathcal{L}^g_\mu.$$

Therefore, $u \in \mathcal{L}^g_\mu$ and $v = w - u \in \mathcal{L}^g_\mu$. Finally, since $\mu + \nu \in \Delta_{lg}$, it follows that

$$[\mathfrak{g}_\mu \otimes A^g_\mu, \mathfrak{g}_\nu \otimes 1] = \mathfrak{g}_{\mu+\nu} \otimes A^g_\mu \subseteq \mathfrak{g}_{\mu+\nu} \otimes A^g.$$

Hence $A^g_\mu \subseteq A^g$. Also,

$$[\mathfrak{g}_{\mu+\nu} \otimes A^g, \mathfrak{g}_{-\nu} \otimes 1] = \mathfrak{g}_\mu \otimes A^g \subseteq \mathfrak{g}_\mu \otimes A^g_\mu,$$

and so $A^g \subseteq A^g_\mu$. Thus our claim (3.4) is settled.

Now, $B = \bigoplus_{g \in G} B^g_\mu$, and $\mathfrak{s}_\mu \otimes B^g_\mu = \mathcal{L}^g_\mu$ if $g \notin L$ since $A^g = 0$. If $\mu, \nu \in \Delta_{sh}$ and $\mu - \nu \in \Delta$, then

$$\mathfrak{s}_\mu \otimes B^g_\mu = [\mathfrak{s}_\nu \otimes B^g_\nu, \mathfrak{g}_{\mu-\nu} \otimes 1] = \mathfrak{s}_\mu \otimes B^g_\nu \quad \text{for all } g \in G.$$

Therefore, $B^g_\mu = B^g_\nu$. Thus by the same argument as in [AG, (5.11)], we get $B^g_\mu = B^g_\nu$ for any $\mu, \nu \in \Delta_{sh}$ and all $g \in G$. So for $g \in G$ we put

$$B^g := B^g_\mu \quad \text{for any choice of } \mu \in \Delta_{sh}.$$

Consequently,

$$B = \bigoplus_{g \in G} B^g, \quad \text{with}$$

$$\mathcal{L}^g_\mu = \mathfrak{s}_\mu \otimes B^g \quad \text{for all } \mu \in \Delta_{sh} \text{ and } g \notin L,$$

and

$$\mathfrak{s}^g_\mu = (\mathfrak{g}_\mu \otimes A^g) \oplus (\mathfrak{s}_\mu \otimes B^g) \quad \text{for all } \mu \in \Delta_{sh} \text{ and } g \in L. \tag{3.6}$$

Let

$$\mathfrak{a} = \bigoplus_{g \in G} \mathfrak{a}^g, \quad \text{where} \quad \mathfrak{a}^g := A^g \oplus B^g$$

($A^g = 0$ if $g \notin L$). Let $S := \operatorname{supp} \mathfrak{a} = \operatorname{supp} \mathcal{L}_\mu$ for $\mu \in \Delta_{sh}$. By (3.6), we have $L \subseteq S$, and so $S + S \supset \operatorname{supp} \mathcal{L}$, which generates G. Hence S generates G.

At this stage we know that $\mathfrak{a}$ is a vector space graded by the group G, and the support of $\mathfrak{a}$ generates G. We need to verify that $\mathfrak{a}$ is a graded algebra. Now when $w = E_{1,2} - E_{2r-1,2r} \in \mathfrak{g}_{\varepsilon_1 - \varepsilon_2}$ and $z = E_{2,2r-1} \in \mathfrak{g}_{2\varepsilon_2}$, we have $[w, z] = E_{1,2r-1} + E_{2,2r} \in \mathfrak{g}_{\varepsilon_1+\varepsilon_2}$ and $w \circ z = E_{1,2r-1} - E_{2,2r} \in \mathfrak{s}_{\varepsilon_1+\varepsilon_2}$, both of which are nonzero. Then the product $[w \otimes a, z \otimes a']$ with $a \in A^g$, $a' \in A^h$ shows that $A^g \circ A^h \subseteq A^{g+h}$ and $[A^g, A^h] \subseteq B^{g+h}$, which combine to say $A^g A^h \subseteq \mathfrak{a}^{g+h}$.

The elements $s = E_{2,1} + E_{2r,2r-1}$ and $s' = E_{1,2r-1} - E_{2,2r}$ belong to $\mathfrak{s}$ as does $t = E_{1,3} + E_{2r-2,2r}$ when $r \geq 3$. Setting $x = E_{1,2} - E_{2r-1,2r} \in \mathfrak{g}$, we have $[x, s] = E_{1,1} - E_{2,2} - E_{2r-1,2r-1} + E_{2r,2r}$ and $x \circ s = E_{1,1} + E_{2,2} - E_{2r-1,2r-1} - E_{2r,2r}$, from which we can deduce that $A^g \circ B^h \subseteq B^{g+h}$ and $[A^g, B^h] \subseteq A^{g+h}$. Thus, $A^g B^h \subseteq \mathfrak{a}^{g+h}$. We can use the fact that $[s, s'] = 2E_{2,2r} \in \mathfrak{g}_{2\varepsilon_2}$ to determine that $B^g \circ B^h \subseteq A^{g+h}$. Now when $r \geq 3$, $s \circ t = E_{2,3} + E_{2r-2,2r-1} \neq 0$, from which we obtain $[B^g, B^h] \subseteq B^{g+h}$. Thus, for $r \geq 3$, we have $B^g B^h \subseteq \mathfrak{a}^{g+h}$. The product $s \circ t$ is identically 0 on $\mathfrak{s}$ when $r = 2$, and all we can deduce in this case is that $B^g \circ B^h \subseteq A^{g+h}$.

These arguments have shown that $\mathfrak{a}$ is graded for $r \geq 3$. By Corollary 2.21 or Remark 2.22, it follows that $(A, \cdot)$ is a Jordan $\langle L \rangle$-graded algebra for $r \geq 2$. The involution σ is clearly graded, so we have (i).

(ii) For this second part, let $\mathcal{L} = \mathfrak{sp}_{2r}(\mathfrak{a})$, where $\mathfrak{a} = A \oplus B = \bigoplus_{g \in G} (A^g \oplus B^g)$ is a G-graded algebra with symmetric elements A and skew elements B relative to a graded involution. Assume $\mathfrak{a}$ is associative; or in the $r = 3$ case, an alternative algebra such that A lies in the nucleus of $\mathfrak{a}$; or in the case $r = 2$, a Jordan algebra. For $g \in G$, set $\mathcal{L}_\mu^g := (\mathfrak{g}_\mu \otimes A^g) \oplus (\mathfrak{s}_\mu \otimes B^g)$ if $\mu \in \Delta_{sh}$, and $\mathcal{L}_\mu^g := \mathfrak{g}_\mu \otimes A^g$ if $\mu \in \Delta_{lg}$. Then $\mathcal{L}$ admits a compatible G-grading, $\mathcal{L} = \bigoplus_{g \in G} \mathcal{L}^g$, with $\mathcal{L}^g = \bigoplus_{\mu \in \Delta \cup \{0\}} \mathcal{L}_\mu^g$ and $\mathcal{L}_0^g = \sum_{\mu \in \Delta} \sum_{g = g' + g''} [\mathcal{L}_\mu^{g'}, \mathcal{L}_{-\mu}^{g''}]$, so that $\mathcal{L}$ is a (C_r, G)-graded Lie algebra. □

4. The coordinate algebra of a division (C_r, G)-graded Lie algebra

In this section we investigate Lie algebras that are (C_r, G)-graded and satisfy the division property (see Remarks 2.2); that is, the so-called *division* (C_r, G)*-graded Lie algebras*. Our main result will be that the coordinate algebra of such a Lie algebra is a division G-graded algebra where by that we mean the following.

Definition 4.1. A G-graded unital (associative, alternative, or Jordan) algebra $\mathcal{A}$ is said to be *division G-graded* (or have the division property) if all nonzero homogeneous elements are invertible. If $\mathcal{A}$ is a division G-graded algebra such that $\dim \mathcal{A}^g \leq 1$ for all homogeneous spaces $\mathcal{A}^g$, then $\mathcal{A}$ is a *G-torus*. A $\mathbb{Z}^n$-torus is referred to as an *n-torus* or simply a *torus*.

Examples 4.2. (1) An associative G-torus is nothing but a twisted group algebra $\mathbb{F}^t[G]$. Thus, an associative n-torus (also known in the literature as a *quantum torus*) is a Laurent polynomial ring $\mathbb{F}_{\underline{q}}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ in n variables with multiplication given by $t_i t_j = q_{i,j} t_j t_i$ where $\underline{q} = (q_{i,j})$ is an $n \times n$ matrix with entries in $\mathbb{F}^\times$ such that $q_{i,i} = 1$ for all i and $q_{j,i} = q_{i,j}^{-1}$.

(2) Alternative tori were classified in [BGKN] for fields $\mathbb{F}$ such that every element of $\mathbb{F}$ has a square root in $\mathbb{F}$, and in [Y2] for arbitrary $\mathbb{F}$. An alternative torus is either a quantum torus or an octonion torus (sometimes called a Cayley torus). In the second case, it is the Cayley-Dickson algebra $\mathbb{O}_n$ over the ring of Laurent polynomials $\mathbb{F}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ in n variables for some $n \geq 3$ obtained by successively applying the Cayley-Dickson process with elements x_1, x_2, x_3, such that $x_i^2 = t_i$, where the t_i are the structure constants of the process. A graded involution σ of $\mathbb{O}_n$ whose symmetric elements lie in the nucleus, must be the standard involution, i.e., the involution determined by $x_i \mapsto -x_i$ for $1 \leq i \leq 3$, and $t_i \mapsto t_i$ for $4 \leq i \leq n$.

(3) If a division G-graded Jordan algebra $\mathfrak{a} = \bigoplus_{g \in G} \mathfrak{a}^g$ has a decomposition $\mathfrak{a}^g = A^g \oplus B^g$ for each $g \in G$ so that $A = \bigoplus_{g \in G} A^g$ is a commutative associative subalgebra, and $\mathfrak{a} = A \oplus B$ is a Jordan algebra over A of a symmetric bilinear form on the graded A-module $B := \bigoplus_{g \in G} B^g$, then we say that $\mathfrak{a}$ is a *division G-graded Jordan algebra of Clifford type*. The algebra $\mathfrak{a}$ has a natural involution σ, with $\sigma(a) = a$ for all $a \in A$ and $\sigma(b) = -b$ for all $b \in B$. We call this σ the standard involution. If $\mathfrak{a}$ is a Jordan G-torus, then it is said to be a *Clifford G-torus*.

Suppose that $\mathfrak{L}$ is a division (C_r, G)-graded Lie algebra with grading subalgebra $\mathfrak{g}$. We may assume that $\mathfrak{L}$ is centerless, as the coordinate algebras of $\mathfrak{L}$ and $\mathfrak{L}/\mathfrak{Z}(\mathfrak{L})$ are the same because the center $\mathfrak{Z}(\mathfrak{L})$ is contained in the sum of the trivial $\mathfrak{g}$-submodules of $\mathfrak{L}$. By the previous section, the coordinate algebra $\mathfrak{a} = A \oplus B$ of $\mathfrak{L}$ is G-graded.

Let $0 \neq a+b \in A^g \oplus B^g = \mathfrak{a}^g$ for $a \in A^g$, $b \in B^g$ and $g \in S = \operatorname{supp} \mathfrak{a}$ (Note $a = 0$ if $g \notin L$). Let $\mu := \varepsilon_1 - \varepsilon_2 \in \Delta_{sh}$, $e := E_{1,2} - E_{3,4}$, $e' := \frac{1}{2}(E_{2,1} - E_{4,3})$ $s := E_{1,2} + E_{3,4}$ and $s' := \frac{1}{2}(E_{2,1} + E_{4,3})$. Then for $r \geq 2$, we have

$$\begin{aligned}
[e, e'] &= [s, s'] = \frac{1}{2}\mu^\vee \quad \text{(recall we are assuming } \mathcal{Z}(\mathcal{L}) = 0),\\
e \circ s' &= s \circ e', \qquad \text{(which is linearly independent of } \mu^\vee),\\
[e, s'] &= [s, e'] \neq 0, \quad \text{and}\\
\mathfrak{tr}(ee') &= \mathfrak{tr}(ss') \neq 0.
\end{aligned}$$

Also, one can check that if $r \geq 3$, then

$$e \circ e' = s \circ s', \quad \text{(which is linearly independent of } [e, s']),$$

and $e \circ e' = s \circ s' = 0$ if $r = 2$.

Now, $e_\mu \otimes a + s_\mu \otimes b \in \mathcal{L}_\mu^g$, and by the division property of $\mathcal{L}$, there exists $y \in \mathcal{L}_{-\mu}^{-g}$ such that $[e_\mu \otimes a + s_\mu \otimes b, y] = \mu^\vee$. Since $\mathcal{L}_{-\mu}^{-g} = (\mathfrak{g}_{-\mu} \otimes A^{-g}) \oplus (s' \otimes B^{-g})$, we have that $y = e' \otimes a' + s' \otimes b'$ for suitable elements $a' \in A^{-g}$ and $b' \in B^{-g}$. Consequently,

$$\begin{aligned}
\mu^\vee \otimes 1 = \mu^\vee &= [e \otimes a + s \otimes b, e' \otimes a' + s' \otimes b']\\
&= [e, e'] \otimes \frac{1}{2} a \circ a' + e \circ e' \otimes \frac{1}{2}[a, a'] + \mathfrak{tr}(ee') D_{a,a'}\\
&\quad + e \circ s' \otimes \frac{1}{2}[a, b'] + [e, s'] \otimes \frac{1}{2} a \circ b'\\
&\quad + s \circ e' \otimes \frac{1}{2}[b, a'] + [s, e'] \otimes \frac{1}{2} b \circ a'\\
&\quad + [s, s'] \otimes \frac{1}{2} b \circ b' + s \circ s' \otimes \frac{1}{2}[b, b'] + \mathfrak{tr}(ss') D_{b,b'},
\end{aligned}$$

from which we deduce that

$$\begin{aligned}
&a \circ a' + b \circ b' = 2,\\
&[a, b'] + [b, a'] = 0 = a \circ b' + b \circ a',\\
&D_{a,a'} + D_{b,b'} = 0, \quad \text{and}\\
&[a, a'] + [b, b'] = 0 \quad (\text{if } r \geq 3)
\end{aligned}$$

So we get $(a+b) \circ (a'+b') = 2$ and $D_{a+b,a'+b'} = D_{a,a'} + D_{b,b'} = 0$. Moreover, if $r \geq 3$, then $[a+b, a'+b'] = 0$. Therefore, $a+b$ is invertible in the Jordan algebra $\mathfrak{a}^+$ if $r \geq 2$, and $a+b$ is also invertible in the alternative algebra $\mathfrak{a}$ if $r \geq 3$. Thus,

(4.3)

(i) $\mathfrak{a}^+$ is a division G-graded Jordan algebra with graded involution for $r \geq 2$.

(ii) $(A, \cdot)$ is a division $\langle L\rangle$-graded Jordan algebra for $r \geq 2$,

(iii) $\mathfrak{a}$ is a division G-graded alternative algebra with graded involution if $r \geq 3$, and

(iv) $\mathfrak{a}$ is a division G-graded associative algebra with graded involution if $r \geq 4$.

In particular, $S = \operatorname{supp} \mathcal{L}_\mu$ for $\mu \in \Delta_{sh}$ and $L = \operatorname{supp} \mathcal{L}_\nu$ for $\nu \in \Delta_{lg}$ are reflection spaces of G (in the sense of [Y3]), and $S = G$ if $r \geq 3$. Thus we have established the following result.

Theorem 4.4. *Let $\mathcal{L}$ be a centerless division (C_r, G)-graded Lie algebra. Then $\mathcal{L} \cong \mathfrak{sp}_{2r}(\mathfrak{a})$ for some division G-graded algebra $\mathfrak{a}$ with graded involution such that (i)-(iv) of* (4.3) *hold. Also, $\mathfrak{sp}_{2r}(\mathfrak{a})$ is a centerless division (C_r, G)-graded Lie algebra for any division G-graded associative algebra $\mathfrak{a}$ with graded involution if $r \geq 2$, for any division G-graded alternative algebra $\mathfrak{a}$ with graded involution so that the symmetric elements are in the nucleus of $\mathfrak{a}$ if $r = 3$, or for any division G-graded Jordan algebra $\mathfrak{a}$ of Clifford type if $r = 2$.*

Proof. All this is apparent from our discussions above, except perhaps for the division property of $\mathcal{L} = \mathfrak{sp}_{2r}(\mathfrak{a})$ in the second statement. For $\mu \in \Delta_{lg}$ and $g \in L$, let $e \in \mathfrak{g}_\mu$ and $e' \in \mathfrak{g}_{-\mu}$ be such that $[e, e'] = \mu^\vee$. Then for $0 \neq v \in \mathcal{L}_\mu^g$, there exists $0 \neq a \in A^g$ such that $v = e \otimes a$. Taking $w = e' \otimes a^{-1} \in \mathcal{L}_{-\mu}^{-g}$, we get $[v, w] = \mu^\vee$. (Note that $[a, a^{-1}] = aa^{-1} - a^{-1}a = 0$.)

For $\mu \in \Delta_{sh}$, it is easy to see the existence of the elements $e \in \mathfrak{g}_\mu$, $e' \in \mathfrak{g}_{-\mu}$, $s \in \mathfrak{s}_\mu$ and $s' \in \mathfrak{s}_{-\mu}$ satisfying (4.1). Then for $g \in S$ and $0 \neq v \in \mathcal{L}_\mu^g$, there exist $a \in A^g$ and $b \in B^g$ such that $v = e \otimes a + s \otimes b$. Taking $w = e' \otimes a' + s' \otimes b' \in \mathcal{L}_{-\mu}^{-g}$, where $(a + b)^{-1} = a' + b'$, we get $[v, w] = \mu^\vee$. Hence $\mathcal{L}$ is division graded. □

Remark 4.5. The argument above affords a more direct and easier proof that the division property holds for the coordinate algebra of a centerless division (C_r, G)-graded Lie algebra than the one given in [Se, pp. 99-101], which treats only a particular case of this result; namely, that the coordinate algebra of a finite-dimensional simple Lie algebra of relative type C_r is a division algebra.

5. The coordinate algebra of a Lie G-torus of type C_2

We apply the following identities to determine the coordinate algebra $\mathfrak{a} = A \oplus B$ of a Lie G-torus of type C_2. Seligman [Se, pp. 88-95] used these same identities in his classification of the finite-dimensional simple Lie algebras of characteristic zero graded by the root system C_2, but they are valid in any C_2-graded Lie algebra. In expressing them, we have translated them into our notation using $\circ$ and $[\cdot,\cdot]$ and have written the inner derivations as left operators rather than right operators as in [Se]. Each identity carries two numbers - the left being the reference in [Se] and the right being our own equation label.

$$(37') \quad a \circ (a'' \circ a') - a'' \circ (a \circ a') = [a, [a'', a']] - [a'', [a, a']], \tag{5.1}$$

$$(38'') \quad [a, a' \circ a''] = [a, a'] \circ a'' - [a'', a] \circ a', \tag{5.2}$$

$$(39) \quad [[a', a''], a] = a' \circ (a'' \circ a) - a'' \circ (a \circ a') \tag{5.3}$$

$$(39') \quad = 4D_{a',a''},$$

$$(40) \quad D_{[a,a'],b} = D_{[b,a'],a} - D_{[b,a],a'}, \tag{5.4}$$

$$(41'') \quad [b, a \circ a'] = [b, a] \circ a' - [a', b] \circ a, \tag{5.5}$$

$$(42') \quad [b \circ a, a'] = [b, a \circ a'] + b \circ [a, a'] - [b, a] \circ a', \tag{5.6}$$

$$(42'') \quad a' \circ [b, a] = b \circ [a, a'] + [b \circ a', a], \tag{5.7}$$

$$(43') \quad 4D_{a,a'}b = [a, [a', b]] - [a', [a, b]], \tag{5.8}$$

$$(44') \quad [a, [b, a']] = (b \circ a) \circ a' - b \circ (a \circ a'), \tag{5.9}$$

$$(46) \quad D_{b \circ b', a} + D_{b' \circ a, b} + D_{b \circ a, b'} = 0, \tag{5.10}$$

$$(51) \quad b \circ (b' \circ a) - b' \circ (b \circ a) = 4D_{b,b'}a = [b, [b', a]] - [b', [b, a]], \tag{5.11}$$

$$(52) \quad 2a \circ (b \circ b') = b \circ (b' \circ a) + b' \circ (b \circ a) + [b, [b', a]] + [b', [b, a]], \tag{5.12}$$

$$(53) \quad [a, b \circ b'] = [a, b] \circ b' - [b', a] \circ b, \tag{5.13}$$

$$(56') \quad [b, b' \circ b''] + [b', b'' \circ b] + [b'', b \circ b'] = 0, \tag{5.14}$$

for $a, a', a'' \in A$ and $b, b' \in B$.

First we establish a general lemma for any C_2-graded Lie algebra.

Lemma 5.15. *Suppose that* $\mathfrak{a} = A \oplus B$ *is the coordinate algebra of a* C_2*-graded Lie algebra. For* $b, b' \in B$*, suppose that there exist elements* $a_1, a_2, a_3, a_4 \in A$ *such that* $b = \frac{1}{2}[a_1, a_2]$ *and* $b' = \frac{1}{2}[a_3, a_4]$*. Then* $D_{b,b'} = [D_{a_1,a_2}, D_{a_3,a_4}]$*. Hence,* $D_{b,b'}$ *restricted to the Jordan algebra* $(A, \cdot)$ *is an inner derivation.*

Proof. We know that $D_{b,b'}$ is an inner derivation of the Jordan algebra $(\mathfrak{a}^+, \cdot)$. What this result asserts is that $D_{b,b'}$ acts as an inner derivation of the Jordan algebra $(A, \cdot)$.

By (5.3), we have $[b, a] = \frac{1}{2}[[a_1, a_2], a]] = 2D_{a_1,a_2}a$ and $[b', a] = 2D_{a_3,a_4}a$. Then, by the same reason, $[b, D_{a_3,a_4}a] = 2D_{a_1,a_2}D_{a_3,a_4}a$ and $[b', D_{a_1,a_2}a] = 2D_{a_3,a_4}D_{a_1,a_2}a$. Therefore, by (5.11),

$$\begin{aligned} D_{b,b'}a &= \frac{1}{4}\big([b, [b', a]] - [b', [b, a]]\big) \\ &= \frac{1}{2}\big([b, D_{a_3,a_4}a] - [b', D_{a_1,a_2}a]\big) \\ &= D_{a_1,a_2}D_{a_3,a_4}a - D_{a_3,a_4}D_{a_1,a_2}a \\ &= [D_{a_1,a_2}, D_{a_3,a_4}]a. \quad \square \end{aligned}$$

Next we impose the assumptions that the Lie algebra is (C_2, G)-graded and satisfies the division property.

Lemma 5.16. *Let* $\mathfrak{a} = A \oplus B$ *be the coordinate algebra of a division* (C_2, G)*-graded Lie algebra. Assume* $0 \neq a, a' \in A$ *and* $0 \neq b \in B$ *are homogeneous. Then*

(i) $a \circ a' \neq 0$ *or* $[a, a'] \neq 0$*;*
(ii) $a \circ b \neq 0$ *or* $[a, b] \neq 0$*.*

Proof. For (i), suppose that $a \circ a' = 0 = [a, a']$. Then by (5.1), we have

$$a \circ (a' \circ a'') = [a, [a'', a']].$$

But then substituting ${a'}^{-1}$ for a'', gives $4a = 0$, a contradiction.

For (ii), suppose that $a \circ b = 0 = [a, b]$. Then by (5.9), we have

$$[a, [b, a']] = -b \circ (a \circ a').$$

Letting $a' = a^{-1}$ gives $[a, [b, a^{-1}]] = -4b$. But, by (5.8), we have

$$0 = 4D_{a,a^{-1}}b = [a, [a^{-1}, b]] - [a^{-1}[a, b]].$$

Hence, $-4b = [a, [b, a^{-1}]] = [a^{-1}, [a, b]] = 0$, a contradiction. □

Recall that for a C_2-graded Lie algebra, the skew product on B may be arbitrarily defined. Here we will make a precise definition of that skew product when the Lie algebra is a Lie G-torus of type C_2 . This then will enable us to determine the structure of the corresponding coordinate algebra $\mathfrak{a} = A \oplus B = \bigoplus_{g \in B} (A^g \oplus B^g)$. The Lie G-torus condition forces $\dim_{\mathbb{F}} \mathfrak{a}^g \leq 1$ for all $g \in G$, where $\mathfrak{a}^g = A^g \oplus B^g$, and so this implies the helpful fact that $A^g = 0$ or $B^g = 0$.

Lemma 5.17. *Let $\mathfrak{a} = A \oplus B$ be the coordinate algebra of a Lie G-torus of type C_2. Set $B_0 := \{b \in B \mid [b, A] = 0\}$. Then,*

$$B = [A, A] \oplus B_0.$$

Moreover, $[A, A]$ and B_0 are graded, and for any homogeneous element $b \in [A, A]$, there exist homogeneous elements $a_1, a_2 \in A$ such that $b = [a_1, a_2]$.

Proof. Clearly $[A, A]$ and B_0 are graded spaces. Suppose that $0 \neq b \in B^g \setminus B_0$. Then, there exists some $a \in A^h$ such that $[b, a] \neq 0$. Note that $a^{-1} \circ [b, a] \in A^g = 0$ by one-dimensionality. Hence, by Lemma 5.16, we have $[a^{-1}, [b, a]] \neq 0$. Since B^g is one-dimensional, there must exist some $\vartheta \in \mathbb{F}$ such that $b = \vartheta[a^{-1}, [b, a]]$. Thus if one takes $a_1 = \vartheta a^{-1}$ and $a_2 = [b, a]$, then $b = [a_1, a_2]$, and it follows that $B = [A, A] + B_0$.

Suppose that $b = [a_h, a_k] \in B_0$, where $0 \neq a_h \in A^h$ and $0 \neq a_k \in A^k$. Then $[b, a_h^{-1}] = 0$, and hence $b \circ a_h^{-1} \neq 0$ by Lemma 5.16. Thus, by the one-dimensionality of A^k, there exists some $\xi \in F$ such that $a_k = \xi b \circ a_h^{-1}$. We apply identity (5.7) with $a = a_h$ and $a' = \xi a_h^{-1}$ to obtain $0 = b \circ [a_h, \xi a_h^{-1}] + [b \circ \xi a_h^{-1}, a_h] = [b \circ \xi a_h^{-1}, a_h] = [a_k, a_h] = b$, and so $[A, A] \cap B_0 = 0$. □

Set $S := \operatorname{supp} \mathfrak{a}$, $S_+ := \operatorname{supp} A$, and $S_- := \operatorname{supp} B$. Then $S = S_+ \sqcup S_-$, which is a disjoint union by the one-dimensionality condition on the graded spaces of $\mathfrak{a}$.

Lemma 5.18. *Let $\mathfrak{a} = A \oplus B$ be the coordinate algebra of a Lie G-torus of type C_2. Then the following are equivalent:*

(i) *S_+ is a subgroup of G;*
(ii) $[A, B] = 0$*;*
(iii) $[A, A] = 0$*;*

(iv) *The product* $\cdot$ *on* A *coincides with the product from* $\mathfrak{a}$*;*
(v) $(A, \cdot)$ *is associative.*

Proof. It follows from Lemma 5.17 that $[A, B] = 0$ if and only if $[A, A] = 0$. In this case, the product of $\mathfrak{a}$ and $\cdot$ coincide on A, and by (5.3), $(A, \cdot)$ is associative. Then, by the division property, the support S_+ of the division graded associative algebra $(A, \cdot)$ is a subgroup of G. Thus, we only need to prove that $[A, B] = 0$ if S_+ is a subgroup. For $g \in S_+$, $h \in S_-$, and $b \in B^h$, we have $[A^g, b] \subseteq A^{g+h}$. But since S_+ is a subgroup, $g + h \notin S_+$. Hence, $A^{g+h} = 0$, and so $[A, B] = 0$. $\square$

At this juncture, it is convenient to divide our considerations into two cases, namely,

(i) S_+ is a subgroup;
(ii) S_+ is not a subgroup.

Case (i): When S_+ is a subgroup, we will show that if we set $[B, B] = 0$, then $\mathfrak{a} = \mathfrak{a}^+$ is a Clifford G-torus. By Lemma 5.18, we know that the product of $\mathfrak{a}$ coincides with $\cdot$ on A, and $(A, \cdot)$ is a commutative, associative algebra. Now by (5.9), we have $a' \cdot (a \cdot b) \cdot a' = (a \cdot a') \cdot b$, i.e., B is a graded $(A, \cdot)$-module by the action $\cdot$. Also, (5.11) and (5.2) imply that

$$b \circ (b' \circ a) = b' \circ (b \circ a) = a \circ (b \circ b'),$$

and so $\circ$ (and also $\cdot$) defines a symmetric A-bilinear form on B. Note that the form is nondegenerate by the division property. Thus, if we specify that $[B, B] = 0$, then $\mathfrak{a} = \mathfrak{a}^+$ is a Jordan algebra of a symmetric bilinear form $\cdot$ on B over A. (Observe we did not use the fact that $\mathfrak{a}^+$ is a Jordan algebra in showing this.) Therefore, we have that $\mathfrak{a}$ is a Clifford G-torus.

Case (ii): When S_+ is not a subgroup, we define a linear map $\varphi : [A, A] \longrightarrow \mathrm{IDer}(A, \circ)$ into the inner derivations of the Jordan algebra $(A, \circ)$, by requiring that $\varphi(b) \in \mathrm{IDer}(A, \circ)$ be given by $\varphi(b)(a) = [b, a]$ for all $b \in [A, A]$ and $a \in A$. By (5.3), the image of φ is indeed in $\mathrm{IDer}(A, \circ)$, and φ is surjective. Moreover, by Lemma 5.17, φ is injective. We will use the bijection φ to define a skew product $[b, b']$ for $b, b' \in B$ in the following way. If $b, b' \in [A, A]$, then by Lemma 5.15, $D_{b,b'} \in \mathrm{IDer}(A, \circ)$. Hence, there is a unique element in $[A, A]$, denote it $[b, b']$, so that $[b, b'] := \varphi^{-1}(4D_{b,b'})$. If $b \in B_0$ or $b' \in B_0$, we specify that $[b, b'] := 0$. Thus we have the following relation:

$$4D_{b,b'}a = [[b, b'], a] \tag{5.19}$$

(see (5.11) for $b \in B_0$ or $b' \in B_0$), which is a well-known identity for an associative algebra.

Now we can prove that $\mathfrak{a}$ is associative in exactly the same way as in [Se, pp. 105-111]. Indeed, we have established all the properties needed in Seligman's argument to show that the associative law holds in $\mathfrak{a}$ except for the simplicity of our Lie algebra $\mathfrak{L}$. However, a centerless Lie G-torus is graded simple; that is, it has no nontrivial graded ideals (see [Y3, Lem. 4.4]), and simplicity may be replaced by graded simplicity with no harm to the argument. For the convenience of the reader, in the paragraphs to follow we present an alternative proof of associativity which differs somewhat from Seligman's original argument. The ultimate conclusion of Case (ii) will be that $\mathfrak{a}$ is an associative G-torus with graded involution.

The D-mappings are derivations not only relative to the symmetric product $\circ$ but also relative to the skew product $[\cdot,\cdot]$ we have defined above. First we prove the following claim about $D_{B,B}B$. (Seligman proved this claim using the Lie algebra $\mathfrak{L}$, but we prove it using just the coordinate algebra $\mathfrak{a}$.)

Claim 5.20. *When $S_+ = \operatorname{supp} A$ is not a subgroup, then $D_{B_0,B_0}B_0 = 0 = D_{B_0,B_0}B$, and therefore the following hold:*

(i) $D_{B_0,B_0} = 0 = D_{B_0,B}$,

(ii) $D_{B,B}B_0 = 0$,

(iii) $D_{B,B} = D_{[A,A],[A,A]} \subset D_{A,A}$.

Thus $D_{B,B}B \subseteq D_{A,A}B \subseteq D_{A,A}[A,A] \subseteq [A,A]$.

Proof. From Lemma 5.17 we have

$$D_{B,B} = D_{[A,A]\oplus B_0,[A,A]\oplus B_0}.$$

By (5.4), $D_{[A,A],B_0} = 0$, so the above becomes

$$D_{[A,A],[A,A]} + D_{B_0,B_0} \subseteq D_{A,A} + D_{B_0,B_0},$$

again using (5.4). Now (5.11) implies $D_{B_0,B_0}A = 0$, and so $D_{B_0,B_0}[A,A] = 0$. Hence, $D_{B_0,B_0}B = D_{B_0,B_0}B_0 \subseteq B_0$. Since $D_{B,B_0} = D_{B_0,B_0}$ by the above, we have $D_{B,B_0}B = D_{B_0,B_0}B_0$; however, (5.8) gives $D_{A,A}B_0 = 0$, so that $D_{B,B}B_0 = D_{B_0,B_0}B_0$ as well.

Now set

$$K := B \circ D_{B_0,B_0}B_0 + D_{B_0,B_0}B_0.$$

We claim K is an ideal of $\mathfrak{a}^+$. To see this, note that

$$\begin{aligned} B \circ D_{B_0,B_0}B_0 &\subseteq D_{B_0,B_0}(B \circ B_0) + (D_{B_0,B_0}B_0) \circ B_0 \\ &\subseteq D_{B_0,B_0}A + (D_{B_0,B_0}B_0) \circ B_0 \subseteq (D_{B_0,B_0}B_0) \circ B_0, \end{aligned}$$

and so

$$B \circ (B \circ D_{B_0,B_0}B_0) \subseteq B \circ (B_0 \circ D_{B_0,B_0}B_0),$$

which is contained in

$$B_0 \circ (B \circ D_{B_0,B_0}B_0) + D_{B,B_0}D_{B_0,B_0}B_0 \subseteq B_0 \circ (B_0 \circ D_{B_0,B_0}B_0) + D_{B_0,B_0}B_0.$$

Now $B_0 \circ (B_0 \circ D_{B_0,B_0}B_0) + D_{B_0,B_0}B_0$ lies in $(D_{B_0,B_0}B_0) \circ (B_0 \circ B_0) + D_{B_0,B_0}B_0$, since

$$\begin{aligned} B_0 \circ (B_0 \circ D_{B_0,B_0}B_0) &= D_{D_{B_0,B_0}B_0,B_0}B_0 + (D_{B_0,B_0}B_0) \circ (B_0 \circ B_0) \\ &\subseteq D_{B_0,B_0}B_0 + (D_{B_0,B_0}B_0) \circ (B_0 \circ B_0). \end{aligned}$$

But

$$\begin{aligned} (D_{B_0,B_0}B_0) \circ A &\subseteq D_{B_0,B_0}(B_0 \circ A) + B_0 \circ D_{B_0,B_0}A = D_{B_0,B_0}(B_0 \circ A) \\ &\subseteq D_{B_0,B_0}B = D_{B_0,B_0}B_0, \end{aligned} \tag{5.21}$$

so the above shows

$$B \circ (B \circ D_{B_0,B_0}B_0) \subset D_{B_0,B_0}B_0.$$

Thus to verify that K is an ideal, it suffices to show $A \circ (B \circ D_{B_0,B_0}B_0) \subseteq B \circ D_{B_0,B_0}B_0$. But $B \circ D_{B_0,B_0}B_0 = B_0 \circ D_{B_0,B_0}B_0$, so the required inclusion follows from (5.2) and (5.21). Consequently, K is an ideal in $\mathfrak{a}^+$ as claimed.

Clearly K is graded, and $\mathfrak{a}^+$ is graded simple. Hence, $K = 0$ or $K = \mathfrak{a}^+$. Suppose $K = \mathfrak{a}^+$. Then $A = B \circ D_{B_0,B_0}B_0$. But we have $[B, B \circ D_{B_0,B_0}B_0] = [[A,A], B \circ D_{B_0,B_0}B_0] = [[A,A], (D_{B_0,B_0}B_0) \circ B_0]$, and by (5.3), this is contained in

$$D_{A,A}((D_{B_0,B_0}B_0) \circ B_0) = 0,$$

since $D_{A,A}B_0 = 0$. That is, we have $[B, A] = [B, B \circ D_{B_0,B_0}B_0] = 0$, which is not our case. Therefore $K = 0$, and all the statements in Claim 5.20 follow. $\square$

Now, we want to show that the associator $(\alpha, \alpha', \alpha'') := (\alpha\alpha')\alpha'' - \alpha(\alpha'\alpha'') = 0$, and for this purpose, it suffices to verify that the following two identities

$$(\alpha \circ \alpha') \circ \alpha'' - \alpha \circ (\alpha' \circ \alpha'') = [\alpha, [\alpha', \alpha'']] - [[\alpha, \alpha'], \alpha''] \tag{5.22}$$

$$[\alpha, \alpha'] \circ \alpha'' - \alpha \circ [\alpha', \alpha''] = [\alpha, \alpha' \circ \alpha''] - [\alpha \circ \alpha', \alpha'']. \tag{5.23}$$

are satisfied for all $\alpha, \alpha', \alpha'' \in \mathfrak{a}$. By (5.1), relation (5.22) holds for $\alpha, \alpha', \alpha'' \in A$. Note that from the definition of $D_{\alpha,\alpha'}$ in (2.7), we have $D_{\alpha'',\alpha}\alpha' = \alpha'' \circ (\alpha \circ \alpha') - \alpha \circ (\alpha'' \circ \alpha')$, which is the left-hand side of (5.22). In addition, we have (5.22) for $\alpha, \alpha'' \in B$ and $\alpha' \in A$ by (5.11). Interchanging a and a' in (5.9) and subtracting the two relations gives (5.22) for $\alpha, \alpha'' \in A$ and $\alpha' \in B$. Thus the cases remaining to establish (5.22) are:

(a) $(a \circ b) \circ b' - a \circ (b \circ b') = [a, [b, b']] - [[a, b], b']$
(b) $(a \circ a') \circ b - a \circ (a' \circ b) = [a, [a', b]] - [[a, a'], b]$
(c) $(b \circ b') \circ b'' - b \circ (b' \circ b'') = [b, [b', b'']] - [[b, b'], b'']$

for all $a, a', a'' \in A$ and $b, b', b'' \in B$.

For (a), starting with (5.2), we have

$$\begin{aligned} 2a \circ (b \circ b') &= b \circ (b' \circ a) + b' \circ (b \circ a) + [b, [b', a]] + [b', [b, a]] \\ &= b \circ (b' \circ a) + b' \circ (b \circ a) \\ &\qquad + [[b, b'], a] + 2[b', [b, a]] \quad \text{by (5.19) and (5.11)} \\ &= 2b \circ (b' \circ a) + 2[b', [b, a]] \\ &\qquad \text{by (5.19) and the definition of } D_{b,b'}. \end{aligned}$$

Hence, $a \circ (b \circ b') = b \circ (b' \circ a) + [b', [b, a]]$, and thus,

$$\begin{aligned} (a \circ b) \circ b' - a \circ (b \circ b') &= (a \circ b) \circ b' - b \circ (b' \circ a) - [b', [b, a]] \\ &= -[[b, b'], a] - [b', [b, a]] \quad \text{by (5.19)}, \\ &= [a, [b, b']] - [[a, b], b']. \end{aligned}$$

Now applying (5.9), we see that (b) holds once we check that

$$[[a, a'], b] = [a, [a', b]] - [a', [a, b]]. \tag{5.24}$$

Recall that $[[a, a'], b]$ is a unique element in B so that $[[[a, a'], b], a''] = 4D_{[[a,a'],b]}a''$ for all $a'' \in A$. But then

$$\begin{aligned} [[[a, [a', b]] - [a', [a, b]], b], a''] &= [[[a, [a', b]], b], a''] - [[[a', [a, b]], b], a''] \\ &= 4D_{[[a,[a',b]],b]}a'' - 4D_{[a',[a,b]],b]}a'' \text{ by (5.23)} \\ &= 4D_{[[a,a'],b]}a'' \quad \text{by (5.4)}. \end{aligned}$$

As a result, we obtain (5.24).

Finally for (c), we observe that $(b \circ b') \circ b'' - b \circ (b' \circ b'') = D_{b'',b} b'$ is in $[A, A]$ by Claim 5.20. So, by the decomposition of B in Lemma 5.17, it suffices to show that, for all $a \in A$, $[D_{b'',b} b', a] + [[[b, b'], b''], a] = [[b, [b', b'']], a]$, or that $D_{b'',b}[b', a] - [b' D_{b'',b} a] + D_{[b,b'],b''} a = D_{b,[b',b'']} a$. By (5.11), this amounts to verifying that

$$D_{b'',b}[b', a] - [b', D_{b'',b} a] + [[b, b'], [b'', a]] - [b'', [[b, b'], a]] \\ = [b, [[b', b''], a]] - [[b', b''], [b, a]],$$

or

$$D_{b'',b}[b', a] - [b', D_{b'',b} a] + D_{b,b'}[b'', a] - [b'', D_{b,b'} a] = [b, D_{b',b''} a] - D_{b',b''}[b, a],$$

or

$$[D_{b'',b} b' + D_{b,b'} b'' + D_{b',b''} b, a] = 0.$$

But $D_{b'',b} b' + D_{b,b'} b'' + D_{b',b''} b = 0$ (simply by the definition of the inner derivation), and hence (c) is established.

Now for identity (5.23), interchanging α and α'' in (5.2) will show that this identity holds for $\alpha, \alpha', \alpha'' \in A$. Also, (5.7) implies the case with $\alpha, \alpha' \in A$ and $\alpha'' \in B$, and the case with $\alpha', \alpha'' \in B$ and $\alpha \in A$. What remains to be checked is that the following equations hold for all $a, a', a'' \in A$ and $b, b', b'' \in B$:

(d) $[a, b] \circ a' - a \circ [b, a'] = [a, b \circ a'] - [a \circ b, a']$
(e) $[b, a] \circ b' - b \circ [a, b'] = [b, a \circ b'] - [b \circ a, b']$
(f) $[b, b'] \circ a - b \circ [b', a] = [b, b' \circ a] - [b \circ b', a]$
(g) $[b, b'] \circ b'' - b \circ [b', b''] = [b, b' \circ b''] - [b \circ b', b'']$.

Reversing the roles of a and a' in (5.7) and adding gives

$$[b \circ a, a'] + [b \circ a', a] + a \circ [b, a'] + a' \circ [b, a] = 2[b, a \circ a'].$$

Then by (5.5), we have (d).

In view of (5.13), verification of (e) reduces to showing

$$[b \circ a, b'] = [a, b \circ b'] + [b, b' \circ a],$$

or, since all terms belong to $[A, A]$, that for all $a'' \in A$,

$$[[b \circ a, b'], a''] + [[b' \circ a, b], a''] = [[a, b \circ b'], a''].$$

By definition, the left side equals $4D_{b\circ a,b'}a'' + 4D_{b'\circ a,b}a''$, which by (5.10) is $4D_{a,b\circ b'}a''$. The relation in (e) now follows from (5.3).

To establish (f), it will suffice by the decomposition $B = [A,A] \oplus B_0$ to treat two separate cases:

(1) $b' = b_0 \in B_0$;

(2) $b' = [a', a'']$ for some $a', a'' \in A$.

Now when $b' = b_0$, we have $[b, b'] = 0$, and our relation reduces to showing

$$[b \circ b_0, a] = [b, b_0 \circ a].$$

Since both members are in $[A, A]$, it suffices to prove that

$$[[b \circ b_0, a], a'] = [[b, b \circ a], a'].$$

for $a' \in A$. By (5.3), the left side is $D_{b\circ b_0,a}a'$, while the right equals $D_{b,b_0\circ a}a'$ by the definition of $[\cdot,\cdot]$ on B. Equation (5.10) shows that the difference of the two is $D_{b\circ a,b_0}a' = 0$, since $D_{B,B_0} = 0$ by Claim 5.20. Thus (f) holds if $b' \in B_0$.

Suppose then that $b' = [a', a'']$, where $a', a'' \in A$. Here (f) reads:

$$[b,[a',a'']] \circ a - b \circ [[a',a''],a] = [b,[a',a''] \circ a] - [b \circ [a',a''],a]. \tag{5.25}$$

We have $[b,[a',a'']] = (b \circ a') \circ a'' + [[b,a'],a''] - b \circ (a' \circ a'')$ from (5.22). Substitution shows the first term on the left-hand side of (5.25) to be:

$$((b \circ a') \circ a'') \circ a + [[b,a'],a''] \circ a - (b \circ (a' \circ a'')) \circ a.$$

By (5.3), the second term on the left of (5.25) is

$$-b \circ D_{a',a''}a = -D_{a',a''}(b \circ a) + (D_{a',a''}b) \circ a$$

by the derivation property. This expression is equal to

$$-[a',[a'',b \circ a]] + [a'',[a',b \circ a]] + [a',[a'',b]] \circ a - [a'',[a',b]] \circ a$$

by (5.8). Hence, the left-hand side of (5.25) becomes

$$-[a',[a'',b\circ a]] + [a'',[a',b\circ a]] + ((b\circ a')\circ a'' - b\circ(a'\circ a'') + [a',[a'',b]])\circ a, \tag{5.26}$$

which is equal to $-[a',[a'',b \circ a]] + [a'',[a',b \circ a]]$ by (5.9). Thus (f) reduces to showing

$$-[a',[a'',b \circ a]] + [a'',[a',b \circ a]] = [b,[a',a''] \circ a] - [b \circ [a',a''],a]. \tag{5.27}$$

All terms in (5.27) lie in $[A, A]$, so it is sufficient to show, for all $a''' \in A$, that

$$[-[a', [a'', b \circ a]] + [a'', [a', b \circ a]], a'''] = [[b, [a', a''] \circ a] - [b \circ [a', a''], a], a'''].$$

By (5.3) and our definition of $[\cdot, \cdot]$ on B, this identity is equivalent to $-D_{a',[a'',b\circ a]} + D_{a'',[a',b\circ a]} = D_{b,[a',a'']\circ a} - D_{b\circ[a',a''],a}$. Then (5.10) can be quoted to give $-D_{a',[a'',b\circ a]} + D_{a'',[a',b\circ a]} = D_{a\circ b,[a',a'']}$, and then (f) is a direct consequence of (5.4).

Finally, we tackle (g). As in case (1), where $b' \in B_0$, we must show $[b, b' \circ b''] + [b'', b \circ b'] = 0$ for all $b, b'' \in B$. However, this is immediate from (5.14). For case (2), where $b' = [a, a']$, we use (b) of (5.22) to substitute for $[[a, a'], b]$ and $[[a, a'], b'']$, then apply (5.9) to obtain $[b, b'] = (a \circ b) \circ a' - a \circ (b \circ a')$ and $[b, b''] = a \circ (b'' \circ a') - (a \circ b'') \circ a'$. Now showing (g) reduces, by ((5.14), to showing

$$[b', b'' \circ b] + [b, b'] \circ b'' - [b', b''] \circ b = 0$$

for $b' = [a, a']$, or that

$$[[a, a'], b'' \circ b] + [b, [a, a']] \circ b'' - [[a, a'], b''] \circ b = 0. \tag{5.28}$$

Now $[[a, a'], b'' \circ b] = D_{a,a'}(b'' \circ b)$ by (5.3), and this says

$$\begin{aligned}(D_{a,a'}b'') \circ b + (D_{a,a'}b) \circ b''\\ = [a', [b'', a]] \circ b - [a, [b'', a']] \circ b + [a', [b, a]] \circ b'' - [a, [b, a']] \circ b''\end{aligned}$$

by (5.8) Thus, the left side of (5.28) becomes

$$\begin{aligned}&\Big([a', [b'', a]] \circ b - [a, [b'', a']] - a \circ (b'' \circ a') + (a \circ b'') \circ a'\Big) \circ b\\ &\quad + \Big([a', [b, a]] \circ b'' - [a, [b, a']] + (a \circ b) \circ a' - a \circ (b \circ a')\Big) \circ b'',\end{aligned}$$

which is 0 by (5.22). Thus, (g) is proved, and we have the desired conclusion that $\mathfrak{a}$ is associative.

Remark 5.29. When $S_+ = \operatorname{supp} A$ is not a subgroup of G, we have $[A, B_0] = 0 = [B, B_0]$, so B_0 is contained in the center of the associative algebra $\mathfrak{a} = A \oplus B$.

Remark 5.30. When S_+ is not a subgroup, then we claim that $G = \langle S_+ \rangle$ is forced. To see this, let $G' := \langle S_+ \rangle$ and suppose that $G' \neq G$. Then, there exists some $g \in S_- \setminus G'$. Take $0 \neq b \in \mathfrak{a}^g = B^g$. Then, $[A, b] = 0$. Indeed, let $0 \neq a \in \mathfrak{a}^h = A^h$. Then, $[a, b] \in A^{h+g}$, but $h + g \notin G'$ since $h \in G'$. Hence, by Lemma 5.16, $b \circ a \neq 0$ for any homogeneous $0 \neq a \in A^h$. For the same reason, $(b \circ a) \circ a' \neq 0$ for any homogeneous $0 \neq a' \in A$ since $b \circ a \in B^{h+g}$ and $h + g \notin G'$. However, there exist nonzero homogeneous elements $a, a' \in A$ such that $a \circ a' = 0$ since S_+ is not a subgroup. This is a contradiction by (5.9);

$$(b \circ a) \circ a' = b \circ (a \circ a')$$

for any $a, a' \in A$.

Consequently, for an associative G-torus $\mathfrak{a} = A \oplus B$ with involution, if the set S_+ does not generate G, then S_+ is a subgroup, and $\mathfrak{a}$ becomes a Clifford G-torus upon defining $[B, B] = 0$.

Combining all our results, we arrive at our main theorem.

Theorem 5.31. *A centerless Lie G-torus $\mathcal{L}$ of type C_r is isomorphic to a Lie algebra $\mathfrak{sp}_{2r}(\mathfrak{a})$, where $\mathfrak{a}$ is:*

- *an associative G-torus with graded involution if $r \geq 4$,*
- *an alternative G-torus with graded involution whose symmetric elements are in the nucleus of $\mathfrak{a}$ if $r = 3$,*
- *an associative G-torus with graded involution or a Clifford G-torus if $r = 2$.*

This result generalizes the classification of the core of extended affine Lie algebras of type C_r in [AG], as the core is a Lie torus, i.e., a Lie G-torus for $G = \mathbb{Z}^n$. In this case one can use more concrete terminology in describing $\mathcal{L}$.

Corollary 5.32. (Compare [AG, Thm. 4.87].) *A centerless Lie torus $\mathcal{L}$ of type C_r is isomorphic to a Lie algebra $\mathfrak{sp}_{2r}(\mathfrak{a})$, where $\mathfrak{a}$ is:*

- *a quantum torus with graded involution if $r \geq 4$,*
- *a quantum torus with graded involution or an octonion torus with standard involution if $r = 3$,*
- *a quantum torus with graded involution or a Clifford torus if $r = 2$.*

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Department of Mathematics, University of Wisconsin, Madison, WI 53706-1388 USA

benkart@math.wisc.edu

Department of Mathematics, North Dakota State University, Fargo, ND 58105-5075 USA

yoji.yoshii@ndsu.nodak.edu