BANACH-LIE ALGEBRAS WITH EXTREMAL ELEMENTS

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Dedicated to Professor El Amin Kaidi on the occasion of his 60th birthday

Abstract. In this paper strongly prime Banach-Lie algebras with extremal elements are described in a way similar to that of the classical Banach-Lie algebras of compact operators on Hilbert spaces.

1. Introduction

A finite-dimensional Lie algebra L over an algebraically closed field $\mathbb{F}$ of characteristic 0 is semisimple if and only if it is non-degenerate, $[x,[x,L]] = 0$ implies $x = 0$ for all x in L, and linearly spanned by its extremal elements, i.e., elements e in L such that $[e,[e,L]] = \mathbb{F}e$. Similarly, an infinite-dimensional simple Lie algebra over $\mathbb{F}$ is finitary, according to Baranov's definition [**1**], if and only if it is non-degenerate and contains extremal elements [**6**, Corollary 5.5]. It should be noted that the notion of extremal element in Lie algebras is similar to those of rank-one element in associative algebras ($eAe = \mathbb{F}e$, $e \neq 0$) and reduced element in Jordan algebras ($U_eJ = \mathbb{F}e$, $e \neq 0$) [**14**].

Banach-Lie algebras of compact operators on infinite-dimensional complex Hilbert spaces [**5**], and topologically simple L^*-algebras [**4**], are natural examples of prime non-degenerate (here called strongly prime) Banach-Lie algebras containing extremal elements: any of these Banach-Lie algebras contains a minimal ideal, its socle, which is a simple finitary Lie algebra.

In Theorem 7.3 we describe the strongly prime infinite-dimensional complex Banach-Lie algebras with extremal elements as Lie algebras of bounded linear operators on Banach spaces (which are self-dual or form part of a Banach pairing) containing an ideal which is a simple

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finitary Lie algebra. This result could be regarded as a Lie version of those given in [**3**, Section 31, Theorem 6] for primitive Banach algebras with minimal one-sided ideals, and in [**15**, Theorem 1.1] for prime non-degenerate Jordan-Banach algebras with nonzero socle. In fact, we must give an special credit to this last paper, where we have found inspiration for some of the arguments used in the proof of our main result. Finally, we show that via conjugations and anti-conjugations, Banach-Lie algebras of compact operators on Hilbert spaces can be also modelled in terms of Banach pairings and self-dual Banach spaces.

2. Preliminaries

2.1. Throughout this section we will be dealing with Lie algebras L, with $[x, y]$ denoting the Lie bracket and ad_x the adjoint map determined by x, over a field $\mathbb{F}$ of characteristic 0 [**13**].

2.2. An element $x \in L$ is an *absolute zero divisor* if $\mathrm{ad}_x^2 = 0$; L is *non-degenerate* if it has no nonzero absolute zero divisors, *semiprime* if $[I, I] = 0$ implies $I = 0$, and *prime* if $[I, J] = 0$ implies $I = 0$ or $J = 0$, for any ideals I, J of L. A Lie algebra is *strongly prime* if it is prime and non-degenerate, and *simple* if it is nonabelian and contains no proper ideals.

2.3. Every ideal of a non-degenerate (strongly prime) Lie algebra is non-degenerate (strongly prime) [**16**, Lemma 4] and [**10**, (0.4), (1.5)].

2.4. The *annihilator* or *centralizer* of a subset S of L is the set $\mathrm{Ann}_L\, S$ consisting of the elements $x \in L$ such that $[x, S] = 0$. By the Jacobi identity, $\mathrm{Ann}_L\, S$ is a subalgebra of L and an ideal whenever S is so. Clearly, $\mathrm{Ann}_L\, L = Z(L)$, the center of L. If L is semiprime, then $I \cap \mathrm{Ann}_L\, I = 0$ for any ideal I of L. The annihilator of a non-degenerate ideal I of L has the following quadratic expression: $\mathrm{Ann}_L\, I = \{a \in L \mid [a, [a, I]] = 0\}$ [**7**, (2.5)].

Lemma 2.5. *Let L be a Lie algebra containing a nonabelian minimal ideal M. Then L is strongly prime if and only if $[a, [M, a]] \neq 0$ for any nonzero $a \in L$.*

Proof. Suppose that L is strongly prime and that $[a,[M,a]] = 0$ for some $a \in L$. Then $a \in \operatorname{Ann}_L M = 0$ by 2.4. Suppose conversely that $[a,[M,a]] \neq 0$ for any nonzero $a \in L$. Then L is non-degenerate, and since M is minimal, for any ideal I of L either $M \subset I$ or $M \cap I = 0$; but $M \cap I = 0$ implies $[x,[M,x]] = 0$ for any $x \in I$ and hence $I = 0$ by hypothesis, so $M \subset I$ for any nonzero ideal I of L, which clearly implies that L is prime since M is not abelian. □

2.6. An *inner ideal* of a Lie algebra L is a subspace B of L such that $[B,[B,L]] \subset B$ [**2**]. An *abelian inner ideal* is an inner ideal which is also an abelian subalgebra. An element $x \in L$ is said to be *extremal* if it generates a one-dimensional inner ideal, that is, $\operatorname{ad}_x^2 L = \mathbb{F}x$.

2.7. The *socle* of a non-degenerate Lie algebra L is defined as the sum of all minimal inner ideals of L. By [**6**, Theorem 2.5], $\operatorname{Soc} L$ is an ideal of L and a direct sum $\operatorname{Soc} L = \bigoplus_\alpha M_\alpha$ of simple ideals M_α of L. Furthermore each simple component M_α of $\operatorname{Soc} L$ is either inner simple or contains an abelian minimal inner ideal [**2**, Theorem 1.12].

2.8. An element x in L is called a *Jordan element* if $\operatorname{ad}_x^3 = 0$. Clearly, any element of an abelian inner ideal is a Jordan element. Conversely, by [**2**, (1.8)], any Jordan element b yields the abelian inner ideal $[b,[b,L]]$. A good reason for this terminology is the following analogue of the fundamental identity for Jordan algebras:

$$\operatorname{ad}^2_{\operatorname{ad}^2_x y} = \operatorname{ad}^2_x \operatorname{ad}^2_y \operatorname{ad}^2_x$$

for any $y \in L$. Another reason is given in the next proposition [**9**, Theorem 2.4].

Proposition 2.9. *Let $a \in L$ be a Jordan element. Then L with the new product defined by $x \cdot_a y := \frac{1}{2}[[x,a],y]$ is a nonassociative algebra denoted by $L^{(a)}$, such that*

(i) $\operatorname{Ker}_L a := \{x \in L | [a,[a,x]] = 0\}$ *is an ideal of $L^{(a)}$.*

(ii) $L_a := L^{(a)}/\operatorname{Ker}_L a$ *is a Jordan algebra, called* the Jordan algebra of L at a.

2.10. Let L be a complex Lie algebra. By an *algebra-norm* of L we mean any norm $||\cdot||$ on the complex vector space L making continuous the bracket product, i.e., there exists a positive number k such that $||[x,y]|| \leq k||x||||y||$ for all $x,y \in L$. A *normed* Lie algebra is a complex Lie algebra L endowed with an algebra norm. If the norm is complete, then L is called a Banach-Lie algebra.

3. Lie algebras with extremal elements

All the vector spaces considered in this section are infinite-dimensional over an algebraically closed field $\mathbb{F}$ of characteristic 0.

3.1. Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a pair of dual vectors spaces over $\mathbb{F}$, i.e., X, Y are vector spaces over $\mathbb{F}$, and $\langle\cdot,\cdot\rangle : X \times Y \to \mathbb{F}$ is a non-degenerate bilinear form. (Note that any vector space X gives rise to the canonical pair (X, X^*), where X^* is the dual of X.) We associate with $(X, Y, \langle\cdot,\cdot\rangle)$ the following algebras:

(i) The associative algebra $\mathcal{L}_Y(X)$ of all the linear operators $a : X \to X$ having a (unique) adjoint $a^{\#} : Y \to Y$, i.e., $\langle ax, y\rangle = \langle x, a^{\#}y\rangle$ for all $x \in X$, $y \in Y$. Note that $\mathcal{L}_{X^*}(X) = \operatorname{End} X$.
(ii) The ideal $\mathcal{F}_Y(X)$ of all linear operators $a \in \mathcal{L}_Y(X)$ having finite rank.
(iii) The *general linear algebra* $\mathfrak{gl}_Y(X) := \mathcal{L}_Y(X)^{(-)}$.
(iv) The *finitary linear algebra* $\mathfrak{fgl}_Y(X) := \mathcal{F}_Y(X)^{(-)}$.
(v) The *special linear algebra* $\mathfrak{fsl}_Y(X) := [\mathfrak{fgl}_Y(X), \mathfrak{fgl}_Y(X)]$. Clearly, $\mathfrak{fgl}_Y(X)$ and $\mathfrak{fsl}_Y(X)$ are ideals of $\mathfrak{gl}_Y(X)$.

In the case that X coincides with Y, we will write $\mathfrak{gl}(X, \langle\cdot,\cdot\rangle)$, $\mathfrak{fgl}(X, \langle\cdot,\cdot\rangle)$ and $\mathfrak{fsl}(X, \langle\cdot,\cdot\rangle)$ instead of $\mathfrak{gl}_X(X)$, $\mathfrak{fgl}_X(X)$ and $\mathfrak{fsl}_X(X)$.

3.2. Given $x \in X$ and $y \in Y$, let y^*x denote the linear operator defined by $y^*x(x') = \langle x', y\rangle x$, for all $x' \in X$. It is easy to see that $y^*x \in \mathcal{F}_Y(X)$, with adjoint x^*y. Moreover, $y^*x \in \mathfrak{fsl}_Y(X)$ if and only if $\langle x, y\rangle = 0$ [**8**, Theorem 1.7].

The following proposition reviews some well-known results on the structure of the three Lie algebras listed above. Nevertheless, we provide an elementary proof for the sake of completeness and to present some arguments which will be frequently used in what follows.

Proposition 3.3. *Assume that $(X, Y, \langle\cdot,\cdot\rangle)$ is an infinite-dimensional pair of dual vector spaces over $\mathbb{F}$. Then*

(i) $\mathfrak{fsl}_Y(X)$ *is an infinite-dimensional centroid-simple Lie algebra over* $\mathbb{F}$.
(ii) *If L is a subalgebra of $\mathfrak{gl}_Y(X)$ containing $\mathfrak{fsl}_Y(X)$, then L has extremal elements and $Z(L)$ is either* 0 *or* $\mathbb{F}\,\mathrm{Id}_X$.
(iii) *If further $L \cap \mathbb{F}\,\mathrm{Id}_X = 0$, then L is strongly prime.*

Proof. (i) See [**1**, Proposition 6.1].

(ii) If $a \in L$ is not a multiple of the identity map on X, then there exists $x \in X$ such that the vectors x and ax are linearly independent. Take $y \in Y$ such that $\langle x, y\rangle = 0$ and $\langle ax, y\rangle = 1$. Then $y^*x \in \mathfrak{fsl}_Y(X) \subset L$ and $[a, y^*x] = y^*ax - (a^\# y)^*x \neq 0$, so a is not a central element. We also note that $b = y^*x$ is an extremal element of L [**7**, Proposition 6.4(i)].

(iii) Since $\mathfrak{fsl}_Y(X)$ is a simple ideal of L, to prove that L is strongly prime it suffices to show by 2.5 that for any $a \in \mathfrak{gl}_Y(X)$ which is not a multiple of the identity, $[[a, \mathfrak{fsl}_Y(X)], a] \neq 0$. Take $x \in X$ such that x and ax are linearly independent, and let $y \in Y$ be such that $\langle x, y\rangle = 0$ and $\langle ax, y\rangle = 1$. Then the vectors y and $a^\# y$ are also linearly independent; otherwise, $a^\# y = \alpha y$ would imply $\langle ax, y\rangle = \langle x, a^\# y\rangle = \langle x, \alpha y\rangle = 0$, which is a contradiction. Put $b := y^*x$. Since $\langle x, y\rangle = 0$, $b \in \mathfrak{fsl}_Y(X)$. Moreover, $[[a, b], a] = 2aba - a^2b - ba^2 = 2a(y^*x)a - a^2(y^*x) - (y^*x)a^2 = 2(a^\# y)^*ax - y^*a^2x - (a^{2\#}y)^*x$. Take $x' \in X$ such that $\langle x', y\rangle = 0$ and $\langle x', a^\# y\rangle = 1$ (which is possible because y and $a^\# y$ are linearly independent) and compute its image under the operator $[[a, b], a]$. We get $[[a, b], a]x' = 2ax - \langle x', a^{2\#}y\rangle x \neq 0$, since x and ax are linearly independent. $\square$

3.4. Let X be a vector space over $\mathbb{F}$ endowed with a non-degenerate symmetric (respectively, alternate) bilinear form $\langle\cdot,\cdot\rangle$. Then $(X, X, \langle\cdot,\cdot\rangle)$ is a pair of dual vector spaces and the adjoint becomes an involution, denoted by $*$, in the associative algebra $\mathcal{L}(X) := \mathcal{L}_X(X)$, making the ideal $\mathcal{F}(X)$ $*$-invariant. Associated with the self-dual vector space X we have the following Lie algebras:

(i) The *orthogonal algebra* $\mathfrak{o}(X, \langle\cdot,\cdot\rangle) := \mathrm{Skew}(\mathcal{L}(X), *)$ if $\langle\cdot,\cdot\rangle$ is symmetric. Respectively, the *symplectic algebra* $\mathfrak{sp}(X, \langle\cdot,\cdot\rangle) := \mathrm{Skew}(\mathcal{L}(X), *)$ if $\langle\cdot,\cdot\rangle$ is alternate.

(ii) The *finitary orthogonal algebra* $\mathfrak{fo}(X, \langle\cdot,\cdot\rangle) := \mathrm{Skew}(\mathcal{F}(X), *) = [\mathrm{Skew}(\mathcal{F}(X), *), \mathrm{Skew}(\mathcal{F}(X), *)]$ if $\langle\cdot,\cdot\rangle$ is symmetric. Respectively, the *finitary symplectic algebra* $\mathfrak{fsp}(X, \langle\cdot,\cdot\rangle) := \mathrm{Skew}(\mathcal{F}(X), *) = [\mathrm{Skew}(\mathcal{F}(X), *), \mathrm{Skew}(\mathcal{F}(X), *)]$, if $\langle\cdot,\cdot\rangle$ is alternate.

3.5. If $\langle\cdot,\cdot\rangle$ is symmetric, then for any $x, y \in X$ the linear operator $[x, y] := x^*y - y^*x$ belongs to $\mathfrak{fo}(X, \langle\cdot,\cdot\rangle)$. In fact, these operators linearly span $\mathfrak{fo}(X, \langle\cdot,\cdot\rangle)$. If $\langle\cdot,\cdot\rangle$ is alternate, then $\mathfrak{fsp}(X, \langle\cdot,\cdot\rangle)$ is the linear span of the operators x^*x.

3.6. Let $\langle\cdot,\cdot\rangle$ be symmetric or alternate. For a *hyperbolic pair* we mean a pair (x, y) of isotropic vectors of X such that $\langle x, y\rangle = 1$. A *hyperbolic plane* is any 2-dimensional subspace of X having a basis consisting of a hyperbolic pair. Since $\mathbb{F}$ is algebraic closed, a 2-dimensional subspace H of X is a hyperbolic plane if and only if it is non-degenerate.

Lemma 3.7. *Let $a \in \mathfrak{o}(X, \langle\cdot,\cdot\rangle)$ be such that any isotropic vector $x \in X$ is an eigenvector for a. Then $a = 0$.*

Proof. Let (x_1, y_1) and (x_2, y_2) be mutually orthogonal hyperbolic pairs. Note that $ax_i = \lambda_i x_i$ implies $ay_i = -\lambda_i y_i$. Moreover, since both $x_1 + x_2$ and $x_1 + y_2$ are isotropic, $a(x_1 + x_2) = \lambda_1 x_1 + \lambda_2 x_2$ implies $\lambda_1 = \lambda_2$, and $a(x_1 + y_2) = \lambda_1 x_1 - \lambda_2 y_2$ implies $\lambda_1 = -\lambda_2$. Thus $\lambda_1 = \lambda_2 = 0$. Hence $a = 0$, since X is spanned by isotropic vectors. $\square$

The following two propositions are the analogues of 3.3 for Lie algebras of linear operators of the orthogonal and the symplectic types. As in the case of 3.3, we provide elementary proofs of these results.

Proposition 3.8. *Let $(X, \langle\cdot,\cdot\rangle)$ be an infinite-dimensional self-dual vector space with symmetric inner product. Then* (i) $\mathfrak{fo}(X, \langle\cdot,\cdot\rangle)$ *is an infinite-dimensional centroid-simple Lie algebra over $\mathbb{F}$.* (ii) *Any subalgebra L of $\mathfrak{o}(X, \langle\cdot,\cdot\rangle)$ containing $\mathfrak{fo}(X, \langle\cdot,\cdot\rangle)$ is strongly prime and has extremal elements.*

Proof. (i) See [**1**, Proposition 6.4]. (ii) Since $\mathfrak{fo}(X, \langle\cdot,\cdot\rangle)$ is a simple ideal of L, to prove that L is strongly prime it suffices to show by 2.5 that for any nonzero $a \in \mathfrak{o}(X, \langle\cdot,\cdot\rangle)$, $[[a, \mathfrak{fo}(X, \langle\cdot,\cdot\rangle)], a] \neq 0$. By 3.7, there exists an isotropic vector $x \in X$ such that x and ax are linearly independent. Let $z \in X$ be such that $\langle x, z\rangle = 1$ and $\langle ax, z\rangle = 0$. Then $H := \mathbb{F}x + \mathbb{F}z$ is a hyperbolic plane, and since $\langle ax, x\rangle = 0$, $ax \in H^{\perp}$. Thus the vectors x, z and ax are linearly independent. Take $y \in X$ such that $y \in H^{\perp}$ and $\langle y, ax\rangle = 1$, and put $b := [x, y] = x^*y - y^*x$. Then $b \in \mathfrak{fo}(X, \langle\cdot,\cdot\rangle) \subset L$ and $[[a, b], a] = 2a(x^*y - y^*x)a - a^2(x^*y - y^*x) - (x^*y - y^*x)a^2 = -2(ax)^*ay + 2(ay)^*ax - x^*a^2y + y^*a^2x - (a^2x)^*y + (a^2y)^*x$. If ax is isotropic, then $[[a, b], a]x = -2ax + \langle x, a^2y\rangle x \neq 0$ since the vectors x and ax are linearly independent. Suppose then that ax is anisotropic. Since the field $\mathbb{F}$ is algebraically closed we may assume that $\langle ax, ax\rangle = 1$. Taking $y = ax$ we obtain $[[a, b], a]x = -2ax - \langle x, a^2x\rangle ax + \langle x, a^3x\rangle x = -2ax + ax = -ax \neq 0$, since $\langle x, a^3x\rangle = -\langle ax, a(ax)\rangle = 0$. Therefore, in both cases $[[a, b], a] \neq 0$ as required. Finally, let $x, y \in X$ be two nonzero isotropic and mutually orthogonal vectors. Then it follows from [**7**, Proposition 6.4(ii)] that $[x, y]$ is an extremal element of L. □

Proposition 3.9. *Let $(X, \langle\cdot,\cdot\rangle)$ be an infinite-dimensional self-dual vector space with alternate inner product. Then* (i) $\mathfrak{fsp}(X, \langle\cdot,\cdot\rangle)$ *is an infinite-dimensional centroid-simple Lie algebra over* $\mathbb{F}$. (ii) *Any subalgebra L of $\mathfrak{s}p(X, \langle\cdot,\cdot\rangle)$ containing $\mathfrak{fsp}(X, \langle\cdot,\cdot\rangle)$ is strongly prime and has extremal elements.*

Proof. (i) See [**1**, Proposition 6.4]. (ii) Since $\mathfrak{fsp}(X, \langle\cdot,\cdot\rangle)$ is a simple ideal of L, to prove that L is strongly prime it suffices to show by 2.5 that for any nonzero $a \in \mathfrak{s}p(X, \langle\cdot,\cdot\rangle)$, $[[a, \mathfrak{fsp}(X, \langle\cdot,\cdot\rangle)], a] \neq 0$. Let a be a nonzero element of L. Then a is not a multiple of the identity, so there exists $x \in X$ such that x and ax are linearly independent. Let $y \in X$ be such that $\langle x, y\rangle = 0$ and $\langle ax, y\rangle = 1$, and put $b := x^*x$. Then $b \in \mathfrak{fsp}(X, \langle\cdot,\cdot\rangle) \subset L$ and $[[a, b], a] = 2a(x^*x)a - a^2x^*x - x^*xa^2 = -2(ax)^*ax - x^*a^2x - (a^2x)^*x$. Hence $[[a, b], a]y = 2ax - \langle y, a^2x\rangle x \neq 0$, since x and ax are linearly independent. Finally, it follows from [**7**, Proposition 6.4(iii)] that x^*x is an extremal element of L. □

Theorem 3.10. *Let L be an infinite-dimensional Lie algebra over $\mathbb{F}$. Then L is strongly prime and contains an extremal element if and only if it is, up to isomorphism, one of the following:*

(i) $\mathfrak{fsl}_Y(X) \leq L \leq \mathfrak{gl}_Y(X)$ *with* $L \cap \mathbb{F}\,\mathrm{Id}_X = 0$, *where* (X, Y) *is an infinite-dimensional pair of dual vector spaces over* $\mathbb{F}$.

(ii) $\mathfrak{fo}(X, \langle\cdot,\cdot\rangle) \leq L \leq \mathfrak{o}(X, \langle\cdot,\cdot\rangle)$, *where* $(X, \langle\cdot,\cdot\rangle$ *is an infinite-dimensional self-dual vector space over* $\mathbb{F}$ *with symmetric inner product.*

(iii $\mathfrak{fsp}(X, \langle\cdot,\cdot\rangle) \leq L \leq \mathfrak{s}p(X, \langle\cdot,\cdot\rangle)$, *where* $(X, \langle\cdot,\cdot\rangle$ *is an infinite-dimensional self-dual vector space over* $\mathbb{F}$ *with alternate inner product.*

Proof. By 3.3, 3.8 and 3.9, the Lie algebras L listed above are strongly prime and contain extremal elements. Suppose conversely that L is an infinite-dimensional strongly prime Lie algebra over $\mathbb{F}$ containing extremal elements. Then we have by [**6**, Corollary 5.5] that $\mathrm{Soc}\, L$ is a simple finitary Lie algebra over $\mathbb{F}$. Hence, by Baranov's classification [**1**, Corollary 1.2], $\mathrm{Soc}\, L$ is isomorphic to one the Lie algebras $\mathfrak{fsl}_Y(X)$, $\mathfrak{fo}(X, \langle\cdot,\cdot\rangle)$ or $\mathfrak{fsp}(X, \langle\cdot,\cdot\rangle)$. Use now the fact that $L \subset \mathrm{Der}\,\mathrm{Soc}\, L$, via the adjoint representation, and the computation (see [**7**, Theorem 6.2]) of the algebra of derivations of each one of the finitary simple Lie algebras to complete the proof. □

Remarks 3.11. The above theorem was proved in [**7**, Theorem 6.7] by assuming the existence of Jordan reduced elements, a more restrictive notion of that of extremal element and that requieres the existence of a nontrivial 3-grading.

4. Banach-Lie algebras of special type

4.1. Following [**3**], a *Banach pairing* is a pair of dual vector spaces $(X, Y, \langle\cdot,\cdot\rangle)$ over $\mathbb{C}$ such that both X and Y are endowed with prefixed complete norms making the bilinear form $\langle\cdot,\cdot\rangle$ continuous. As a consequence of the closed graph theorem, every $a \in \mathcal{L}_Y(X)$ is a norm-continuous operator on X, so $\mathcal{L}_Y(X)$ is a subalgebra of the Banach (associative) algebra $\mathrm{BL}(X)$ of all bounded linear operators on X. Although $\mathcal{L}_Y(X)$ needs not be complete for the operator norm, it has a natural structure of Banach algebra under the norm $|\cdot|'$ defined by $|a|' = \max\{|a|, |a^{\#}|\}$, where $|\cdot|$ denotes the operator norm. As a consequence, we have

Proposition 4.2. *Let* $(X, Y, < .,. >)$ *be a Banach pairing. Then* $\mathfrak{gl}_Y(X)$ *is a Banach-Lie algebra for the norm defined by* $|a|' = \max\{|a|, |a^\#|\}$, *with* $|\cdot|$ *denoting the operator norm.*

A similar result to the statement of the next proposition was proved in [**15**, Case 1 or Proposition 2.1] for Jordan-Banach algebras. In fact, the proof we give here follows the pattern of that for the Jordan case.

Proposition 4.3. *Let* $(X, Y, < .,. >)$ *be an infinite-dimensional pair of dual vector spaces over* $\mathbb{C}$, *and let* L *be a Lie algebra such that*

$$\mathfrak{fsl}_Y(X) \leq L \leq \mathfrak{gl}_Y(X) \text{ and } L \cap \mathbb{C}\,\mathrm{Id}_X = 0.$$

Suppose further that L *is a Banach-Lie algebra for a norm* $||\cdot||$. *Then* $(X, Y, < .,. >)$ *can be endowed with a Banach pairing structure making the injection of* $(L, ||\cdot||)$ *into* $(\mathfrak{gl}_Y(X), |\cdot|')$ *continuous.*

Proof. Let $x \in X$ and $y \in Y$ be such that $\langle x, y\rangle = 0$, and let $a \in L$. Then $y^*x \in \mathfrak{fsl}_Y(X) \subset L$ and we have

$$[[y^*x, a], y^*x] = 2\langle ax, y\rangle y^*x, \text{ and hence } |\langle ax, y\rangle| \leq k||y^*x||||a||, \quad (1)$$

for some positive number k. Indeed, since $(y^*x)(y^*x) = \langle x, y\rangle y^*x = 0$, we have $[[y^*x, a], y^*x] = 2(y^*x)a(y^*x) = 2(y^*x)(y^*ax) = 2\langle ax, y\rangle y^*x$.

Given $x_0 \in X$ and $y_0 \in Y$ such that $\langle x_0, y_0\rangle = 1$, we have

$$X = \mathbb{C}x_0 \oplus \{y_0\}^\perp,\ \{x_0\}^{\perp\perp} = \mathbb{C}x_0,\ Y = \mathbb{C}y_0 \oplus \{x_0\}^\perp \text{ and } \{y_0\}^{\perp\perp} = \mathbb{C}y_0. \quad (2)$$

From now on fix $x_0 \in X$ and $y_0 \in Y$ as above, and let $\pi : X \to \{y_0\}^\perp$ be the projection of X onto $\{y_0\}^\perp$ along $\mathbb{C}x_0$, i.e., $\pi(\alpha x_0 + x_1) = x_1$, for all $\alpha \in \mathbb{C}$ and $x_1 \in \{y_0\}^\perp$. Let $\varphi : L \to \{y_0\}^\perp$ be the mapping defined by $\varphi(a) = \pi(ax_0)$ for all $a \in L$. Then

$$\varphi \text{ is onto and } \mathrm{Ker}(\varphi) = \bigcap_{y_1 \in \{x_0\}^\perp} \mathrm{Ker}(\mathrm{ad}^2_{y_1^* x_0}). \quad (3)$$

Indeed, given $x_1 \in \{y_0\}^\perp$, take $a = y_0^* x_1$. Since $< x_1, y_0 >= 0$, $a \in \mathfrak{fsl}(X) \subset L$ and satisfies $\varphi(a) = \pi(ax_0) = \pi(< x_0, y_0 > x_1) = \pi(x_1) = x_1$ by the definition of π, which proves that φ is onto. The equality relative to $\mathrm{Ker}(\varphi)$ follows from the following chain of equivalences, for any $a \in L$, obtained by using (1) and (2):

$$\varphi(a) = 0 \Leftrightarrow ax_0 \in \mathbb{C}x_0 = \{x_0\}^{\perp\perp} \Leftrightarrow a \in \bigcap_{y_1 \in \{x_0\}^{\perp}} \mathrm{Ker}(\mathrm{ad}^2_{y_1^* x_0}).$$

Since the operators ad^2_b are continuous for any $b \in L$, it follows from (3) that $\mathrm{Ker}(\varphi)$ is a closed subspace of L, so we can carry over the complete quotient norm of $L/\mathrm{Ker}(\varphi)$ to $\{y_0\}^{\perp}$ to define a complete norm on the subspace $\{y_0\}^{\perp}$

$$p(x_1) := \inf\{||a|| : a \in L \text{ and } \pi(ax_0) = x_1\}, \tag{4}$$

for all $x_1 \in \{y_0\}^{\perp}$. Since $X = \mathbb{C}x_0 \oplus \{y_0\}^{\perp}$ by (2), we can extend the norm p to a complete norm, also denoted by p, on the whole X

$$p(\alpha x_0 + x_1) := |\alpha| + p(x_1), \text{ for all } \alpha \in \mathbb{C} \text{ and } x_1 \in \{y_0\}^{\perp}. \tag{5}$$

Since the adjoint involution defines an isomorphism of $\mathfrak{gl}_Y(X)$ onto $\mathfrak{gl}_X(Y)$, we can consider the Banach-Lie algebra $L^{\#}$ with norm $||a^{\#}|| := ||a||$ for any $a \in L$. Then we obtain as above a complete norm q on Y given by

$$q(\beta y_0 + y_1) := |\beta| + q(y_1), \tag{6}$$

for all $\beta \in \mathbb{C}$ and $y_1 \in \{x_0\}^{\perp}$, where $q(y_1) = \inf\{||b|| : b \in L \text{ and } \rho(b^{\#} y_0) = y_1\}$, with $\rho(\beta y_0 + y_1) = y_1$ being the projection of $Y = \mathbb{C}y_0 \oplus \{x_0\}^{\perp}$ onto $\{x_0\}^{\perp}$.

The bilinear form $\langle\cdot,\cdot\rangle : X \times Y \to \mathbb{C}$ is continuous for the norms p and q defined above. Since both (X, p) and (Y, q) are Banach spaces, it suffices to show that $\langle\cdot,\cdot\rangle$ is separately continuous. Fix $y \in Y$ and let x be any vector of X. By (2), $y = \beta y_0 + y_1$ and $x = \alpha x_0 + x_1$, where $\alpha, \beta \in \mathbb{C}$, $y_1 \in \{x_0\}^{\perp}$ and $x_1 \in \{y_0\}^{\perp}$. By (3), there exists $a \in L$ such that $\pi(ax_0) = x_1$. Thus $\langle x, y\rangle = \alpha\beta + \langle x_1, y_1\rangle = \alpha\beta + \langle ax_0, y_1\rangle$. Taking modules on both sides of this equation and using (1) we get

$$|\langle x, y\rangle| \le |\alpha||\beta| + |\langle ax_0, y_1\rangle| \le |\alpha||\beta| + k||y_1^* x_0||||a||$$

for some positive number k. Then taking the infimum over all $a \in L$ such that $\pi(ax_0) = x_1$ we obtain

$$|\langle x, y\rangle| \le |\alpha||\beta| + k||y_1^* x_0||p(x_1) \le \max\{|\beta|,\, k||y_1^* x_0||\}p(x),$$

which proves that fixed $y \in Y$ the linear form $x \mapsto \langle x, y\rangle$ is continuous. Dually, fixed $x \in X$, the linear form $y \mapsto \langle x, y\rangle$ is continuous.

Let us finally see that the injection of $(L, ||\cdot||)$ into $(\mathfrak{gl}_Y(X), |\cdot|')$ is continuous. By the closed-graph theorem, we need only show that if $\{a_n\} \to 0$ in $(L, ||\cdot||)$ and $\{a_n\} \to a$ in $\mathfrak{gl}_Y(X), |\cdot|'))$, then $a = 0$. Fix a nonzero vector $x \in X$ and let $y \in \{x\}^{\perp}$. Then the inequality $|\langle a_n x, y\rangle| \leq k||y^*x||||a_n||$ implies $\{\langle a_n x, y\rangle\} \to 0$. On the other hand, since $\{a_n\} \to a$ with respect to the operator norm, $\{a_n x\} \to ax$ in (X, p). Hence $\{\langle a_n x, y\rangle\} \to \langle ax, y\rangle$ by the continuity of the bilinear form just proved. Therefore $\langle ax, y\rangle = 0$ for all $y \in \{x\}^{\perp}$, so $ax \in \{x\}^{\perp\perp} = \mathbb{C}x$. Since x is any nonzero vector of X, this proves that a is a multiple of the identity on X, so $a \in L \cap \mathbb{C}\,\mathrm{Id}_X = 0$, which completes the proof. □

5. Banach-Lie algebras of orthogonal type

5.1. A Banach pairing $(X, Y, \langle\cdot,\cdot\rangle)$ with $X = Y$ is called a *self-dual Banach space* and is denoted by $(X, \langle\cdot,\cdot\rangle)$. Note that in our definition of self-dual Banach space, $\langle\cdot,\cdot\rangle$ is assumed to be bilinear, while in other contexts [**3**, Definition 36.3], $\langle\cdot,\cdot\rangle$ is assumed to be sesquilinear.

Proposition 5.2. *Let $(X, , \langle\cdot,\cdot\rangle)$ be an infinite-dimensional self-dual complex vector space with respect to a symmetric bilinear form, and let L be a Lie algebra such that $\mathfrak{fo}(X, \langle\cdot,\cdot\rangle) \leq L \leq \mathfrak{o}(X, \langle\cdot,\cdot\rangle)$. Suppose further that L is a Banach-Lie algebra for a norm $||\cdot||$. Then X can be equipped with a complete norm making the bilinear form $\langle\cdot,\cdot\rangle$ and the injection of $(L, ||\cdot||)$ into $(\mathfrak{o}(X, \langle\cdot,\cdot\rangle), |\cdot|)$ continuous.*

Proof. Recall that for any $x, y \in X$, $[x, y] = x^*y - y^*x \in \mathfrak{fo}(X, \langle\cdot,\cdot\rangle)$. Moreover, $[x, y] = 0$ if and only if x, y are linearly dependent.

Let $x \in X$ be a nonzero isotropic vector and let H be a hyperbolic plane of X containing x. By [**8**, (12)], for any $a \in \mathfrak{o}(X, \langle\cdot,\cdot\rangle)$ and any isotropic vector $z \in H^{\perp}$,

$$\mathrm{ad}^2_{[x,z]}\, a = 2\langle ax, z\rangle[x, z], \text{ and hence } |\langle ax, z\rangle| \leq k||[x, z]||||a|| \qquad (7)$$

for some positive constant k. Let x, a and H be as above, and denote by $\mathrm{Iso}(H^\perp)$ the set of all the isotropic vectors of $H^\perp$. We have

$$a \in \bigcap_{z \in \mathrm{Iso}(H^\perp)} \mathrm{Ker}(\mathrm{ad}^2_{[x,z]}) \Leftrightarrow ax \in \mathbb{C}x. \tag{8}$$

Indeed, it follows from (7) that $a \in \mathrm{Ker}(\mathrm{ad}^2_{[x,z]})$ for all $z \in \mathrm{Iso}(H^\perp)$ if and only if ax is orthogonal to any $z \in \mathrm{Iso}(H^\perp)$, equivalently, if and only if $ax \in H^{\perp\perp} = H$, since any vector of $H^\perp$ is a sum of isotropic vectors. Now $ax = \alpha x + \beta y$, together with $\langle ax, x\rangle = 0$ and $\langle x, y\rangle = 1$, implies $ax \in \mathbb{C}x$; the reverse implication is clear.

Fix a hyperbolic plane H of X and a hyperbolic basis (x, y) of H. Since $\langle x, y\rangle = 1$, $X = \mathbb{C}x \oplus \{y\}^\perp$ by (2), so we can consider the corresponding projection $\pi : X \to \{y\}^\perp$. Define $\varphi : L \to \{y\}^\perp$ by $\varphi(a) = \pi(ax)$ for all $a \in L$. Then

$$\mathrm{Ker}(\varphi) = \bigcap_{z \in \mathrm{Iso}(H^\perp)} \mathrm{Ker}(\mathrm{ad}^2_{[x,z]}) \text{ and } \mathrm{Im}(\varphi) = H^\perp. \tag{9}$$

By (8), $\varphi(a) = 0 \Leftrightarrow ax \in \mathbb{C}x \Leftrightarrow a \in \bigcap_{z \in \mathrm{Iso}(H^\perp)} \mathrm{Ker}(\mathrm{ad}^2_{[x,z]})$. On the other hand, given $z \in H^\perp$, take $a = [y, z]$. Then $a \in \mathfrak{fo}(X, \langle\cdot,\cdot\rangle) \subset L$ and it satisfies $ax = (y^*z)x - (z^*y)x = \langle x, y\rangle z - \langle x, z\rangle y = z$. Conversely, since $\langle ax, x\rangle = 0$ for any $a \in \mathfrak{o}(X, \langle\cdot,\cdot\rangle)$, $ax \in \{x\}^\perp = \mathbb{C}x + H^\perp$, and hence $\pi(ax) \in H^\perp$.

Since the operators ad^2_b are continuous for any $b \in L$, it follows from (9) that $\mathrm{Ker}(\varphi)$ is a closed subspace of L, so we can carry over the complete quotient norm of $L/\mathrm{Ker}(\varphi)$ to $H^\perp$, i.e, we define a complete norm on the subspace $H^\perp$ by

$$p(z) := \inf\{||a|| : a \in L \text{ and } \pi(ax) = z\}, \tag{10}$$

for all $z \in H^\perp$. We extend p to a complete norm, also denoted by p, of the whole $X = H \oplus H^\perp$as follows

$$p(h + z) := \sqrt{q(h)^2 + p(z)^2}, \tag{11}$$

for all $h = \alpha x + \beta y \in H$ and $z \in H^\perp$, where $q(h) = \max\{|\alpha|, |\beta|\}$.

Let us now show that the symmetric bilinear form $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{C}$ is continuous for the norm p. Since the sum $X = H \oplus H^\perp$ is orthogonal and H is finite-dimensional, we need only see that the restriction of

$\langle\cdot,\cdot\rangle$ to $H^\perp \times H^\perp$ is continuous. Moreover, since $H^\perp$ is a Banach space, it suffices to show that it is separately continuous. Fix a nonzero vector $v \in H^\perp$ and let $\tilde{v}$ denote the linear form $z \mapsto \langle z, v\rangle$, $z \in H^\perp$. Since φ is onto by (9), for any $z \in H^\perp$ there exists $a \in L$ such that $\pi(ax) = z$. Suppose first that v is isotropic. Then we have by (7) that $|\langle ax, z\rangle| \le k||[x,z]||||a||$. Taking the infimum over all $a \in L$ such that $\pi(ax) = z$ we obtain $|\langle z, v\rangle| \le k||[x,z]||p(z)$, which proves the continuity of the mapping $\tilde{v}$ when v is isotropic. The general case of an arbitrary $v \in H^\perp$ reduces to the previous isotropic one taking into account that any $v \in H^\perp$ is a sum of isotropic vectors of $H^\perp$.

Let us finally see that the injection of $(L, ||\cdot||)$ into $(\mathfrak{o}(X, \langle\cdot,\cdot\rangle), |\cdot|)$ is continuous. Assume that $\{a_n\}$ converges to 0 with respect to $||\cdot||$, and to some $a \in \mathfrak{o}(X, \langle\cdot,\cdot\rangle)$ with respect to the operator norm $|\cdot|$. By the closed-graph theorem, we need only show that $a = 0$, which in virtue of 3.7 is equivalent to verify that any isotropic vector is an eigenvector for a. Fix a nonzero isotropic vector $u \in X$, let H be a hyperbolic plane containing u, and let z be an arbitrary isotropic vector of $H^\perp$. By (7), $|\langle au, z\rangle| \le k||[u,z]||||a||$ for some positive constant k. Then $\{a_n\} \to 0$ in $(L, ||\cdot||)$ yields $\langle a_n u, z\rangle \to 0$. Hence $\langle au, z\rangle = 0$ by the continuity of the bilinear form $\langle\cdot,\cdot\rangle$ and the convergence of $\{a_n\}$ to a in $(\mathfrak{o}(X, \langle\cdot,\cdot\rangle), |\cdot|)$. Since any vector of $H^\perp$ is a sum of isotropic vectors, we have actually proved that $au \in H^{\perp\perp} = H$, and since also $\langle au, u\rangle = 0$, $au \in H \cap \{u\}^\perp = \mathbb{C}u$, as required. □

6. Banach-Lie algebras of symplectic type

Proposition 6.1. *Let $(X, , \langle\cdot,\cdot\rangle)$ be an infinite-dimensional self-dual complex vector space with respect to an alternate bilinear form, and let L be a Lie algebra such that $\mathfrak{fsp}(X, \langle\cdot,\cdot\rangle) \le L \le \mathfrak{s}p(X, \langle\cdot,\cdot\rangle)$. Suppose further that L is a Banach-Lie algebra for a norm $||\cdot||$. Then X can be equipped with a complete norm making the bilinear form $\langle\cdot,\cdot\rangle$ and the injection of $(L, ||\cdot||)$ into $(\mathfrak{s}p, \langle\cdot,\cdot\rangle(X), |\cdot|)$ continuous.*

Proof. For $x, y \in X$, set $[x,y] := x^*y + y^*x$. Since $(x^*y)^* = -y^*x$, $[x,y] \in \mathfrak{fsp}(X, \langle\cdot,\cdot\rangle)$. In particular, $x^*x = \frac{1}{2}[x,x] \in \mathfrak{fsp}(X, \langle\cdot,\cdot\rangle)$ for all $x \in X$.

Let $x, z \in X$ be orthogonal and $a \in \mathfrak{sp}(X, \langle\cdot,\cdot\rangle)$. Then

$$\frac{1}{2}\operatorname{ad}^2_{[x,z]} a = \langle z, ax\rangle[x,z] + \langle z, az\rangle x^*x + \langle x, ax\rangle z^*z. \tag{12}$$

Set $b = [x, z]$. Since any vector of X is isotropic and since x, z are orthogonal, $b^2 = 0$. Hence

$$\frac{1}{2}\operatorname{ad}^2_b a = -bab = -b(x^*az + z^*ax) = -x^*(baz) - z^*(bax).$$

Since $ba = (x^*z + z^*x)a = -(ax)^*z - (az)^*x$, $baz = -\langle z, ax\rangle z - \langle z, az\rangle x$ and $bax = -\langle x, ax\rangle z - \langle x, az\rangle x$. Thus $\frac{1}{2}\operatorname{ad}^2_b a = \langle z, ax\rangle x^*z + \langle z, az\rangle x^*x + \langle x, ax\rangle z^*z + \langle x, az\rangle z^*x = \langle z, ax\rangle[x,z] + \langle z, az\rangle x^*x + \langle x, ax\rangle z^*z$, as claimed. In particular, for any $x \in X$ and $a \in \mathfrak{sp}(X, \langle\cdot,\cdot\rangle)$, we have

$$\operatorname{ad}^2_{x^*x} a = 2\langle x, ax\rangle x^*x. \tag{13}$$

Let H be a hyperbolic plane of X and x a nonzero vector in H. For any $z \in H^\perp$, let ξ_z denote the linear operator $\operatorname{ad}^2_{[x,z]} : L \to L$. Then for $a \in L$, we have

$$ax \in \mathbb{C}x \Leftrightarrow a \in \bigcap_{z\in H^\perp} \xi_z^{-1}(\mathbb{C}x^*x). \tag{14}$$

By (12), $a \in \xi_z^{-1}(\mathbb{C}x^*x)$ for any $z \in H^\perp$ if and only if $ax \in H^{\perp\perp} \cap \{x\}^\perp = H \cap \{x\}^\perp = \mathbb{C}x$, as required.

Fix a hyperbolic plane H of X and a hyperbolic basis (x, y) of H. Since $\langle x, y\rangle = 1$, $X = \mathbb{C}x \oplus \{y\}^\perp$ by (2), so we can consider the corresponding projection $\pi : X \to \{y\}^\perp$. As in (3), define $\varphi : L \to \{y\}^\perp$ by $\varphi(a) = \pi(ax)$ for all $a \in L$. Then

$$\operatorname{Ker}(\varphi) = \bigcap_{z\in H^\perp} \xi_z^{-1}(\mathbb{C}x^*x) \text{ and } \operatorname{Im}(\varphi) = \{y\}^\perp = \mathbb{C}y \oplus H^\perp. \tag{15}$$

By (14), $\pi(ax) = 0 \Leftrightarrow ax \in \mathbb{C}x \Leftrightarrow a \in \bigcap_{z\in H^\perp} \xi_z^{-1}(\mathbb{C}x^*x)$, which proves the equality relative to the kernel. On the other hand, since $\{y\}^\perp = \mathbb{C}y \oplus H^\perp$, $y = y^*y(x)$ and $[y, z](x) = z$ for any $z \in H^\perp$ imply that $\operatorname{Im}(\varphi) = \{y\}^\perp$.

Since the operators ad^2_b are continuous for any $b \in L$, and since finite-dimensional subspaces are closed, it follows from (15) that $\operatorname{Ker}(\varphi)$ is a closed subspace of L, so we can carry over the complete quotient norm of $L/\operatorname{Ker}(\varphi)$ to $\{y\}^\perp$ as follows

$$p(u) := \inf\{||a|| : a \in L \text{ and } \pi(ax) = u\}, \tag{16}$$

for all $u \in \{y\}^\perp$. Since $X = \mathbb{C}x \oplus \{y\}^\perp$ by (2), we can extend the norm p to a complete norm, also denoted by p, on the whole X

$$p(\alpha x + u) := |\alpha| + p(u), \text{ for all } \alpha \in \mathbb{C} \text{ and } u \in \{y\}^\perp. \tag{17}$$

Let us now show that the alternate form $\langle \cdot, \cdot \rangle$ is continuous with respect to the norm p defined in (17). Again it is enough to prove that for any $w \in X$, the linear form defined by $\tilde{w}(x') = \langle w, x' \rangle$ for all $x' \in X$ is continuous. Since $X = \mathbb{C}x \oplus \{y\}^\perp = \mathbb{C}x \oplus \mathbb{C}y \oplus H^\perp$, we can divide the proof into three cases.

Consider first the case that $w = x$. For any $\alpha, \beta \in \mathbb{C}$ and any $z \in H^\perp$, we have $\tilde{x}(\alpha x+\beta y+z) = \langle x, \alpha x+\beta y+z\rangle = \beta$. Now by (15) there exist $\lambda \in \mathbb{C}$ and $a \in L$ such that $ax = \lambda x + \beta y + z$, and hence $\langle x, ax\rangle = \beta$. Then it follows from (13) that $|\beta| = |\langle x, ax\rangle| \leq k||x^*x||||a||$ for some positive number k. Hence

$$|\tilde{x}(\alpha x+\beta y+z)| = |\beta| \leq k||x^*x||p(\beta y+z) \leq \max\{k||x^*x||, 1\}p(\alpha x+\beta y+z),$$

which proves the continuity of $\tilde{x}$. The case that $w = y$ is straightforward: for any $\alpha \in \mathbb{C}$ and for any $u \in \{y\}^\perp$, we have $|\tilde{y}(\alpha x + u)| = |\alpha| \leq |\alpha| + p(u) = p(\alpha x + u)$. The remaining case that $w \in H^\perp$ goes as follows. Since $X = \mathbb{C}x + \{y\}^\perp$ and $\tilde{w}|_{\mathbb{C}x} = 0$, it suffices to show that the restriction of $\tilde{w}$ to $\{y\}^\perp$ is continuous. Let $v \in \{y\}^\perp$ and take by (15) a in L such that $\pi(ax) = v$. Now we have by (12),

$$\frac{1}{2} \operatorname{ad}^2_{[x,w]} a = \langle w, ax\rangle[x, w] + \langle w, aw\rangle x^*x + \langle x, ax\rangle w^*w.$$

Then taking norms in the above formula we obtain

$$|\langle w, ax\rangle|||[x, w]|| \leq ||\frac{1}{2} \operatorname{ad}^2_{[x,w]} a|| + |\langle w, aw\rangle|||x^*x|| + |\langle x, ax\rangle|||w^*w||,$$

where by the continuity of the Lie product,

$$||\frac{1}{2} \operatorname{ad}^2_{[x,w]} a|| \leq k||[x, w]||^2||a||$$

for some positive number k, and where by (13) and the above formula,

$$|\langle w, aw\rangle| \leq \frac{1}{||w^*w||}||\frac{1}{2}\operatorname{ad}^2_{w^*w} a|| \leq \frac{k}{2}||w^*w||||a||.$$

Similarly, $|\langle x, ax\rangle| \leq \frac{k}{2}||x^*x||||a||$. Hence

$$|\tilde{w}(v)| = |\langle w, \pi(ax)\rangle| = |\langle w, ax\rangle| \leq k(x,w)||a||$$

for some positive number $k(x,w)$ depending on x and w but independent of a, and taking now the infimum over all $a \in L$ such that $\pi(ax) = v$ we get $|\tilde{w}(v)| \leq k(x,w)p(v)$, which proves the continuity of $\tilde{w}$.

Let us finally see that the injection of $(L, ||\cdot||)$ into $(\mathfrak{sp}(X, \langle\cdot,\cdot\rangle), |\cdot|)$ is continuous. Let $\{a_n\} \to 0$ in $(L||\cdot||)$ and $a_n \to a$ in $(\mathfrak{sp}(X, \langle\cdot,\cdot\rangle), |\cdot|)$, and fix $x \in X$. It follows from (13) that $\{\langle x, a_n x\rangle\} \to 0$, and hence $\langle x, ax\rangle = 0$, by the continuity of the bilinear form and the convergence of $\{a_n\}$ to a with respect to the operator norm $|\cdot|$. Thus ax is orthogonal to x for any $x \in X$, which implies $a = 0$. □

7. Banach-Lie algebras with extremal elements

Lemma 7.1. *Let L be a Banach-Lie algebra and let $a \in L$ be a Jordan element of L. Then the Jordan algebra L_a attached to a becomes a Banach-Jordan algebra for the quotient norm.*

Proof. Straightforward. □

Lemma 7.2. *Let L be a non-degenerate Banach-Lie algebra. Then L has extremal elements if and only if it contains abelian minimal inner ideal.*

Proof. We need only prove that every abelian minimal inner ideal B of L is one-dimensional. Take a nonzero element $b \in B$. Then $B = \operatorname{ad}^2_b L$ and there exists $c \in L$ such that $[b, [c, b]] = 2b$. Then, by [**9**, 2.15(ii)], L_b is a unital Jordan algebra having $\overline{c}$ as a unity element, with no proper inner ideals [**9**, (2.14)]. Hence L_b is a division Jordan algebra. Since L_b is also a Banach-Jordan algebra by 7.1, $L_b = \mathbb{C}\overline{c}$, so $B = \operatorname{ad}^2_b L = \operatorname{ad}^2_b \mathbb{C}c = \mathbb{C}b$. □

Theorem 7.3. *Let* $(L, ||\cdot||)$ *be an infinite-dimensional Banach-Lie algebra. Then* L *is strongly prime and contains extremal elements if and only if any one of the following statements holds:*

(i) *There exists an infinite-dimensional Banach pairing* $(X, Y, \langle\cdot\rangle)$ *such that* $\mathfrak{fsl}_Y(X) \leq L \leq \mathfrak{gl}_Y(X)$ *and* $L \cap \mathbb{C}\,\mathrm{Id}_X = 0$ *and the injection* $(L, ||\cdot||)$ *into* $(\mathfrak{gl}_Y(X), |\cdot|')$ *is continuous.*

(ii) *There exists an infinite-dimensional self-dual Banach space* $(X, \langle\cdot,\cdot\rangle)$ *with respect to a symmetric bilinear form such that* $\mathfrak{fo}(X, \langle\cdot,\cdot\rangle) \leq L \leq \mathfrak{o}(X, \langle\cdot,\cdot\rangle)$*, and the injection of* $(L, ||\cdot||)$ *into* $(\mathfrak{o}(X, \langle\cdot,\cdot\rangle), |\cdot|)$ *is continuous.*

(iii) *There exists an infinite-dimensional self-dual Banach space* $(X, \langle\cdot,\cdot\rangle)$ *with respect to an alternate bilinear form such that* $\mathfrak{fsp}(X, \langle\cdot,\cdot\rangle) \leq L \leq \mathfrak{s}p(X, \langle\cdot,\cdot\rangle)$*, and the injection of* $(L, ||\cdot||)$ *into* $(\mathfrak{o}(X, \langle\cdot,\cdot\rangle), |\cdot|)$ *is continuous.*

Proof. By 3.10, each one of the Banach-Lie algebras L listed above is strongly prime and contains extremal elements. Suppose conversely that L is an infinite-dimensional strongly prime Banach-Lie algebra with extremal elements. Then, again by 3.10, L can be represented as a Lie algebra of either special type (i), orthogonal type (ii), or symplectic type (iii). Now we apply 4.3, 5.2 and 6.1, respectively. □

7.4. A Banach-Lie algebra is said to be *topologically simple* if it is not abelian and it does not contain proper closed ideals. Topologically simple non-degenerate Lie algebras are strongly prime. Natural examples of topologically simple Banach-Lie algebras containing extremal elements are $\overline{\mathfrak{fsl}_Y(X)}$, $\overline{\mathfrak{fo}(X, \langle\cdot,\cdot\rangle)}$ and $\overline{\mathfrak{fsp}(X, \langle\cdot,\cdot\rangle)}$, where in all the cases *overline* denotes the norm-operator closure. From 7.3, we obtain the following partial converse.

Corollary 7.5. *Any infinite-dimensional Banach-Lie algebras* $(L, ||\cdot||)$ *which is non-degenerate, topologically simple and contains extremal elements admits one of the following representations:*

(i) $\mathfrak{fsl}_Y(X) \leq (L, ||\cdot||) \leq (\overline{\mathfrak{fsl}_Y(X)}, |\cdot|')$,

(ii) $\mathfrak{fo}(X, \langle\cdot,\cdot\rangle) \leq (L, ||\cdot||) \leq ((\overline{\mathfrak{fo}(X, \langle\cdot,\cdot\rangle)}, |\cdot|)$

(iii) $\mathfrak{fsp}(X, \langle\cdot,\rangle) \leq (L, ||\cdot||) \leq ((\overline{\mathfrak{fsp}(X, \langle\cdot,\cdot\rangle)}, |\cdot|)$

where in all the cases the injections are continuous.

Note, however, that the Banach-Lie algebras $(L, ||\cdot||)$ listed in (i), (ii) and (iii) of are not necessarily topologically simple.

8. Banach-Lie algebras of compact operators on Hilbert spaces

Let H be an infinite-dimensional complex Hilbert space. Following the monograph of P. De La Harpe [**5**], we denote by $\mathfrak{gl}(H, \mathcal{C}_\infty)$ the Banach-Lie algebra of all compact operators on H, by $\mathfrak{o}(H, \mathcal{C}_\infty)$ the orthogonal Banach-Lie algebra of compact operators on H, and by $\mathfrak{fsp}(H, \mathcal{C}_\infty)$ the symplectic Banach-Lie algebra of compact operators on H. In this section we show that these Banach-Lie algebras of compact operators on Hilbert spaces can be described in terms of Banach pairings and self-dual Banach spaces, via conjugations and anti-conjugations.

8.1. Let H be a complex Hilbert space with inner product denoted by $(\cdot,\cdot)$. A *conjugation* of H is a conjugate linear isometry which is also involutive, i.e., a map $\theta : H \to H$ satisfying $(x, \theta y) = (y, \theta x)$ and $\theta^2 x = x$ for all $x, y \in H$. An *anti-conjugation* is a map $\zeta : H \to H$ satisfying $(x, \zeta y) = -(y, \zeta x)$ and $\zeta^2 x = -x$ for all $x, y \in H$. By [**12**, (7.5.6)], the existence and uniqueness (up to linear isometries) of conjugations and anti-conjugations is granted in infinite-dimensional Hilbert spaces.

Proposition 8.2. *Let $(H, (\cdot,\cdot))$ be an infinite-dimensional complex Hilbert space and fix a conjugation $\theta : H \to H$. Then.*

(i) *For all $x, y \in H$, $\langle x, y\rangle := (x, \theta y)$ defines a non-degenerate symmetric bilinear form which makes H into a self-dual Banach space with respect to the Hilbert norm.*
(ii) *$\mathfrak{gl}(H, \langle\cdot,\cdot\rangle)$ coincides with the Banach-Lie algebra of bounded linear operators $BL(H)^{(-)}$. In fact, $a^\# = \theta a^* \theta$ for all $a \in BL(H)$, where a^* is the Hilbert adjoint.*
(iii) *$\mathfrak{gl}(H, \mathcal{C}_\infty) = \overline{\mathfrak{fgl}(H, \langle\cdot,\cdot\rangle)} = \overline{\mathfrak{fsl}(H, \langle\cdot,\cdot\rangle)}$ and therefore it is a topologically simple non-degenerate Banach-Lie algebra with extremal elements.*

(iv) $\mathfrak{o}(H, \mathcal{C}_\infty) = \overline{\mathfrak{fo}(H, \langle\cdot,\cdot\rangle)}$ *and therefore it is a topologically simple non-degenerate Banach-Lie algebra with extremal elements.*

Proof. (i) For any $x, y \in H$, we have $\langle x, y\rangle = (x, \theta y) = (y, \theta x) = \langle y, x\rangle$, which proves that the bilinear form $\langle\cdot,\cdot\rangle$ is symmetric; the nondegeneracy is clear and the continuity follows from the Cauchy-Schwartz inequality and the fact that θ is an isometry.

(ii) By the closed graph theorem, for a linear operator a on H the following are equivalent: (1) $a \in \mathfrak{gl}(H, \langle\cdot,\cdot\rangle)$, (2) $a \in BL(H)$, (3) a has an adjoint with respect to the Hilbert product. Moreover, in this case,

$$\langle ax, y\rangle = (ax, \theta y) = (x, a^*\theta y) = (x, \theta\theta a^*\theta y) = \langle x, \theta a^*\theta y\rangle.$$

(iii) and (iv). By a well-known result of P. R. Halmos [**11**], $\mathfrak{gl}(H, \mathcal{C}_\infty)$ and $\mathfrak{o}(H, \mathcal{C}_\infty)$ coincide with their derived ideals, and hence with the closures, with respect to the operator norm, of the special Lie algebra $\mathfrak{fsl}(H, \langle\cdot,\cdot\rangle)$ and the finitary orthogonal algebra $\mathfrak{fo}(H, \langle\cdot,\cdot\rangle)$ respectively, since compact operators on Hilbert spaces are limits of sequences of finite-rank operators with respect to the norm topology. Then, it follows from 7.3 and 7.4 that both $\mathfrak{gl}(H, \mathcal{C}_\infty)$ and $\mathfrak{o}(H, \mathcal{C}_\infty)$ are topologically simple non-degenerate Banach-Lie algebras with extremal elements. □

Proposition 8.3. *Let $(H, (\cdot,\cdot))$ be an infinite-dimensional complex Hilbert space and fix an anti-conjugation $\zeta : H \to H$. Then*

(i) $\langle x, y\rangle := (x, \zeta y)$ *for all $x, y \in H$ defines a non-degenerate alternate bilinear form which makes H into a self-dual Banach space with respect to the Hilbert norm.*

(ii) $\mathfrak{fsp}(H, \mathcal{C}_\infty) = \overline{\mathfrak{fsp}(H, \langle\cdot,\cdot\rangle)}$, *and therefore it is a topologically simple non-degenerate Banach-Lie algebra with extremal elements.*

Proof. It is similar to that of 8.2. □

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