LIE PI-ALGEBRAS WITH AN ALGEBRAIC ADJOINT REPRESENTATION REVISITED

ANTONIO FERNÁNDEZ LÓPEZ

Abstract. Zelmanov's theorem on Lie PI-algebras with an algebraic adjoint representation over a field Φ of characteristic zero is here revisited. Recent results on Jordan structures in Lie algebras allows us to give a new and shorter proof when Φ is algebraically closed.

1. Introduction

In his celebrated paper [18], Zelmanov proves that a Lie algebra L over a field Φ of characteristic zero with an algebraic adjoint representation and satisfying a polynomial identity is locally finite-dimensional, thus yielding the solution of the Kurosh problem for Lie algebras.

In this note we show that if, additionally, L is nondegenerate ($\mathrm{ad}_x^2 L = 0 \Rightarrow x = 0$, $x \in L$) and Φ is algebraically closed, then L is actually a subdirect product of a family of simple Lie algebras of bounded finite dimension. This result enables us to shorten and simplify the proof of theorem above *under the additional assumption that Φ is algebraically closed.*

To be more precise, let us give a brief outline of Zelmanov's original proof and comment on the changes and new tools introduced in our approach. First Zelmanov observes that it suffices to show that if L is nondegenerate and nonzero, then it contains a nonzero locally finite-dimensional ideal. Now the proof splits into two cases. If L is Engel, that is, any element of L is ad-nilpotent, he proves [18, Proposition 1] that L is locally nilpotent, so he can assume that L contains a non-Engel element, say a. Then he takes an algebraically closed extension F of Φ of sufficiently large cardinality, makes the scalar extension $L \otimes_\Phi F$, and takes the quotient algebra of $L \otimes_\Phi F$ by its Kostrikin radical, thus obtaining a Lie algebra $\overline{L}$ (over an algebraically closed field F which is large for $\overline{L}$) which is nondegenerate, satisfies a polynomial identity, and has a nontrivial finite grading (that induced by the non-Engel element a). Moreover, he proves that L can be embedded in $\overline{L}$. At this point the proof becomes quite involved. Let $\overline{L}_\alpha$ be an extreme subspace of the grading in $\overline{L}$ defined by ad_a. From the fact that $\overline{L}$ satisfies a polynomial identity, he derives that the Jordan pair $V = (\overline{L}_\alpha, \overline{L}_{-\alpha})$ is PI. Moreover, V is nondegenerate, and since F is large for V, it follows from [18, Theorem JP1 and Lemma JP1] that V is actually a semiprimitive Jordan pair. By using deep results on Lie algebras with finite gradings (the most difficult part of the paper [18, pages 543-548]) and after a skilful manipulation of the primitive ideals of V, Zelmanov then proves: (i) the ideal I of $\overline{L}$ generated by $\overline{L}_\alpha$ can be embedded in a subdirect product of a family of simple Lie algebras of finite bounded dimension, and (ii) L intersects I nontrivially. Hence he obtains that $L \cap I$ satisfies all the identities of some finite-dimensional Lie algebra. The last step

Key words and phrases. Lie algebra, Jordan algebra, polynomial identity, algebraic adjoint representation, Kostrikin radical, Jordan element, socle.

Supported by the MEC and Fondos FEDER, MTM2007-61978 and MTM2010-19482.

of the proof is the following result of independent interest [18, Lemmas 5, 6 and 7] (or [12, Theorem 5.4.6]): A Lie algebra (over a field of characteristic zero) with an algebraic adjoint representation and satisfying all the identities that hold in some finite-dimensional Lie algebra is locally finite-dimensional. This completes the proof.

In our approach, we don't need to take a large scalar extension, nor to deal with semiprimitive Jordan pairs. Instead we apply a recent result proved in [8, Theorem 3.10] (actually this result, at least in its germinal state, could be attributed to Zelmanov himself) which reduces the proof to the case that L is prime, thus avoiding the most difficult part of the original proof. Then, as a consequence of the Kostrikin lemma (in the Engel case) and of the existence of a nontrivial finite grading (in the non-Engel case), we obtain that L contains a nonzero Jordan element, say x, $(\mathrm{ad}_x^3 L = 0)$. Then we take the Jordan algebra L_x of L at x [5]. This Jordan algebra inherits primeness and nondegeneracy of L. Moreover, it is algebraic and PI. Using the structure theory of Jordan PI-algebras and the transference of inner ideals from L_x to L, we obtain that L contains an extremal element, say y, $(\mathrm{ad}_y L = \Phi y)$. Then the socle of L [4] is a locally finite-dimensional simple Lie algebra, and since L satisfies a polynomial identity, $\mathrm{Soc}\, L$ is actually finite-dimensional, its dimension being bounded by a function of the degree of the polynomial identity. Finally, $L = \mathrm{Soc}\, L$ since any derivation of a simple finite-dimensional Lie algebra over a field of characteristic zero is inner.

2. Lie algebras and Jordan algebras

1. Throughout this note, and unless specified otherwise, we will be dealing with Lie algebras L [10] and [11], with $[x, y]$ denoting the Lie bracket and ad_x the adjoint map determined by x, and with *linear* Jordan algebras J [13], with Jordan product $x \cdot y$, multiplication operators $m_x : y \mapsto x \cdot y$, and quadratic operators $U_x = 2m_x^2 - m_{x^2}$, over a field Φ of characteristic 0. We set

$$[x_1] := x_1 \quad \text{and} \quad [x_1, x_2, \ldots, x_n] := [x_1, [x_2, \ldots, x_n]]$$

for $n > 1$ and $x_1, x_2, \ldots, x_n \in L$. Similarly, we set

$$x_1 \cdot x_2 \cdots x_n := x_1 \cdot (x_2 \cdots x_n)$$

for $n > 1$ and $x_1, x_2, \ldots, x_n \in J$.

Any associative algebra A gives rise to a Lie algebra A^-, with Lie bracket $[x, y] := xy - yx$, and a linear Jordan algebra A^+, with Jordan product $x \cdot y := 1/2(xy + yx)$. A Jordan algebra J is said to be *special* if it is isomorphic to a subalgebra of A^+ for some associative algebra A.

2. An *inner ideal* of J is a vector subspace B of J such that $U_B J \subseteq B$. Similarly, an *inner ideal* of L is a vector subspace B of L such that $[B, [B, L]] \subseteq B$. An *abelian inner ideal* of L is an inner ideal B which is also an abelian subalgebra, i.e., $[B, B] = 0$.

3. An element $x \in L$ is called *Engel* if ad_x is a nilpotent operator. In this case, the nilpotence index of ad_x is called the *index* of x. Engel elements of index at most 3 are called *Jordan elements.* Clearly, any element of an abelian inner ideal is a Jordan element. Conversely, by [2, Lemma 1.8], any Jordan element x generates the abelian inner ideal $\mathrm{ad}_x^2 L$. A good reason for this terminology is the following analogue of the fundamental identity for Jordan algebras:

$$\mathrm{ad}^2_{\mathrm{ad}_x^2 y} = \mathrm{ad}_x^2 \mathrm{ad}_y^2 \mathrm{ad}_x^2$$

which holds for any Jordan element x and any $y \in L$ [2, Lemma 1.7(iii)]. Another reason will be given Section 4.

4. A well-known lemma due to Kostrikin [12, Lemma 2.1.1] provides a method to construct Jordan elements by means of Engel elements, namely, if $x \in L$ is an Engel element of index n then, for any $a \in L$, $\mathrm{ad}_x^{n-1}a$ is Engel of index $\leq n-1$. Recently, García and Gómez have given the following refinement of this result [7, Theorem 2.3 and Corollary 2.4].

Lemma 2.1. *If $x \in L$ is an Engel element of index n, then $\mathrm{ad}_x^{n-1}L$ is an abelian inner ideal of L. Hence, $\mathrm{ad}_x^{n-1}a$ is a Jordan element for any $a \in L$.*

5. Let Λ be a torsion free abelian group and let L be a Lie algebra. A Λ-grading $L = \sum_{\lambda\in\Lambda} L_\lambda$ of L is said to be *finite* if the set $\Lambda^* = \{\lambda \in \Lambda : L_\lambda \neq 0\}$ is finite, and *nontrivial* if Λ^* contains a nonzero element. Notice that if a Λ-grading is finite and nontrivial, then the subgroup $G = G(\Lambda^*)$ of Λ generated by Λ^* is free of finite rank, and therefore it is isomorphic to $\mathbb{Z}^r$ for some positive integer r.

Proposition 2.2. *Let L be a Lie algebra with a nontrivial finite Λ-grading. Then L contains a nonzero Jordan element.*

Proof. Take a basis $\{\lambda_1, \dots, \lambda_r\}$ of G such that for some $\alpha \in \Lambda^*$, $\alpha = n_{\alpha_1}\lambda_1 + \dots + n_{\alpha_r}\lambda_r$ with $n_{\alpha_1} \neq 0$, and let $\pi : G \to \mathbb{Z}$ be the homomorphism defined by putting $\pi(\lambda_1) = 1$ and $\pi(\lambda_i) = 0$ for $1 < i \leq r$. We may also assume that $|\pi(\beta)| \leq |\pi(\alpha)|$ $(\beta \in \Lambda^*)$. Then any $x \in L_\alpha$ is a Jordan element: for any $\beta \in \Lambda^*$, $\mathrm{ad}_x^3 L_\beta \subseteq L_{3\alpha+\beta} = 0$ since $|\pi(3\alpha + \beta)| > |\pi(\alpha)|$. □

Corollary 2.3. *Let L be a Lie algebra over an algebraically closed field Φ of characteristic zero. If L has a nonzero element whose adjoint is algebraic, then L contains a nonzero Jordan element.*

Proof. Let a be a nonzero element of L such that ad_a is algebraic. If a is Engel of index, say n, we have by Lemma 2.1 that $\mathrm{ad}_a^{n-1}b$ is a nonzero Jordan element for some $b \in L$. Otherwise, by [10, Lemma 2.4.2(B)], ad_a yields a nontrivial finite $(\Phi, +)$-grading on L given by

$$L_\lambda = \{x \in L : (\mathrm{ad}_a - \lambda 1_L)^m x = 0 \text{ for some } m \geq 1\},$$

with $L_\lambda = 0$ if $\lambda \in \Phi$ is not an eigenvalue of ad_a. By Proposition 2.2, L contains a nonzero Jordan element. □

6. An element $x \in J$ is called an *absolute zero divisor* if $U_x = 0$. We say J is *nondegenerate* if it has no nonzero absolute zero divisors, *semiprime* if $U_B B = 0$ implies $B = 0$, and *prime* if $U_B C = 0$ implies $B = 0$ or $C = 0$, for any ideals B, C of J. Similarly, given a Lie algebra L, $x \in L$ is an *absolute zero divisor* of L if $\mathrm{ad}_x^2 = 0$, L is *nondegenerate* if it has no nonzero absolute zero divisors, *semiprime* if $[B, B] = 0$ implies $B = 0$, and *prime* if $[B, C] = 0$ implies $B = 0$ or $C = 0$, for any ideals B, C of L. A Jordan or Lie algebra is *strongly prime* if it is prime and nondegenerate. *Simplicity,* for both Jordan and Lie algebras, means nonzero product and the absence of nonzero proper ideals.

7. Following [12, Definition 5.4.1], the smallest ideal of a Lie algebra L whose associated quotient algebra is nondegenerate is called the *Kostrikin radical* of L, denoted by $K(L)$. Put $K_0(L) = 0$ and let $K_1(L)$ be the ideal generated by all absolute zero divisors. Using transfinite induction, a nondecreasing chain of ideals $K_\alpha(L)$ is defined by putting $K_\alpha(L) = \bigcup_{\beta<\alpha} K_\beta(L)$

if α is a limit ordinal, and $K_\alpha(L)/K_{\alpha-1}(L) = K_1(L/K_{\alpha-1}(L))$ otherwise. It is obvious that $K(L) = \bigcup_\alpha K_\alpha(L)$. The Jordan counterpart of the Kostrikin radical is the *McCrimmon radical* (also called *degenerate radical*) $Mc(J)$ [13, page 92].

The following result, proved by Grishkov in [9], can be found translated to English in [12, Theorem 5.4.2].

Theorem 2.4. *Let L be a Lie algebra over a field of characteristic zero. Then $K_1(L)$ is locally nilpotent. Hence, simple Lie algebras over a field of characteristic zero are nondegenerate.*

The following characterization of the Kostrikin radical was proved in [8, Theorem 3.10].

Theorem 2.5. *The Kostrikin radical $K(L)$ of a Lie algebra L over a field of characteristic zero is the intersection of all strongly prime ideals of L. Therefore, L is nondegenerate if, and only if, it is a subdirect product of strongly prime Lie algebras.*

8. The *socle* of a Jordan algebra is the sum of all its minimal inner ideals [14]. The *socle* of a Lie algebra L, $\operatorname{Soc} L$, is defined as the sum of all minimal inner ideals of L [4]. By [14, Theorem 17] (for Jordan algebras) and [4, Theorem 2.5] (for Lie algebras), the socle of a nondegenerate Jordan algebra (Lie algebra) is the direct sum of its minimal ideals, each of which is a simple Jordan algebra (Lie algebra).

9. Let L be a Lie algebra over a field Φ. Recall that a nonzero element $x \in L$ is said to be *extremal* if $\operatorname{ad}_x^2 L = \Phi x$, that is, if it generates a one-dimensional inner ideal. A Lie algebra is said to be *central* if its centroid coincides with the ground field. Note also that any simple Lie algebra is central over its centroid.

Proposition 2.6. *Let L be a simple Lie algebra over a field Φ. If L contains an extremal element then L is central.*

Proof. See [4, Lemma 5.4]. □

10. The adjoint representation of a Lie algebra L is said to be *algebraic* if ad_x is an algebraic operator for each x in L. It was proved in [15] that a Lie algebra whose adjoint representation is algebraic contains a maximal locally finite-dimensional ideal and the quotient algebra over this ideal has no nonzero locally finite-dimensional ideals. A similar result also holds for *Engel Lie algebras* (any element is Engel) with respect to the so-called locally nilpotent radical [12, Proposition 1.3.2].

3. Polynomial identities

Let $L(X)$ denote the free Lie Φ-algebra over a countable set of indeterminates X. By using the Jacobi identity, we see that any monomial of $L(X)$ can be written as a linear combination of *standard monomials* $[x_{i_1}, \dots, x_{i_m}]$ (see (1) for notation). For each positive integer n, let S_n denote the set of all permutations of $1, \dots, n$.

Lemma 3.1. *Let n be a positive integer. Then there exists a function $f_n : S_n \to \{0, 1, -1\}$ such that, for any $x_1, \dots, x_n, x_{n+1}$ in X,*

$$\operatorname{ad}_{[x_1,\dots,x_n]} x_{n+1} = \sum_{\sigma \in S_n} f_n(\sigma)[x_{\sigma(1)}, \dots, x_{\sigma(n)}, x_{n+1}].$$

Proof. By induction on n. The case $n=1$ is trivial. Now

$$\mathrm{ad}_{[x_1,x_2,\ldots,x_{n+1}]}x_{n+2} = \mathrm{ad}_{[x_1,[x_2,\ldots,x_{n+1}]]}x_{n+2} = \mathrm{ad}_{x_1}\mathrm{ad}_{[x_2,\ldots,x_{n+1}]}x_{n+2} - \mathrm{ad}_{[x_2,\ldots,x_{n+1}]}\mathrm{ad}_{x_1}x_{n+2}.$$

Hence, by the induction hypothesis,

$$\begin{aligned}\mathrm{ad}_{[x_1,\ldots,x_{n+1}]}x_{n+2} &= \sum_{\sigma\in S_n} f_n(\sigma)[x_1,x_{\sigma(2)},\ldots,x_{\sigma(n+1)},x_{n+2}] \\ &- \sum_{\sigma\in S_n} f_n(\sigma)[x_{\sigma(2)},\ldots,x_{\sigma(n+1)},x_1,x_{n+2}] \\ &= \sum_{\tau\in S_{n+1}} f_{n+1}(\tau)[x_{\tau(1)},\ldots,x_{\tau(n+1)},x_{n+2}].\end{aligned}$$

□

Let $p=p(x_1,\ldots,x_n)$ be an element of a free Lie Φ-algebra $L(X)$. We say that a Lie algebra L satisfies the identity $p=0$ if $p(a_1,\ldots,a_n)=0$ for any $a_1,\ldots,a_n$ in L. A Lie algebra satisfying a nontrivial polynomial identity is called a *Lie PI-algebra.*

Proposition 3.2. *Any Lie PI-algebra L satisfies a multilinear identity $p=0$, where*

$$p(x_1,\ldots,x_n,x_{n+1}) = \sum_{\sigma\in S_n}\alpha_\sigma[x_{\sigma(1)},\ldots,x_{\sigma(n)},x_{n+1}] \quad (\alpha_\sigma\in\Phi).$$

Proof. As pointed out above, we may assume that L satisfies a polynomial identity $p=0$, where p is a linear combination of standard monomials $[x_{i_1},\ldots,x_{i_m}]$. Then, as in the case of an associative PI-algebra (see [3, Proposition 7.5.3]), we can assume that p is multilinear. Finally, by Lemma 3.1, we can replace p by a polynomial having the required form. □

Recall that a Jordan polynomial $p=p(x_1,\ldots,x_n)$ of the free Jordan Φ-algebra $J(X)$ is said to be an *s-identity* if it is satisfied by all special Jordan algebras, but not by all Jordan algebras. A Jordan algebra J satisfying a polynomial identity which is not an s-identity is called a Jordan PI-algebra.

Proposition 3.3. *A nonzero Jordan polynomial of the form*

$$p(x_1,\ldots,x_n,x_{n+1}) = \sum_{\sigma\in S_n}\alpha_\sigma x_{\sigma(1)}\cdots x_{\sigma(n)}\cdot x_{n+1} \quad (\alpha_\sigma\in\Phi)$$

is never an s-identity.

Proof. By relabeling the variables we may assume that $\alpha_\sigma=1$ for $\sigma=\mathrm{Id}$. Let $Y=\{y_1,y_2,\ldots\}$ be a countable set. Denote by S the free semigroup generated by $Y\cup\{0\}$ satisfying the relations $y_iy_j=0\,(j\neq i+1)$ and $y_i0=0y_i=0\,(i\geq 1)$. Let A be the associative algebra defined by taking $S-\{0\}$ as a basis. It is easy to verify that $2^np(y_1,\ldots,y_n,y_{n+1})=y_1\ldots y_ny_{n+1}\neq 0$. Thus the special Jordan algebra A^+ does not satisfies the identity $p(x_1,\ldots,x_n,x_{n+1})=0$, and therefore $p(x_1,\ldots,x_n,x_{n+1})$ is not an s-identity. □

4. The Jordan algebras of a Lie algebra

In [5] a Jordan algebra was attached to any Jordan element of a Lie algebra. As will be proved in the propositions below, many properties of a Lie algebra can be transferred to its Jordan algebras, as well as the nature of the Jordan element in question is reflected on the structure of the attached Jordan algebra. These facts turn out to be crucial for applications of Jordan theory to Lie algebras.

Proposition 4.1. *Let a be a Jordan element of a Lie algebra L over a field Φ of characteristic $\neq 2, 3$. Then L with the new product defined by $x \cdot_a y := \frac{1}{2}[[x, a], y]$ is a nonassociative algebra denoted by $L^{(a)}$, such that*

(i) $\mathrm{Ker}_L a := \{x \in L : [a, [a, x]] = 0\}$ *is an ideal of* $L^{(a)}$.
(ii) $L_a := L^{(a)}/\mathrm{Ker}_L a$ *is a Jordan algebra, called* the Jordan algebra of L at a.

Proof. [5, Theorem 2.4]. □

Proposition 4.2. *Let a be a Jordan element of a Lie algebra L.*

(i) *If L is nondegenerate (strongly prime), then L_a is nondegenerate (strongly prime).*
(ii) *If every element of L is Engel, then L_a is nil.*
(iii) *If L has an algebraic adjoint representation, then L_a is algebraic.*
(iv) *If L is PI, then L_a is PI.*

Proof. Inheritance of nondegeneracy (strong primeness) was proved in [5, Proposition 2.15(i)] ([6, Theorem 2.2(i)]); (ii) and (iii) are consequence of the identity $\overline{x}^n = \overline{\mathrm{ad}^{n-1}_{[x,a]}x}$ which holds for any $x \in L$ and any positive integer n, with $x \to \overline{x}$ denoting the linear mapping of L onto L_a. It only remains to prove (iv). By Proposition 3.2, L satisfies a multilinear $p = 0$, where

$$p(x_1, \ldots, x_n, x_{n+1}) = \sum_{\sigma \in S_n} \alpha_\sigma [x_{\sigma(1)}, \ldots, x_{\sigma(n)}, x_{n+1}].$$

By replacing each x_i by $[x_i, a]$ we obtain that the Jordan polynomial

$$q(x_1, \ldots, x_n, x_{n+1}) = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)} \cdot x_{n+1}$$

vanishes on L_a. Since by Proposition 3.3 this polynomial is not an s-identity, L_a is PI. □

Proposition 4.3. *Let L be a nondegenerate Lie algebra with an algebraic adjoint representation over an algebraically closed field Φ of characteristic zero. Then any abelian minimal inner ideal B of L is one-dimensional, so any nonzero element of B is extremal*

Proof. Let x be a nonzero element of B. Then $\mathrm{ad}^2_x L = B$ and x is a Jordan element of L. By [5, (2.14)] together with the minimality of B, the Jordan algebra L_x of L at x has no nonzero proper inner ideals, that is, it is a division Jordan algebra, and by [5, Proposition 2.15(ii)], any $y \in L$ such that $[[x, y], x] = 2x$ yields the identity element $\overline{y}$ of L_x. Since L_x is algebraic (Proposition 4.2) and Φ is algebraically closed, $\overline{L} = L_x = \Phi\overline{y}$. Hence $B = \mathrm{ad}^2_x L = \Phi x$. □

5. The theorem

Proposition 5.1. *Let $L \neq 0$ be a strongly prime Lie algebra over an algebraically closed field Φ of characteristic zero. If L has an algebraic adjoint representation and satisfies a polynomial identity of degree n, then L is isomorphic to one of the algebras $G_2, F_4, E_6, E_7, E_8, A_r$, B_r, C_r, or D_r, $r \leq [n/2]$.*

Proof. By Corollary 2.3, L contains a nonzero Jordan element, say a. Then we have by Proposition 4.2 that L_a is a strongly prime algebraic Jordan PI-algebra. Using just the fact that L_a is strongly prime and PI, it follows from [17, Theorems 5 and 7] that the centre $Z(L_a)$ of L_a is a nonzero integral domain and the central localization $Z(L_a)^{-1}L_a$ is a simple unital Jordan algebra containing minimal inner ideal. Moreover, since L_a is algebraic over Φ and Φ is algebraically closed, $Z(L_a)$ is the field Φ itself and L_a is itself a simple unital Jordan algebra with minimal inner ideals. But minimal inner ideals of L_a give rise to abelian minimal inner ideals of L [5, (2.14)], so L contains abelian minimal inner ideals and hence extremal elements by Proposition 4.3. But any extremal element in a nondegenerate Lie algebra generates a minimal ideal by [4, Theorem 1.15], which is locally finite-dimensional by [18, Lemma 15]. Since L is prime, it follows from the structure of the socle [4, Theorem 2.5(i)] that $\operatorname{Soc} L$ has a unique simple component. Therefore $\operatorname{Soc} L$ is a locally finite-dimensional simple Lie algebra. But Bakhturin proved in [1] that a simple Lie algebra (over a field of characteristic zero) which is locally finite-dimensional and satisfies a nontrivial identity over its centroid is finite-dimensional over its centroid. Since by Proposition 2.6 the centroid of $\operatorname{Soc} L$ coincides with Φ, $\operatorname{Soc} L$ is actually finite-dimensional. Moreover, since no matrix algebra $M_r(\Phi)$ satisfies an identity of degree less than $2r$ and the Lie algebra $M_r(\Phi)^-$ can be embedded in each one of the simple Lie algebras A_r, B_r, C_r and D_r, $\operatorname{Soc} L$ is necessarily one of the simple finite-dimensional Lie algebra listed in the claim of the proposition. Finally, again by primeness, L can be embedded in $\operatorname{Der}(\operatorname{Soc} L)$ via the adjoint representation. Hence $L = \operatorname{Soc} L$ because every derivation of a simple finite-dimensional Lie algebra over a field of characteristic zero is inner. □

Theorem 5.2. *Let L be a nondegenerate Lie algebra over an algebraically closed field Φ of characteristic zero. If L has an algebraic adjoint representation and satisfies a polynomial identity of degree n, then L is a subdirect product of a family of finite-dimensional simple Lie algebras, each of which isomorphic to one of the algebras $G_2, F_4, E_6, E_7, E_8, A_r$, B_r, C_r, or D_r, where $r \leq [n/2]$.*

Proof. By Theorem 2.5, L is a subdirect product of a family $\{L_i\}_{i \in I}$ of strongly prime Lie algebras L_i. Moreover, each L_i has an algebraic adjoint representation and satisfies a polynomial identity of degree n. Hence, by Proposition 5.1, each L_i is isomorphic to either $G_2, F_4, E_6, E_7, E_8, A_r$, B_r, C_r, or D_r, where $r \leq [n/2]$. □

Corollary 5.3. *Let L be a Lie PI-algebra over an algebraically closed field Φ of characteristic zero. If L has an algebraic adjoint representation, then L is locally finite-dimensional.*

Proof. After factorizing by the largest locally finite-dimensional ideal (10), it suffices to prove that L contains a nonzero locally finite-dimensional ideal. Moreover, since $K_1(L)$ is locally nilpotent by Theorem 2.4, it is locally finite-dimensional, so we may suppose that L is nondegenerate. Then we have by Proposition 5.2 that L is a subdirect product of a family of finite-dimensional simple Lie algebras each of which isomorphic to either $G_2, F_4, E_6, E_7, E_8, A_r$, B_r,

C_r, or D_r, for some positive integer r. This implies that L satisfies all the identities of a finite-dimensional Lie algebra and hence it is locally-finite dimensional by [18, Lemma 7] (or [12, Theorem 5.4.6]). □

Remark 5.4. In our proof of Corollary 5.3, Φ being algebraically closed was required in order to guarantee the existence of nontrivial Jordan elements, cf. Corollary 2.3, and to assure that an algebraic domain over Φ agrees with Φ, cf. Proposition 5.1. However, this is not actually necessary when one considers the analogous result for Engel PI-algebras [18, Proposition 1]. As a further illustration of our methods we give an alternative proof of this proposition.

Proposition 5.5. (Zelmanov) *Any Engel Lie PI-algebra L over a field of characteristic zero is locally nilpotent.*

Proof. Suppose that L is not locally nilpotent. As in Zelmanov's original proof, after factorizing L by its locally nilpotent radical we may assume that L is nonzero and nondegenerate. By Lemma 2.1, L contains a nonzero Jordan element a, and by Proposition 4.2, L_a is a nondegenerate Lie PI-algebra which is also nil. Then, by [16, Theorem 4], $L_a = Mc(L_a) = 0$, and hence $\mathrm{ad}_a^2 L = 0$, which is a contradiction since L is nondegenerate. □

Acknowledgement. The author would like to thank Esther García, Miguel Gómez Lozano, Ottmar Loos, and Angel Rodríguez Palacios for helpful discussions about this paper.

References

[1] Y. A. Bakhturin, Simple Lie algebras satisfying a nontrivial identity, *Serdica* **2** (1976), 241-246.
[2] G. Benkart, On inner ideals and ad-nilpotent elements of Lie algebras, *Trans. Amer. Math. Soc.* **232** (1977), 61-81.
[3] P. M. Cohn, *Further Algebra and Applications*, Springer-Verlag, Berlin, Heidelberg, New York, 2003.
[4] C. Draper, A. Fernández López, E. García, and M. Gómez Lozano, The socle of a non-degenerate Lie algebra, *J. Algebra* **319** (2008), 2372-2394.
[5] A. Fernández López, E. García, and M. Gómez Lozano, The Jordan algebras of a Lie algebra, *J. Algebra* **308** (2007), 164-177.
[6] E. García, and M. Gómez Lozano, An elemental characterization of strong primeness in Lie algebra, *J. Algebra* **312** (2007), 132-141.
[7] E. García and M. Gómez Lozano, A note on a result of Kostrikin, *Comm. Algebra* **37** (2009), 2405-2409.
[8] E. García and M. Gómez Lozano, A characterization of the Kostrikin radical of a Lie algebra, *Jordan Theory Preprint Archives* **287** (2010).
[9] A. N. Grishkov, Local nilpotency of an ideal of a Lie algebra generated by second-order elements, *Sibirsk. Math. Zh.* **23** (1982), no. 1, 241-246 (Russian).
[10] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Grad. Texts in Math. 9, Springer-Verlag, Heidelberg-Berlin- New York, 1972.
[11] N. Jacobson, *Lie Algebras*, Interscience Publishers, New York, 1962.
[12] A. I. Kostrikin, *Around Burnside.* A Series of Modern Surveys in Mathematics. Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, Hong Kong, 1990.
[13] K. McCrimmon, *A Taste on Jordan Algebras*, Universitex. Springer-Verlag, Berlin, Heidelberg, New York, 2004.
[14] J. M. Osborn and M. L. Racine, Jordan rings with nonzero socle, *Trans. Amer. Math. Soc.* **251** (1979), 375-387.
[15] B. I. Plotkin, Algebraic sets of elements in groups and Lie algebras, *Uspekhi Math. Nauk.* **13** (1958), no. 6 (84), 133-138 (Russian).
[16] E. I. Zelmanov, Absolute zero-divisors an algebraic Jordan algebras, *Siberian Math. J.* **23** (1982), 100-116.
[17] E. I. Zelmanov, Prime Jordan algebras, II, *Siberian Math. J.* **24** (1983), 89-104.

[18] E. I. Zelmanov, Lie algebras with an algebraic adjoint representation *Math. USSR Sbornik* **49** (1984), 537-552.

Departamento de Álgebra, Geometría y Topología, Universidad de Málaga, Spain
E-mail address: emalfer@uma.es