A SURVEY ON ALBERT ALGEBRAS

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Abstract. A fairly complete account will be given of what is presently known about Albert algebras over commutative rings. In particular, we sketch a novel approach to the two Tits constructions of cubic Jordan algebras that yields new insights even when the base ring is a field. The paper concludes with a discussion of cohomological invariants and with a number of open problems.

Contents

1. Introduction

This survey article grew out of a series of lectures given during the Fields Institute workshop on exceptional algebras and groups at the University of Ottawa, April 19−22, 2012. The principal aim of these lectures was to provide a rather complete account of what is presently known about Albert algebras and their cubic companions. My hope is that I succeeded in preparing the ground for an adequate understanding of the connection with exceptional groups, particularly those of type F_4, that could (and actually did) arise in other lectures of the conference.

Still, due to severe time constraints, choices had to made and many important topics, like, e.g., twisted compositions ([102, 103, 72]), had to be excluded; for the same reason, proofs had to be mostly omitted. On the other hand, a substantial amount of the material could be presented not just over fields but, in fact, over arbitrary commutative rings. In particular, this holds true for our approach to the two Tits constructions of cubic Jordan algebras that yields new insights even when the base ring is a field.

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Treating a topic like Albert algebras in a systematic sort of way has the disadvantage that it takes a whole series of long-winding preparations before the proper subject matter can be addressed. In order to mitigate this unpleasant effect, I have therefore inserted a preliminary section where I describe results on Albert algebras and their applications that have been obtained during the first fifteen to twenty years after their inception. The emphasis here is not so much on conceptual precision but, rather, on giving a first, albeit sketchy, impression of what the subject is all about. This will change in subsequent sections of the paper, where I start basically from scratch in order to describe some of what I regard as the essential ingredients of the theory.

Though in writing up this survey I have tried my best to keep track of the historical development, I will surely have overlooked quite a few important contributions to the subject that should have been quoted at the proper place. I apologize in advance for all these omissions.

2. Prologue: from quantum mechanics to algebra

Albert algebras left their mark on mathematics and physics, though not under this name, actually under no name at all, as early as 1934 when Jordan, von Neumann and Wigner developed a structure theory for what in modern terminology are called finite-dimensional euclidean Jordan algebras [39]. In the course of their investigation, the question arose of whether a certain commutative non-associative real algebra of dimension 27 is a Jordan algebra, and whether it is in some sense exceptional. Unable to settle this question themselves, the authors turned to Albert for help, who in due course provided an affirmative answer in an immediate follow-up [1] to [39]. In order to appreciate Albert's result more fully, it will be imperative to put it on a broader algebraic footing. For this reason, the reals will henceforth be replaced by an arbitrary field F of characteristic not 2 remaining fixed throughout the rest of this section. Then Albert's theorem may be based on the notion of a Jordan algebra. In the present context we prefer instead to use the more elaborate term *linear* Jordan algebra, in order to distinguish it from the concept of a *quadratic* Jordan algebra to be discussed in §4 below.

2.1. Linear Jordan algebras. By a *linear Jordan algebra* over F we mean a non-associative F-algebra J satisfying the identities

$$xy = yx \qquad \text{(commutative law)}, \tag{1}$$

$$x(x^2y) = x^2(xy) \qquad \text{(Jordan identity)} \tag{2}$$

for all $x, y \in J$.

2.2. Examples. (a) Let A be an associative algebra over F. Then the F-vector space A becomes a linear Jordan algebra over F under the *symmetric product* defined by

$$x \bullet y := \frac{1}{2}(xy + yx) \qquad (x, y \in A) \tag{1}$$

in terms of the multiplication of A. This linear Jordan algebra is denoted by A^+.

(b) Let (B, τ) be an associative F-algebra with involution, so B is an associative algebra over F and $\tau\colon B \to B$ is an *involution*, i.e., an anti-automorphism of period 2 over F. Then

$$H(B, \tau) := \{x \in B \mid \tau(x) = x\} \tag{2}$$

is a subalgebra of B^+ and hence a linear Jordan algebra.

2.3. Special and exceptional linear Jordan algebras. A linear Jordan algebra over F is said to be *special* if it is isomorphic to a subalgebra of A^+, for some associative algebra A over F. Linear Jordan algebras that are not special are called *exceptional*. The examples presented in 2.2 are all special linear Jordan algebras.

2.4. **Octonion algebras.** For a formal definition of this concept in a considerably more general setting, we refer the reader to 5.5 below. Here we will be content with listing a few properties, all of them valid in octonion algebras but some of them redundant, that will be crucial for understanding the subsequent development of this section.

Accordingly, let C be an octonion algebra over F. Then C is a non-associative F-algebra with the following properties.

(a) $\dim_F(C) = 8$.
(b) C is *unital*, i.e., it has an identity element which we denote by 1_C.
(c) There is a non-degenerate quadratic form $n_C \colon C \to F$, called the *norm* of C, uniquely determined by the condition that it *permits composition*: $n_C(uv) = n_C(u)n_C(v)$ for all $u, v \in C$.
(d) There is an involution $\iota_C \colon C \to C$, $u \mapsto \bar{u}$, called the *conjugation* of C, uniquely determined by the condition that it is *scalar*: $u\bar{u} = n_C(u)1_C$ for all $u \in C$. We have

$$H(C, \iota_C) = F1_C. \tag{1}$$

In general, there will be many non-isomorphic octonion algebras over a given field. But, up to isomorphism, there is exactly one containing zero divisors. Such an octonion algebra is said to *split*. If F is algebraically (or only separably) closed, then every octonion algebra over F is split. On the other hand, there are precisely two non-isomorphic octonion algebras over the reals, the split one and $\mathbb{O}$, the classical algebra of Graves-Cayley octonions.

2.5. **The conjugate transpose involution.** Let C be an octonion algebra over F. Then ordinary matrix multiplication converts $\mathrm{Mat}_3(C)$, the space of 3×3-matrices with entries in C, into a non-associative F-algebra with identity element $\mathbf{1}_3$, the 3×3 unit matrix. A straightforward verification shows that the map

$$\mathrm{Mat}_3(C) \longrightarrow \mathrm{Mat}_3(C), \quad x \longmapsto \bar{x}^t,$$

is an involution of $\mathrm{Mat}_3(C)$, called its *conjugate transpose* involution. Hence

$$\mathrm{Her}_3(C) := \{x \in \mathrm{Mat}_3(C) \mid x = \bar{x}^t\} \tag{1}$$

becomes a commutative F-algebra, again with identity element $\mathbf{1}_3$, under the symmetric matrix product

$$x \bullet y := \frac{1}{2}(xy + yx) \qquad (x, y \in \mathrm{Her}_3(C)). \tag{2}$$

By (2.4.1), this algebra has dimension 27 over F. For $F := \mathbb{R}$, $C = \mathbb{O}$, it is the one considered in [39, 1].

2.6. **Theorem.** ([1]) *Let C be an octonion algebra over F. Then* $\mathrm{Her}_3(C)$ *is a central simple exceptional linear Jordan algebra over F.* □

Remark. Albert's original proof in [1] is given only for the special case $F = \mathbb{R}$, $C = \mathbb{O}$, but carries over almost verbatim to the case of an arbitrary octonion algebra over an arbitrary field (of characteristic not 2).

2.7. **Albert algebras.** For the time being, let C be the split octonion algebra over F. By an *Albert algebra* over F we mean an *F-form* of $\mathrm{Her}_3(C)$, i.e., a non-associative F-algebra that becomes isomorphic to $\mathrm{Her}_3(C)$ after extending scalars to the separable closure of F.

Note that the octonion algebras over F are precisely the F-forms of C. In analogy to this, we have

2.8. **Theorem.** ([2]) *The finite-dimensional central simple exceptional linear Jordan algebras over F are precisely the Albert algebras over F.* □

Albert algebras are intimately tied up with exceptional (algebraic or Lie) groups and Lie algebras. This was shown rather early on by looking at derivations.

2.9. **Derivations.** Let A be a non-associative algebra over F. By a *derivation* of A we mean a linear map $D\colon\ A \to A$ satisfying the product rule for derivatives:

$$D(xy) = (Dx)y + x(Dy) \qquad (x, y \in A). \tag{1}$$

The derivations of A form a Lie algebra, denoted by $\mathrm{Der}(A)$, under the commutator product. If A is finite-dimensional, then $\mathrm{Der}(A)$ is the Lie algebra of $\mathbf{Aut}(A)$, the automorphism group of A viewed as a group scheme over F [42, (20.4)(8), (21.5)(9)].

After E. Cartan [13] had shown that the derivation algebra of the Graves-Cayley octonions is the compact Lie algebra of type G_2 over the reals, it seemed natural to expect similar results for Albert algebras. Historically, the first ones along these lines are the following.

2.10. **Theorem.** ([14]) *Let J be an Albert algebra over a field F of characteristic zero. Then the derivation algebra of J is a central simple Lie algebra of type F_4 over F.*

Instead of a proof. Passing to the algebraic closure of F, we may assume that F is algebraically closed and then have $J = \mathrm{Her}_3(C)$, where C stands for the split octonion algebra over F. Now one shows that the Lie algebra $\mathrm{Der}(J)$ is simple of dimension 52. Since, by the Cartan-Killing classification of simple Lie algebras over algebraically closed fields of characteristic zero, the Lie algebras over F having the properties above are precisely those of type F_4, the theorem follows.

By a slight modification of the procedure just described, one can connect Albert algebras also with exceptional Lie algebras (or groups) of type E_6.

2.11. **The structure algebra.** Let J be an Albert algebra over F. We write $L_x\colon\ J \to J$, $y \mapsto xy$, for the left multiplication operator on J affected by $x \in J$ and put

$$L_J^0 := \{L_x \mid x \in J,\ \mathrm{tr}(L_x) = 0\}, \tag{1}$$

where tr stands for the trace of a linear endomorphism acting on a finite-dimensional vector space. Then it follows easily that

$$\mathfrak{str}(J) := L_J^0 + \mathrm{Der}(J) \tag{2}$$

is a Lie algebra of linear transformations, called the *structure algebra* of J, and the sum on the right is a direct sum of vector spaces. Hence $\mathfrak{str}(J)$ has dimension 78. Once it has been shown that $\mathfrak{str}(J)$ is also simple as a Lie algebra, the arguments indicating the proof of Thm. 2.10 can be repeated almost verbatim and then lead to the following result.

2.12. **Theorem.** ([14]) *Let J be an Albert algebra over a field F of characteristic zero. Then the structure algebra of J is a central simple Lie algebra of Type E_6 over F.* $\square$

Remark. The structure algebra of an Albert algebra is the Lie algebra of the structure group scheme, to be defined in 4.15 below.

Even more important than the connection of Albert algebras with derivations is the one with automorphisms. More specifically, using methods form Galois cohomology (see [42, (29.8) and (31.47)] for details), one derives the following fundamental result,

2.13. **Theorem.** *The assignment $C \mapsto \mathbf{Aut}(C)$ (resp. $J \mapsto \mathbf{Aut}(J)$) defines a bijection from the set of isomorphism classes of octonion (resp. Albert) algebras over F onto the set of isomorphism classes of absolutely simple simply connected algebraic groups of type G_2 (resp. F_4) over F.* $\square$

3. Notation and reminders

Throughout the remainder of this paper, we let k be an arbitrary commutative ring. A (non-associative) k-algebra A, i.e., a k-module together with a k-bilinear multiplication (the *algebra structure* of A), subject to no further restrictions, is said to be *unital* if it contains a unit (or identity) element, which will then be denoted by $1 = 1_A$. By a *unital* subalgebra we mean a subalgebra of a unital algebra containing its unit. A *unital homomorphism* of unital k-algebras is a k-algebra homomorphism taking 1 into 1.

We denote by k-**alg** the category of unital commutative associative k-algebras, morphisms being unital k-algebra homomorphisms. The identity transformation of a k-module M will be denoted by $\mathbf{1}_M$. Given $R \in k$-**alg**, we write $M_R := M \otimes R$ for the *base change* (or *scalar extension*) of M from k to R, unadorned tensor products always being taken over k. It is an R-module in a natural way, and the assignment $x \mapsto x_R := x \otimes 1_R$ gives a k-linear map from M to M_R which in general is neither injective nor surjective.

3.1. **Quadratic maps.** Let M, N be k-modules. A map $Q\colon\ M \to N$ is said to *quadratic* if Q is homogeneous of degree 2, so $Q(\alpha x) = \alpha^2 Q(x)$ for all $\alpha \in k$, $x \in M$, and the induced map

$$\partial Q\colon\ M \times M \longrightarrow N, \quad (x, y) \longmapsto \partial Q(x, y) := Q(x + y) - Q(x) - Q(y),$$

is (symmetric) k-bilinear, called the *bilinearization* or *polarization* of Q. For convenience, we systematically simplify notation by writing $Q(x, y)$ instead of $\partial Q(x, y)$. Note that $Q(x, x) = 2Q(x)$, so Q may be recovered from ∂Q if 2 is a unit in k. Given any $R \in k$-**alg**, a quadratic map $Q\colon\ M \to N$ has a unique extension to a quadratic map $Q_R\colon\ M_R \to N_R$ over R. In the special case $N := k$, we speak of a quadratic *form* (over k).

3.2. **Projective modules.** Recall that a k-module M is *projective* if it is a direct summand of a free k-module. The following fact will be particularly useful in the present context. Writing $\mathrm{Spec}(k)$ for the set of prime ideals in k, $k_\mathfrak{p}$ for the localization of k at $\mathfrak{p} \in \mathrm{Spec}(k)$ and $M_\mathfrak{p} := M_{k_\mathfrak{p}}$ for the corresponding base change of M, the following conditions are equivalent (cf., e.g., [10, II, §5, Thm. 1]).

(i) M is finitely generated projective.
(ii) For all $\mathfrak{p} \in \mathrm{Spec}(k)$, the $k_\mathfrak{p}$-module $M_\mathfrak{p}$ is free of finite rank $r_\mathfrak{p}$, and the map $\mathrm{Spec}(k) \to \mathbf{Z}$, $\mathfrak{p} \mapsto r_\mathfrak{p}$, is locally constant with respect to the Zariski topology of $\mathrm{Spec}(k)$.

If in this case, the rank of the free $k_\mathfrak{p}$-module $M_\mathfrak{p}$ does not depend on $\mathfrak{p} \in \mathrm{Spec}(k)$, then we say M *has a rank* and call this number the *rank* of M.

3.3. **Regularity conditions on quadratic forms.** Let $q\colon\ M \to k$ be a quadratic form. Then ∂q is a symmetric bilinear form on M, inducing canonically a linear map

$$M \longrightarrow M^* := \mathrm{Hom}_k(M, k), \quad x \longmapsto q(x, -),$$

whose kernel,

$$\mathrm{Rad}(\partial q) := \{x \in M \mid \forall\, y \in M :\ q(x, y) = 0\},$$

is called the *bilinear radical* of q.

If k is a field, it is customary to call q *non-degenerate* if, for $x \in M$, the relations $q(x) = q(x, y) = 0$ for all $y \in M$ imply $x = 0$. Note that, for $\mathrm{char}(k) \neq 2$, non-degeneracy of q is equivalent to ∂q being non-degenerate in the usual sense.

Now return to the case that k is an arbitrary commutative ring. Deviating slightly from a terminology introduced by Loos [45, 3.2], a quadratic form $q\colon\ M \to k$ is said to be *separable* if M is projective as a k-module, and for all *fields* $F \in k$-**alg**, the extended quadratic form $q_F\colon\ M_F \to F$ over F is non-degenerate in the sense just defined. By contrast, q will be called *non-singular* if M is finitely generated projective as a k-module and the homomorphism $M \to M^*$ induced by ∂q is in fact an isomorphism.

Both of these concepts are invariant under base change: if q is separable (resp. non-singular), so is q_R, for any $R \in k$-**alg**.

4. Jordan algebras

As we have seen in §2, an adequate conceptual framework for investigating Albert algebras over fields of characteristic not 2 is provided by the theory of linear Jordan algebras. But since we intend to study Albert algebras not just over fields (of characteristic not 2) but, more generally, over arbitrary commutative rings, the theory of linear Jordan algebras is no longer appropriate and has to be replaced by a suitable generalization. Historically speaking, such a generalization has been obtained in two steps.

As a first step, one simply notes that the main ingredients pertaining to the theory of linear Jordan algebras can be preserved quite easily if the base field F of characteristic not 2 is replaced by a commutative ring containing $\frac{1}{2}$ (equivalently, making 2 invertible). We illustrate this by sketching the concept of

4.1. **Linear Jordan algebras.** Assume our base ring k contains $\frac{1}{2}$. By a *linear Jordan algebra* over k, we mean a non-associative k-algebra J which is commutative and satisfies the Jordan identity (2.1.2). Unital linear Jordan algebras over k together with unital k-algebra homomorphisms form a category, denoted by k-**linjord**. Moreover, linear Jordan algebras are stable under base change: if J is a linear Jordan algebra over k, then so is J_R over R, for all $R \in k$-**alg**.

If A is an associative k-algebra, then the symmetric matrix product (2.2.1) converts the k-module A into a linear Jordan algebra over k denoted by A^+. If (B, τ) is an associative k-algebra with involution, i.e., with an anti-automorphism of period 2 over k, then (2.2.2) defines a subalgebra $H(B, \tau) \subseteq B^+$, which is therefore a linear Jordan algebra over k.

Passing on to a version of the theory that no longer requires $\frac{1}{2}$ in the base ring turns out to be much more delicate. The key to this passage is

4.2. **The U-operator of a linear Jordan algebra.** Assuming that k contains $\frac{1}{2}$, let J be a linear Jordan algebra over k and write $L_x\colon J \to J$, $y \mapsto xy$, for the left multiplication operator of $x \in J$. Then the U-operator of J is defined as the map

$$U\colon J \longrightarrow \mathrm{End}_k(J), \quad x \longmapsto U_x := 2L_x^2 - L_{x^2}, \tag{1}$$

which is obviously quadratic and, via its bilinearization, gives rise to the *Jordan triple product*

$$\{xyz\} := U_{x,z}y = (U_{x+z} - U_x - U_z)y = 2\big(x(zy) + z(xy) - (xz)y\big). \tag{2}$$

Given $x, y \in J$, we define the "V-operator"

$$V_{x,y}\colon J \longrightarrow J, \quad z \longmapsto \{xyz\}, \tag{3}$$

which is obviously linear.

4.3. **Examples.** Let A be an associative algebra over k. Then the U-operator of the linear Jordan algebra A^+ is given by

$$U_x y = xyx \qquad (x, y \in A) \tag{1}$$

in terms of the multiplication of A. Note that this formula is inherited by all subalgebras of A^+, so (1) also describes the U-operator of $H(B, \tau)$, for any associative k-algebra (B, τ) with involution.

The U- and V-operators as well as the Jordan triple product satisfy a number of elementary, but highly non-trivial, identities, among which we single out the following as particularly important.

4.4. **Theorem.** ([55, 59]) *If k contains $\frac{1}{2}$ and J is a unital linear Jordan algebra over k, then the identities*

$$U_{1_J} = \mathbf{1}_J, \tag{1}$$

$$U_{U_x y} = U_x U_y U_x \qquad \text{(fundamental formula)}, \tag{2}$$

$$U_x V_{y,x} = V_{x,y} U_x \tag{3}$$

hold in all scalar extensions of J. □

We are now prepared to formalize the concept of a Jordan algebra in a completely different manner. It turns out that this concept

- works quite well over arbitrary base rings, including the ring $\mathbf{Z}$ of rational integers,
- is isomorphic to the concept of a unital linear Jordan algebra if the base ring contains $\frac{1}{2}$,
- takes the bold step of axiomatizing a class of algebra structures which are no longer bilinear but, instead, linear in one variable and quadratic in the other.

4.5. **Quadratic Jordan algebras.** ([50]) By a *quadratic Jordan algebra* over k we mean a k-module J together with a distinguished element $1_J \in J$ (the *unit*) and a quadratic map $U\colon\ J \to \mathrm{End}_k(J)$, $x \mapsto U_x$, (the *U-operator*) such that, setting

$$\{xyz\} := V_{x,y}z := U_{x,z}y = (U_{x+z} - U_x - U_z)y \tag{1}$$

(the *Jordan triple product*, obviously trilinear and symmetric in the outer variables), the following identities hold under all scalar extensions.

$$U_{1_J} = \mathbf{1}_J, \tag{2}$$

$$U_{U_x y} = U_x U_y U_x \qquad \text{(fundamental formula)}, \tag{3}$$

$$U_x V_{y,x} = V_{x,y} U_x. \tag{4}$$

A *homomorphism* of quadratic Jordan algebras is a linear map preserving units and U-operators, hence also the Jordan triple product. Thus quadratic Jordan algebras over k form a category denoted by k-**quadjord**.

Philosophically speaking, the U-operator serves as the exact analogue of the left (or right) multiplication in associative (or alternative) algebras. Specializing one of the variable in (1) to 1_J, the Jordan triple product collapses to the bilinear and commutative *Jordan circle product*

$$x \circ y := \{x1_J y\} = \{1_J xy\} = \{xy1_J\}. \tag{5}$$

4.6. **Connecting linear and quadratic Jordan algebras.** Assume that our base ring k contains $\frac{1}{2}$.

(a) Let J be a unital linear Jordan algebra over k. Then the identity element of J and its U-operator as defined in 4.2 convert J into a quadratic Jordan algebra over k (Thm. 4.4), denoted by J_{quad}. If $\varphi\colon\ J \to J'$ is a unital homomorphism of unital linear Jordan algebras over k, then it is also one of quadratic Jordan algebras: $\varphi\colon\ J_{\mathrm{quad}} \to J'_{\mathrm{quad}}$. Thus we obtain a functor from k-**linjord** to k-**quadjord**.

(b) Conversely, let J be a quadratic Jordan algebra over k. Then the bilinear product

$$xy := \frac{1}{2} x \circ y$$

derived from the circle product of J makes J a linear Jordan algebra over k [38], denoted by J_{lin}. If $\varphi\colon\ J \to J'$ is a homomorphism of quadratic Jordan algebras over k, then it is also one of unital linear Jordan algebras: $\varphi\colon\ J_{\mathrm{lin}} \to J'_{\mathrm{lin}}$. Thus we obtain a functor from k-**quadjord** to k-**linjord**.

4.7. **Theorem.** *The functors*

$$k\textbf{-linjord} \longrightarrow k\textbf{-quadjord} \longrightarrow k\textbf{-linjord}$$

defined in 4.6 *are isomorphisms of categories and inverse to each other.* □

4.8. **Conventions.** From now on, we use the term *Jordan algebra* to designate a quadratic Jordan algebra, and we write

$$k\textbf{-jord} := k\textbf{-quadjord}$$

for the category of (quadratic) Jordan algebras over k. Also, if k contains $\frac{1}{2}$, we identify the categories k-**jord** and k-**linjord** by means of Thm. 4.7.

In the remainder of this section, we merely sketch some of the most basic concepts from the theory of Jordan algebras. For more details, see [38].

4.9. **Ideals, quotients, simplicity.** Let J be a Jordan algebra over k. A submodule $I \subseteq J$ is called an *ideal* if, in obvious notation,

$$U_I J + U_J I \subseteq I.$$

In this case, the defining identities (4.5.2)−(4.5.4) together with their linearizations imply $\{JJI\} \subseteq I$ and the quotient module J/I carries the unique structure of a Jordan algebra over k such that the natural map $J \to J/I$ is a homomorphism of Jordan algebras. A Jordan algebra is called *simple* if it is non-zero and contains only the trivial ideals. A Jordan algebra over a field F is called *absolutely simple* if the base change J_K is simple for all field extensions K/F. For $\mathrm{char}(F) \neq 2$ and $\dim_F(J) < \infty$, this is equivalent to J being central simple.

4.10. **Examples of Jordan algebras.** The examples of linear Jordan algebra discussed in 2.2 and 4.1 can be transferred to the quadratic setting by making use of Ex. 4.3 in the following way.

(a) Let A be a unital associative algebra over k. Then the k-module A together with the unit element 1_A and the U-operator defined by $U_x y := xyx$ is a Jordan algebra over k, denoted by A^+. The Jordan triple product in A^+ is given by $\{xyz\} = xyz + zyx$, the Jordan circle product by $x \circ y = xy + yx$.

(b) Let (B, τ) be a unital associative k-algebra with involution, so B is a unital associative algebra over k and $\tau\colon B \to B$ is an *involution*, i.e., a k-linear map of period 2 and an anti-automorphism of the algebra structure:

$$\tau\big(\tau(x)\big) = x, \quad \tau(xy) = \tau(y)\tau(x)$$

for all $x, y \in B$. Then $\tau\colon B^+ \to B^+$ is an automorphism of period 2, and the τ-symmetric elements of B, i.e.,

$$H(B, \tau) := \{x \in B \mid \tau(x) = x\}$$

form a (unital) subalgebra of B^+, hence, in particular, a Jordan algebra, called the *Jordan algebra of τ-symmetric elements of B* .

There is yet another class of Jordan algebras we haven't encountered before.

(c) Let $\mathcal{Q} = (M, q, e)$ be a *pointed quadratic form* over k, so M is a k-module, $q\colon M \to k$ is a quadratic form, and $e \in M$ is a distinguished element (the *base point*) satisfying $q(e) = 1$. Defining the *conjugation* of $\mathcal{Q}$ by

$$\iota_{\mathcal{Q}}\colon M \longrightarrow M, \quad x \longmapsto \bar{x} := q(e, x)e - x,$$

which is a linear map fixing e and having period 2, the k-module M together with the unit e and the U-operator

$$U_x y := q(x, \bar{y})x - q(x)\bar{y}$$

becomes a Jordan algebra over k, denoted by $J(\mathcal{Q})$ and said to be *associated* with $\mathcal{Q}$.

Remark. The assertions of (a) (resp. (b)) remain valid if the associative algebra A (resp. B) is replaced by an alternative one.

4.11. **Special and exceptional Jordan algebras.** A Jordan algebra is said to be *special* if there exists a unital associative algebra A and an injective homomorphism $J \to A^+$ of Jordan algebras. Jordan algebras that are not special are called *exceptional*. These notions are equivalent to the ones of 2.3 for linear Jordan algebras. The Jordan algebras 4.10 (a),(b) are obviously special, while the ones in (c) are special if k is a field but not in general [38].

4.12. **Powers and Invertibility.** Let J be a Jordan algebra over k and $x \in J$.

(a) Powers $x^n \in J$ with integer coefficients $n \geq 0$ are defined inductively by $x^0 = 1_J$, $x^1 = x$, $x^{n+2} = U_x x^n$. One then obtains the expected formulas

$$U_{x^m} x^n = x^{2m+n}, \quad \{x^m x^n x^p\} = 2x^{m+n+p}$$

for all integers $m, n, p \geq 0$.

(b) An element $x \in J$ is said to be *invertible* if there exists an element $y \in J$ such that $U_x y = x$, $U_x y^2 = 1_J$. In this case, y is unique and called the *inverse* of x in J, written as x^{-1}. The set of invertible elements of J will be denoted by $J^\times$ It is easy to see that *the following conditions are equivalent.*

(i) *x is invertible.*
(ii) *U_x is bijective.*
(iii) $1_J \in \mathrm{Im}(U_x)$.

In this case $x^{-1} = U_x^{-1} x$. Moreover, if $x, y \in J$ are invertible, so is $U_x y$ with inverse $(U_x y)^{-1} = U_{x^{-1}} y^{-1}$.

(c) If A is a unital associative (or alternative) k-algebra, then invertibility and inverses in A and A^+ are the same. Similarly, if (B, τ) is a unital associative (or alternative) algebra with involution, then invertibility in the Jordan algebra $H(B, \tau)$ amounts to the same as invertibility in the ambient associative (or alternative) algebra B, and again the inverses are the same.

(d) J is said to be a *Jordan division algebra* if $J \neq \{0\}$ and all its non-zero elements are invertible. Hence if A (resp. B) are as in (c), then A^+ is a Jordan division algebra if and only if A is an associative (or alternative) division algebra; similarly, if B is an associative (or alternative) division algebra, then $H(B, \tau)$ is a Jordan division algebra, but not conversely.

The group of left multiplications by invertible elements in a unital associative algebra for trivial reasons acts transitively on its invertible elements. By contrast, the U-operators belonging to invertible elements of a Jordan algebra J do not in general form a group, and the group they generate will in general not be transitive on the invertible elements of J. Fortunately, there is a substitute for this deficiency.

4.13. **Isotopes.** Let J be a Jordan algebra over k and $p \in J$ be an invertible element. Then the k-module J together with the new unit $1_{J^{(p)}} = p^{-1}$ and the new U-operator $U_x^{(p)} := U_x U_p$ is a Jordan algebra over k, called the *p-isotope* of J and denoted by $J^{(p)}$. We clearly have $J^{(1_J)} = J$, $J^{(p)\times} = J^\times$ and $(J^{(p)})^{(q)} = J^{(U_p q)}$ for all $q \in J^\times$. Calling a Jordan algebra J' *isotopic* to J if $J' \cong J^{(p)}$ for some $p \in J^\times$, we therefore obtain an equivalence relation on the category of Jordan algebras over k.

4.14. **Examples of isotopes.** (a) Let A be a unital associative algebra over k and $p \in A^{+\times} = A^\times$. Then right multiplication by p in A gives an isomorphism

$$R_p \colon (A^+)^{(p)} \xrightarrow{\sim} A^+,$$

of Jordan algebras.

(b) In general, however, isotopy is not the same as isomorphism. For example, let (B, τ)

be a unital associative algebra with involution and $p \in H(B,\tau)^\times$. Then the formula $\tau^{(p)}(x) := p^{-1}\tau(x)p$ defines a new involution on B, and right multiplication by p in B leads to an isomorphism

$$R_p\colon\ H(B,\tau)^{(p)} \xrightarrow{\sim} H(B,\tau^{(p)})$$

of Jordan algebras. Using this, it is easy to construct examples (e.g., with B a quaternion algebra over a field) where $H(B,\tau)$ and $H(B,\tau)^{(p)}$ are *not* isomorphic.

The fact that isotopy of Jordan algebras does not break down to isomorphism gives rise to an important class of algebraic groups.

4.15. **The structure group.** Let J be a Jordan algebra over k. Then *for all $\eta \in \mathrm{GL}(J)$, the following conditions are equivalent.*

(i) *η is an isomorphism from J to $J^{(p)}$, for some $p \in J^\times$.*
(ii) *There exists an element $\eta^\sharp \in \mathrm{GL}(J)$ such that $U_{\eta(y)} = \eta U_y \eta^\sharp$ for all $y \in J$.*

The elements of $\mathrm{GL}(J)$ satisfying one (hence both) of these conditions form a subgroup of $\mathrm{GL}(J)$, called the *structure group* of J and denoted by $\mathrm{Str}(J)$. By (ii) and the fundamental formula (4.5.3), the elements U_x, $x \in J^\times$, generate a subgroup of $\mathrm{Str}(J)$ which we call the *inner structure group* of J and denote by $\mathrm{Instr}(J)$.

5. Octonions

We have seen in §2 that octonions are an indispensable tool for the study of Albert algebras over fields of characteristic not 2. In the present section, we define them in a precise manner over arbitrary commutative rings and describe their most important properties. But rather than presenting them in an isolated sort of way (as we did in 2.4), it makes much more sense to regard them as members of a wider class of algebras called composition algebras.

The following definition has been suggested by Loos. Over fields, it is partially in line with the one in [42, 33.B], the important difference being that we insist on a unit element, while loc. cit. does not.

5.1. **Composition algebras.** By a *composition algebra* over k we mean a unital non-associative k-algebra C satisfying the following conditions.

(a) C is finitely generated projective of rank $r > 0$ as a k-module.
(b) There exists a separable quadratic form $n_C\colon C \to k$ that *permits composition*: $n_C(xy) = n_C(x)n_C(y)$ for all $x, y \in C$.

The quadratic form n_C in (b) is uniquely determined and is called the *norm* of C. Moreover, we call $t_C := n_C(1_C, -)$ the *trace* and

$$\iota_C\colon\ C \longrightarrow C, \quad x \longmapsto \bar{x} := t_C(x)1_C - x,$$

the *conjugation* of C; it is easily seen to be a linear map of period 2.

The simplest example of a composition algebra is the base ring k itself, with norm, trace, conjugation respectively given by $n_k(\alpha) = \alpha^2$, $t_k(\alpha) = 2\alpha$, $\bar{\alpha} = \alpha$ for all $\alpha \in k$.

5.2. **Properties of composition algebras.** Let C be a composition algebra over k. The following properties may be found in [47, 54, 69].

(a) Composition algebras are stable under base change: C_R is a composition algebra over R for all $R \in k$-**alg**.
(b) C has rank $1, 2, 4$ or 8.
(c) C is alternative, i.e., the *associator* $[x, y, z] := (xy)z - x(yz)$ is alternating in $x, y, z \in C$, equivalently, all subalgebras of C on two generators are associative.
(d) n_C is non-singular unless the rank of C is 1 and $\frac{1}{2} \notin k$.
(e) The conjugation of C is an involution, so we have

$$\bar{\bar{x}} = x, \quad \overline{xy} = \bar{y}\,\bar{x}$$

for all $x, y \in C$.

(f) The trace $t_C\colon C \to k$ is an *associative* linear form:

$$t_C\big((xy)z\big) = t_C\big(x(yz)\big) =: t_C(xyz)$$

for all $x, y, z \in C$.

(g) C satisfies the quadratic equations

$$x^2 - t_C(x)x + n_C(x)1_C = 0$$

for all $x \in C$, equivalently, the conjugation of C is a *scalar* involution:

$$x\bar{x} = n_C(x)1_C, \quad x + \bar{x} = t_C(x)1_C$$

for all $x \in C$.

(h) $x \in C$ is invertible in the alternative algebra C (same definition as in associative algebras) iff $n_C(x) \in k$ is invertible in k; in this case

$$x^{-1} = n_C(x)^{-1}\bar{x}.$$

We now turn to examples of composition algebras. In addition to the base ring itself, they may be described as follows.

5.3. **Quadratic étale algebras.** By a *quadratic étale algebra* over k we mean a composition algebra of rank 2. Quadratic étale algebras are commutative associative. If E is a quadratic étale k-algebra, then its norm and trace are given by

$$n_E(x) = \det(L_x), \quad t_E(x) := \mathrm{tr}(L_x),$$

where $L_x\colon R \to R$, $y \mapsto xy$, stands for the left multiplication operator of $x \in E$. The most elementary example of a quadratic étale algebra is $E^0 := k \oplus k$ (direct sum of ideals), with norm, trace, conjugation given by

$$n_{E^0}(\alpha \oplus \beta) = \alpha\beta, \quad t_{E^0}(\alpha \oplus \beta) = \alpha + \beta, \quad \overline{\alpha \oplus \beta} = \beta \oplus \alpha$$

for all $\alpha, \beta \in k$.

5.4. **Quaternion algebras.** By a *quaternion algebra* over k we mean a composition algebra of rank 4. Quaternion algebras are associative but not commutative. If E is a quadratic étale k-algebra with conjugation $a \mapsto \bar{a}$ and $\mu \in k$ is a unit, then $[E, \mu)$, the free k-algebra generated by E and an additional element j subject to the relations $j^2 = \mu 1_E$, $ja = \bar{a}j$ $(a \in E)$, is a quaternion algebra over k. *Conversely, every quaternion algebra over k may be written in this way provided k is a semi-local ring, e.g., a field.* The most elementary example is provided by the algebra $\mathrm{Mat}_2(k)$ of 2×2- matrices over k whose norm, trace and conjugation are respectively given by the ordinary determinant, the ordinary trace and the *symplectic involution*

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \overline{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}} := \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$$

of 2×2-matrices.

5.5. **Octonion algebras.** By an *octonion algebra* over k we mean a composition algebra of rank 8. Octonion algebras are alternative but no longer associative. The construction of octonion algebras is not quite as straightforward as in the two previous cases.

5.6. **The hermitian vector product.** Let E be a quadratic étale k-algebra and (M, h) a *ternary hermitian space* over E, so M is a finitely generated projective *right* E-module of rank 3, $h\colon M \times M \to E$ is a hermitian form (anti-linear in the first, linear in the second variable), and the assignment $x \mapsto h(x, -)$ determines an E-module *iso*morphism from M to M^*, the ι_E-twisted E-dual of M. Suppose further we are given an *orientation*[1] of (M, h), i.e., an isomorphism

$$\Delta\colon \bigwedge^3(M, h) \xrightarrow{\sim} \big(E, (a, b) \mapsto \bar{a}b\big)$$

[1]This terminology has been adopted from [8].

of unary hermitian spaces, which may not exist but if it does is unique up to a factor of norm 1 in E. Then the equation

$$h(x \times_{h,\Delta} y, z) = \Delta(x \wedge y \wedge z)$$

defines a bi-additive alternating map $(x, y) \mapsto x \times_{h,\Delta} y$ from $M \times M$ to M that is anti-linear in each variable and is called the *hermitian vector product* induced by (M, h) and the orientation Δ.

5.7. **Theorem.** ([104]) *With the notation and assumptions of* 5.6, *the k-module $E \oplus M$ becomes an octonion algebra*

$$C := \mathrm{Zor}(E, M, h, \Delta) \tag{1}$$

over k under the multiplication

$$(a \oplus x)(b \oplus y) = \big(ab - h(x, y)\big) \oplus \big(y\bar{a} + xb + x \times_{h,\Delta} y\big)$$

whose unit element, norm, trace, conjugation are respectively given by

$$\begin{aligned} 1_C &= 1_E \oplus 0, \\ n_C(a \oplus x) &= n_E(a) + h(x, x), \\ t_C(a \oplus x) &= t_E(a), \\ \overline{a \oplus x} &= \bar{a} \oplus (-x) \end{aligned}$$

for all $a \in E$ and all $x \in M$. Conversely, every octonion algebra over k containing E as a composition subalgebra arises in this manner. □

Remark. The significance of this result derives from the fact that, if k is a semi-local ring or $2 \in \mathrm{Jac}(k)$, the Jacobson radical of k (e.g., if $2 = 0$ in k), then every composition algebra of rank > 1 is easily seen to contain a quadratic étale subalgebra, while in general this need not be so ([43, 17, 16, 15]).

5.8. **Zorn vector matrices.** Applying Thm. 5.7 to the special case that $E := E^0 = k \oplus k$ as in 5.3, the E-module $M := E^3$ is free of rank 3 and the hermitian form h is given by $h(x, y) := \bar{x}^t y$, the canonical identification $\bigwedge^3(M, h) = (E, (a, b) \mapsto \bar{a}b)$ yields an orientation Δ of (M, h) such that the octonion k-algebra C of (5.7.1) identifies canonically with the k-module

$$\mathrm{Zor}(k) := \begin{pmatrix} k & k^3 \\ k^3 & k \end{pmatrix}$$

under the multiplication

$$\begin{pmatrix} \alpha & u' \\ u & \alpha' \end{pmatrix} \begin{pmatrix} \beta & v' \\ v & \beta' \end{pmatrix} := \begin{pmatrix} \alpha\beta - u'^t v & \alpha v' + \beta' u' + u \times v \\ \beta u + \alpha' v + u' \times v' & -u^t v' + \alpha' \beta' \end{pmatrix}$$

for all $\alpha, \alpha', \beta, \beta' \in k$ and all $u, u', v, v' \in k^3$. We speak of the *algebra of Zorn vector matrices* over k in this context. Unit element, norm, trace and conjugation of $C := \mathrm{Zor}(k)$ are given by

$$1_C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad n_C(x) = \alpha\alpha' + u'^t u, \quad t_C(x) = \alpha + \alpha', \quad \bar{x} = \begin{pmatrix} \alpha' & -u' \\ -u & \alpha \end{pmatrix}$$

for

$$x = \begin{pmatrix} \alpha & u' \\ u & \alpha' \end{pmatrix} \in C.$$

In particular, the norm of C is hyperbolic.

5.9. **Splitness.** A composition algebra C of rank r over k is said to be *split* if one of the following conditions holds.

(i) $r = 1$.
(ii) $r = 2$ and $C \cong E^0 := k \oplus k$ as in 5.3.
(iii) $r = 4$ and $C \cong \mathrm{Mat}_2(k)$ as in 5.4.
(iv) $r = 8$ and $C \cong \mathrm{Zor}(k)$ as in 5.8.

There is an important generalization of this concept, called *generic splitness*, that will be discussed in 5.16 below.

5.10. **Reminder: faithful flatness and étale algebras.** Recall that a k-module M is said to be *flat* if the functor $- \otimes M$ preserves exact sequences; it is said to be *faithfully flat* provided a sequence of k-modules is exact if and only if it becomes so after applying the functor $- \otimes M$.

A k-algebra $E \in k\text{-}\mathbf{alg}$ is said to be *finitely presented* if there exist a positive integer n and a short exact sequence

$$0 \longrightarrow I \longrightarrow k[\mathbf{t}_1, \ldots, \mathbf{t}_n] \longrightarrow E \longrightarrow 0 ,$$

of k-algebras, where $\mathbf{t}_1, \ldots, \mathbf{t}_n$ are independent variables and $I \subseteq k[\mathbf{t}_1, \ldots, \mathbf{t}_n]$ is a *finitely generated* ideal. E is said to be *étale* if it is finitely presented and satisfies the following equivalent conditions [33].

(i) For all $k' \in k\text{-}\mathbf{alg}$ and all ideals $I \subseteq k'$ satisfying $I^2 = \{0\}$, the natural map

$$\mathrm{Hom}_{k\text{-}\mathbf{alg}}(E, k') \longrightarrow \mathrm{Hom}_{k\text{-}\mathbf{alg}}(E, k'/I)$$

is bijective.

(ii) E is flat over k and, for all $\mathfrak{p} \in \mathrm{Spec}(k)$, the extended algebra $E \otimes \kappa(\mathfrak{p})$ over $\kappa(\mathfrak{p})$, the quotient field of $k/\mathfrak{p}$, is a (possibly infinite) direct product of finite separable extension fields of k.

In particular, quadratic étale k-algebras (cf. 5.3) are étale in this sense, as are cubic étale ones (cf. 10.9 below).

5.11. **Theorem.** ([48]) *Let C be a k-algebra. For C to be an octonion algebra over k it is necessary and sufficient that there exist a faithfully flat étale k-algebra R such that $C_R \cong \mathrm{Zor}(R)$ is a split octonion algebra over R.* $\square$

5.12. **Composition algebras over fields.** If $k = F$ is a field, composition algebras in general, and octonions in particular, behave in an especially nice way.

(a) Composition algebras over F are either split quadratic étale or they are simple algebras in the sense that they contain only the trivial ideals. In fact, the quaternion algebras over F are precisely the finite-dimensional central simple associative algebras of degree 2 while, by a result of Kleinfeld [40], octonion algebras over fields are the only (unital) simple alternative rings that are not associative.

(b) Composition algebras over fields are classified by their norms, so if we are given two composition algebras C, C' over F, then

$$C \cong C' \Longleftrightarrow n_C \cong n_{C'}.$$

(c) As an application of (b), it follows easily that *for any composition algebra C of dimension at least 2 over F, the following conditions are equivalent.*

(i) *C is split.*
(ii) *C has zero divisors, so some $x \neq 0 \neq y$ in C have $xy = 0$.*
(iii) *The norm of C is isotropic.*
(iv) *The norm of C is hyperbolic.*

It is a natural question to ask which of the results assembled in 5.12 can be (fully or partially) generalized from fields to arbitrary commutative rings. For example, the trivial fact that quadratic étale algebras are always classified by their norms has a natural (but non-trivial) extension to quaternion algebras.

5.13. **Theorem.** ([41]) *Quaternion algebras over commutative rings are classified by their norms.* □

Trying to accomplish the same for octonion algebras turns out to be much more delicate, as the results discussed in the remainder of this section will attest[2]. In order to describe these results in more detail, a short digression will be necessary.

5.14. **Isotopes of alternative algebras.** Let A be a unital alternative algebra over k and suppose $p, q \in A$ are invertible. Write $A^{(p,q)}$ for the non-associative k-algebra living on the k-module A under the multiplication

$$x \cdot_{p,q} y := (xp)(qy) \qquad (x, y \in A). \tag{1}$$

McCrimmon [53] has shown that $A^{(p,q)}$ is again a unital alternative algebra over k, called the (p, q)-*isotope* of A, with identity element $1^{(p,q)} = 1_{A^{(p,q)}} = (pq)^{-1}$.

For example, if C is a composition algebra, then so is $C^{(p,q)}$, for all invertible elements $p, q \in C$; in fact, one checks easily using (1) that $n_{C^{(p,q)}} = n_C(pq)n_C$. It follows that the left multiplication operator $L_{pq}\colon C^{(p,q)} \to C$, $x \mapsto (pq)x$, is an isometry from $n_{C^{(p,q)}}$ to n_C preserving units but almost never an isomorphism. [53] also contains examples showing that isotopes of alternative algebras are in general not isomorphic. But these examples display pathologies in characteristic 3, for which alternative algebras are notorious, and it has long been suspected that examples avoiding such pathologies do not exist. This suspicion, however, was confounded by the following remarkable results of Gille and Alsaody-Gille, respectively.

5.15. **Theorem.** (a) ([31]) *There exist non-isomorphic octonion algebras over appropriate commutative base rings whose norms are isometric.*

(b) ([7]) *Two octonion algebras C and C' over k have isometric norms if and only if C' is isomorphic to an isotope of C.* □

Remark. Part (a) of this theorem says that octonion algebras over rings are *not* classified by their norms. In part (b), sufficiency is trivial, as has been indicated in 5.14. Combining (a) and (b), we see that there exist octonion algebras over appropriate base rings which are isotopic but not isomorphic. In fact, the examples exhibited in [31] are the co-ordinate algebras of some affine varieties over a field of arbitrary characteristic.

Other striking results on octonion algebras over rings have recently been obtained by Asok-Hoyois-Wendt [8]. They are concerned with the following generalization of splitness.

5.16. **Generic splitness.** Let k be an integral domain with quotient field K. A composition algebra C over k is said to be *generically split* if C_K, the base change of C from k to K, is split. For example, if C is a *Zorn algebra*, i.e., has the form (5.7.1) with $E \cong k \oplus k$ split quadratic étale, then C is clearly generically split, by 5.12 (c). The converse, however, does not hold. In fact, [8] establishes a number of far-reaching connections of octonion algebras with algebraic homotopy theory and then proceeds, basically in the authors' own words, to

(i) study classification of generically split octonion algebras over schemes,
(ii) analyze when generically split octonion algebras may be realized as Zorn algebras,
(iii) study when generically split octonion algebras are determined by their norm forms.

[2] I am grateful to Erhard Neher for having brought these results to my attention.

6. Cubic norm structures

With the introduction of cubic norm structures, we perform the last step in paving the way for the definition of Albert algebras. They require a small preparation of their own.

6.1. **Polynomial laws.** For the time being we are working over a field F. Given finite-dimensional vector spaces V, W over F, with bases $v_1, \dots, v_n, w_1, \dots, w_m$, respectively, any chain of polynomials $p_1, \dots, p_m \in F[\mathbf{t}_1, \dots, \mathbf{t}_n]$ defines a family of set maps $f_R \colon V_R \to W_R$, one for each $R \in k$-**alg**, given by

$$f_R(\sum_{j=1}^{n} r_j v_{jR}) := \sum_{i=1}^{m} p_i(r_1, \dots, r_n) w_{iR} \qquad (r_1, \dots, r_n \in R), \tag{1}$$

It is clear that the family $f := (f_R)_{R \in k\text{-}\mathbf{alg}}$ determines the polynomials p_i uniquely. By abuse of language, we speak of f as a *polynomial map* from V to W, written as $f \colon V \to W$. This notion is obviously independent of the bases chosen. Moreover, it is readily checked that the set maps $f_R \colon V_R \to W_R$ defined by (1) vary functorially (in the obvious sense, cf. (2) below) with $R \in k$-**alg**. This key property of polynomial maps is the starting point of the theory of polynomial laws due to Roby [90]; for an alternate approach, see [21].

Returning to our base ring k, we associate with any k-module M a (covariant) functor $\mathbf{M} : k\text{-}\mathbf{alg} \to \mathbf{set}$ (where **set** stands for the category of sets) by setting $\mathbf{M}(R) = M_R$ as sets for $R \in k$-**alg** and $\mathbf{M}(\varphi) = \mathbf{1}_M \otimes \varphi : M_R \to M_S$ as set maps for morphisms $\varphi : R \to S$ in k-**alg**. We then define a *polynomial law f from M to N* (over k) as a natural transformation $f : \mathbf{M} \to \mathbf{N}$. This means that, for all $R \in k$-**alg**, we are given set maps $f_R : M_R \to N_R$ varying functorially with R, so whenever $\varphi : R \to S$ is a morphism in k-**alg**, we obtain a commutative diagram

$$\begin{tikzcd} M_R \arrow[r, "f_R"] \arrow[d, "\mathbf{1}_M \otimes \varphi"'] & N_R \arrow[d, "\mathbf{1}_N \otimes \varphi"] \\ M_S \arrow[r, "f_S"'] & N_S. \end{tikzcd} \tag{2}$$

A polynomial law from M to N will be symbolized by $f \colon M \to N$, in spite of the fact that it is *not* a map from M to N in the usual sense. But it induces one, namely $f_k : M \to N$, which, however, need not determine f uniquely. On the other hand, the standard differential calculus for polynomial maps (cf., e.g., [11] or [36]) carries over to polynomial laws virtually without change.

Polynomial laws from M to k are said to be *scalar*. The totality of scalar polynomial laws on M is a unital commutative associative k-algebra, denoted by $\mathrm{Pol}_k(M)$ and isomorphic to the polynomial ring $k[\mathbf{t}_1, \dots, \mathbf{t}_n]$ if M is a free k-module of rank n.

A polynomial law $f \colon M \to N$ is said to be *homogeneous of degree d* if $f_R(rx) = r^d f_R(x)$ for all $R \in k$-**alg**, $r \in R$, $x \in M_R$. Homogeneous polynomial laws of degree 1 (resp. 2) identify canonically with linear (resp. quadratic) maps in the usual sense. Scalar homogeneous polynomial laws are called *forms*. We speak of linear, quadratic, cubic, quartic, ... forms instead of forms of degree $d = 1, 2, 3, 4, \dots$.

6.2. **The concept of a cubic norm structure.** Combining the approach of [51] with the terminology of [78], we define a *cubic norm structure* over k as a k-module X together with

(i) a distinguished element $1 = 1_X \in X$ (the *base point*), which we will assume to be *unimodular* in the sense that $\lambda(1_X) = 1_k$ for some linear form λ on X, equivalently, the submodule $k1_X \subseteq X$ is free of rank 1 and a direct summand,

(ii) a quadratic map $\sharp = \sharp_X \colon X \to X$, $x \mapsto x^\sharp$ (the *adjoint*),

(iii) a cubic form $N = N_X \colon X \to k$ (the *norm*), such that the following identities hold in all scalar extensions.

$$N(1) = 1, \quad 1^\sharp = 1 \qquad \text{(base point identities)}, \tag{1}$$

$$x^{\sharp\sharp} = N(x)x \qquad \text{(adjoint identity)}, \tag{2}$$

$$(\partial_y N)(x) = (DN)(x)y = T(x^\sharp, y) \qquad \text{(gradient identity)}, \tag{3}$$

$$1 \times x = T(x)1 - x \qquad \text{(unit identity)}. \tag{4}$$

Here $x \times y = (x+y)^\sharp - x^\sharp - y^\sharp$ is the bilinearization of the adjoint, and $T = T_X\colon\ X \times X \to k$ is the *bilinear trace*, i.e., up to a sign the logarithmic Hessian of N at 1,

$$T(y,z) = -(D^2 \log N)(1)(y,z) = (\partial_y N)(1)(\partial_z N)(1) - (\partial_y \partial_z N)(1), \tag{5}$$

giving rise to the *linear trace* $T_X(x) = T(x) = T(x,1)$. Defining

$$S := S_X\colon\ X \longrightarrow k, \quad x \longmapsto S(x) := T(x^\sharp), \tag{6}$$

we obtain a quadratic form, called the *quadratic trace* of X.

It is clear that cubic norm structures are stable under base change.

6.3. **Theorem.** ([51]) *With the notation and assumptions of* 6.2, *the unit element* $1_J := 1_X$ *and the* U*-operator defined by*

$$U_x y := T(x,y)x - x^\sharp \times y \tag{1}$$

for all $x, y \in X$ *give the* k*-module* X *the structure of a Jordan algebra* $J = J(X)$ *such that the relations*

$$x^3 - T(x)x^2 + S(x)x - N(x)1_J = 0 = x^4 - T(x)x^3 + S(x)x^2 - N(x)x, \tag{2}$$

$$x^\sharp = x^2 - T(x)x + S(x)1_J \tag{3}$$

hold in all scalar extensions. Moreover, N *is unital and permits Jordan composition,*

$$N(1_J) = 1, \quad N(U_x y) = N(x)^2 N(y) \tag{4}$$

in all scalar extensions. Finally, an arbitrary element $x \in X$ *is invertible in* J *if and only if* $N(x) \in k$ *is invertible in* k*, in which case*

$$x^{-1} = N(x)^{-1} x^\sharp. \tag{5}$$

□

Remark. (a) The preceding construction is clearly compatible with arbitrary base change. (b) One is tempted to multiply the first equation of (2) by x in order to derive the second. But this is allowed only in linear Jordan algebras, i.e., in the presence of $\frac{1}{2}$ (in which case the second equation is indeed a consequence of the first) but not in general.

6.4. **Cubic Jordan algebras.** By a *cubic Jordan algebra* over k we mean a Jordan k-algebra J together with a cubic form $N_J\colon\ J \to k$ (the *norm*) such that there exists a cubic norm structure X with $J = J(X)$ and $N_J = N_X$. In this case, we call $T_J := T_X$ the (*bi-*)*linear trace* and $S_J := S_X$ the *quadratic trace* of J. Cubic Jordan algebras are clearly invariant under base change. In the sequel, we rarely distinguish carefully between a cubic norm structure and its associated cubic Jordan algebra.

A cubic Jordan algebra J over k is said to be *non-singular* if it is finitely generated projective as a k-module and the bilinear trace $T_J\colon\ J \times J \to k$ is non-singular as a symmetric bilinear form, so it induces an isomorphism from the k-module J onto its dual in the usual way.

6.5. **Examples.** We will encounter many more examples later on. For the time being, we settle with the following simple cases.

(a) Consider the k-module $X := \mathrm{Mat}_3(k)$, equipped with the identity matrix $\mathbf{1}_3$ as base point, the usual adjoint as adjoint, and the determinant as norm. Then X is a cubic norm structure satisfying $J(X) = \mathrm{Mat}_3(k)^+$. *Thus* $\mathrm{Mat}_3(k)^+$ *together with the determinant is a cubic Jordan algebra over* k.

(b) Let $\mathcal{Q} = (M, q, e)$ be a pointed quadratic form over k. Then the k-module $X := k \oplus M$ together with the base point $1 := 1_k \oplus e$, the adjoint and the norm respectively given by

$$(\alpha \oplus u)^\sharp := q(u) \oplus (\alpha \bar{u}), \quad N_X(\alpha \oplus u) := \alpha q(u)$$

in all scalar extensions is a cubic norm structure over k which satisfies $J(X) = k \oplus J(\mathcal{Q})$ as a direct sum of ideals.

Cubic Jordan algebras are invariant under isotopy:

6.6. **Theorem.** ([51]) *With the notation and assumptions of* 6.2, *let* $p \in J(X)^\times$. *Then the new*

base point $1^{(p)} := p^{-1}$,
adjoint $x^{\sharp^{(p)}} := N(p)U_{p^{-1}}x^\sharp$,
norm $N^{(p)}(x) := N(p)N(x)$

make the k-module X into a new cubic norm structure, denoted by $X^{(p)}$ and called the p-isotope *of X. Moreover, $J(X^{(p)}) = J(X)^{(p)}$ is the p-isotope of $J(X)$.* □

7. First properties of Albert algebras

We are finally in a position to define Albert algebras.

7.1. **The concept of an Albert algebra.** By an *Albert algebra* over k we mean a cubic Jordan k-algebra J satisfying the following conditions.

(a) J is finitely generated projective of rank 27 as a k-module.
(b) For all fields $F \in k$-**alg**, the extended cubic Jordan algebra J_F over the field F is simple.

Albert algebras are clearly stable under base change: if J is an Albert algebra over k, then J_R is one over R, for all $R \in k$-**alg**. Moreover, over fields of characteristic not 2, Albert algebras in the sense of 2.7 are the same as the ones in the sense of 7.1. This follows immediately from Thm. 2.8 combined with Cor. 7.10 (c) below.

Before stating a few elementary properties of Albert algebras, we require a small notational digression into polynomial laws.

7.2. **Polynomials over the ring of scalar polynomial laws.** Let J be a Jordan k-algebra and $\mathbf{t}$ a variable. An element $\mathbf{p}(\mathbf{t}) \in \mathrm{Pol}_k(J)[\mathbf{t}]$ has the form

$$\mathbf{p}(\mathbf{t}) = \sum_{i=0}^{d} f_i \mathbf{t}^i \qquad (f_i \in \mathrm{Pol}_k(J),\ 0 \le i \le d).$$

Given $x \in J_R$, $R \in k$-**alg**, we may thus form the "ordinary" polynomial

$$\mathbf{p}(\mathbf{t}; x) := \sum_{i=0}^{d} (f_i)_R(x)\mathbf{t}^i \in R[\mathbf{t}],$$

in which we may replace the variable $\mathbf{t}$ by x:

$$\mathbf{p}(x; x) := \sum_{i=0}^{d} (f_i)_R(x)x^i \in J_R.$$

7.3. **First properties of Albert algebras.** (a) Albert algebras are *generically algebraic of degree* 3 in the sense of [46]: if J is an Albert algebra over k, then the polynomial

$$\mathbf{m}_J(\mathbf{t}) := \mathbf{t}^3 - T_J\mathbf{t}^2 + S_J\mathbf{t} - N_J \in \mathrm{Pol}_k(J)[\mathbf{t}]$$

is the unique monic polynomial of least degree satisfying $\mathbf{m}_J(x;x) = (\mathbf{t}\mathbf{m}_J)(x;x) = 0$ for all $x \in J_R$, $R \in k$-**alg**. In particular, the cubic norm structure belonging to J (cf. 6.4) is uniquely determined by the Jordan algebra structure of J alone.

(b) Homomorphisms between Albert algebras are isomorphisms and automatically preserve norms, adjoints and traces.

(c) Isotopes of Albert algebras are Albert algebras.

Before deriving further properties of Albert algebras, we give a few important examples.

7.4. **Twisted hermitian matrices.** Let C be a composition algebra over k, with norm n_C, trace t_C and conjugation ι_C, $u \mapsto \bar{u}$. One can show that $1_C \in C$ is unimodular, so we obtain a natural identification $k = k1_C \subseteq C$ that is compatible with base change. Now suppose

$$\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \gamma_3) \in \mathrm{GL}_3(k)$$

is an invertible diagonal matrix. Then the map

$$\mathrm{Mat}_3(C) \longrightarrow \mathrm{Mat}_3(C), \quad x \longmapsto \Gamma^{-1}\bar{x}^t\Gamma,$$

is an involution, called the Γ-*twisted conjugate transpose involution*. The elements of $\mathrm{Mat}_3(C)$ that remain fixed under this involution (i.e., are Γ-hermitian) *and have diagonal entries in* k form a k-submodule of $\mathrm{Mat}_3(C)$ that is finitely generated projective of rank $3(\mathrm{rk}(C)+1)$ and will be denoted by

$$\mathrm{Her}_3(C, \Gamma).$$

In particular, we put

$$\mathrm{Her}_3(C) := \mathrm{Her}_3(C, \mathbf{1}_3).$$

Note that for $\frac{1}{2} \in k$, the condition of the diagonal entries being scalars is automatic. Writing e_{ij} for the ordinary matrix units, there is a natural set of generators for the k-module $\mathrm{Her}_3(C, \Gamma)$ furnished by the quantities

$$u[jl] := \gamma_l u e_{jl} + \gamma_j \bar{u} e_{lj}$$

for $u \in C$ and $j, l = 1, 2, 3$ distinct. Indeed, a straightforward verification shows that $x \in \mathrm{Mat}_3(C)$ belongs to $\mathrm{Her}_3(C, \Gamma)$ if and only if it can be written in the form (necessarily unique)

$$x = \sum \alpha_i e_{ii} + \sum u_i[jl], \qquad (\alpha_i \in k,\ u_i \in C,\ i = 1, 2, 3), \tag{1}$$

where we systematically subscribe to the convention that summations like the ones on the right of (1) extend over all cyclic permutations (ijl) of (123), i.e., over $(123), (231), (312)$.

7.5. **Theorem.** ([51]) *With the notation and assumptions of* 7.4, *base point, adjoint and norm given respectively by the formulas*

$$1 = \sum e_{ii} \tag{1}$$

$$x^\sharp = \sum \big(\alpha_j\alpha_l - \gamma_j\gamma_l n_C(u_i)\big)e_{ii} + \sum (\gamma_i \overline{u_j u_l} - \alpha_i u_i)[jl], \tag{2}$$

$$N(x) = \alpha_1\alpha_2\alpha_3 - \sum \gamma_j\gamma_l\alpha_i n_C(u_i) + \gamma_1\gamma_2\gamma_3 t_C(u_1u_2u_3) \tag{3}$$

in all scalar extensions[3] *convert the k-module* $\mathrm{Her}_3(C, \Gamma)$ *into a cubic norm structure over k whose bilinear trace has the form*

$$T(x, y) = \sum \alpha_i\beta_i + \sum \gamma_j\gamma_l n_C(u_i, v_i) \tag{4}$$

[3]Note that the last term of (3) by 5.2 (f) is unambiguous.

for x as in (7.4.1) and $y = \sum \beta_i e_{ii} + \sum v_i[jl] \in \mathrm{Her}_3(C,\Gamma)$, $\beta_i \in k$, $v_i \in C$, $i = 1,2,3$. □

7.6. **Remark.** (a) The cubic norm structure (or cubic Jordan algebra) constructed in Theorem 7.5 will also be denoted by $\mathrm{Her}_3(C,\Gamma)$. The Jordan circle product of $x, y \in \mathrm{Her}_3(C,\Gamma)$ may be expressed as $x \circ y = xy + yx$ in terms of ordinary matrix multiplication. In particular, over fields of characteristic not 2, the Jordan structure of $\mathrm{Her}_3(C,\Gamma)$ for C an octonion algebra and $\Gamma = \mathbf{1}_3$ as described here is compatible with the one of 2.5.

(b) *Multiplying Γ with an invertible scalar, or its diagonal entries with invertible squares, doesn't change the isomorphism class of* $\mathrm{Her}_3(C,\Gamma)$. More precisely, given $\delta, \delta_i \in k^\times$, $1 \le i \le 3$, and setting

$$\Gamma' := \mathrm{diag}(\delta_1^2\gamma_1, \delta_2^2\gamma_2, \delta_3^2\gamma_3),$$

the assignments

$$\begin{aligned} \sum \alpha_i e_{ii} + \sum u_i[jl] &\longmapsto \sum \alpha_i e_{ii} + \sum (\delta u_i)[jl], \\ \sum \alpha_i e_{ii} + \sum u_i[jl] &\longmapsto \sum \alpha_i e_{ii} + \sum (\delta_j \delta_l u_i)[jl] \end{aligned}$$

give isomorphisms

$$\mathrm{Her}_3(C,\delta\Gamma) \xrightarrow{\sim} \mathrm{Her}_3(C,\Gamma), \quad \mathrm{Her}_3(C,\Gamma') \xrightarrow{\sim} \mathrm{Her}_3(C,\Gamma).$$

Using the first isomorphism, we see that *we may always assume* $\gamma_1 = 1$, while combining it with the second, we see that *we may always assume* $\det(\Gamma) = 1$.

(c) *The Jordan algebra* $\mathrm{Her}_3(C,\Gamma)$ *is isotopic to the Jordan algebra* $\mathrm{Her}_3(C)$. More precisely,

$$p := \sum \gamma_i e_{ii} \in \mathrm{Her}_3(C,\Gamma)^\times,$$

and the assignment

$$\sum \alpha_i e_{ii} + \sum u_i[jl] \longmapsto \sum (\gamma_i \alpha_i) e_{ii} + \sum (\gamma_j \gamma_l u_i)[jl]$$

determines an isomorphism from the isotope $\mathrm{Her}_3(C,\Gamma)^{(p)}$ onto $\mathrm{Her}_3(C)$.

(d) If the composition algebra C is associative, i.e., has rank at most 4, then $\mathrm{Her}_3(C,\Gamma)$ is a subalgebra of $\mathrm{Mat}_3(C)^+$, so its U-operator has the form $U_x y = xyx$ in terms of the ordinary matrix product. In particular, $\mathrm{Her}_3(C,\Gamma)$ *is a special Jordan algebra.* The case of an octonion algebra will be stated separately.

7.7. **Corollary.** *Let C be an octonion algebra over k and $\Gamma \in \mathrm{GL}_3(k)$ a diagonal matrix. Then* $\mathrm{Her}_3(C,\Gamma)$ *is an Albert algebra over k.*

Proof. As a k-module, $J := \mathrm{Her}_3(C,\Gamma)$ is finitely generated projective of rank $3(8+1) = 27$. Moreover, J is a cubic Jordan algebra which is simple if k is a field ([52]). □

7.8. **Reduced cubic Jordan algebras.** A cubic Jordan algebra J over k is said to be *reduced* if it isomorphic to $\mathrm{Her}_3(C,\Gamma)$ for some composition k-algebra C and some diagonal matrix $\Gamma \in \mathrm{GL}_3(k)$[4]. Thus, reduced *Albert* algebras have this form with C an *octonion* algebra. We speak of *split cubic Jordan algebras* (resp. of *split Albert algebras*) over k if they have the form $\mathrm{Her}_3(C)$, C a split composition algebra, (resp. $\mathrm{Her}_3(\mathrm{Zor}(k))$).

7.9. **Theorem.** *For J to be an Albert algebra over k it is necessary and sufficient that J be a Jordan k-algebra and there exist a faithfully flat étale k-algebra R such that $J_R \cong \mathrm{Her}_3(\mathrm{Zor}(R))$ is a split Albert algebra over R.*

Instead of a proof. The condition is clearly sufficient. To prove necessity, it will be enough, by Theorem 5.11, to find a faithfully flat étale k-algebra R making J_R a reduced Albert algebra over R. This can be accomplished by standard arguments, similar to the ones employed in the proof of the aforementioned theorem.

[4]This is an ad-hoc definition. For a more intrinsic notion see, e.g., [36].

Remark. It would be desirable to establish the existence of a faithfully flat étale k-algebra R as above in a single step, without recourse to Theorem 5.11. Conceivably, this could be accomplished by appealing to Neher's theory of grids [61].

7.10. **Corollary.** (a) *Albert algebras are non-singular cubic Jordan algebras. Moreover, their quadratic traces are separable quadratic forms.*

(b) *Let $\varphi\colon J \to J'$ be a linear bijection of non-singular cubic Jordan algebras that preserves norms and units. Then φ is an isomorphism of the underlying cubic norm structures, hence of cubic Jordan algebras as well.*

(c) *Albert algebras are exceptional Jordan algebras.*

Proof. (a) By Theorem 7.9, it suffices to prove the first assertion for split Albert algebras, where it follows immediately from (7.5.4). The assertion about the quadratic trace is established similarly.

(b) Since φ preserves norms and units, it preserves traces as well. Setting $N := N_J$, $N' := N_{J'}$, ditto for the traces, and applying the chain rule to $N' \circ \varphi = N$, we obtain $(DN')(\varphi(x))\varphi(y) = (DN)(x)y$, and the gradient identity yields

$$T'\big(\varphi(x)^\sharp, \varphi(y)\big) = T(x^\sharp, y) = T'\big(\varphi(x^\sharp), \varphi(y)\big).$$

Since T' is non-singular, we conclude that φ preserves adjoints, hence is an isomorphism of the underlying cubic norm structures. But the U-operator is built up from adjoints and traces, by (6.3.1). Hence φ preserves U-operators and therefore is an isomorphism of Jordan algebras.

(c) By Theorem 7.9 it suffices to prove this for the split Albert algebra. This in turn follows from the fact that *Glennie's identity*

$$\begin{aligned} G_9(X, Y, Z) &:= G(X, Y, Z) - G(Y, X, Z) = 0, \\ G(X, Y, Z) &:= U_X Z \circ U_{X,Y} U_Z Y^2 - U_X U_Z U_{X,Y} U_Y Z \end{aligned} \tag{1}$$

holds in all special Jordan algebras but not in $\mathrm{Her}_3(\mathrm{Zor}(k))$ [38]. *Thus Albert algebras are not even homomorphic images of special Jordan algebras.* In the presence of $\frac{1}{2}$, this can also be proved more directly by following Albert's original arguments in [1]. $\square$

This is the appropriate place to remind the reader of one of the most fundamental contributions of Zelmanov to the theory of Jordan algebras without finiteness conditions.

7.11. **Theorem.** ([56]) *A simple Jordan algebra is either special or an Albert algebra over some field.* $\square$

8. Reduced cubic Jordan algebras over fields

In this section, we will be concerned with a class of cubic Jordan algebras that are particularly well understood. For example, we will see that reduced cubic Jordan algebras over fields have a nice set of classifying invariants with natural interpretations in terms of Galois cohomology. In most cases, the references given below were originally confined to reduced Albert algebras only but, basically without change, allow an immediate extension to arbitrary cubic Jordan algebras.

Throughout this section, we fix a field F and two reduced cubic Jordan algebras J, J' over F, written in the form $J \cong \mathrm{Her}_3(C, \Gamma)$, $J' \cong \mathrm{Her}_3(C', \Gamma')$ for some composition F-algebras C, C' and some diagonal matrices

$$\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \gamma_3), \quad \Gamma' = \mathrm{diag}(\gamma'_1, \gamma'_2, \gamma'_3)$$

belonging to $\mathrm{GL}_3(F)$. Our first result says that *the elements of* Γ *may be multiplied by invertible norms of* C *without changing the isomorphism class of* J. More precisely, the following statement holds.

8.1. **Proposition.** *Given invertible elements $a_2, a_3 \in C$, we have*

$$J \cong \mathrm{Her}_3(C, \Gamma_1), \quad \Gamma_1 := \mathrm{diag}\,\big(n_C(a_2a_3)^{-1}\gamma_1, n_C(a_2)\gamma_2, n_C(a_3)\gamma_3\big).$$

In particular, $J \cong \mathrm{Her}_3(C)$ if C is split of dimension at least 2.

Instead of a proof. The proof of the first part rests on the fact that composition algebras over fields are classified by their norms (5.12 (b)), hence does not carry over to arbitrary base rings (Thm. 5.15 (a)). The second part follows from the fact that we may assume $\det(\Gamma) = 1$ (7.6 (b)) and that the norm of a split composition algebra over F having dimension at least 2 is hyperbolic (5.12 (c)), hence universal. □

8.2. **Theorem.** ([5, 19, 37]) *The following conditions are equivalent.*

(i) *J and J' are isotopic.*
(ii) *There exists a norm similarity from J to J', i.e., a linear bijection $\varphi\colon J \to J'$ satisfying $N_{J'} \circ \varphi = \alpha N_J$ for some $\alpha \in k^\times$.*
(iii) *C and C' are isomorphic.*

Instead of a proof. (iii) ⇒ (i) follows from 7.6 (c), (i) ⇔ (ii) from Cor. 7.10 (b). The hard part is the implication (i) ⇒ (iii). □

8.3. **The co-ordinate algebra.** By Theorem 8.2, the composition algebra C up to isomorphism is uniquely determined by J. We call C the *co-ordinate* (or *coefficient*) *algebra* of J.

8.4. **Nilpotent elements.** As usual, an element of a Jordan algebra is said to be *nilpotent* if and only if some power with positive integer exponent is equal to zero. For reasons that will be explained in Remark 14.3 below, cubic Jordan algebras containing non-zero nilpotent elements are said to be *isotropic*. There is a simple criterion to detect isotropy in a reduced cubic Jordan algebra over F.

8.5. **Theorem.** ([5]) *J is isotropic if and only if*

$$J \cong \mathrm{Her}_3(C, \Gamma_{\mathrm{nil}}), \quad \Gamma_{\mathrm{nil}} := \mathrm{diag}(1, -1, 1).$$

□

Criteria for isomorphism between J and J' are more delicate. The first one to deal with this question in characteristic not 2 was Springer [101], who considered the quadratic form $x \mapsto T(x^2)$ in terms of the linear trace T. Working in arbitrary characteristic, the key idea, due to Racine [88], consists in looking at the quadratic trace S, $x \mapsto T(x^\sharp)$. The connection between the two may be described as follows.

8.6. **Remark.** (a) The quadratic form $Q(x) := T(x^2)$ bilinearizes to $Q(x, y) = 2T(x, y)$, which shows that Springer's approach mentioned above does not succeed in characteristic two.

(b) Applying the linear trace to (7.5.2), it follows that

$$S_J = [-1] \oplus \mathbf{h} \oplus \langle -1\rangle \,.\, Q_J, \quad Q_J := \langle \gamma_2\gamma_3, \gamma_3\gamma_1, \gamma_1\gamma_2\rangle \otimes n_C, \tag{1}$$

where $[-1]$ stands the one-dimensional quadratic from $-\alpha^2$ and $\mathbf{h}$ for the hyperbolic plane, while $\langle \alpha_1, \ldots, \alpha_n\rangle$ for $\alpha_1, \ldots, \alpha_n \in F$ refers to the symmetric bilinear form over F given on $F^n \times F^n$ by the matrix $\mathrm{diag}(\alpha_1, \ldots, \alpha_n)$. Hence S_J determines Q_J uniquely and conversely. In particular, Q_J is an invariant of J. On the other hand, (7.5.4) and (1) imply that the bilinear trace of J has the form

$$T_J = \langle 1, 1, 1\rangle \oplus \partial Q_J \tag{2}$$

Thus, if $\mathrm{char}(F) \neq 2$, the bilinear trace and the quadratic trace of J are basically the same.

8.7. **Theorem.** ([42, 88, 95, 101]) *The following conditions are equivalent.*

(i) *J and J' are isomorphic.*
(ii) *J and J' have isomorphic co-ordinate algebras and isometric quadratic traces.*
(iii) *J and J' have isometric quadratic traces.*

Instead of a proof. (i) $\Rightarrow$ (ii) follows immediately from Thm. 8.2.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (ii). Suppose $Q_J \cong Q_{J'}$. It is easy to see that Q_J is hyperbolic if and only if C is split, forcing the 3-Pfister forms n_C and $n_{C'}$ to have the same splitting fields. Hence they are isometric[5] [18, Corollary 23.6] (see also [35, Theorem 4.2], [22]). □

8.8. **Invariants of reduced cubic Jordan algebras.** We now assume $\dim_F(C) = 2^r$, $r = 1, 2, 3$, so the case $C = k$ is ruled out. Since composition algebras over F are classified by their norms, it follows from Theorem 8.7 combined with Remark 8.6 that reduced cubic Jordan algebras $J \cong \mathrm{Her}_3(C, \Gamma)$ as above, where we may assume $\gamma_1 = 1$ by Remark 7.6 (b), have the quadratic n-Pfister forms ($n = r, r + 2$)

$$f_r(J) = n_C, \quad f_{r+2}(J) = n_C \oplus Q_J \cong \langle\langle -\gamma_2, -\gamma_3 \rangle\rangle \otimes n_C \tag{1}$$

as classifying invariants. Here a quadratic form Q over F is said to be *n-Pfister* if it can be written in the form

$$Q \cong \langle\langle \alpha_1, \dots, \alpha_{n-1} \rangle\rangle \otimes n_E := \langle\langle \alpha_1 \rangle\rangle \otimes \big(\dots (\langle\langle \alpha_{n-1} \rangle\rangle \otimes n_E) \dots \big) \tag{2}$$

for some $\alpha_1, \dots, \alpha_{n-1} \in F^\times$ and some quadratic étale F-algebra E, where $\langle\langle \alpha \rangle\rangle := \langle 1, -\alpha \rangle$ as binary symmetric bilinear forms. For basic properties of Pfister quadratic forms in arbitrary characteristic, see [18].

In particular, assuming $\mathrm{char}(F) \neq 2$ for simplicity, we deduce from [18, 16.2] (see also [86] for more details and for the history of the subject[6]) that n-Pfister quadratic forms Q as in (2) have the cup product

$$(\alpha_1) \cup \dots \cup (\alpha_{n-1}) \cup [E] \in H^n(F, \mathbf{Z}/2\mathbf{Z}) \tag{3}$$

as a classifying (Galois) cohomological invariant, where (α) stands for the canonical image of $\alpha \in F^\times$ in $H^1(F, \mathbf{Z}/2\mathbf{Z})$ and $[E] \in H^1(F, \mathbf{Z}/2\mathbf{Z})$ for the cohomology class of the quadratic étale F-algebra E (see 13.6 below for more details). From this we conclude that the classifying invariants of reduced cubic Jordan algebras may be identified with

$$f_r(J) \in H^r(F, \mathbf{Z}/2\mathbf{Z}), \quad f_{r+2}(J) \in H^{r+2}(F, \mathbf{Z}/2\mathbf{Z}); \tag{4}$$

they are called the *invariants mod* 2 of J.

9. The first Tits construction

The two Tits constructions provide us with a powerful tool to study Albert algebras that are not reduced. In the present section and the next, we describe an approach to these constructions that is inspired by the Cayley-Dickson construction of composition algebras.

9.1. **The internal Cayley-Dickson construction.** Let C be a composition algebra with norm n_C over k. Suppose $B \subseteq C$ is a unital subalgebra and $l \in C$ is perpendicular to B relative to ∂n_C, so $n_C(B, l) = \{0\}$. Setting $\mu := -n_C(l)$, it is then easily checked that *the multiplication rule*

$$(u_1 + v_1 l)(u_2 + v_2 l) = (u_1 u_2 + \mu \bar{v}_2 v_1) + (v_2 u_1 + v_1 \bar{u}_2) l \tag{1}$$

holds for all $u_i, v_i \in B$, $i = 1, 2$. In particular, $B + Bl$ is the subalgebra of C generated by B and l.

[5] I am grateful to Skip Garibaldi for having pointed out this fact as well as the subsequent references to me.

[6] I am grateful to Detlev Hoffmann for having drawn my attention to this article, but also for many illuminating comments.

9.2. **The external Cayley-Dickson construction.** Abstracting from the preceding set-up, particularly from (9.1.1), we now consider an *associative* composition k-algebra B with norm n_B, trace t_B, conjugation ι_B, and an arbitrary scalar $\mu \in k^\times$. Then *the direct sum $C := B \oplus Bj$ of two copies of B as a k-module becomes a composition algebra $C := \mathrm{Cay}(B, \mu)$ over k under the multiplication*

(1) $$(u_1 + v_1 j)(u_2 + v_2 j) = (u_1 u_2 + \mu \bar{v}_2 v_1) + (v_2 u_1 + v_1 \bar{u}_2)j \quad (u_i, v_i \in B, i = 1, 2),$$

with norm, polarized norm, trace, conjugation given by

(2) $$n_C(u + vj) = n_B(u) - \mu n_B(v),$$

(3) $$n_C(u_1 + v_1 j, u_2 + v_2 j) = n_B(u_1, v_1) - \mu n_B(u_2, v_2),$$

(4) $$t_C(u + vj) = t_B(u),$$

(5) $$\overline{u + vj} = \bar{u} - vj$$

for all $u, u_1, u_2, v, v_1, v_2 \in B$. We say C arises from B, μ by means of the *Cayley-Dickson construction.* For example, if E is a quadratic étale k-algebra, the Cayley-Dickson construction can be performed twice starting from E, so with scalars $\mu_1, \mu_2 \in k^\times$, we obtain in

$$\mathrm{Cay}(E; \mu_1, \mu_2) := \mathrm{Cay}\big(\mathrm{Cay}(E, \mu_1), \mu_2\big)$$

an octonion algebra over k.

The usefulness of the Cayley-Dickson construction is underscored by the

9.3. **Embedding property of composition algebras.** Suppose we are given

(a) a composition algebra C over k (any commutative ring), with norm n_C,
(b) a composition subalgebra $B \subseteq C$
(c) an invertible element $l \in C$ perpendicular to B relative to ∂n_C.

Then B is associative and the inclusion $B \hookrightarrow C$ has a unique extension to a homomorphism

$$\mathrm{Cay}\big(B, -n_C(l)\big) = B \oplus Bj \longrightarrow C$$

sending j to l. Moreover, this homomorphism is an *iso*morphism if $\mathrm{rk}(B) = \frac{1}{2}\,\mathrm{rk}(C)$.

We wish to extend the preceding approach to the level of cubic norm structures and their associated Jordan algebras. In doing so, the role of associative composition algebras used to initiate the Cayley-Dickson construction will be played, somewhat surprisingly, by

9.4. **Cubic alternative algebras.** By a *cubic alternative algebra* over k we mean a unital alternative k-algebra A together with a cubic form $N_A \colon A \to k$ (the *norm*) satisfying the following conditions.

(a) $1_A \in A$ is unimodular.
(b) N_A is unital and permits composition: $N(1_A) = 1$ and the relation

$$N(xy) = N(x)N(y)$$

holds in all scalar extensions.
(c) Defining the (*linear*) *trace* $T_A \colon A \to k$ and the *quadratic trace* $S_A \colon A \to k$ by

$$T_A(x) := (\partial_x N_A)(1_A), \quad S_A(x) := (\partial_{1_A} N_A)(x),$$

the relation

$$x^3 - T_A(x)x^2 + S_A(x)x - N_A(x)1_A = 0$$

holds in all scalar extensions.

Given a cubic alternative k-algebra A, its linear trace turns out to be an associative linear form, so we have

(1) $$T_A\big((xy)z\big) = T_A\big(x(yz)\big) =: T_A(xyz)$$

for all $x, y, z \in A$. Moreover, if we define the *adjoint* of A as the quadratic map

$$\sharp_A \colon A \longrightarrow A, \quad x \longmapsto x^\sharp := x^2 - T_A(x)x + S_A(x)1_A.$$

then the k-module $X := A$ together with the base point $1_X := 1_A$, the adjoint $\sharp_X := \sharp_A$ and the norm $N_X := N_A$ is a cubic norm structure over k, denoted by $X = X(A)$, with linear trace $T_X = T_A$, quadratic trace $S_X := S_A$ and bilinear trace given by $T_X(x, y) = T_A(xy)$ for all $x, y \in A$. Moreover, the associated cubic Jordan algebra is $J(X) = A^+$. We call $X(A)$ the cubic norm structure *associated with* A.

9.5. **Pure elements.** Let X be a cubic norm structure over k with norm $N = N_X$, trace $T = T_X$ and suppose $X_0 \subseteq X$ is a *non-singular* cubic sub-norm structure, so X_0 is a non-singular cubic norm structure in its own right, with norm $N_0 = N_{X_0}$, trace $T_0 = T_{X_0}$, and the inclusion $X_0 \hookrightarrow X$ is a homomorphism of cubic norm structures. Then the orthogonal decomposition

$$X = X_0 \oplus V, \quad V := X_0^\perp,$$

with respect to the bilinear trace of X comes equipped with two additional structural ingredients: there is a canonical bilinear action

$$X_0 \times V \longrightarrow V, \quad (x_0, v) \longmapsto x_0 \,.\, v := -x_0 \times v,$$

and there are quadratic maps $Q \colon V \to X_0$, $H \colon V \to V$ given by

$$v^\sharp = -Q(v) + H(v) \qquad\qquad (v \in V).$$

With these ingredients, an element $l \in X$ is said to be *pure* relative X_0 if

(i) $l \in V$ is invertible in $J(X)$ (equivalently, $N(l) \in k^\times$),
(ii) $l^\sharp \in V$ (equivalently, $Q(l) = 0$),
(iii) $X_0 \,.\, (X_0 \,.\, l) \subseteq X_0 \,.\, l$.

If this is so, we can give the k-module X_0 the structure of a well defined non-associative k-algebra $A_X(X_0, l)$ whose bilinear multiplication x_0y_0 is uniquely determined by the formula

$$(x_0y_0) \,.\, l := x_0 \,.\, (y_0 \,.\, l)$$

for all $x_0, y_0 \in X_0$.

9.6. **Theorem.** (The internal first Tits construction) *With the notation and assumptions of* 9.5, $A := A_X(X_0, l)$ *together with* $N_A := N_0$ *is a cubic alternative k-algebra satisfying* $X(A) = X_0$, *hence* $A^+ = J(X_0)$. *Moreover, with* $\mu := N(l) \in k$, *the relations*

$$N(x_0 + x_1 \,.\, l + x_2 \,.\, l^\sharp) = N_A(x_0) + \mu N_A(x_1) + \mu^2 N_A(x_2) - \mu T_A(x_0x_1x_2), \tag{1}$$

$$(x_0 + x_1 \,.\, l + x_2 \,.\, l^\sharp)^\sharp = (x_0^\sharp - \mu x_1x_2) + (\mu x_2^\sharp - x_0x_1) \,.\, l + (x_1^\sharp - x_2x_0) \,.\, l^\sharp \tag{2}$$

hold in all scalar extensions. □

9.7. **Theorem.** (The external first Tits construction) ([20, 51, 78]) *Let A be a cubic alternative k-algebra with norm N_A, trace T_A, write $X_0 = X(A)$ for the associated cubic norm structure and suppose $\mu \in k$ is an arbitrary scalar. Then the threefold direct sum*

$$X := \mathfrak{T}_1(A, \mu) := A \oplus Aj_1 \oplus Aj_2$$

of A as a k-module is a cubic norm structure with base point, adjoint, norm and bilinear trace given by

$$1_X := 1_A + 0 \cdot j_1 + 0 \cdot j_2, \tag{1}$$

$$x^\sharp := (x_0^\sharp - \mu x_1x_2) + (\mu x_2^\sharp - x_0x_1)j_1 + (x_1^\sharp - x_2x_0)j_2, \tag{2}$$

$$N_X(x) := N_A(x_0) + \mu N_A(x_1) + \mu^2 N_A(x_2) - \mu T_A(x_0x_1x_2), \tag{3}$$

$$T_X(x, y) = T_0(x_0, y_0) + \mu T_0(x_1, y_2) + \mu T(x_2, y_1) \tag{4}$$

for all $x = x_0 + x_1j_1 + x_2j_2, y = y_0 + y_1j_1 + y_2j_2$, $x_i, y_i \in A_R$, $i = 0, 1, 2$, $R \in k$-**alg**. *We say* $\mathfrak{T}_1(A, \mu)$ *arises from* A, μ *by means of the* first Tits construction. □

Remarks. (a) We write $J(A, \mu) := J(\mathfrak{T}_1(A, \mu))$ for the cubic Jordan algebra associated with $\mathfrak{T}_1(A, \mu)$.
(b) Identifying A in $\mathfrak{T}_1(A, \mu)$ through the initial summand makes $X(A) \subseteq X = \mathfrak{T}_1(A, \mu)$ a cubic sub-norm structure. We note that $j_1 \in X$ is pure relative to $X(A)$ if and only if $\mu \in k^\times$. In this case, $A_X(X(A), j_1) = A$ as cubic alternative algebras, and if A (i.e., $X(A)$) is non-singular, then so is $\mathfrak{T}_1(A, \mu)$.

Among the three conditions defining the notion of a pure element in 9.5, the last one seems to be the most delicate. It is therefore important to note that, under certain restrictions concerning the linear algebra of the situation, it turns out to be superfluous.

9.8. **Theorem.** (Embedding property of the first Tits construction) *Let X be a cubic norm structure over k and suppose X is finitely generated projective of rank at most $3n$, $n \in \mathbb{N}$, as a k-module. Suppose further that $X_0 \subseteq X$ is a non-singular cubic sub-norm structure of rank exactly n. For an invertible element $l \in J(X)$ to be pure relative to X_0 it is necessary and sufficient that both l and $l^\sharp$ be orthogonal to X_0. In this case, setting $\mu = N_X(l)$, there exists a unique homomorphism from the first Tits construction $\mathfrak{T}_1(A_X(X_0, l), \mu)$ to X extending the identity of X_0 and sending j_1 to l. Moreover, this homomorphism is an isomorphism, and the cubic norm structure X is non-singular of rank exactly $3n$.* $\square$

Remark. Already the first part of the preceding result, let alone the second, becomes false without the rank condition, even if the base ring is a field.

9.9. **Examples of cubic alternative algebras.** Besides cubic *associative* algebras, like cubic étale algebras (see 10.9 below) or Azumaya algebras of degree 3, examples of cubic *properly* alternative algebras arise naturally as follows. Letting C be a composition algebra over k with norm n_C, we put $A := ke_1 \oplus C$ as a direct sum of ideals, where $ke_1 \cong k$ is a copy of k as a k-algebra, and define the norm $N_A\colon A \to k$ as a cubic form by the formula

$$N_A(re_1 + u) = rn_C(u) \qquad (r \in R,\ u \in C_R,\ R \in k\text{-}\mathbf{alg}).$$

Then A is a cubic alternative algebra with norm N_A over k which is not associative if and only if C is an octonion algebra over k.

One may ask what will happen when this cubic alternative algebra enters into the first Tits construction. Here is the answer.

9.10. **Theorem.** *With the notation and assumptions of* 9.9, *let $\mu \in k^\times$. Then*

$$J(A, \mu) \cong \mathrm{Her}_3(C, \Gamma_{\mathrm{nil}}), \quad \Gamma_{\mathrm{nil}} = \mathrm{diag}(1, -1, 1).$$

In particular, if $k = F$ is a field and C is an octonion algebra over F, then $J(A, \mu)$ is an isotropic Albert algebra over F. $\square$

10. The second Tits construction

The fact that cubic alternative (rather than associative) algebras belong to the natural habitat of the first Tits construction gives rise to a remarkable twist when dealing with the second. We describe this twist in two steps.

10.1. **Unital isotopes of cubic alternative algebras.** Let A be a cubic alternative algebra over k. Isotopes of A in the sense of 5.14 having the special form

$$A^p := A^{(p^{-1}, p)} \tag{1}$$

for some invertible element $p \in A$ are called *unital*. Note that $A^p = A$ as k-modules, while the multiplication of A^p is given by $x \cdot^p y := (xp^{-1})(py)$ and its norm $N_{A^p} = N_A$ agrees with that of A. The construction has the following properties, for all $p, q \in A^\times$.

- $1_{A^p} = 1_A$,
- $(A^p)^+ = A^+$,

- $A^{\alpha p} = A^p$ for all $\alpha \in k^\times$,
- $(A^p)^q = A^{pq}$,
- $A^p = A$ if A is associative.

10.2. **Isotopy involutions.** By a *cubic alternative k-algebra with isotopy involution of the r-th kind* $(r = 1, 2)$ we mean a quadruple $\mathcal{B} = (E, B, \tau, p)$ consisting of

(i) a composition algebra E of rank r over k, called the *core* of $\mathcal{B}$, with conjugation ι_E, $a \mapsto \bar{a}$, so either $E = k$ or E is a quadratic étale k-algebra,
(ii) a cubic alternative E-algebra B,
(iii) an invertible element $p \in B^\times$,
(iv) an ι_E-semi-linear homomorphism $\tau : B \to (B^p)^{\mathrm{op}}$ of unital alternative algebras satisfying the relations
$$\tau(p) = p, \quad \tau^2 = \mathbf{1}_B, \quad N_B \circ \tau = \iota_E \circ N_B.$$

By 10.1 and (iv), $\tau\colon B^+ \to B^+$ is a semi-linear involutorial automorphism, forcing $H(\mathcal{B}) := H(B, \tau) = \{x \in B \mid \tau(x) = x\} \subseteq X(B)$ to be a cubic norm structure over k such that $H(\mathcal{B}) \otimes E \cong X(B)$. Homomorphisms of cubic alternative algebras with isotopy involution of the r-th kind are defined in the obvious way. A scalar $\mu \in E$ is said to be *admissible* relative to $\mathcal{B}$ if $N_B(p) = \mu\bar{\mu}$. Condition (iv) implies in particular first $\tau(xy) = (\tau(y)p^{-1})(p\tau(x))$ and then

$$xp\tau(x) := x\big(p\tau(x)\big) \in H(\mathcal{B}) \tag{1}$$

but NOT $(xp)\tau(x) \in H(\mathcal{B})$. Note that admissible scalars relative to $\mathcal{B}$ are related not only to p but, via p, also to τ. Note also by 10.1 that, if $p \in k1_B$ is a scalar, then isotopy involutions are just ordinary involutions.

We say $\mathcal{B}$ is *non-singular* if B is so as a cubic alternative algebra over E, equivalently, $H(\mathcal{B})$ is so as a cubic norm structure over k. If the core of $\mathcal{B}$ agrees with the centre of B (as an alternative ring), $\mathcal{B}$ is said to be *central*. Finally, $\mathcal{B}$ is said to be *division* if B is an alternative division algebra, so all non-zero elements of B are invertible.

10.3. **Remark.** Let $\mathcal{B} = (E, B, \tau, p)$ be a cubic alternative algebra with isotopy involution of the r-th kind $(r = 1, 2)$ and suppose B is associative, in which case we say $\mathcal{B}$ is *associative*. Then 10.1 implies $B^p = B$, and $\tau\colon B \to B$ is an ordinary involution of the r-th kind (also called a *unitary involution* for $r = 2$). Thus the parameter p in $\mathcal{B}$ can be safely ignored, allowing us to relax the notation to $\mathcal{B} = (E, B, \tau)$ for cubic associative algebras with involution of the r-th kind. In accordance with the terminology of [78], we then speak of (p, μ) as an *admissible scalar* for $\mathcal{B}$ if $p \in H(\mathcal{B})^\times$ and $\mu \in E^\times$ satisfy $N_B(p) = \mu\bar{\mu}$.

10.4. **Theorem.** (The external second Tits construction) *Let $\mathcal{B} = (E, B, \tau, p)$ be a cubic alternative k-algebra with isotopy involution of the r-th kind $(r = 1, 2)$ and suppose $\mu \in E$ is an admissible scalar relative to $\mathcal{B}$. With the convention of* (10.2.1), *the direct sum*

$$X := \mathfrak{T}_2(\mathcal{B}, \mu) := H(\mathcal{B}) \oplus Bj \tag{1}$$

of $H(\mathcal{B})$ and B as k-modules is a cubic norm structure over k with base point, adjoint, norm and bilinear trace respectively given by

$$1_X := 1_B + 0 \cdot j, \tag{2}$$

$$(x_0 + uj)^\sharp := \big(x_0^\sharp - up\tau(u)\big) + \big(\bar{\mu}\tau(u)^\sharp p^{-1} - x_0 u\big)j, \tag{3}$$

$$N_X(x_0 + uj) := N_B(x_0) + \mu N_B(u) + \bar{\mu}\overline{N_B(u)} - T_B\Big(x_0\big(up\tau(u)\big)\Big), \tag{4}$$

$$T_X(x_0 + uj, y_0 + vj) = T_B(x_0, y_0) + T_B\big(up\tau(v)\big) + T_B\big(vp\tau(u)\big) \tag{5}$$

*for all $x_0, y_0 \in H(\mathcal{B}_R)$, $u, v \in B_R$, $R \in k$-**alg**. We say $\mathfrak{T}_2(\mathcal{B}, \mu)$ arises from $\mathcal{B}, \mu$ by means of the* second Tits construction. □

Remarks. (a) We write $J(\mathcal{B}, \mu) := J(\mathfrak{T}_2(\mathcal{B}, \mu))$ for the cubic Jordan algebra associated with $\mathfrak{T}_2(\mathcal{B}, \mu)$.

(b) Identifying $H(\mathcal{B}) \subseteq \mathfrak{T}_2(\mathcal{B}, \mu)$ through the initial summand makes $X(H(\mathcal{B})) \subseteq \mathfrak{T}_2(\mathcal{B}, \mu)$ cubic sub-norm structure.

There is also an internal second Tits construction; it rests on the notion of

10.5. **Étale elements.** Returning to the set-up of 9.5, an element $w \in X$ is said to be *étale* relative to X_0 if it satisfies the conditions

(1) $$w \in V, \quad Q(w) \in J(X_0)^\times, \quad N_X(w)^2 - 4N_{X_0}\big(Q(w)\big) \in k^\times.$$

This implies that

(2) $$E_w := k[\mathbf{t}]/\Big(\mathbf{t}^2 - N_X(w)\mathbf{t} + N_{X_0}\big(Q(w)\big)\Big)$$

is a quadratic étale k-algebra (hence the name) that is generated by an invertible element.

Remark. Suppose $\mathcal{B}$ is non-singular and $r = 2$ in 10.4, so we are dealing with isotopy involutions of the second kind. Then $j \in \mathfrak{T}_2(\mathcal{B}, \mu)$ as in (10.4.1) is an étale element relative to $H(\mathcal{B})$ if and only if E is generated by μ as a k-algebra.

10.6. **Theorem.** (The internal second Tits construction) *With the notation and assumptions of* 10.5, *suppose X_0 has rank n, X is finitely generated projective of rank at most $3n$ as a k-module and $w \in X$ is étale relative to X_0. Then there are a cubic alternative k-algebra $\mathcal{B}$ with isotopy involution of the second kind as in* 10.2 *satisfying $E = E_w$, an admissible scalar $\mu \in E$ relative to $\mathcal{B}$ and an isomorphism*

$$\mathfrak{T}_2(\mathcal{B}, \mu) \xrightarrow{\sim} X$$

sending $H(\mathcal{B})$ to X_0 and j to w.

Instead of a proof. The proof consists in

- changing scalars from k to E, making $X_{0E} \subseteq X_E$ a non-singular cubic sub-norm structure,
- using w to exhibit a pure element l of X_E relative to X_{0E},
- applying Theorem 9.8 to give X_{0E} the structure of a cubic alternative E-algebra A and to identify X_E with the first Tits construction $\mathfrak{T}_1(A, \mu)$, for some $\mu \in E^\times$,
- arriving at the desired conclusion by the method of étale descent. □

10.7. **Examples: core split isotopy involutions of the second kind.** Let A be a cubic alternative k-algebra and $q \in A^\times$.

(a) One checks that

$$\mathcal{B} := (k \oplus k, A \oplus A^{\mathrm{op}}, q \oplus q, \varepsilon_A),$$

ε_A being the switch on $A \oplus A^{\mathrm{op}}$, is a cubic alternative k-algebra with isotopy involution of the second kind whose core splits. Conversely, all cubic alternative algebras with core split isotopy involution of the second kind are easily seen to be of this form.

(b) The admissible scalars relative to $\mathcal{B}$ as in (a) are precisely the elements $\mu = \lambda \oplus \lambda^{-1}N_A(q)$ with $\lambda \in k^\times$, and there is a natural isomorphism

$$\mathfrak{T}_2(\mathcal{B}, \mu) \xrightarrow{\sim} \mathfrak{T}_1(A, \lambda).$$

It follows that

- first Tits constructions are always second Tits constructions,
- second Tits constructions become first Tits constructions after an appropriate quadratic étale extension.

10.8. **Remark.** For an Albert algebra J over k to arise from the first or second Tits construction, it is obviously necessary that J contain a non-singular cubic Jordan subalgebra of rank 9. In general, such subalgebras do not exist [64, 63]. Vladimir Chernousov (oral communication during the workshop) has raised the question of whether their existence is related to the existence of tori (with appropriate properties) in the group scheme of type F_4 corresponding to J.

10.9. **Examples: cubic étale algebras.** Let L be a *cubic étale* k-algebra, so L is a non-singular cubic commutative associative k-algebra that has rank 3 as a finitely generated projective k-module. Suppose further we are given a quadratic étale k-algebra E. Then

$$L * E := (E, L \otimes E, \mathbf{1}_L \otimes \iota_E)$$

is a cubic étale k-algebra with involution of the second kind.

As yet, I do not have complete results on the classification of isotopy involutions, but it should certainly help to note that they are basically the same as isotopes of ordinary involutions. The precise meaning of this statement may be read off from the following proposition and its corollary.

10.10. **Proposition.** *If $\mathcal{B} = (E, B, \tau, p)$ is a cubic alternative algebra with isotopy involution of the r-th kind over k, then so is*

$$\mathcal{B}^q := (E, B^q, \tau^q, p^q), \quad \tau^q(x) := q^{-1}\tau(qx), \quad p^q := pq$$

for every $q \in H(\mathcal{B})^\times$. We call $\mathcal{B}^q$ the q-isotope *of $\mathcal{B}$ and have*

$$H(\mathcal{B}^q) = H(\mathcal{B})q.$$

Moreover,

$$(\mathcal{B}^q)^{q'} = \mathcal{B}^{qq'} \qquad\qquad (q' \in H(\mathcal{B}^q)^\times).$$

□

10.11. **Corollary.** *With $q := p^{-1}$, the map τ^q is an ordinary involution of the r-th kind on B^q, and $\tau = (\tau^q)^p$ is the p-isotope of τ^q.* □

Isotopes of isotopy involutions also relate naturally to isotopes of second Tits constructions.

10.12. **Proposition.** (cf. [78]) *Let $\mathcal{B} = (E, B, \tau, p)$ be a cubic alternative algebra with isotopy involution of the r-th kind over k and $q \in H(\mathcal{B})^\times$. If $\mu \in E$ is an admissible scalar relative to $\mathcal{B}$, then $N_B(q)^{-1}\mu$ is an admissible scalar relative to $\mathcal{B}^q$ and the map*

$$\mathfrak{T}_2(\mathcal{B}^q, N_B(q)^{-1}\mu) \xrightarrow{\sim} \mathfrak{T}_2(\mathcal{B}, \mu)^{(q^{-1})}, \quad x_0 + uj \longmapsto qx_0 + uj$$

is an isomorphism of cubic norm structures. □

11. Searching for étale elements

The value of the results derived in the preceding section, particularly of Theorem 10.6, hinges on existence criteria for étale elements. Here we are able to guarantee good results only if the base ring is a field. The following important observation, however, holds in full generality.

11.1. **Theorem.** *Let E be a quadratic étale k-algebra, (M, h) a ternary hermitian space over E and Δ an orientation of (M, h) (cf. 5.6). Suppose further we are given a diagonal matrix $\Gamma \in \mathrm{GL}_3(k)$. Then*

$$J := \mathrm{Her}_3(C, \Gamma),$$

with $C := \mathrm{Zor}(E, M, h, \Delta)$ (cf. (5.7.1)) is a reduced Albert algebra over k containing $J_0 := \mathrm{Her}_3(E, \Gamma)$ as a non-singular cubic Jordan subalgebra, and the following conditions are equivalent.

(i) *J contains étale elements relative to J_0.*
(ii) *M is free (of rank 3) as an E-module and $E = k[a]$ for some invertible element $a \in E$.* □

Remark. It will not always be possible to generate a quadratic étale algebra by an invertible element, even if the base ring is a field. In that case, the sole counter example is $E = k \oplus k$ over $k = \mathbb{F}_2$.

Combining Theorem 11.1 with a Zariski density argument and the fact that finite-dimensional absolutely simple Jordan algebras over a finite field are reduced, we obtain

11.2. **Corollary.** *Let F be a field, J an Albert algebra over F and $J_0 \subseteq J$ an absolutely simple nine-dimensional subalgebra. Then precisely one of the following holds.*

(a) *J contains étale elements relative to J_0.*

(b) *$J_0 \cong \mathrm{Mat}_3(F)^+$ and $F = \mathbb{F}_2$.*

Moreover, if $J_0 \cong \mathrm{Mat}_3(F)^+$, then J contains pure elements relative to J_0. $\square$

11.3. **Corollary.** ([52], [77]) *Let J be an Albert algebra over a field F and suppose $J_0 \subseteq J$ is an absolutely simple nine-dimensional subalgebra. Then there exist a non-singular cubic associative F-algebra $\mathcal{B}$ with involution of the second kind as well as an admissible scalar (p, μ) relative to $\mathcal{B}$ such that $J \cong J(\mathcal{B}, p, \mu)$ under an isomorphism matching J_0 with $H(\mathcal{B})$.* $\square$

Treating the analogous situation on the nine-dimensional level, we obtain:

11.4. **Proposition.** *Let E be a quadratic étale k-algebra, $\Gamma \in \mathrm{GL}_3(k)$ a diagonal matrix and $J := \mathrm{Her}_3(E, \Gamma)$ the corresponding reduced cubic Jordan algebra over k. Then the diagonal matrices $L := \mathrm{Diag}_3(k)$ form a cubic étale subalgebra of J, and the following conditions are equivalent.*

(i) *J contains étale elements relative to L.*

(ii) *$E = k[a]$ for some invertible element $a \in E$.* $\square$

But since cubic étale algebras over finite fields need not be split, the analogue of Corollary 11.2 does not hold on the nine-dimensional level, while the corresponding analogue of Corollary 11.3, though true, is more difficult to ascertain if the base field is finite. In order to formulate this result, we require two preparations.

11.5. **The product of quadratic étale algebras.** Quadratic étale algebras over a field F are classified by $H^1(F, \mathbf{Z}/2\mathbf{Z})$ (see 13.6 below). Since the latter carries a natural abelian group structure, so do the isomorphism classes of the former. Explicitly, if E, E' are two quadratic étale F-algebras, so is

$$E \cdot E' := H(E \otimes E', \iota_E \otimes \iota_{E'}),$$

and the composition $(E, E') \mapsto E \cdot E'$ of quadratic étale F-algebras corresponds to the additive group structure of $H^1(F, \mathbf{Z}/2\mathbf{Z})$.

11.6. **The discriminant of a cubic étale algebra.** Let L be a cubic étale algebra over a field F. Then precisely one of the following holds.

(a) L is *reduced*: $L \cong F \oplus K$, for some quadratic étale F-algebra K, necessarily unique. We say that L is *split* if K is.

(b) L/F is a separable cubic field extension, which is either cyclic or has the separable cubic field extension L_K/K cyclic for some separable quadratic field extension K/F, again necessarily unique.

We call the quadratic étale F-algebra

$$\mathrm{Disc}(L) := \begin{cases} K & \text{if } L \text{ is as in (a)}, \\ F \oplus F & \text{if } L/F \text{ is a cyclic cubic field extension}, \\ K & \text{if } L/F \text{ is a separable non-cyclic cubic field extension} \\ & \text{and } K \text{ is as in (b)}, \end{cases}$$

the *discriminant* of L. This terminology is justified by the fact that, if $\mathrm{char}(F) \neq 2$, then $\mathrm{Disc}(L) = F(\sqrt{d})$, where d is the ordinary discriminant of the minimum polynomial of some generator of L over F. But notice $d = 1$ for $\mathrm{char}(F) = 2$.

11.7. **Theorem.** ([42, 75, 85]) *Let $\mathcal{B} = (E, B, \tau)$ be a central simple cubic associative algebra with involution of the second kind over a field F and suppose $L \subseteq H(\mathcal{B})$ is a cubic étale subalgebra. Then, using the notation of* 10.9, 11.6, *there exists an admissible scalar (p, μ) relative to $L * (E \cdot \mathrm{Disc}(L))$ such that the inclusion $L \hookrightarrow H(\mathcal{B})$ extends to an isomorphism*

$$J\Big(L * \big(E \cdot \mathrm{Disc}(L)\big), p, \mu\Big) \xrightarrow{\sim} H(\mathcal{B}).$$

□

12. Cubic Jordan division algebras

Cubic Jordan algebras over fields have recently been shown to parametrize Tits hexagons [60]. In the present section, we deal with the special case of cubic Jordan division algebras, which are basically the same as hexagonal systems in the sense of [107]; *Albert division algebras*, i.e., Albert algebras that are Jordan division algebras at the same time, form a particularly important subclass. Fixing a cubic Jordan algebra J, with norm N, adjoint $x \mapsto x^\sharp$, trace T, quadratic trace S, over an arbitrary field F throughout this section, we begin with a digression into the nil radical.

12.1. **The nil radical and separability.** The *nil radical* of J is defined as the unique ideal $\mathrm{Nil}(J) \subseteq J$ that is maximal with respect to the property of containing only nilpotent elements (cf. 8.4). By [76, Thm. 3.6] we have

$$\mathrm{Nil}(J) = \big\{x \in J \mid N(x) = 0,\ \ T(x, J) = T(x^\sharp, J) = \{0\}\big\}. \tag{1}$$

J is said to be *semi-simple* (resp. *separable*) if $\mathrm{Nil}(J) = \{0\}$ (resp. the base change J_K is semi-simple for all field extensions K/F). By (1), non-singular cubic Jordan algebras are always separable.

12.2. **Theorem.** ([88]) *The following conditions are equivalent.*

- (i) *J is separable.*
- (ii) *One of the following holds.*
  - (a) *There exists a separable pointed quadratic form $\mathcal{Q}$ over F such that J is isomorphic to the cubic Jordan algebra $F \oplus J(\mathcal{Q})$ of* 6.5 (b).
  - (b) *J is reduced: there exist a composition algebra C over F and a diagonal matrix $\Gamma \in \mathrm{GL}_3(F)$ such that $J \cong \mathrm{Her}_3(C, \Gamma)$.*
  - (c) *J is a separable Jordan division algebra.*

□

Remark. With the obvious adjustments, Racine's theorem holds (and was phrased as such) under slightly more general conditions, replacing separability by the absence of absolute zero divisors: $U_x = 0 \Rightarrow x = 0$.

12.3. **Corollary.** *An absolutely simple cubic Jordan algebra over F is either reduced or a division algebra.* □

One advantage of working with cubic Jordan algebras over fields is that subspaces stabilized by the adjoint automatically become cubic Jordan subalgebras. This aspect must be borne in mind in the first of the following technicalities but also later on.

12.4. **Lemma.** ([12]) *Let J be a cubic Jordan algebra over F. Then the cubic Jordan subalgebra $J' \subseteq J$ generated by elements $x, y \in J$ is spanned by*

$$1, x, x^\sharp, y, y^\sharp, x \times y, x^\sharp \times y, x \times y^\sharp, x^\sharp \times y^\sharp$$

as a vector space over F. In particular, $\dim_F(J') \leq 9$.

For a proof of this result, see also [73, Prop. 6.6].

12.5. **Lemma.** *Let $\mathcal{B} = (E, B, \tau, p)$ be a cubic alternative F-algebra with isotopy involution of the r-th kind ($r = 1, 2$) and suppose $\mu \in E$ is an admissible scalar. Then the following conditions are equivalent.*

(i) *The second Tits construction $J(\mathcal{B}, \mu)$ is a Jordan division algebra.*
(ii) *$H(\mathcal{B})$ is a Jordan division algebra and $\mu \notin N_B(B^\times)$.*

□

12.6. **Lemma.** *A cubic Jordan division algebra over F is either non-singular or a purely inseparable field extension of characteristic 3 and exponent at most 1.* □

12.7. **Theorem.** (Enumeration of cubic Jordan division algebras) *A Jordan F-algebra $J \neq F$ is a cubic Jordan division algebra if and only if one of the following conditions holds.*

(a) *J/F is a purely inseparable field extension of characteristic 3 and exponent 1.*
(b) *J/F is a separable cubic field extension.*
(c) *$J \cong D^+$ for some central associative division algebra D of degree 3 over F.*
(d) *$J \cong H(E, D, \tau)$, for some central associative division algebra (E, D, τ) of degree 3 over F with involution of the second kind.*
(e) *$J \cong J(D, \mu)$ for some central associative division algebra D of degree 3 over F and some scalar $\mu \in F^\times \setminus N_D(D^\times)$.*
(f) *$J \cong J(\mathcal{D}, p, \mu)$ for some central cubic associative division algebra $\mathcal{D} = (E, D, \tau)$ with involution of the second kind over F and some admissible scalar (p, μ) relative to $\mathcal{D}$ with $\mu \notin N_D(D^\times)$.*

In cases (e),(f), *J is an Albert division algebra, and conversely.*

Sketch of proof. Let J be a cubic Jordan division algebra over F. If J is singular, we are in case (a), by Lemma 12.6, so we may assume J is non-singular. If $\dim_F(J) \leq 3$, we are clearly in case (b), so we may assume $\dim_F(J) > 3$. Then, by the theorem of Chevalley (cf. [44]), the base field is infinite, and Theorem 12.2 combined with a descent argument shows that J is absolutely simple of dimension $6, 9, 15$, or 27.

First suppose $\dim_F(J) = 6$. Since J is non-singular, it contains a separable cubic subfield, from which it is easily seen to arise by means of the second Tits construction in such a way that condition (ii) of Lemma 12.5 is violated. Hence J cannot be a division algebra. This contradiction shows that J has dimension at least 9.

If $\dim_F(J) = 9$, Thm. 12.2 implies that J is an F-form of $\mathrm{Her}_3(F \oplus F) \cong A^+$, $A := \mathrm{Mat}_3(F)$, and since the $\mathbf{Z}$-automorphisms of A^+ are either automorphisms or anti-automorphisms of A, a Galois descent argument shows that we are in cases (c),(d), so we may assume $\dim_F(J) > 9$. Any separable cubic subfield $E \subsetneq J$ together with an element $y \in J \setminus E$ by Lemma 12.4 and by what we have seen already generates a separable subalgebra $J' \subseteq J$ of dimension 9. Hence Prop. 12.8 below shows $\dim_F(J) = 27$, so J is an Albert algebra, by Cor. 11.3 and Lemma 12.5 necessarily of the form described in (e),(f). □

12.8. **Proposition.** *Let $J' \subset J$ be a proper separable division subalgebra of finite dimension n. Then $\dim_F(J) \geq 2n$.*

Proof. Since J' is non-singular by Lemma 12.6, we obtain an orthogonal decomposition $J = J' \oplus J'^\perp$ relative to the bilinear trace. The formalism of 9.5 yields a bilinear action $J' \times J'^\perp \to J'^\perp$, $(x, v) \mapsto x.v$. We have $J'^\perp \neq \{0\}$, and fixing $0 \neq v \in J'^\perp$, obtain an induced linear map $J' \to J'^{\perp}$, $x \mapsto x.v$, which is easily seen to be injective since J' is a division algebra. This proves $\dim_F(J'^{\perp}) \geq n$, and the assertion follows. $\square$

12.9. **Historical remarks.** It took the Jordan community a considerable while to realize that Albert division algebras do indeed exist, see, e.g., [94] (resp. [87]) for some explicit (resp. implicit) details on this topic. The first examples of Albert division algebras were constructed by Albert [3], who investigated them further in [4]. After Springer [102] (see also Springer-Veldkamp [103]) had provided an alternate approach to the subject by means of *twisted compositions*, it was Tits who presented his two constructions (without proof) at the Oberwolfach conference on Jordan algebras in 1967; a thorough treatment in book form was subsequently given by Jacobson [36]. It should also be mentioned that the formula (9.7.3) for the norm of a first Tits construction with the cubic associative algebra $A := \mathrm{Mat}_3(k)$ and the scalar $\mu := 1$ as input, leading to the split Albert algebra in the process, is already in [23, 26.8].

All these investigations were confined to base fields of characteristic not 2. McCrimmon [51, 52] removed this restriction. Later on, the two Tits constructions were put in broader perspective through the Tits *process* developed by Petersson-Racine [77, 78], which was subsequently applied first by Tits-Weiss [107] to the theory of Moufang polygons and later on by Mühlherr-Weiss [60] to the one of Tits polygons.

Returning to Albert's investigations [3, 4] over fields of characteristic not 2, it is quite clear that he understood the term *Jordan division algebra* in the linear sense, i.e., as referring to a (finite-dimensional) (linear) Jordan algebra J whose (linear) Jordan product has no zero divisors. The question is how cubic Jordan division algebras in our sense fit into this picture. Here is the simple answer.

12.10. **Proposition.** ([68]) *For a finite-dimensional cubic Jordan algebra J over a field of characteristic not 2 to be a Jordan division algebra it is necessary and sufficient that it be a linear Jordan division algebra, i.e., that its bilinear Jordan product have no zero divisors.*

Proof. Sufficiency is easy. To prove necessity, suppose J is a Jordan division algebra and $a, b \in J$ satisfy $a.b = 0$, the dot referring to the bilinear Jordan product. The case of a field extension being obvious, we may assume by Lemma 12.4 and Thm. 12.7 that J has dimension 9, hence is as in (c) or (d) of that theorem. Passing to a separable quadratic extension if necessary we may in fact assume $J = D^+$ for some central associative division algebra D of degree 3 over the base field. Then $a.b = 0$ is equivalent to the relation $ab = -ba$ in terms of the associative product of D. Taking norms, we conclude $N_D(a)N_D(b) = -N_D(a)N_D(b)$, and since we are in characteristic not 2, one of the elements a, b must be zero. $\square$

13. Invariants

After having enumerated non-singular cubic Jordan algebras in Thms. 12.2, 12.7, we now turn to the problem of classification, with special emphasis on Albert algebras. The most promising approach to this problem is by means of invariants. In Section 8, particularly 8.8, we have already encountered the classifying invariants f_r and f_{r+2} for *reduced* cubic Jordan algebras $\mathrm{Her}_3(C, \Gamma)$ (C a composition algebra of dimension 2^r, $r = 1, 2, 3$), with nice cohomological interpretations to boot. We speak of the *invariants mod* 2 in this context.

Our first aim in this section will be to show that these invariants survive also for cubic Jordan *division* algebras. In the case of Albert algebras, Serre [96, Thm. 10] (see also [27, Thm. 22.4]) has done so by appealing to the algebraic theory of quadratic forms, particularly to the Arason-Pfister theorem (cf. [18, Cor. 23.9]), combined with a descent property of Pfister forms due to Rost [92].

Here we will describe an approach that is more Jordan theoretic in nature. Fixing an arbitrary base field F and an absolutely simple cubic Jordan algebra J over F, our approach is based on the following concept.

13.1. **Reduced models.** A field extension L/F is called a *reducing field* of J if the scalar extension J_L is a reduced cubic Jordan algebra over L. By a *reduced model* of J we mean a reduced cubic Jordan algebra J_{red} over F which becomes isomorphic to J whenever scalars are extended to an arbitrary reducing field of J: $(J_{\mathrm{red}})_L \cong J_L$ for all field extensions L/F having J_L reduced. It is clear that, once existence and uniqueness of the reduced model have been established, the invariants mod 2 of J simply can be defined as the ones of J_{red}.

13.2. **Theorem.** ([83]) *Reduced models exist and are unique up to isomorphism.*

Instead of a proof. Uniqueness follows from Theorem 8.7 and Springer's theorem [18, Cor. 18.5], which implies that two non-singular quadratic forms over F that become isometric after an odd degree field extension must have been so all along. Existence can be established in a way that yields some insight into the structure of the reduced model at the same time.

13.3. **The octonion algebra of an involution.** Let $\mathcal{B} = (E, B, \tau)$ be a central simple associative algebra of degree 3 with involution of the second kind over F. Then $(1_B, 1_E)$ is an admissible scalar relative to $\mathcal{B}$ and, following [81], we deduce from Corollary 12.3 combined with Lemma 12.5 that $J(\mathcal{B}, 1_B, 1_E)$ is a reduced Albert algebra over F. Its co-ordinate algebra (cf. 8.3) is an octonion algebra over F denoted by $C := \mathrm{Oct}(\mathcal{B})$, while its norm is a 3-Pfister quadratic form thoroughly investigated by Haile-Knus-Rost-Tignol [34]. They showed, in particular, (see also [71]) that this 3-Pfister quadratic form is a classifying invariant for involutions (of the second kind) on B. Therefore we will also call $\mathrm{Oct}(\mathcal{B})$ the *octonion algebra of* τ. In the following description of its norm, we make use of the product of quadratic étale algebras and of the discriminant of a cubic étale algebra as explained in 11.5, 11.6.

13.4. **Theorem.** ([42, 81, 83]) *Let $\mathcal{B} = (E, B, \tau)$ be a central simple associative algebra of degree* 3 *with involution of the second kind over F and suppose $L \subseteq H(\mathcal{B})$ is a cubic étale subalgebra.*

(a) *Writing $d_{E/F} \in F^\times$ for the ordinary discriminant of E as a quadratic étale F-algebra, we have*

$$n_{\mathrm{Oct}(\mathcal{B})} \cong n_{E \cdot \mathrm{Disc}(L)} \oplus d_{E/F}(S_{H(\mathcal{B})}|_{L^\perp}).$$

(b) *$E \subseteq \mathrm{Oct}(\mathcal{B})$ is a quadratic étale subalgebra, and scalars $\gamma_1, \gamma_2 \in F^\times$ satisfy the relation* $\mathrm{Oct}(\mathcal{B}) = \mathrm{Cay}(E; -\gamma_1, -\gamma_2)$ *if and only if*

$$H(\mathcal{B})_{\mathrm{red}} = \mathrm{Her}_3(E, \Gamma), \quad \Gamma = \mathrm{diag}(\gamma_1, \gamma_2, 1)$$

is a reduced model of $H(\mathcal{B})$. □

13.5. **Corollary.** ([83]) *Let J be an Albert algebra over F, realized as $J = J(\mathcal{B}, p, \mu)$ by means of the second Tits construction, where $\mathcal{B} = (E, B, \tau)$ is a central simple associative algebra of degree* 3 *with involution of the second kind over F and (p, μ) is an admissible scalar relative to $\mathcal{B}$. Then $E \subseteq \mathrm{Oct}(\mathcal{B}^q)$, $q := p^{-1}$, is a quadratic étale subalgebra and, for any $\gamma_1, \gamma_2 \in F^\times$ such that* $\mathrm{Oct}(\mathcal{B}^q) = \mathrm{Cay}(E; -\gamma_1, -\gamma_2)$,

$$J_{\mathrm{red}} = \mathrm{Her}_3(\mathrm{Oct}(\mathcal{B}^q), \Gamma), \quad \Gamma = \mathrm{diag}(\gamma_1, \gamma_2, 1),$$

is a reduced model of J. □

There exists yet another cohomological invariant of Albert algebras, called the *invariant mod* 3, that, contrary to the previous ones, doesn't seem to allow a non-cohomological interpretation. In order to define it, we require a short digression into Galois cohomology, see, e.g., [99], [42] or [32].

13.6. **Facts from Galois cohomology.** (a) Let $\mathbf{G}$ be a commutative group scheme of finite type over our base field F, so $\mathbf{G}$ is a functor from F-**alg** to abelian groups, represented by a finitely generated commutative associative F-algebra. Writing F_s for the separable closure of F, we put

$$H^i(F, \mathbf{G}) := H^i\big(\operatorname{Gal}(F_s/F), \mathbf{G}(F_s)\big)$$

for all integers $i \geq 0$ and

$$H^*(F, \mathbf{G}) := \bigoplus_{i\geq 0} H^i(F, \mathbf{G}),$$

which becomes a graded ring under the cup product. If we are given a field extension K/F, there are natural homomorphisms

$$\begin{aligned} \operatorname{res} := \operatorname{res}_{K/F}\colon\ & H^*(F, \mathbf{G}) \longrightarrow H^*(K, \mathbf{G}) \qquad &\text{(restriction)} \\ \operatorname{cor} := \operatorname{cor}_{K/F}\colon\ & H^*(K, \mathbf{G}) \longrightarrow H^*(F, \mathbf{G}) \qquad &\text{(corestriction)} \end{aligned}$$

of graded rings, and if K/F is finite algebraic, then

$$\operatorname{cor} \circ \operatorname{res} = [K : F]\mathbf{1}.$$

In particular, *if a prime p kills $\mathbf{G}$ but does not divide $[K : F]$, then the restriction map* $\operatorname{res}\colon H^*(F, \mathbf{G}) \to H^*(K, \mathbf{G})$ *is injective.*

(b) For a positive integer n, we consider the constant group scheme $\mathbf{Z}/n\mathbf{Z}$ with trivial Galois action and conclude that

$$H^1(F, \mathbf{Z}/n\mathbf{Z}) = \operatorname{Hom}_G(G, \mathbf{Z}/n\mathbf{Z}), \quad G := \operatorname{Gal}(F_s/F),$$

the right-hand side of the first equation referring to *continuous* G-homomorphisms, classifies *cyclic étale F-algebras of degree* n, i.e., pairs (L, ρ) where L is an étale algebra of dimension n over F and $\rho\colon L \to L$ is an F-automorphism having order n and fixed algebra $F1_L$. The cohomology class of (L, ρ) will be denoted by $[L, \rho] \in H^1(F, \mathbf{Z}/n\mathbf{Z})$. It is easy to see that a cyclic étale F-algebra (L, ρ) of degree n has either L/F a cyclic field extension, with ρ a generator of its Galois group, or is *split*, i.e., isomorphic to

$$\big(F^n, (\alpha_1, \alpha_2, \ldots, \alpha_n) \longmapsto (\alpha_2, \ldots, \alpha_n, \alpha_1)\big).$$

(c) Let $\mathbf{G}_{\mathrm{m}}$ the commutative group scheme of units given by $\mathbf{G}_{\mathrm{m}}(R) = R^\times$ for $R \in F$-**alg**. Then

$$\operatorname{Br}(F) := H^2(F, \mathbf{G}_{\mathrm{m}})$$

is the *Brauer group* of F. It has a natural interpretation as the group of similarity classes $[D]$ of central simple (associative) F-algebras D.

(d) The Brauer group is known to be a torsion group. For a positive integer n, we write

$${}_n\operatorname{Br}(F) := \{\alpha \in \operatorname{Br}(F) \mid n\alpha = 0\}$$

for its n-torsion part, and if n is not divisible by the characteristic of F, then

$${}_n\operatorname{Br}(F) = H^2(F, \boldsymbol{\mu}_n),$$

where $\boldsymbol{\mu}_n$ stands for the group scheme of n-th roots of 1, given by

$$\boldsymbol{\mu}_n(R) := \{r \in R \mid r^n = 1\}, \quad R \in F\text{-}\mathbf{alg}.$$

In particular, if D is a central simple associative algebra of degree n over F, i.e., an F-form of $\operatorname{Mat}_n(F)$, then $[D] \in H^2(F, \boldsymbol{\mu}_n)$.

(e) Let n be a positive integer not divisible by $\operatorname{char}(F)$, (L, ρ) a cyclic étale F-algebra of degree n and $\gamma \in F^\times$. We write $D := (L, \rho, \gamma)$ for the associative F-algebra generated by L and an element w subject to the relations $w^n = 1_D$, $wa = \rho(a)w$ $(a \in L)$. It is known that D is a central simple algebra of dimension n^2 over F; we speak of a *cyclic algebra* in this context. In cohomological terms we have

$$[D] = [L, \rho, \gamma] = [L, \rho] \cup [\gamma] \in H^2(F, \mathbf{Z}/n\mathbf{Z} \otimes \boldsymbol{\mu}_n) = H^2(F, \boldsymbol{\mu}_n),$$

where $\gamma \mapsto [\gamma]$ stands for the natural map $F^\times \to H^1(F, \boldsymbol{\mu}_n)$ induced by the n-th power map $\mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}}$.

13.7. **Theorem.** ([91]) *If F has characteristic not* 3*, there exists a cohomological invariant assigning to each Albert algebra J over F its* invariant mod 3, *i.e., a unique element*

$$g_3(J) \in H^3(F, \mathbf{Z}/3\mathbf{Z})$$

which only depends on the isomorphism class of J and satisfies the following two conditions.

(a) *If $J \cong J(D, \mu)$ for some central simple associative F-algebra D of degree* 3 *and some $\mu \in F^\times$, then*

$$g_3(J) = [D] \cup [\mu] \in H^3(F, \boldsymbol{\mu}_3 \otimes \boldsymbol{\mu}_3) = H^3(F, \mathbf{Z}/3\mathbf{Z}).$$

(b) *g_3 commutes with base change, i.e.,*

$$g_3(J \otimes K) = \mathrm{res}_{K/F}\big(g_3(J)\big)$$

for any field extension K/F.

Moreover,

(c) *g_3 detects division algebras in the sense that an Albert algebra J over F is a division algebra if and only if $g_3(J) \neq 0$.*

□

13.8. **Remark.** (a) In part (a) of the theorem it is important (though trivial) to note that, given a primitive cube root $\zeta \in F_s$ of 1, the identification

$$\boldsymbol{\mu}_3(F_s) \otimes \boldsymbol{\mu}_3(F_s) = \mathbf{Z}/3\mathbf{Z}, \quad \zeta^i \otimes \zeta^j = ij \bmod 3 \qquad (i, j \in \mathbf{Z})$$

does not depend on the choice of ζ.

(b) The invariant mod 3 of Albert algebras originally goes back to a suggestion of Serre [95], [98, pp. 212-222]. Its existence was first proved by Rost [91], with an elementary proof working also in characteristic 2 subsequently provided by Petersson-Racine [82]. The characterization of Albert *division* algebras by the invariant mod 3 rests on a theorem of Merkurjev-Suslin [58], see also [32]. The approach to the invariant mod 3 described in Theorem 13.7 does not work in characteristic 3; in this case, one has to proceed in a different manner due to Serre [97, 84]. Nowadays the invariant mod 3 of Albert algebras fits into the more general framework of the Rost invariant for algebraic groups (groups of type F_4 in the present case), see [27] for a systematic treatment of this topic.

13.9. **Corollary.** *Let J be an Albert division algebra over F. Then J_K is an Albert division algebra over K, for any finite algebraic field extension K/F of degree not divisible by* 3.

Proof. For simplicity we assume $\mathrm{char}(F) \neq 3$. Since $g_3(J) \neq 0$ by Theorem 13.7 (c), and the restriction map from $H^3(F, \mathbf{Z}/3\mathbf{Z})$ to $H^3(K, \mathbf{Z}/3\mathbf{Z})$ is injective by 13.6 (a), we conclude $g_3(J_K) \neq 0$ from Theorem 13.7 (b). Hence J_K is a division algebra. □

13.10. **Symplectic involutions.** Assume $\mathrm{char}(F) \neq 2$ and let (B, τ) be a central simple associative algebra of degree 8 with symplectic involution (of the first kind) over F. Writing t for the generic trace of the Jordan algebra $H(B, \tau)$ and picking an element $e \in X := \mathrm{Ker}(t)$ satisfying $t(e^3) \neq 0$, Allison and Faulkner [6] have shown that X carries the structure of an Albert algebra $J(B, \tau, e)$ in a natural way.

13.11. **Corollary.** *The Albert algebra $J := J(B, \tau, e)$ considered in* 13.10 *is reduced.*

Instead of a proof. (B, τ) has a splitting field of degree $1, 2, 4$, or 8, and one checks easily that the scalar extension J_K is not a division algebra. Hence neither is J, by Corollary 13.9. □

13.12. **Corollary.** ([80]) *Let D be a central simple associative F-algebra of degree 3 and $\mu, \mu' \in F^\times$. For the Albert algebras $J(D,\mu)$, $J(D,\mu')$ to be isomorphic it is necessary and sufficient that $\mu' = \mu N_D(u)$, for some $u \in D^\times$.*

Proof. The condition is easily seen to be sufficient. Conversely, suppose $J(D,\mu)$ and $J(D,\mu')$ are isomorphic. Then they have the same invariant mod 3, and the bi-linearity of the cup product implies

$$g_3\big(J(D,\mu\mu'^{-1})\big) = [D] \cup ([\mu] - [\mu']) = 0.$$

Hence $J(D,\mu\mu'^{-1})$ is not a division algebra by Theorem 13.7 (c), and $\mu\mu'^{-1} \in N_D(D^\times)$ by Lemma 12.5. □

The analogue of this corollary for second Tits constructions is more delicate:

13.13. **Theorem.** ([62, 71]) *Let $\mathcal{D} = (E, D, \tau)$ be a central simple associative algebra of degree 3 with involution of the second kind over F and suppose (p,μ), (p',μ') are admissible scalars relative to $\mathcal{D}$. Then the Albert algebras $J(\mathcal{D},p,\mu)$ and $J(\mathcal{D},p',\mu')$ are isomorphic if and only if $p' = up\tau(u)$ and $\mu' = \mu N_D(u)$ for some $u \in D^\times$.* □

13.14. **The Skolem-Noether problem for Albert algebras.** Let J be an Albert algebra over F. The *Skolem-Noether problem* for J asks whether any isomorphism between separable cubic subalgebras of J can be extended to an automorphism of J. Combining Cor. 13.12 with Thms. 12.7, 13.13 and standard arguments already in [36], a partial affirmative answer to this question can be given as follows.

13.15. **Theorem.** ([36, 62]) *Let J be an Albert algebra over F and J_1, J_2 be separable cubic subalgebras of J which are not cubic étale. Then every isomorphism from J_1 to J_2 can be extended to an automorphism of J.* □

On the other hand, it is known since Albert-Jacobson [5] that the Skolem-Noether problem for cubic étale subalgebras of Albert algebras has a negative answer. Fortunately, however, there is a substitute for this deficiency:

13.16. **Theorem.** ([29]) *Let E_1, E_2 be cubic étale subalgebras of an Albert algebra J over F and suppose $\varphi\colon E_1 \to E_2$ is an isomorphism. Then there exists an element $w \in E_1$ of norm 1 such that $\varphi \circ R_w$ can be extended to an element of the structure group of J, where R_w stands for the right multiplication by w in E_1.* □

As an application of this result, it has been shown in [29] that exceptional groups of type ${}^3D_4$ allow outer automorphisms with particularly nice properties. We refer to [29, 1.1] for details.

14. Open problems

Let F be an arbitrary field. In the preceding sections we have encountered three cohomological invariants of Albert algebras, namely, f_3, f_5, belonging to $H^3(F;\mathbf{Z}/2\mathbf{Z})$, $H^5(F;\mathbf{Z}/2\mathbf{Z})$, respectively, which make sense also in characteristic 2 ([18]), and g_3, belonging to $H^3(F,\mathbf{Z}/3\mathbf{Z})$, which makes sense also in characteristic 3 (Remark 13.8 (b)). Our starting point in this section will be the following result.

14.1. **Theorem.** ([27, 25]) *f_3 and f_5 are basically the only invariants* mod 2 *and g_3 is basically the only invariant* mod 3 *of Albert algebras over fields of characteristic not* 2, 3. □

In view of this result, it is natural to ask the following question:

14.2. **Question.** ([95],[96, p. 465]) *Do the invariants* mod 2 *and* 3 *classify Albert algebras up to isomorphism?*

14.3. **Remark.** By the results of Section 8, Question 14.2 has an affirmative answer when dealing with *reduced* Albert algebras. In particular, an Albert algebra is split if and only if all its invariants f_3, f_5, g_3 are zero. Furthermore, $\mathbf{Aut}(J)$, the group scheme of type F_4 attached to an Albert algebra J over F, is isotropic if and only if the invariants f_5 and g_3 are zero, which in turn is equivalent to J containing non-zero nilpotent elements, justifying the terminology of 8.4 .

A less obvious partial answer to Question. 14.2 reads as follows.

14.4. **Theorem.** ([93]) *Let J, J' be Albert algebras over F and suppose their invariants* $\bmod 2$ *and* 3 *are the same. If F has characteristic not* $2, 3$*, there exist field extensions K/F of degree dividing* 3 *and L/F of degree not divisible by* 3 *such that $J_K \cong J'_K$ and $J_L \cong J'_L$.* □

A complete answer to Question 14.2 being fairly well out of reach at the moment, one might try to answer it for specific subclasses of Albert algebras (other than reduced ones), e.g., for first Tits constructions. This makes sense because first Tits constructions can be characterized in terms of invariants.

14.5. **Theorem.** ([74, 42, 71]) *For an Albert algebra J over F, the following conditions are equivalent.*

(i) *J is a first Tits construction.*
(ii) *The reduced model of J is split.*
(iii) $f_3(J) = 0$.

□

Since f_5 is always a multiple of f_3 by (8.8.1), Question 14.2 when phrased for first Tits constructions reads as follows.

14.6. **Question.** *Does the invariant* $\bmod 3$ *classify first Tits construction Albert algebras up to isomorphism?*

It has been shown in [107] that isomorphism classes of Moufang hexagons are in a one-to-one correspondence with isotopy classes of cubic Jordan division algebras. It is therefore natural to look for classifying invariants of Albert algebras up to isotopy. Before doing so, however, it has to be decided which ones among the invariants f_3, f_5, g_3 are actually *isotopy invariants*. Our answer will be based upon the following result.

14.7. **Theorem.** ([74]) *Let J, J' be Albert algebras over F such that J is a first Tits construction and J' is isotopic to J. Then J' is isomorphic to J.* □

14.8. **Corollary.** *f_3 and g_3 are isotopy invariants of Albert algebras.*

Proof. Let J be an Albert algebra over F. We first deal with f_3, where we may assume that J is a division algebra (Thm. 8.2). Picking any separable cubic subfield $L \subseteq J$, the extended algebra J_L becomes reduced over L, and since the restriction $H^3(F, \mathbf{Z}/2\mathbf{Z}) \to H^3(L, \mathbf{Z}/2\mathbf{Z})$ is injective, the assertion follows. Next we turn to g_3, where we may assume that J is not a first Tits construction (Thm. 14.7). But then it becomes one after an appropriate separable quadratic field extension (10.7), and the assertion follows as before. □

Now the isotopy version of Question14.2 can be phrased as follows.

14.9. **Question.** ([95]) *Do the invariants f_3 and g_3 classify Albert algebras up to isotopy?*

14.10. **Proposition.** *If f_3, f_5 and g_3 classify Albert algebras up to isomorphism, then f_3 and g_3 classify them up to isotopy.*

Instead of a proof. Using the theory of distinguished involutions ([42, 71]), one shows that every Albert algebra over F has an isotope whose f_5-invariant is zero. □

14.11. **The Kneser-Tits problem.** (cf. [24], [30]) Let $\mathbf{G}$ be a simply connected absolutely quasi-simple F-isotropic algebraic group. The *Kneser-Tits problem* asks whether $\mathbf{G}(F)$, the group of F-points of $\mathbf{G}$, is *projectively simple* in the sense that it becomes simple (as an abstract group) modulo its centre. When phrased for certain forms of E_8 having F-rank 2 (cf. [106]), the Kneser-Tits problem can be translated into the setting of Albert algebras and then leads to the *Tits-Weiss conjecture* ([107, p. 418]).

14.12. **The Tits-Weiss conjecture.** *The structure group* (cf. 4.15) *of an Albert algebra over F is generated by U-operators U_x, $x \in J^\times$, and by scalar multiplications $y \mapsto \alpha y$, $\alpha \in F^\times$.*

A partial affirmative solution to this conjecture reads as follows..

14.13. **Theorem.** ([105]) *The Tits-Weiss conjecture is true for Albert division algebras that are* pure *first Tits constructions.* □

Remark. (a) [105] shows in addition that the Tits-Weiss conjecture has an affirmative answer also for *reduced* Albert algebras.

(b) By a *pure* first Tits construction we mean of course an Albert algebra that cannot be obtained from the second. The significance of the preceding result is underscored by the fact that pure first Tits constructions exist in abundance. For example, all Albert division algebras over the iterated Laurent series field in several variables with complex coefficients are pure first Tits constructions ([79]), see also 14.20 below for more details.

14.14. **Essential dimension.** ([57, 89]) Roughly speaking, the essential dimension of an algebraic object over a field F is the minimal number of parameters needed to describe the object uniquely up to isomorphism. In order to make this more precise, we denote by F-**field** the full subcategory of F-**alg** whose objects are fields and consider a covariant functor

$$\Phi\colon F\text{-}\mathbf{field} \longrightarrow \mathbf{set},$$

where **set** stands for the category of sets. Given a field extension K/F (i.e., an object F-**field**), a morphism $\alpha\colon K_0 \to K$ in F-**field** (making K an extension field of K_0) and some $x \in \Phi(K)$, we say that x *is defined over* K_0 (or K_0 *is a field of definition for* x) if x belongs to the image of the set map $\Phi(\alpha)\colon \Phi(K_0) \to \Phi(K)$, so some $x_0 \in \Phi(K_0)$ has $(x_0)_K := \Phi(\alpha)(x_0) = x$. Then we define the *essential dimension* of x (relative to Φ) as

$$\mathrm{ed}(x) := \mathrm{ed}_\Phi(x) := \min \mathrm{tr.deg}_F(K_0),$$

where the minimum is taken over all fields of definition K_0 for x. Moreover, we define the *essential dimension* of Φ as

$$\mathrm{ed}(\Phi) := \max \mathrm{ed}(x),$$

where the maximum is taken over all field extensions K/F and all $x \in \Phi(K)$.

The preceding formalism applies in particular to the functor

$$\mathbf{Alb}\colon F\text{-}\mathbf{field} \longrightarrow \mathbf{set}$$

that assigns to each field extension K/F the set of isomorphism classes of Albert algebras over K, and to each morphism $\alpha\colon K \to K'$ in F-**alg** the set map $\mathbf{Alb}(\alpha)\colon \mathbf{Alb}(K) \to \mathbf{Alb}(K')$ induced by the base change of Albert algebras from K to K'. The functor **Alb** is equivalent to the G-torsor functor over F for G the split group of type F_4.

The essential dimension of the functor **Alb** is not known, nor is the one, aside from trivial cases, of individual Albert algebras. On the positive side, we can record the following two results.

14.15. **Theorem.** ([49]) $\mathrm{ed}(\mathbf{Alb}) \leq 7$ *for* $\mathrm{char}(F) \neq 2, 3$. □

14.16. **Theorem.** ([9]) $\mathrm{ed}(\mathbf{Alb}) \geq 5$ *in all characteristics.* □

It is not known whether Thm. 14.15 survives in the bad characteristics 2 and 3. A much cruder estimate covering these cases as well reads as follows[7].

14.17. **Theorem.** ([26]) $\mathrm{ed}(\mathbf{Alb}) \leq 19$ *in all characteristics.* □

14.18. **Wild Albert algebras.** [28] investigated wild Pfister quadratic forms over Henselian fields and also connected their results with Milnor's K-theory mod 2. In doing so, they took advantage of the fact that Pfister quadratic forms are classified by their cohomological invariants (see 8.8 above).

It is tempting to try a similar approach for wild Albert algebras over Henselian fields. The fact that we do not know at present whether Albert algebras are classified by their cohomological invariants should not serve as a deterrent but, on the contrary, as an incentive to do so. Indeed, working on this set-up could lead to new insights into the classification problem 14.2.

14.19. **The Jacobson embedding theorem.** Jacobson [36] has shown that every element of a split Albert algebra over a field F of characteristic not 2 can be embedded into a (unital) subalgebra isomorphic to $\mathrm{Mat}_3(F)^+$. In view of Jacobson's result, which has been extended to base fields of arbitrary characteristic [73], it is natural to ask the following question: *can every element of a first Tits construction Albert division algebra be embedded into a subalgebra isomorphic to D^+, for some central associative division algebra D of degree 3?* The answer to this question doesn't seem to be known.

14.20. **Special fields.** What do we know about Albert algebras over special fields? The reader may wish to consult the matrix (1) below from which one can depict, over the various fields in column 1, the number of non-isomorphic (resp. non-isotopic) reduced Albert algebras in column 2 (resp. 3) and the number of non-isomorphic (resp. non-isotopic) Albert division algebras in column 4 (resp. 5).

$$\begin{array}{ccccc}
\text{field} & \text{isom-red} & \text{isot-red} & \text{isom-div} & \text{isot-div} \\
\mathbf{C} & 1 & 1 & 0 & 0 \\
\mathbf{R} & 3 & 2 & 0 & 0 \\
\mathbf{F}_q & 1 & 1 & 0 & 0 \\
[F:\mathbf{Q}_p] < \infty & 1 & 1 & 0 & 0 \\
[F:\mathbf{Q}] < \infty & 3^{\#(F\hookrightarrow\mathbf{R})} & 2^{\#(F\hookrightarrow\mathbf{R})} & 0 & 0 \\
\mathbf{C}((\mathbf{t}_1,\dots,\mathbf{t}_n)) & \alpha_n(\mathbf{C}) & \beta_n(\mathbf{C}) & \gamma_n(\mathbf{C}) & \gamma_n(\mathbf{C}) \\
\mathbf{R}((\mathbf{t}_1,\dots,\mathbf{t}_n)) & \alpha_n(\mathbf{R}) & \beta_n(\mathbf{R}) & 0 & 0,
\end{array} \tag{1}$$

Here $\#(F \hookrightarrow \mathbf{R})$ stands for the number of real embeddings of a number field F, while $\mathbf{C}((\mathbf{t}_1,\dots,\mathbf{t}_n))$ (resp. $\mathbf{R}((\mathbf{t}_1,\dots,\mathbf{t}_n))$) refers to the field of iterated formal Laurent series in n variables with complex (resp. real) coefficients. Finally, the numerical entries in the

[7]I am grateful to Mark MacDonald for having drawn my attention to this.

two bottom rows are defined by

$$\alpha_n(\mathbf{C}) = \frac{1}{2^{10}\cdot 3^2\cdot 7}\Big(2^{5n} - 31\cdot 2^{4n} + 347\cdot 2^{3n+1} - 491\cdot 2^{2n+3} + 115\cdot 2^{n+6} + 59\cdot 2^{10}\Big),$$

$$\beta_n(\mathbf{C}) = \frac{1}{2^3\cdot 3\cdot 7}\Big(2^{3n} - 7\big(2^{2n} - 2^{n+1}\big) + 160\Big),$$

$$\gamma_n(\mathbf{C}) = \frac{1}{2^4\cdot 3^3\cdot 13}\Big(3^{3n} - 13\big(3^{2n} - 3^{n+1}\big) - 27\Big),$$

$$\alpha_n(\mathbf{R}) = \frac{1}{2^5\cdot 3^2\cdot 7}\Big(2^{5n} + 49\cdot 2^{4n} + 323\cdot 2^{3n+1} + 209\cdot 2^{2n+3} + 7\cdot 2^{n+8} + 1818\Big),$$

$$\beta_n(\mathbf{R}) = \frac{1}{21}\Big(2^{3n} + 7\big(2^{2n} + 2^{n+1}\big) + 20\Big).$$

The entries in rows $1, 2$ of (1) are standard, while the ones in row 3 (resp. 4) follow from the theorem of Chevalley-Warning (cf. [44]) (resp. from standard properties of quadratic forms over $\mathfrak{p}$-adic fields combined with a theorem of Springer [100]), which imply that Albert algebras over the fields in question are split. Moreover, the numerical entries in row 5 are due to Albert-Jacobson [5], while the ones in rows $6, 7$ are contained in, or follow easily from, [65, 66, 67][8]. Finally, the number of non-isomorphic isotropic Albert algebras over the special fields at hand can also be read off from (1) since, by Theorems 8.2, 8.5, it agrees with the entries of column 3.

14.21. **The cyclicity problem.** We close this paper with a question that, fittingly, was raised by Albert [4] himself: *Does every Albert division algebra contain a cyclic cubic subfield?* A positive answer to this question would not only have a significant impact on Albert's own approach to the subject but also on the theory of cyclic and twisted compositions [102, 103, 42].

While we do know that Albert's question has an affirmative answer if (i) the base field has characteristic not 3 and contains the cube roots of 1 [75] or (ii) the base field has characteristic 3 [70], dealing with this question in full generality seems to be rather delicate since, for example, the answer to its analogue for cubic Jordan division algebras of dimension 9 is negative. This follows from examples constructed in [75] (see also [34]) which live over the field $\mathbf{R}((\mathbf{t}_1, \dots, \mathbf{t}_n))$ in the last row of (14.20.1). Unfortunately, this field is of no interest to Albert's original question since, again by the last row of (14.20.1), it fails to admit Albert division algebras.

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[8]I am grateful to Jean-Pierre Serre [97] for having pointed out to me a mistake in the computations of [67], which is responsible for the (erroneous) formula describing $\gamma_n(\mathbf{C})$ in [67] to differ from the (correct) one presented here by a factor 2: in my original computations, I had overlooked the fact that, given a central associative division algebra D of degree 3, the algebras D and D^{op} are never isomorphic, while the corresponding Jordan algebras always are.

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