THE BEHAVIOUR OF ALBERT DIVISION ALGEBRAS UNDER FIELD EXTENSIONS OF DEGREE COPRIME TO 3

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Abstract. We show that an Albert division algebra over a field remains a division algebra under any field extensions of degree coprime to 3.

Let F be a field. An *Albert algebra* over F is an exceptional simple Jordan algebra of degree 3, i.e. an F-form of the Jordan algebra of 3-by-3 hermitian matrices with diagonal entries in F and off-diagonal entries in the split octonion algebra $\mathrm{Zor}(F)$. For details, the reader is referred to for instance [P-R1, 2] (or [KMRT, p. 524 ff.] for $\mathrm{char}\, F$ not 2 or 3).

An Albert division algebra remains a division algebra under any field extension of F of degree coprime to 3. This result is well-known for people working with Albert algebras, but it seems that it has not been written up anywhere explicitly. Its proof relies on the Serre-Rost invariant, a theorem of Merkurjev-Suslin [P-R1, Theorem 1.8] and standard facts from Galois-cohomology, and is implicitly contained in the results of Petersson-Racine ([P-R1] if $\mathrm{char}\, F \neq 3$ and [P-R2] if $\mathrm{char}\, F = 3$). We give an alternative proof, which instead only needs the following result on the reduced norm residue groups of a central simple algebra:

Lemma 1. *([D, p. 157, Lemma 3]) Let A be a central simple algebra over F and L/F a finite field extension of degree m, then the embedding $F \subset L$ induces an injection*

$$F^\times / N_{A/F} \hookrightarrow L^\times / N_{A \otimes_F L/L}$$

provided that m and the index of A are coprime.

Theorem 1. *Let J be an Albert algebra over F. If J is a division algebra, then it stays a division algebra under each field extension of F of degree coprime to 3.*

Proof. Every Albert algebra over F can be obtained either by a first Tits construction or by a second Tits construction. Assume first that J is an Albert algebra which is a first Tits construction $J(A, a)$ starting with A^+, where A is a central simple associative algebra of degree 3, $a \in F^\times$. J is a division algebra if and only if $a \notin N_{A/F}(A^\times)$ (cf. for instance [P-R1, Proposition 2.6 (a)]). So let J be a division

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algebra and K a field extension of F of degree coprime to 3. Given $a \in F^{\times}$, for any field extension K of F of degree coprime to d, $a \in N_{A/F}(A^{\times})$ if and only if $a \in N_{A_K/K}(A_K^{\times})$ by Lemma 1. Thus J stays a division algebra over K.

Now let J be a Tits process starting with $H(B,\tau)$ where L/F is a separable quadratic field extension and B a simple associative algebra with unitary involution τ and center L. (The case that $L \cong F \times F$ is split implies that $H(B,\tau) = H(E \times E, \epsilon) \cong E^+$ with E a division algebra over F and that hence J again can be constructed as a first Tits construction.) Suppose J is a division algebra, then its norm $N_{J/F}$ is anisotropic.

Let first K be a field extension of degree coprime to 3 which contains L as a subfield. Then J_L is an Albert algebra over L which is isomorphic to a first Tits construction starting with the central simple algebra B, and $B^+ \cong H(B,\tau) \otimes_F L$. Since cubic forms remain anisotropic under quadratic field extensions [La, VII, Ex. 7]], its norm $(N_{J/F})_L = N_{J_L/L}$ is anisotropic and remains anisotropic under each field extension with degree coprime to 3 by above. Now since $[K : F]$ is coprime to 3, so is $[K : L]$ and we see that $N_{J/F}$ stays anisotropic over K. Thus J_K is a division algebra.

Now let K be a field extension of degree m coprime to 3 which does not contain L as a subfield. Then K and L are linearly disjoint and the composite $L' = K \cdot L$ is a field. Moreover, $[L' : K] = 2$ and $[L' : L] = m$. We have

$$J \otimes_F L' \cong (J \otimes_F L) \otimes_L L' \cong (J \otimes_F K) \otimes_K L'.$$

Since $J \otimes_F L$ is a first Tits construction and a division algebra, it stays a division algebra under the extension L' of L which has degree coprime to 3. Hence J_K is a division algebra. $\square$

References

[D] Draxl, P., "Skew Fields", Cambridge University Press, England (1983).

[La] Lang, S., "Algebra". Addison-Wesley, Reading (1965).

[KMRT] Knus, M.A., Merkurjev, A., Rost, M., Tignol, J.-P., "The Book of Involutions". AMS Coll. Publications, Vol. 44 (1998).

[P-R1] Petersson, H. P., Racine, M., *An elementary approach to the Serre-Rost invariant of Albert algebras*. Indag. Math., N.S., 7(3)(1996), 343-365.

[P-R2] Petersson, H. P., Racine, M., *The Serre-Rost invariant of Albert algebras in characteristic three*. Indag. Math., N.S., 8(4)(1997), 543-548.

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