ADDENDUM TO: "FACTORING SKEW POLYNOMIALS OVER HAMILTON'S QUATERNION ALGEBRA AND THE COMPLEX NUMBERS" [J. ALGEBRA 427 (2015), 20-29]

S. PUMPLÜN

Abstract. Let D be the quaternion division algebra over a real closed field F. Then every non-constant polynomial in a skew-polynomial ring $D[t;\sigma,\delta]$ decomposes into a product of linear factors, and thus has a zero. This improves [8, Theorem 2].

Introduction

We use the notation of [8]. In particular, let F be a *real closed field*, that is, a formally real field such that every polynomial of odd degree with coefficients in F has at least one root in F, and for every element $a \in F$ there is $b \in F$ such that $a = b^2$ or $a = -b^2$. Equivalently, F is a real closed field if it is not algebraically closed but if the field extension $F(\sqrt{-1})$ is algebraically closed. Then every division algebra over F has dimension 1, 2, 4 or 8 and up to isomorphism, there are exactly three associative division algebras over F, one each of dimension 1, 2, and 4 (cf. [1], [5]). Let D be the quaternion division algebra over F, σ an injective ring homomorphism of D and δ a σ-derivation. After a linear change of variables, the skew-polynomial ring $D[t;\sigma,\delta]$ is either a twisted polynomial ring or a differential polynomial ring [3, Theorem 1.1.21]. For the special case that $\sigma = id$ and $\delta = 0$, we obtain the usual ring of left polynomials $D[t] = D[t;id,0]$, often also denoted $D_L[t]$ (cf. Gordon and Motzkin [2]).

Generalizing a result by Jimenez and Pérez-Izquierdo [4, Proposition 3], we can improve [8, Theorem 2] and show that every non-constant polynomial in $D[t;\sigma,\delta]$ decomposes into a product of linear factors, and thus has a zero.

For the elementary proof, we employ the nonassociative algebras S_f defined by Petit [7].

1. The Factorization Theorem

Proposition 1. *(i) Let F be a field of characteristic not 2 and D a quaternion division algebra over F. Suppose that any quadratic quaternionic equation*

$$t^2 + tb - a = 0, \quad a, b \in D, a \neq 0$$

has a non-zero solution. Then there are no eight-dimensional nonassociative division algebras over F with two associative nuclei (left, middle or right) isomorphic to D.

Date: 24.2.2015.

1991 *Mathematics Subject Classification.* Primary: 16S36; Secondary: 17A35, 12D05.

Key words and phrases. Skew-polynomials, Ore rings, factorization theorem, roots of polynomials, real closed field.

(ii) Let F be a real closed field and D the quaternion division algebra over F. Then there are no eight-dimensional nonassociative division algebras over F with two associative nuclei (left, middle or right) isomorphic to D.

Proof. (i) The proof is analogous to the one given for [4, Proposition 3].
(ii) If F is a real closed field, an equation of the type $t^2+tb-a=0, \quad a,b\in D, a\neq 0$ always has a non-zero solution [6]. □

Using the result quoted above, we are able to improve [8, Theorem 2]:

Theorem 2. *(Factorization Theorem) Let D be the quaternion division algebra over a real closed field. Then every polynomial $f\in D[t;\sigma,\delta]$ decomposes into a product of linear polynomials. In particular, $f(t)$ has a root.*
(ii) If σ is a ring automorphism and $f=p_1\cdots p_s=p_1'\cdots p_t'$ are two such decompositions, then $s=t$ and there is a permutation $\pi\in S_s$ such that

$$p'_{\pi(i)}\sim p_i$$

for all i, $1\leq i\leq s$.

Proof. (i) By [8, Theorem 2], every polynomial $f\in D[t;\sigma,\delta]$ of degree $m>2$ decomposes into a product of linear or quadratic irreducible polynomials. No irreducible quadratic polynomial is a two-sided element in $D[t;\sigma,\delta]$.

If f is an irreducible quadratic polynomial, then S_f must be a division algebra over F of dimension 8 and hence is nonassociative with $\mathrm{Nuc}_l(S_f)=\mathrm{Nuc}_m(S_f)=D$, contradiction Proposition 1.

In particular, $f(t)$ has a root in D: for $f(t)=g(t)(at-b)$, $f(a^{-1}b)=g(a^{-1}b)(aa^{-1}b-b)=0$.
(ii) By [3, Theorem 1.2.9], this decomposition is unique up to a permutation of the factors and similarity. □

References

[1] E. Darpö, E. Dieterich, M. Herschend, *In which dmensions does a division algebra over a given ground field exist?*, Enseign. Math. (2) 51 (3-4) (2005), 255-263.

[2] B. Gordon, T. S. Motzkin, *On the zeros of polynomials over division rings*, Trans. AMS 116 (1965), 218-226.

[3] N. Jacobson, "Finite-dimensional division algebras over fields," Springer Verlag, Berlin-Heidelberg-New York, 1996.

[4] Jimenez, C., Pérez-Izquierdo, J. M., *Ternary derivations of finite-dimensional real division algebras.* Linear Alg. and Its Appl. 428 (2008), 2192 – 2219.

[5] T. Y. Lam, " A first course in noncommutative algebra", second edition. Graduate Texts in Mathematics, 131. Springer-Verlag, New York, 2001.

[6] I. Niven, *Equations in quaternions*, The Amer. Math. Monthly 48 (10) (1941), 654-661.

[7] J.-C. Petit, *Sur certains quasi-corps généralisant un type d'anneau-quotient*, Séminaire Dubreil. Algèbre et théorie des nombres 20 (1966 - 67), 1-18.

[8] Pumplün, S., *Factoring skew polynomials over Hamilton's quaternion algebra and the complex numbers*, J. Algebra 427 (2015), 20-29.

E-mail address: susanne.pumpluen@nottingham.ac.uk

School of Mathematical Sciences, University of Nottingham, University Park, Nottingham NG7 2RD, United Kingdom