LECTURES ON ALGEBRAS AND TRIPLE SYSTEMS

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Preface

These are notes from a course given during the academic year 1971/72 at the University of Virginia. My aim was to give an introduction to the theory of algebras and triple system which should be accessible to anyone who had had the basic algebra course. I attempted to give a selfcontained exposition and provide a background that will enable the reader to understand much of the current work with triple systems. I also attempted to develop most of the results for arbitrary rings of scalars, resp. without restriction on the characteristic of the field of scalars. For this purpose the definitions of Jordan triple systems and alternative triple systems had to be changed; the present axioms were suggested to me by Kevin McCrimmon.

The material of Chapter I and II is standard and can be found in [7] and [25] (the numbers in brackets refer to the bibliography at the end of these notes). The introduction of the Jacobson radical via symmetry principle, shifting principle and addition formula will show the pattern after which one can introduce the radical in many kinds of algebras and triple systems. The equivalence relation in 2.4. is studied in more detail in [11].

Associative triple systems of the first kind were studied by Lister and the a.t.s. of the second kind at first by Hestenes [3] and some of his students; the structure theory I present in these notes is due to Loos [16], (his assumption char $\neq$ 2 is not really necessary).

For results on Lie algebras (Chapter V) one may consult [4].

The theory of Lie triple systems has been developed by Lister [14]; some generalizations and different proofs are due to myself [20]. The method of computing the Killing form can be found in [1].

For all results on linear Jordan algebras I refer to [1], [5] and [24]. The short proof of the fundamental formula is due to McCrimmon and myself (independently), see [6] and [23]. The material in Chapter IX (quadratic Jordan algebras) is taken from an unpublished manuscript of McCrimmon. (See also [6].)

Jordan triple systems made their first appearance in the literature (not yet with a name) in [9]. They were first studied per se by myself ([20], [21] and [22]). A structure theory for finite dimensional Jts over an algebraically closed field of char $\neq 2$ has been developed by Loos in [15]. The classification of minimal inner ideals (10.6) is modeled after the known results in Jordan algebras and is still incomplete for Jts's. The remarks on regularity in 10.7. are standard (see [8], [27]).

The construction of Lie algebras from Jordan triple systems was discovered by M. Koecher [9]. It turned out that Koecher's construction in the case in which the Jts is a unital Jordan algebra (viewed as a Jts) is a special case of a construction given previously by J. Tits in [26]. In my approach to the Koecher-Tits-construction I tried to make the construction look very natural by emphasising the connection between Jordan triple systems and Lie triple systems. This enabled me to apply the results of Chapter VI. For the significance of Jts and the Koecher-Tits algebras in differential geometry I refer to [10].

The basic formulas in chapter XIII are due to M. Koecher [10]. I provided different proofs, generalized to arbitrary rings of scalars and simplified my previous results in [24]

The theory of the Jacobson radical of a Jts includes a radical theory for many other structures (for example Jordan algebras, alternative algebras, associative and alternative triple systems). The nice result concerning strongly semiprime ideals (Theorem 13.13) is due to Lewand [12].

In Chapter XIV I give a simple proof of the von Neumann-regularity of a semi simple Jts with dcc and thus generalize results of McCrimmon's [19]. Theorem 14.4 is due to Helwig-Hirzebruch [2].

The Peirce decomposition is modeled after corresponding results for quadratic Jordan algebras, ([6] and [18]). The concept and a first structure theory of alternative triple systems is due to Loos [17]. He discovered the connection between alternative and Jordan triple systems. In the given presentation I followed closely his notes [17], but generalized most of the results to arbitrary rings of scalars (and thus was forced to do some lengthy computations, see 16.3, 16.4). The final classification is taken from Loos's notes.

I wish to thank John Faulkner and Kevin McCrimmon for their continued interest and many helpful suggestions and improvements. Finally I would like to thank Georgia Murphy, Suyin Liang and Bok Soon Park for typing the manuscript.

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CONTENTS

I. Nonassociative Algebras.

1.1. Let Φ be a commutative ring with 1. A unitary Φ-module $\mathfrak{A}$ together with a bilinear map (multiplication) $\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$, $(a,b) \mapsto ab$, is called an <u>algebra</u> over Φ (or Φ-algebra). An algebra $\mathfrak{A}$ is called <u>commutative</u> if $ab = ba$ for all $a,b \in \mathfrak{A}$; it is called <u>associative</u>, if $(ab)c = a(bc)$ for all $a,b,c \in \mathfrak{A}$.

<u>Examples.</u> 1). Any Φ-module $\mathfrak{A}$ together with $(a,b) \mapsto 0$ is an algebra.

2). If $\mathfrak{M}$ is a Φ-module, then $\mathrm{End}_\Phi \mathfrak{M}$ together with the usual composition of mappings is an algebra, the algebra of endomorphisms of $\mathfrak{M}$. $\mathrm{End}_\Phi \mathfrak{M}$ is associative (but in general not commutative).

3). $\Phi^{(n,n)}$, the Φ-module of $n \times n$ matrices over Φ together with usual matrix multiplication is an associative algebra.

4). In an associative algebra $\mathfrak{A}$ one often considers the commutator $[a,b] := ab - ba$. $\mathfrak{A}$ together with the map $(a,b) \mapsto [a,b]$ is an algebra, denoted by $\mathfrak{A}^-$. One easily checks

$$[a,a] = 0$$
$$[[a,b],c] + [[b,c],a] + [[c,a],b] = 0$$
$$\text{for all } a,b,c \in \mathfrak{A}^-.$$

5). If one considers the anticommutator $a \circ b := ab + ba$ in $\mathfrak{A}$ ($\mathfrak{A}$ associative) then $\mathfrak{A}$ together with $(a,b) \mapsto a \circ b$ is denoted by $\mathfrak{A}^+$ and one checks

$$a \circ b = b \circ a$$

$$a \circ ((a \circ a) \circ b) = (a \circ a) \circ (a \circ b) \text{ for all } a,b \in \mathfrak{A}^+.$$

An algebra $\mathfrak{L}$ (over Φ) is called a Lie algebra, if

(L.1) $xx = 0$

(L.2) $(xy)z + (yz)x + (zx)y = 0$ (Jacobi identity)

for all $x,y,z \in \mathfrak{L}$.

Example: If $\mathfrak{A}$ is associative then $\mathfrak{A}^-$ is a Lie algebra and any submodule of $\mathfrak{A}$ closed under $[x,y]$ is a Lie algebra.

An algebra $\mathfrak{J}$ is called a Jordan algebra, if

(J.1) $xy = yx$

(J.2) $(xx)(xy) = x((xx)y)$

for all $x,y \in \mathfrak{J}$.

Example: Any submodule of an associative algebra which is closed under $x \circ y$ is a Jordan algebra, in particular $\mathfrak{A}^+$ is a Jordan algebra.

1.2. Let $\mathfrak{A}$ be any nonassociative (that means not necessarily associative) Φ-algebra. For submodules $\mathfrak{U}, \mathfrak{V} \subset \mathfrak{A}$ we use the notations $\mathfrak{U}+\mathfrak{V}$ and $\mathfrak{U}\mathfrak{V}$ for the submodules generated by all $u + v$ rsp. uv, $u \in \mathfrak{U}, v \in \mathfrak{V}$. A submodule $\mathfrak{U}$ is a subalgebra, if $\mathfrak{U}\mathfrak{U} \subset \mathfrak{U}$, it is an ideal, if $\mathfrak{A}\mathfrak{U} + \mathfrak{U}\mathfrak{A} \subset \mathfrak{U}$. An ideal $\mathfrak{F}$ of $\mathfrak{A}$ is called a proper ideal, if $\mathfrak{F} \neq 0$ and $\mathfrak{F} \neq \mathfrak{A}$. $\mathfrak{A}$ is simple, if $\mathfrak{A}$ has no proper ideal and $\mathfrak{A}\mathfrak{A} \neq 0$.

If $\mathfrak{U}$ is an ideal in $\mathfrak{A}$, then one defines in a natural way in the quotient module

$$\bar{\mathfrak{A}} = \mathfrak{A}/\mathfrak{U}$$

a multiplication

$$(a + \mathfrak{N})(b + \mathfrak{N}) := ab + \mathfrak{N}\ .$$

$\bar{\mathfrak{A}}$ together with this multiplication is called the <u>quotient algebra</u> of $\mathfrak{A}$ mod $\mathfrak{N}$.

A <u>homomorphism</u> of Φ-algebras $\mathfrak{A}$, $\mathfrak{A}'$ is a Φ-linear map $f\colon\ \mathfrak{A} \to \mathfrak{A}'$ such that $f(ab) = f(a)f(b)$ for all $a,b \in \mathfrak{A}$.

Isomorphisms and automorphisms are defined in the usual way.

We have the standard results.

<u>Theorem 1.</u> (i) <u>A subset $\mathfrak{L} \subset \mathfrak{A}$ is an ideal, iff $\mathfrak{L}$ is the kernel of some homomorphism.</u>

(ii) <u>If $f\colon\ \mathfrak{A} \to \mathfrak{A}'$ is a homomorphism (of algebras) then</u> $f(\mathfrak{A}) \cong \mathfrak{A}/\text{kernel } f$

(iii) <u>If $\mathfrak{N}, \mathfrak{O}$ are ideals in $\mathfrak{A}$, then</u>

$$\mathfrak{N} + \mathfrak{O}/\mathfrak{O} \cong \mathfrak{N}/\mathfrak{N} \cap \mathfrak{O}\ .$$

1.3. Let $\mathfrak{A}$ be an algebra. A linear map $D\colon\ \mathfrak{A} \to \mathfrak{A}$ is called a <u>derivation</u> of $\mathfrak{A}$, if

$$D(ab) = (Da)b + a(Db) \qquad \text{for all } a,b \in \mathfrak{A}\ .$$

One easily checks that for derivations D_1, D_2 the commutator $[D_1,D_2]$ again is a derivation, hence the Φ-module of all derivations of $\mathfrak{A}$ together with the map $(D_1,D_2) \mapsto [D_1,D_2]$ is a Lie algebra (a subalgebra of $(\text{End}_{\Phi}\mathfrak{A})^-$). It is denoted by $\mathfrak{D}(\mathfrak{A})$ and called the <u>derivation algebra</u> of $\mathfrak{A}$.

1.4. For $a \in \mathfrak{A}$ we define endomorphisms $L(a)$ and $R(a)$ of $\mathfrak{A}$ by

$L(a)\colon x \mapsto ax$; $R(a)\colon x \mapsto xa$

i.e. $L(a)x = ax$, $R(a)x = xa$.

We call $L(a)$ rsp $R(a)$ the <u>left</u> (<u>rsp. right</u>) <u>multiplication</u> of a.

With these notations we rewrite some definitions.

a) $\mathfrak{A}$ is associative, i.e., $(xy)z = x(yz)$ for all $x,y,z \in \mathfrak{A}$ is equivalent to either

(i) $L(xy) = L(x)L(y)$

(ii) $R(yz) = R(z)R(y)$

(iii) $L(x)R(z) = R(z)L(x)$

(for all $x,y,z \in \mathfrak{A}$).

An easy computation shows that $L(x) - R(x)$, $x \in \mathfrak{A}$, are derivations of $\mathfrak{A}$.

b) Let $\mathcal{L}$ be a Lie algebra. In (L.1) we replace x by x + y and obtain $0 = (x + y)(x + y) = xx + xy + yx + yy = xy + yx$, or

$$xy = - yx.$$

with this the Jacobi identity may be written as

$$(xy)z = x(yz) - y(xz).$$

In terms of the left and right multiplications the last two equations are equivalent to

(L.1') $$L(x) = - R(x)$$

(L.2') $$L(xy) = [L(x),L(y)]$$

for all $x,y \in \mathcal{L}$.

c) Looking at ($\mathcal{J}$.1) and ($\mathcal{J}$.2) we see that an algebra $\mathfrak{J}$ is a Jordan algebra, iff

($\mathcal{J}$.1') $$L(x) = R(x)$$

($\mathcal{J}$.2') $$L(x)L(xx) = L(xx)L(x)$$

for all $x \in \mathfrak{J}$.

d) A linear map $D: \mathfrak{A} \to \mathfrak{A}$ is a derivation, iff

$$L(Dy) = [D,L(y)] \qquad \text{for all } y \in \mathfrak{A}.$$

(L.2') shows that in a Lie algebra all left multiplications are derivations.

1.5. For any algebra $\mathfrak{A}$ one defines the "derived series"

$$\mathfrak{A} = \mathfrak{A}^{(0)} \supset \mathfrak{A}^{(1)} \supset \mathfrak{A}^{(2)} \ldots \supset \mathfrak{A}^{(k)} \ldots$$

by $\mathfrak{A}^{(0)} = \mathfrak{A}$, $\mathfrak{A}^{(k+1)} = \mathfrak{A}^{(k)}\mathfrak{A}^{(k)}$.

In general only $\mathfrak{A}^{(0)}$ and $\mathfrak{A}^{(1)}$ are ideals of $\mathfrak{A}$.

Exercise. If $\mathcal{L}$ is a Lie algebra, then $\mathcal{L}^{(k)}$, $k \geq 0$, is an ideal of $\mathcal{L}$.

An algebra $\mathfrak{A}$ is called solvable, if $\mathfrak{A}^{(n)} = 0$ for some n.

Lemma 1. Subalgebras and homomorphic images of solvable algebras are solvable.

Proof. Easy exercise.

Lemma 2. If $\mathfrak{B}$ is an ideal of $\mathfrak{A}$, then $\mathfrak{A}$ is solvable iff $\mathfrak{B}$ and $\mathfrak{A}/\mathfrak{B}$ are solvable.

Proof. One direction follows from lemma 1. By the definition of multiplication in $\mathfrak{A}/\mathfrak{B}$ we get

$$\left(\mathfrak{A}/\mathfrak{B}\right)^{(k)} = \mathfrak{A}^{(k)}/\mathfrak{B}$$

The quotient being solvable implies $\mathfrak{A}^{(k)}/\mathfrak{B} = 0$ for some k, or equivalently $\mathfrak{A}^{(k)} \subset \mathfrak{B}$. But then $\mathfrak{A}^{(k+s)} \subset (\mathfrak{A}^{(k)})^{(s)} \subset \mathfrak{B}^{(s)} = 0$ for some s, since $\mathfrak{B}$ is solvable.

Theorem 2. (i) If $\mathfrak{U}$, $\mathfrak{V}$ are solvable ideals in an algebra $\mathfrak{A}$, then $\mathfrak{U} + \mathfrak{V}$ is a solvable ideal.

(ii) If $\mathfrak{A}$ is Noetherian then $\mathfrak{A}$ has a unique maximal solvable ideal $\mathcal{R}(\mathfrak{A})$ which contains all other solvable ideals and furthermore $\mathcal{R}\left(\mathfrak{A}/\mathcal{R}(\mathfrak{A})\right) = 0$.

(Note: We call $\mathfrak{A}$ Noetherian, if every non-empty set of ideals has a maximal element.)

Proof. By theorem 1 (iii) we have

$$\mathfrak{U} + \mathfrak{B}/\mathfrak{B} \cong \mathfrak{U}/\mathfrak{U} \cap \mathfrak{B} .$$

Since $\mathfrak{U}/\mathfrak{U}\cap\mathfrak{B}$ is a homomorphic image of the solvable ideal $\mathfrak{U}$ it is solvable by lemma 1. Hence $\mathfrak{U}+\mathfrak{B}/\mathfrak{B}$ is solvable, and lemma 2 then shows that $\mathfrak{U}+\mathfrak{B}$ is solvable.

Let $\mathfrak{A}$ be Noetherian and $\mathfrak{R}(\mathfrak{A})$ a maximal element in the set of all solvable ideals in $\mathfrak{A}$ (this set contains the zero ideal). Let $\mathfrak{R}'$ be any solvable ideal; then $\mathfrak{R}(\mathfrak{A}) + \mathfrak{R}'$ is solvable by part (i) of the theorem. Since $\mathfrak{R}(\mathfrak{A}) \subset \mathfrak{R}(\mathfrak{A}) + \mathfrak{R}'$ we have $\mathfrak{R}(\mathfrak{A}) = \mathfrak{R}(\mathfrak{A}) + \mathfrak{R}'$ by the maximality of $\mathfrak{R}(\mathfrak{A})$. This implies $\mathfrak{R}' \subset \mathfrak{R}(\mathfrak{A})$ and if in particular $\mathfrak{R}'$ is maximal solvable then $\mathfrak{R}' = \mathfrak{R}(\mathfrak{A})$. If $\overline{\mathfrak{U}}$ is solvable in $\mathfrak{A}/\mathfrak{R}(\mathfrak{A})$, then $\mathfrak{U}$ is solvable in $\mathfrak{A}$, hence contained in $\mathfrak{R}(\mathfrak{A})$ and consequently $\overline{\mathfrak{U}} = 0$, which shows $\mathfrak{R}(\mathfrak{A}/\mathfrak{R}(\mathfrak{A})) = 0$.

The unique maximal solvable ideal $\mathfrak{R}(\mathfrak{A})$ is called the <u>solvable radical</u> of $\mathfrak{A}$.

1.6. Powers of an element $a \in \mathfrak{A}$ ($\mathfrak{A}$ an arbitrary algebra) are defined recursively by

$$a^1 = a, \quad a^{n+1} = a^n a.$$

In general $a^n a \neq a a^n$.

An algebra $\mathfrak{A}$ is called <u>power-associative</u>, if $a^n a^m = a^{n+m}$ for all $a \in \mathfrak{A}$, $n,m \geq 1$.

$a \in \mathfrak{A}$ is <u>nilpotent</u>, if $a^n = 0$ for some n. (0 is nilpotent).
An ideal $\mathfrak{B} \subset \mathfrak{A}$ is called <u>nil</u>, if all elements in $\mathfrak{B}$ are nilpotent.

<u>Lemma 3. Let $\mathfrak{A}$ be an algebra in which $(a^n)^m = a^{nm}$ for all $a \in \mathfrak{A}$, n,m≥1. If $\mathfrak{B}$, $\mathfrak{C}$ are nil ideals of $\mathfrak{A}$, then $\mathfrak{B} + \mathfrak{C}$ is nil.</u>

<u>Proof</u>. Let $b + c \in \mathfrak{B} + \mathfrak{C}$ ($b \in \mathfrak{B}$, $c \in \mathfrak{C}$), then
$(b + c)^n = b^n + d$ where $d \in \mathfrak{C}$. Since b is nilpotent we get
$(b + c)^n = d$ for some n. Since $d \in \mathfrak{C}$, it is nilpotent and with our assumption it follows
$(b + c)^{nm} = ((b + c)^n)^m = 0$ for some m.

Since the property of an ideal (being nil) is defined elementwise we get the existence of a maximal nil ideal $\mathfrak{N}(\mathfrak{A})$ by Zorn's lemma. The previous lemma shows that if $\mathfrak{A}$ is power associative then $\mathfrak{N}$ is uniquely determined; it is called the <u>nilradical</u> of $\mathfrak{A}$.

1.7. An element $e \in \mathfrak{A}$ (again $\mathfrak{A}$ arbitrary) is called a <u>unit element</u>, if $ea = ae = a$ for all $a \in \mathfrak{A}$, or equivalently, if $L(e) = R(e) = id$, the identity mapping. $c \in \mathfrak{A}$ is called an <u>idempotent</u> of $\mathfrak{A}$ if $c \neq 0$ and $c^2 = c$.

There is a standard construction to imbed any algebra $\mathfrak{A}$ into an algebra $\hat{\mathfrak{A}}$ with unit element. Consider the Φ-module

$$\hat{\mathfrak{A}} = \Phi\cdot 1 \oplus \mathfrak{A} = \{(\alpha, a);\ \alpha \in \Phi,\ a \in \mathfrak{A}\}$$

and define a multiplication in $\hat{\mathfrak{A}}$ by the formula

$$(\alpha, a)(\beta, b) := (\alpha\beta, \alpha b + \beta a + ab),$$

then $\hat{\mathfrak{A}}$ has a unit element (1,0) and $a \mapsto (0,a)$ defines an isomorphism of $\mathfrak{A}$ into $\hat{\mathfrak{A}}$. By means of this isomorphism one identifies

$\mathfrak{A}$ with its image, so $\mathfrak{A}$ is an ideal in $\hat{\mathfrak{A}}$. Instead of $(\alpha, a) \in \hat{\mathfrak{A}}$ we write $\alpha 1 + a$. If $\mathfrak{A}$ is associative, so is $\hat{\mathfrak{A}}$ (easy exercise), but if $\mathfrak{A}$ is a Lie algebra, $\hat{\mathfrak{A}}$ is not a Lie algebra, since a Lie algebra does not have a unit element $\neq 0$.

1.8. An endomorphism $j: \mathfrak{A} \to \mathfrak{A}$ of an algebra is called an *involution*, if

$$j(ab) = j(b)j(a)$$

$$j(j(a)) = a \qquad \text{for all } a,b \in \mathfrak{A}.$$

If $\mathfrak{A}^{op}$ denotes the algebra which has the same module as $\mathfrak{A}$ but multiplication $(x,y) \mapsto x \circ y$ defined by $x \circ y = yx$ for all $x,y \in \mathfrak{A}$, then an involution may be viewed as a isomorphism $j: \mathfrak{A} \to \mathfrak{A}^{op}$. A submodule $\mathfrak{B} \subset \mathfrak{A}$ is j-stable if $j(\mathfrak{B}) \subset \mathfrak{B}$. Let $\mathfrak{A}$ have an involution j. The pair $(\mathfrak{A}, j)$ is called *simple*, if $\mathfrak{A}$ has no proper j-stable ideal and $\mathfrak{A}^2 \neq 0$.

Theorem 3. Let $\mathfrak{A}$ be an algebra with involution j and $(\mathfrak{A}, j)$ simple. Then either

(i) *$\mathfrak{A}$ is simple, or*

(ii) *$\mathfrak{A} \cong \mathfrak{B} \oplus \mathfrak{B}^{op}$, $\mathfrak{B}$ a simple ideal of $\mathfrak{A}$ and*

$$j(b_1, b_2) = (b_2, b_1).$$

Proof. If $\mathfrak{A}$ is not simple, then it has a proper ideal $\mathfrak{B}$. $(0 \neq \mathfrak{B} \neq \mathfrak{A})$. It is obvious that $j(\mathfrak{B}) \cap \mathfrak{B}$ and $j(\mathfrak{B}) + \mathfrak{B}$ are j-stable ideals, consequently $j(\mathfrak{B}) \cap \mathfrak{B} = 0$ and $j(\mathfrak{B}) \oplus \mathfrak{B} = \mathfrak{A}$ since $(\mathfrak{A}, j)$ is simple. By the previous remarks $j(\mathfrak{B})$ may be viewed as isomorphic image of $\mathfrak{B}^{op}$, hence $\mathfrak{A} \cong \mathfrak{B} \oplus \mathfrak{B}^{op}$ and $j(b_1, b_2) = (b_2, b_1)$. If $\mathfrak{C}$ is an ideal of $\mathfrak{B}$ then $\mathfrak{A}\mathfrak{C} \subset \mathfrak{B}\mathfrak{C}$, since $\mathcal{L}^{op}\mathcal{C} \subset \mathcal{L}^{op} \cap \mathcal{L} = 0$. This shows that $\mathcal{C}$ is an ideal in $\mathfrak{A}$ and if $\mathcal{C} \neq 0$ the above construction shows $\mathfrak{A} = \mathcal{C} \oplus j(\mathcal{C})$. This implies $\mathcal{L} = \mathcal{C}$ and $\mathcal{L}$ is simple.

1.9. An important tool in the structure theory of algebras are certain bilinear forms.

Let $\mathfrak{A}$ be an algebra over Φ and $\lambda : \mathfrak{A} \times \mathfrak{A} \to \Phi$ a bilinear form. λ is called <u>associative</u>, if

$$\lambda(xy,z) = \lambda(x,yz)$$

<u>Example</u>. If $\mathfrak{A}$ is a finite dimensional associative algebra over a field, then $(x,y) \mapsto \text{trace } L(xy)$ is an associative bilinear form.

The importance of such forms can be seen from

<u>Theorem 4</u>. <u>(Dieudonné)</u>. <u>Let $\mathfrak{A}$ be a finite dimensional algebra over a field F satisfying</u>

(i) $\mathfrak{A}$ <u>has a symmetric non degenerate associative bilinear form</u> λ,

(ii) <u>if $\mathcal{L} \neq 0$ is an ideal of $\mathfrak{A}$, then $\mathcal{L}^2 \neq 0$.</u>

<u>Then $\mathfrak{A}$ is a direct sum of simple ideals of $\mathfrak{A}$.</u>

<u>Proof</u>. Let $\mathcal{L}$ be a minimal ideal ($\neq 0$) of $\mathfrak{A}$. The associativity of λ shows that $\mathcal{L}^{\perp} = \{x, \ \lambda(x,\mathcal{L}) = 0\}$ is an ideal of $\mathfrak{A}$. Since $\mathcal{L} \cap \mathcal{L}^{\perp}$ is an ideal in $\mathcal{L}$, we get $\mathcal{L} \cap \mathcal{L}^{\perp} = \mathcal{L}$ or $\mathcal{L} \cap \mathcal{L}^{\perp} = 0$, by the choice of $\mathcal{L}$. Suppose the first case holds and let $b, b' \in \mathcal{L}$, $a \in \mathfrak{A}$; then $0 = \lambda(ab,b') = \lambda(a,bb')$. Since λ is non degenerate, $bb' = 0$ and $\mathcal{L}^2 = 0$, contrary to assumption. Hence $\mathcal{L} \cap \mathcal{L}^{\perp} = 0$ and $\mathfrak{A} = \mathcal{L} \oplus \mathcal{L}^{\perp}$ (Here we make use of the finite dimensionality of $\mathfrak{A}$). Any ideal of $\mathcal{L}$ is an ideal of $\mathfrak{A}$ (same argument as in the proof of theorem 3), then by the minimality of

$\mathcal{L}$ it has no proper ideal. Since $\mathcal{L}^2 \neq 0$ by assumption, we see that $\mathcal{L}$ is simple. Since the assumptions (i) and (ii) are true in $\mathcal{L}^{\perp}$ we get by an induction argument the decomposition of $\mathfrak{A}$ as a direct sum of simple ideals.

II. Associative Algebras

2.1. Let $\mathfrak{A}$ be an associative algebra over a ring Φ and assume that $\mathfrak{A}$ has a unit element e. An element $a \in \mathfrak{A}$ is called left invertible (resp. right invertible) if there is an element $b \in \mathfrak{A}$ ($b' \in \mathfrak{A}$) such that $ba = e$ (resp. $ab' = e$). a is invertible if a is left and right invertible.

Lemma 1. The following statements are equivalent,

(i) $a \in \mathfrak{A}$ is invertible,

(ii) there is a unique element $a^{-1} \in \mathfrak{A}$ such that $a^{-1}a = aa^{-1} = e$

(iii) $L(a)$ is invertible (in $\mathrm{end}_{\Phi} \mathfrak{A}$).

Proof. Let $b, b' \in \mathfrak{A}$ be such that $ba = ab' = e$. Then $b = be = b(ab') = (ba)b' = eb' = b'$, consequently (i) $\to$ (ii). If $a^{-1}a = aa^{-1} = e$ then $L(a^{-1})L(a) = L(a)L(a^{-1}) = id$. This shows that $L(a)$ is invertible and $L(a^{-1}) = L(a)^{-1}$, thus (ii) $\to$ (iii). To show (iii) $\to$ (i) assume $L(a)$ invertible, i.e. $L(a)U = UL(a) = id$ for a unique $U \in \mathrm{end}\, \mathfrak{A}$ (apply (i) $\to$ (ii) to $\mathrm{End}\, \mathfrak{A}$). All terms of this equation acting on $e \in \mathfrak{A}$ gives $au = Ua = e$ for $u = Ue$. But then $L(a)L(u) = id$

and $L(a)U = id$, consequently $U = L(u)$ (since the inverse is unique). It follows $au = ua = e$. (Observe that the associative law was used at essential steps).

Lemma 2. *If $u \in \mathfrak{A}$ is nilpotent, then $e - u$ is invertible.*

Proof. Let $u^k = 0$, then put $v = e + u + \ldots + u^{k-1}$ and check $(e - u)v = v(e - u) = e$.

2.2. Lemma 2 leads to the following definition. Let $\mathfrak{A}$ be an associative algebra (not neccessarily with unit element) and $\hat{\mathfrak{A}} = \Phi 1 \oplus \mathfrak{A}$ be the algebra obtained from $\mathfrak{A}$ by adjoining a unit element (see 1.7.).

$x \in \mathfrak{A}$ is called *quasi invertible* (q.i.) with *quasi inverse* y, if $1 - x$ is invertible in $\hat{\mathfrak{A}}$ with inverse $1 + y$. (Remark: If $1 - u$ has left or right inverse $\alpha 1 + v$ in $\hat{\mathfrak{A}}$ then $1 = (\alpha 1 + v)(1 - u) = \alpha 1 + v - \alpha u - vu$ implies $\alpha = 1$.)

Lemma 3. *The following statements are equivalent:*

(i) $x \in \mathfrak{A}$ *is quasi invertible,*

(ii) *there exists* $y \in \mathfrak{A}$ *such that* $y - x = yx = xy$,

(iii) $id - L(x)$ *is invertible.*

In either case the quasi inverse y *is uniquely determined by*

(2.1) $\quad y = (id - L(x))^{-1}x.$

Proof. (ii) $\rightarrow$ (i). Assume $y - x = yx = xy$, then

$$1 = 1 + y - x - yx = (1 + y)(1 - x) \text{ and}$$

$$1 = 1 + y - x - xy = (1 - x)(1 + y).$$

(i) $\rightarrow$ (iii) If $1 - x$ is invertible in $\hat{\mathfrak{A}}$ then by lemma 1 the left multiplication $L(1 - x)$ of $1 - x$ in $\hat{\mathfrak{A}}$ is invertible and consequently the restriction to $\mathfrak{A}$, $\hat{L}(1 - x)|_{\mathfrak{A}} = \mathrm{id} - L(x)$ must be invertible since $\mathfrak{A}$ is an ideal of $\hat{\mathfrak{A}}$. If (iii) holds, set $y := (\mathrm{id} - L(x))^{-1}x$ and obtain $y - x = xy = yx$. (For $xy = yx$ use the fact that $L(x)(\mathrm{id} - L(x))^{-1} = (\mathrm{id} - L(x))^{-1}L(x)$.) Since the inverse of an element is uniquely determined, y is unique and we just saw $y = (\mathrm{id} - L(x))^{-1}x$.

Remarks. 1) Lemma 2 shows that nilpotent elements are quasi invertible.

2) The equivalence (i) <=> (ii) shows that if $\mathfrak{A}$ has a unit element e, then x is q.i. iff e - x is invertible in $\mathfrak{A}$.

2.3. Let $\mathfrak{A}$ be an associative algebra and $u \in \mathfrak{A}$. The map $(x,y) \mapsto xuy$, $x,y \in \mathfrak{A}$ defines another multiplication on $\mathfrak{A}$. The module $\mathfrak{A}$ together with this multiplication is denoted by $\mathfrak{A}_u$ and is called the u-homotope of $\mathfrak{A}$. It is obvious that any homotope of an associative algebra is associative. Lemma 3 shows that x q.i. in $\mathfrak{A}_u$ with quasi inverse y, iff

$$y - x = xuy = yux. \tag{2.2}$$

We introduce the following notations; we say q(x,y) exists, if x is q.i. in $\mathfrak{A}_y$ with quasi inverse q(x,y); if x is q.i. in $\mathfrak{A}$ we denote the quasi inverse of x by q(x,1). Furthermore, we define

$$B(x,y) := \mathrm{id} - L(xy) \tag{2.3}$$

Lemma 4. (Symmetry principle). The following statements are equivalent,

(i) q(x,y) exists,

(ii) q(xy,1) exists,

(iii) $q(y,x)$ <u>exists</u>,

(iv) $q(yx,1)$ <u>exists</u>,

(v) $B(x,y)$ <u>invertible</u>,

(vi) $B(y,x)$ <u>invertible</u>,

<u>In either case</u>

(2.4) $$q(x,y) = B(x,y)^{-1}x$$

<u>Exercise.</u> $q(x,x)$ exists $\rightarrow q(x,1)$ exists.

<u>Proof.</u> (i) $\rightarrow$ (ii). Let $u = q(x,y)$. Then by (2.2) $u - x = xyu = uyx$. Multiply by y from the right to obtain $uy - xy = xyuy = uyxy$, this means that $q(xy,1)$ exists.

(ii) $\rightarrow$ (iii) Let $w = q(xy, 1)$, then

$$w - xy = wxy = xyw, \text{ hence}$$

$$yw - yxy = ywxy = yxyw. \text{ It follows}$$

$$(yw + y) - y = yw = ywxy + yxy = yxyw + yxy = (yw + y)xy = yx(yw + y).$$

But this means that $q(y,x)$ exists.

(iii) $\rightarrow$ (iv) $\rightarrow$ (i) follows from interchanging x and y in the parts we already proved. (ii) $\Leftrightarrow$ (v) follows from lemma 3. Then (2.4) follows from (2.1) in the y-homotope.

<u>Remark</u>. Actually we proved a stronger result, namely if

$$u = q(x,y) \text{ then } uy = q(xy, 1) \text{ and}$$

$$q(y,x) = yq(x,y)y + y.$$

<u>Lemma 5. (Shifting principle)</u>.

<u>If</u> φ, ψ <u>are endomorphisms of</u> $\mathfrak{A}$ <u>such that</u>

$$L(\varphi x)R(\varphi y) = \varphi L(x)R(y)\psi$$

<u>and</u> $$L(\psi x)R(\psi y) = \psi L(x)R(y)\varphi \text{ for all } x,y \in \mathfrak{A},$$

<u>then</u> $q(x,\psi y)$ <u>exists iff</u> $q(\varphi x,y)$ <u>exists</u>.

<u>In either case</u>

$$\varphi q(x, \psi y) = q(\varphi x, y).$$

Proof. Let $u = q(x, \psi y)$, i.e.

$$u - x = u(\psi y)x = x(\psi y)u.$$

Apply φ to obtain (using the assumptions on φ, ψ)

$$\varphi u - \varphi x = \varphi(u\psi yx) = \varphi(x\psi yu)$$

$$= (\varphi u)y(\varphi x) = (\varphi x)y(\varphi u)$$

This shows $\varphi q(x, \psi y) = q(\varphi x, y)$.

Assume $q(\varphi x, y)$ exists, then by the symmetry principle $q(y, \varphi x)$ exists, by the part we already proved we get that $q(\psi y, x)$ exists, again the symmetry principle implies that $q(x, \psi y)$ exists.

Remark: $\varphi = L(a), \psi = R(a)$ and $\varphi = R(b), \psi = L(b), a, b \in \hat{\mathfrak{A}}$ satisfy the hypotheses of the lemma.

Corollary. *If* $a, b \in \hat{\mathfrak{A}}, x, y \in \mathfrak{A}$, *then*

$q(axb, y)$ *exists iff* $q(x, bya)$ *exists.*

Lemma 6. *(Addition formula.) If* $q(x,y)$ *exists, then*

(i) $B(x,y)B(q(x,y),z) = B(x,y+z)$

(ii) $q(q(x,y),z)$ *exists iff* $q(x,y+z)$ *exists. If this is the case then*

$$q(q(x,y),z) = q(x,y+z) \tag{2.5}$$

Proof. Put $u = q(x,y)$. Since $u - x = uyx = xyu$ we get

$$(id - L(xy))(id - L(uz)) = id - L(xy) - L(uz) + L(xyuz)$$

$$= id - L(x(y+z)) = B(x,y+z)$$

This is (i). Since $q(a,b)$ ex. iff $B(a,b)$ invertible, the first part of (ii) can be read off from (i) since $B(x,y)$ is invertible. Using (2.4) and (i) we get

$$q(x,y+z) = B(q(x,y),z)^{-1}B(x,y)^{-1}x = B(q(x,y),z)^{-1}q(x,y)$$
$$= q(q(x,y),z).$$

Now we define

$\mathrm{Rad}\,\mathfrak{A} := \{x \in \mathfrak{A},\ q(x,y) \text{ exists for all } y \in \mathfrak{A}\}$

Note: If $x \in \mathrm{Rad}\,\mathfrak{A}$ then in particular $q(x,1)$ exists (see exercise, p. 13).

Theorem 1. Rad $\mathfrak{A}$ is an ideal in $\mathfrak{A}$ and $\mathrm{Rad}(\mathfrak{A}/\mathrm{Rad}\,\mathfrak{A}) = 0$.

Proof. $x \in \mathrm{Rad}\,\mathfrak{A}$ is equivalent to $B(x,y)$ invertible for all $y \in \mathfrak{A}$, by lemma 4. If $\alpha \in \Phi$, $x \in \mathrm{Rad}\,\mathfrak{A}$ then $\alpha x \in \mathrm{Rad}\,\mathfrak{A}$ follows immediately from $B(\alpha x,y) = B(x,\alpha y)$. If $y,z \in \mathrm{Rad}\,\mathfrak{A}$ then $B(x,y)$ and $B(u,z)$ are invertible for all $x,u \in \mathfrak{A}$ (symmetry principle) in particular $B(q(x,y),z)$ is invertible. The addition formula then shows that $B(x,y+z)$ is invertible for all x, thus $y + z \in \mathrm{Rad}\,\mathfrak{A}$. We proved that Rad $\mathfrak{A}$ is a submodule. If $q(x,y)$ exists for all $y \in \mathfrak{A}$, $q(x,ayb)$ exists for all $a,b \in \hat{\mathfrak{A}}$. But then $q(bxa,y)$ exists (Shifting principle resp. its corollary). Consequently $\hat{\mathfrak{A}}(\mathrm{Rad}\,\mathfrak{A})\hat{\mathfrak{A}} \subset \mathrm{Rad}\,\mathfrak{A}$ and Rad $\mathfrak{A}$ is an ideal. If $\bar{x} \in \mathrm{Rad}\,\bar{\mathfrak{A}}$, $\bar{\mathfrak{A}} = \mathfrak{A}/\mathrm{Rad}\,\mathfrak{A}$, then for every $\bar{y}$ there exists $\bar{u}$ such that $\bar{u} - \bar{x} = \overline{uyx} = \overline{xyu}$ or equivalently $u - x - uyx \in \mathrm{Rad}\,\mathfrak{A}$. But then $B(u - x - uyx, -y) = B(u,-y)B(x,y)$ is invertible and therefore $B(x,y)$ is right invertible, similarly we get that $B(x,y)$ is also left invertible, hence invertible. This is true for all $y \in \mathfrak{A}$, hence $x \in \mathrm{Rad}\,\mathfrak{A}$ and $\bar{x} = 0$.

The ideal Rad $\mathfrak{A}$ is called the Jacobson radical of $\mathfrak{A}$. $\mathfrak{A}$ is called semi simple, if Rad $\mathfrak{A} = 0$.

A submodule $\mathfrak{L}$ of $\mathfrak{A}$ is called a left ideal, if $\mathfrak{A}\mathfrak{L} \subset \mathfrak{L}$, $\mathfrak{L}$ is called quasi invertible (nil) if every element of $\mathfrak{L}$ is quasi invertible (resp. nilpotent). Since a nilpotent element is quasi invertible, every nil module is quasi invertible.

Theorem 2. *If $\mathfrak{L}$ is a quasi invertible left ideal of $\mathfrak{A}$, then $\mathfrak{L} \subset$ Rad $\mathfrak{A}$.*

Proof. Let $b \in \mathfrak{L}$, $x \in \mathfrak{A}$, then $xb \in \mathfrak{L}$ and is quasi invertible by assumption, i.e., $q(xb,1)$ ex. From the symmetry principle we get that $q(b,x)$ exists for all $x \in \mathfrak{A}$, hence $b \in$ Rad $\mathfrak{A}$.

Corollary. *Rad $\mathfrak{A}$ contains every nil left ideal of $\mathfrak{A}$.*

Remark: The sam argument applies to right ideals.

Theorem 3. *If $\mathfrak{L}$ is an ideal of $\mathfrak{A}$, then*

$$\text{Rad}\,\mathfrak{L} = \mathfrak{L} \cap \text{Rad}\,\mathfrak{A}.$$

Proof. Clearly $\mathfrak{L} \cap \text{Rad}\,\mathfrak{A} \subset \text{Rad}\,\mathfrak{L}$ since the quasi inverse of an element of $\mathfrak{L}$ is in $\mathfrak{L}$ (by (2.1)). Conversely let x be an element in Rad $\mathfrak{L}$; then $q(x,b)$ exists for all $b \in \mathfrak{L}$ and therefore $B(x,b)$ is invertible for all $b \in \mathfrak{L}$. Since $B(x,-z)B(x,z) = B(x,z)B(x,-z) = B(x,zxz)$ for all $z \in \mathfrak{A}$ and $zxz \in \mathfrak{L}$ $(x \in \mathfrak{L})$, we get that $B(x,z)$ has to be invertible for all $z \in \mathfrak{A}$, or $x \in \text{Rad}\,\mathfrak{A}$.

Corollary. *Every ideal of a semi simple associative algebra is semi simple.*

Exercise: If $\alpha : \mathfrak{A} \to \mathfrak{A}$ is an automorphism, then

$$\alpha(\text{Rad}\,\mathfrak{A}) = \text{Rad}\,\mathfrak{A}$$

2.4. Using the notion of quasi invertibility we can introduce a relation on $\mathfrak{A}$ by the following definition

$$R := \{ (x,y) \in \mathfrak{A} \times \mathfrak{A}, \quad x = q(y,w) \text{ for some } w \in \mathfrak{A} \}$$

Using lemma 3 we see $(x,y) \in R$ iff $x - y = xwy = ywx$ for some $w \in \mathfrak{A}$.

<u>Theorem 4.</u> <u>R is an equivalence relation on</u> $\mathfrak{A}$.

<u>Proof.</u> $(x,x) \in R$ for all $x \in \mathfrak{A}$ and

$(x,y) \in R \Rightarrow (y,x) \in R$ are obvious.

If $(x,y),(y,z) \in R$, then $x = q(y,w)$ for some $w \in \mathfrak{A}$ and $y = q(z,u)$ for some u. But then from the addition formula we get $x = q(q(z,u),w) = q(z,u+w)$, in particular $(x,z) \in R$.

2.5. <u>The Peirce decomposition.</u> Let $\mathfrak{A}$ be an associative algebra and $c = c^2$ an idempotent in $\mathfrak{A}$. Clearly

$$x = cxc + (cx - cxc) + (xc - cxc) + (x - cx - xc + cxc) \tag{2.6}$$

Define

$\mathfrak{A}_{11} = c\,\mathfrak{A}\,c$, $\mathfrak{A}_{10} = c\,\mathfrak{A}(1-c)$, $\mathfrak{A}_{01} = (1-c)\,\mathfrak{A}\,c$, $\mathfrak{A}_{00} = (1-c)\,\mathfrak{A}\,(1-c)$; it is immediately seen

$c\,\mathfrak{A}_{11} = \mathfrak{A}_{11}c = \mathfrak{A}_{11}$, $c\,\mathfrak{A}_{10} = \mathfrak{A}_{10}$, $\mathfrak{A}_{10}c = 0$, $c\,\mathfrak{A}_{01} = 0$, $\mathfrak{A}_{01}c = \mathfrak{A}_{01}$, $c\,\mathfrak{A}_{00} = \mathfrak{A}_{00}c = 0$. This together with (2.6) shows

$$\mathfrak{A} = \mathfrak{A}_{11} \oplus \mathfrak{A}_{10} \oplus \mathfrak{A}_{01} \oplus \mathfrak{A}_{00} \tag{2.7}$$

The decomposition (2.7) is called the <u>Peirce decomposition</u> of $\mathfrak{A}$ relative to c and $\mathfrak{A}_{ij}$ are the Peirce spaces (resp. modules).

<u>Exercise.</u> $\mathfrak{A}_{ii}\,\mathfrak{A}_{ii} \subset \mathfrak{A}_{ii}$ $(i = 0,1)$, $\mathfrak{A}_{11}\,\mathfrak{A}_{10} \subset \mathfrak{A}_{10}$, $\mathfrak{A}_{11}\,\mathfrak{A}_{01} = 0$, etc.

Lemma 7. If $\mathcal{L} \subset \mathfrak{A}$ is an ideal, then

$$\mathcal{L} = \bigoplus_{i,j = 0,1} (\mathfrak{A}_{ij} \cap \mathcal{L})$$

Proof. The decomposition (2.6) (which is unique) shows that the components of $b \in \mathcal{L}$ in the different Peirce spaces are elements of $\mathcal{L}$ since $\mathcal{L}$ is an ideal.

Theorem 5. (i) $\text{Rad}\, \mathfrak{A} = \oplus(\mathfrak{A}_{ij} \cap \text{Rad}\, \mathfrak{A})$

(ii) $\text{Rad}\, \mathfrak{A}_{ii} = \mathfrak{A}_{ii} \cap \text{Rad}\, \mathfrak{A}$

Proof. Clearly $\mathfrak{A}_{ii} \cap \text{Rad}\, \mathfrak{A} \subset \text{Rad}\, \mathfrak{A}_{ii}$. Assume $i = 1$ and $x \in \text{Rad}\, \mathfrak{A}_{11}$, then $x = cxc$ and $q(x,cyc)$ exist for all $y \in \mathfrak{A}$. But by the symmetry principle this is the case iff $q(cxc,y) = q(x,y)$ exists for all $y \in \mathfrak{A}$, consequently $x \in \text{Rad}\, \mathfrak{A}$.

Example. Let $\mathfrak{A}$ be an associative Φ-algebra with unit element e. Consider the Φ-algebra $\mathfrak{A}^{(n,n)}$ of all nxn matrices with coefficients in $\mathfrak{A}$. The multiplication is the usual matrix multiplication. $\mathfrak{A}^{(n,n)}$ is associative.

The matrix

$$E = \begin{pmatrix} e & & & & \\ & \ddots & & 0 & \\ & & e & & \\ & & & 0 & \\ & 0 & & & \ddots \\ & & & & & 0 \end{pmatrix} \begin{matrix} \}\, p \\ \\ \}\, q \end{matrix}, \quad p + q = n$$

obviously is an idempotent.

For the computation of the Peirce components A_{ij} of

$$A = \left(\begin{array}{c|c} A_1 & \overbrace{A_2}^{q} \\ \hline A_3 & A_4 \end{array}\right)\}p$$

we use (2.6) and get

$A_{11} = EAE = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}$ as component of A in $\mathfrak{A}_{11}$. Similarly

$$A_{10} = \begin{pmatrix} 0 & A_2 \\ 0 & 0 \end{pmatrix}, \quad A_{01} = \begin{pmatrix} 0 & 0 \\ A_3 & 0 \end{pmatrix}, \quad A_{00} = \begin{pmatrix} 0 & 0 \\ 0 & A_4 \end{pmatrix}.$$

2.6. <u>$\mathfrak{A}$-modules.</u> Let $\mathfrak{A}$ be an associative Φ-algebra and $\mathfrak{M}$ a unital Φ-module. $\mathfrak{M}$ together with a map $\mathfrak{A} \times \mathfrak{M} \to \mathfrak{M}$, $(a,m) \mapsto a\cdot m$ is called a left $\mathfrak{A}$-module, if $(a,m) \to a\cdot m$ is Φ-bilinear and if $x\cdot(y\cdot m) = (xy)\cdot m$ for all $x,y \in \mathfrak{A}$, $m \in \mathfrak{M}$.

<u>Example</u>. Any left ideal in $\mathfrak{A}$ is an $\mathfrak{A}$-module.

<u>Remark:</u> If $\mathfrak{L}$ is a subalgebra of $\mathfrak{A}$, then $\mathfrak{M}$ together with the induced map $\mathfrak{L} \times \mathfrak{M} \to \mathfrak{M}$, $(b,m) \mapsto b\cdot m$, is a $\mathfrak{L}$-module. Right $\mathfrak{A}$-modules are defined accordingly.

2.7. An associative algebra is called (left) <u>Artinian</u>, if any non-empty set of left ideals has a minimal element.

<u>Exercise</u>. $\mathfrak{A}$ is left Artinian, iff any descending chain of left ideals of $\mathfrak{A}$, $\mathfrak{L}_1 \supset \mathfrak{L}_2 \supset \cdots \supset \mathfrak{L}_k \supset \cdots$, becomes stationary, i.e., $\mathfrak{L}_n = \mathfrak{L}_{n+j}$, $j \geq 1$ for some n. Since we will be mainly concerned with the radical of Artinian algebras, we shall only prove the fundamental result about the radical in an Artinian algebra. We need a definition. If $\mathfrak{A}$ is an associative algebra and $\mathfrak{L}$ a subalgebra, then the powers of $\mathfrak{L}$ are defined recursively by $\mathfrak{L}^1 = \mathfrak{L}$, $\mathfrak{L}^{k+1} = \mathfrak{L}^k\mathfrak{L}$. $\mathfrak{L}$ is called <u>nilpotent</u> if $\mathfrak{L}^n = 0$ for some $n \geq 1$.

<u>Theorem 6.</u> <u>If $\mathfrak{A}$ is Artinian, then</u> Rad $\mathfrak{A}$ <u>is nilpotent</u>.

<u>Proof.</u> We set $\mathfrak{R} := \mathrm{Rad}\,\mathfrak{A}$ and consider the descending chain

$$\mathfrak{R} \supset \mathfrak{R}^2 \supset \cdots \mathfrak{R}^m \supset \cdots$$

then $\mathcal{R}^k = \mathcal{R}^n$ for some k and all $n \geq k$.

Suppose $\mathcal{R}^k \neq 0$ and let

$S = \{ 0 \neq \mathfrak{U}$, $\mathfrak{U}$ is left ideal of $\mathfrak{A}$ and $\mathcal{R}^k \mathfrak{U} \neq 0 \}$.

$S \neq \emptyset$ since $\mathcal{R}^k \mathcal{R} = \mathcal{R}^{k+1} = \mathcal{R}^k \neq 0$ implies $\mathcal{R} \in S$. Let $\mathfrak{L}$ be a minimal element of S ($\mathfrak{A}$ is Artinian). $\mathcal{R}^k \mathfrak{L} \neq 0$ implies that there exists an element $b \in \mathfrak{L}$ such that $\mathcal{R}^k b \neq 0$. Clearly $\mathcal{R}^k b \subset \mathfrak{L}$. Since $\mathcal{R}^k (\mathcal{R}^k b) = \mathcal{R}^{2k} b = \mathcal{R}^k b \neq 0$ we get that $\mathcal{R}^k b \in S$. Consequently $\mathcal{R}^k b = \mathfrak{L}$ since $\mathfrak{L}$ is minimal in S. Now we have $b = rb$ for some $r \in \mathcal{R}^k \subset \mathcal{R}$ or equivalently $(1 - r)\, b = 0$. But $r \in \mathcal{R}$ implies $1 - r$ invertible in $\hat{\mathfrak{A}}$ (see remark on p. 15), thus $b = 0$ which is a contradiction to $\mathcal{R}^k b \neq 0$. Hence $\mathcal{R}^k = 0$.

Corollary. A simple Artinian associative algebra $\mathfrak{A}$ is semi simple.

Proof. If Rad $\mathfrak{A}$ is not trivial then $\mathfrak{A} = \text{Rad}\,\mathfrak{A}$, since Rad $\mathfrak{A}$ is an ideal and $\mathfrak{A}$ is simple. By the preceding theorem we get that $\mathfrak{A}$ is nilpotent. Then $\mathfrak{A}\mathfrak{A} \subsetneqq \mathfrak{A}$ (otherwise $\mathfrak{A}^k = \mathfrak{A}$ for all k and then $\mathfrak{A} = 0$). Since $\mathfrak{A}\mathfrak{A}$ is an ideal in $\mathfrak{A}$ it has to be zero. This is a contradiction to $\mathfrak{A}\mathfrak{A} \neq 0$.

We state without proof the main results on semi simple associative Artinian algebras.

Theorem 7: An Artinian algebra is semi simple, iff it is the direct sum of a finite number of simple Artinian algebras.

An associative algebra is a division algebra, if every element $\neq 0$ is invertible.

Theorem 8. If $\mathfrak{A}$ is a simple Artinian algebra over a field K, then $\mathfrak{A}$ is isomorphic to the K-algebra of all nxn matrices over a K-division algebra (for some n).

A semi simple Artinian algebra has a unit element.

Exercise. Let $\mathfrak{A}$ be an Artinian algebra. Show that $\mathfrak{A}$ is semisimple iff any ideal $\mathfrak{B}_1$ in $\mathfrak{A}$ has a direct (ideal) complement $\mathfrak{B}_2$, i.e., $\mathfrak{A} = \mathfrak{B}_1 \oplus \mathfrak{B}_2$, $\mathfrak{B}_2$ an ideal of $\mathfrak{A}$. If $\mathfrak{A}$ has an involution j and $\mathfrak{B}_1$ is j-invariant, then $\mathfrak{B}_2$ is j-invariant. (Hint: decompose the unit element e in $\mathfrak{A}$ as $e = e_1 + e_2$, show e_i unit element in $\mathfrak{B}_i$ and $j(e_i) = e_i$.).

III. Triple Systems.

3.1. A unital Φ-module $\mathfrak{T}$ together with a trilinear map $\mathfrak{T} \times \mathfrak{T} \times \mathfrak{T} \to \mathfrak{T}$, $(x,y,z) \mapsto \langle xyz \rangle$ is called a triple system.

Examples. 1) Let $\mathfrak{T} = \Phi^{(p,q)}$ be the Φ-module of rectangular pxq-matrices. If $A,B,C \in \mathfrak{T}$, then AB^tC is in $\mathfrak{T}$, where B^t denotes the transposed of B. Since $(A,B,C) \mapsto \langle ABC \rangle := AB^tC$ is trilinear, $\mathfrak{T}$ together with this "triple product" is a triple system.

2) If $\mathfrak{A}$ is any (non associative) Φ-algebra. Then $\mathfrak{A}$ together with the map $(x,y,z) \mapsto \langle xyz \rangle := (xy)z$ is a triple system and any submodule closed under $(xy)z$ is a triple system. Note: if $\mathfrak{A}$ has a unit element e then $xz = \langle xez \rangle$ and the structure of $\mathfrak{A}$ as an algebra can be completely recovered from the triple system structure on $\mathfrak{A}$.

3) Most important examples for the situation just described are the following. Let $\mathfrak{A}$ be a Φ-algebra and $j: \mathfrak{A} \to \mathfrak{A}$ an involutorial automorphism (i.e. $j(ab) = j(a)j(b)$, $j^2 = id$) then $\mathfrak{A}_\varepsilon = \{x \in \mathfrak{A}, j(x) = \varepsilon x\}$, $\varepsilon = \pm 1$, are closed under $(x,y,z) \mapsto (xy)z$, but in general $\mathfrak{A}_-$ is not a subalgebra.

The above examples show that a theory of triple systems of course includes a theory of algebras and "minus spaces" of algebras relative to involutorial automorphisms.

For submodules $\mathfrak{U}, \mathfrak{V}, \mathfrak{W} \subset \mathcal{T}$, we denote by $\langle \mathfrak{U}\mathfrak{V}\mathfrak{W} \rangle$ the submodule of $\mathcal{T}$ generated by all "triple products" $\langle uvw \rangle$, $u \in \mathfrak{U}, v \in \mathfrak{V}, w \in \mathfrak{W}$. A submodule $\mathfrak{U}$ is a <u>subsystem</u> if $\langle \mathfrak{U}\mathfrak{U}\mathfrak{U} \rangle \subset \mathfrak{U}$, it is an <u>ideal</u>, if $\langle \mathfrak{U}\mathcal{T}\mathcal{T} \rangle + \langle \mathcal{T}\mathfrak{U}\mathcal{T} \rangle + \langle \mathcal{T}\mathcal{T}\mathfrak{U} \rangle \subset \mathfrak{U}$. A Φ-linear map $f: \mathcal{T} \to \mathcal{T}'$ is a <u>homomorphism</u> of triple systems $\mathcal{T}, \mathcal{T}'$, if $f(\langle xyz \rangle) = \langle f(x) f(y) f(z) \rangle$ for all $x, y, z \in \mathcal{T}$. Isomorphisms and automorphisms are defined the usual way and the standard results hold. (The proofs are the same as for algebras.) If $\mathfrak{U}$ is an ideal in a triple system $\mathcal{T}$, then $\bar{\mathcal{T}} = \mathcal{T}/\mathfrak{U}$ together with

$$\langle (x + \mathfrak{U})(y + \mathfrak{U})(z + \mathfrak{U}) \rangle := \langle xyz \rangle + \mathfrak{U}$$

again is a triple system.

<u>Theorem 1.</u> (i) <u>$\mathfrak{U} \subset \mathcal{T}$ is an ideal, iff $\mathfrak{U}$ is the kernel of some homomorphism.</u>

(ii) <u>If $f: \mathcal{T} \to \mathcal{T}'$ is a homomorphism, then</u> $f(\mathcal{T}) \cong \mathcal{T}/\text{kernel } f$

(iii) <u>If $\mathfrak{U}, \mathfrak{V}$ are ideals of $\mathcal{T}$, then</u>

$$\mathfrak{U} + \mathfrak{V}/\mathfrak{V} \cong \mathfrak{U}/\mathfrak{U} \cap \mathfrak{V} .$$

A triple system $\mathcal{T}$ is called simple if $\langle \mathcal{T}\mathcal{T}\mathcal{T} \rangle \neq 0$ and $\mathcal{T}$ has no proper ideals.

3.2. The <u>derivatives</u> of a triple system $\mathcal{T}$ are defined recursively

$$\mathcal{T}^{(0)} = \mathcal{T}, \quad \mathcal{T}^{(n+1)} = \langle \mathcal{T}^{(n)} \mathcal{T}^{(n)} \mathcal{T}^{(n)} \rangle .$$

$\mathcal{T}$ is solvable, if $\mathcal{T}^{(n)} = 0$ for some n.

<u>Exercise</u>. State and prove the corresponding results to 1.5.
If $\mathcal{T}$ is Noetherian then there exists a unique maximal solvable ideal $\operatorname{Rad}\mathcal{T}$ in $\mathcal{T}$, the <u>solvable radical</u> of $\mathcal{T}$. $\operatorname{Rad}(\mathcal{T}/_{\operatorname{Rad}\mathcal{T}}) = 0$, and if $\operatorname{Rad}(\mathcal{T}/_{\mathcal{U}}) = 0$ then $\operatorname{Rad}\mathcal{T} \subset \mathcal{U}$. Powers of an element $a \in \mathcal{T}$ are defined recursively

$$a^1 := a, \quad a^{2(n+1)+1} := \langle a^{2n+1} a a \rangle$$

<u>Note:</u> Only odd power are defined.
$a \in \mathcal{T}$ is nilpotent, if $a^{2n+1} = 0$ for some n. A subsystem $\mathcal{U} \subset \mathcal{T}$ is nil, if every element in $\mathcal{U}$ is nilpotent. If $(a^{2n+1})^{2m+1} = a^{(2n+1)(2m+1)}$ for all $m,n > 0$ and all $a \in \mathcal{T}$, then there exists a unique maximal nil ideal in $\mathcal{T}$, the <u>nilradical</u> of $\mathcal{T}$.

<u>Exercise</u>. Prove existence and uniqueness of the nilradical.

3.3. Similar to the definition of left and right multiplication in algebras we define bilinear maps
$L,R,P\colon \mathcal{T} \times \mathcal{T} \to \operatorname{End}\mathcal{T}$, $L\colon (x,y) \mapsto L(x,y)$ $R\colon (x,y) \mapsto R(x,y)$, $P\colon (x,y) \mapsto P(x,y)$, by $L(x,y)z = \langle xyz \rangle$, $R(x,y)z = \langle zyx \rangle$, $P(x,y)z = \langle xzy \rangle$. Then

$$\langle xyz \rangle = L(x,y)z = R(z,y)x = P(x,z)y.$$

<u>Caution:</u> Observe the reversed order in $\langle xyz \rangle = R(z,y)x$.
Derivations are defined the obvious way. $D \in \operatorname{End}_\Phi \mathcal{T}$ is a derivation of $\mathcal{T}$, if

(3.1) $\quad D\langle xyz \rangle = \langle (Dx)yz \rangle + \langle x(Dy)z \rangle + \langle xy(Dz) \rangle$

for all $x,y,z \in \mathcal{T}$, or equivalently

(3.2) $\quad [D, L(x,y)] = L(Dx,y) + L(x,Dy)$ for all $x,y \in \mathcal{T}$.

Again $\mathfrak{D}(\mathfrak{T})$ the Φ-module of all derivations of $\mathfrak{T}$ is a subalgebra of $(\mathrm{End}_\Phi \mathfrak{T})^-$

Exercise. If $\mathfrak{A}$ is a triple system coming from an algebra (see example 2) then any algebra derivation or homomorphism is a derivation or homomorphism of the triple system.

3.4. There is still another aspect of triple systems we want to mention. Let $\mathfrak{T}$ be an arbitrary triple system over $\Phi, L(x,y)z = \langle xyz \rangle$. Then by definition $(x,y) \mapsto L(x,y)$ is a bilinear map of $\mathfrak{T} \times \mathfrak{T}$ into $\mathrm{End}_\Phi \mathfrak{T}$. But from the definition of the tensor product of Φ-modules, we get a unique linear map

$$S: \mathfrak{T} \otimes \mathfrak{T} \to \mathrm{End}\,\mathfrak{T} \text{ , such that}$$

$$S(x \otimes y) = L(x,y).$$

And obviously any linear map of $\mathfrak{T} \otimes \mathfrak{T} \to \mathrm{End}\,\mathfrak{T}$ defines a triple system structure on $\mathfrak{T}$.

Now we restrict to a special case. Assume $\mathfrak{T}$ is finite dimensional over a field F. Then $\mathfrak{T} \otimes \mathfrak{T} \cong \mathrm{End}\,\mathfrak{T}$, but there are many ways to obtain this isomorphism. We assume, that λ is a non degenerate symmetric bilinear form on $\mathfrak{T}$. We define $xy^* \in \mathrm{End}\,\mathfrak{T}$ by

$$(xy^*)z := \lambda(z,y)x$$

It is easy to prove and is left as an exercise, $x \otimes y \to xy^*$ defines an isomorphism (of vector spaces) $\mathfrak{T} \otimes \mathfrak{T}$ and $\mathrm{End}\,\mathfrak{T}$, in particular

(i) $\{xy^*, x,y \in \mathfrak{T}\}$ generates $\mathrm{End}\,\mathfrak{T}$. Furthermore

(3.3) (ii) $\mathrm{trace}\; xy^* = \lambda(x,y)$

(iii) $(xy^*)^* = yx^*$

(iv) $A(xy^*)B^* = Ax(By)^*$ for all $x,y \in \mathfrak{T}$, $A,B \in \mathrm{End}\,\mathfrak{T}$

where A^* denotes the adjoint of A relative to λ.

As in the case of algebras (see 1.9.), associative bilinear-forms might be useful.

There are more possibilities to define associative bilinear forms on $\mathcal{T}$. One possible definition is as follows: λ is called associative, if

(3.5) (i) $\lambda(\langle xyz\rangle,u) = \lambda(x,\langle uzy\rangle) = \lambda(z,\langle yxu\rangle)$ for all $x,y,z,u \in \mathcal{T}$. Assume λ non degenerate, symmetric and associative. Then (3.5) is equivalent to

(3.5') $L(x,y)^* = L(y,x);R(z,y)^* = R(y,z)$. If $A \in \operatorname{End}\mathcal{T}$ then there exists a unique $S(A) \in \operatorname{End}\mathcal{T}$ such that

(3.6) $\operatorname{trace} AL(x,y) = \lambda(S(A)x,y)$ (since λ is non degenerate).

Next we show

$$(3.7) \qquad S(uv^*) = L(u,v)$$

where $uv^*z = \lambda(z,v)u$ (see (3.3)).

$$\begin{aligned}\lambda(S(uv^*)x,y) &= \operatorname{trace} uv^*L(x,y) = \operatorname{tr} L(x,y)uv^* \\ &= \lambda(\langle xyu\rangle,v) = \lambda(x,\langle vuy\rangle) \\ &= \lambda(\langle uvx\rangle,y) = \lambda(L(u,v)x,y) .\end{aligned}$$

(3.6) and (3.7) imply $\operatorname{trace} AS(xy^*)$ $\operatorname{trace} S(A)xy^*$, consequently

(3.8) $\operatorname{trace} S(A)B = \operatorname{trace} AS(B)$.

<u>Exercise:</u> Define $S'(A)$ by $\operatorname{trace} AR(x,y) = \lambda(S'(A)x,y)$ and show $\operatorname{tr} S'(A)B = \operatorname{tr} AS'(B)$.

IV. Associative Triple Systems.

4.1. As we have seen in example 2) of the previous chapter, one can associate to any class of algebras a corresponding class of triple systems by considering the triple composition $(a,b,c) \to \langle abc\rangle = (ab)c$, where $(a,b) \to ab$ is the product in the algebra. Starting with associative algebras we come to the definition:

A triple system $\mathcal{T}$ is <u>associative</u> (of the first kind), if

(4.1) $\langle xy\langle uvw\rangle\rangle = \langle\langle xyu\rangle vw\rangle = \langle x\langle yuv\rangle w\rangle$ for all $x,y,u,v,w \in \mathcal{T}$.

In terms of left and right multiplications (4.1) is equivalent to either

$$
\begin{aligned}
(4.2) \qquad & L(x,y)L(u,v) = L(\langle xyu\rangle,v) = L(x,\langle yuv\rangle) \\
& R(w,v)R(u,y) = R(\langle uvw\rangle,y) = R(w,\langle yuv\rangle) \\
& L(x,y)R(w,v) = R(w,v)L(x,y) = P(x,w)P(y,v)
\end{aligned}
$$

<u>Example.</u> Any associative algebra $\mathcal{A}$ together with $(x,y,z) \mapsto (xy)z$ is an associative triple system of the first kind, and so is any submodule of $\mathcal{A}$ closed under $(xy)z$.

Let $\mathcal{E} := \mathrm{End}_\Phi \mathcal{T} \oplus (\mathrm{End}_\Phi \mathcal{T})^{op}$ the direct sum of the algebra of endomorphisms of $\mathcal{T}$ with its opposite algebra. Consider $\mathcal{L}_0$ the submodule of $\mathcal{E}$ generated by all $\lambda(x,y) := (L(x,y),R(y,x))$ then (4.2) and (4.3) show

$$
\begin{aligned}
\lambda(x,y)\lambda(u,v) &= (L(x,y),R(y,x))(L(u,v),R(v,u)) \\
&= (L(x,y)L(u,v),R(v,u)R(y,x)) \\
&= (L(\langle xyu\rangle,v),R(v,\langle xyu\rangle)) \\
&= (L(x,\langle yuv\rangle),R(\langle yuv\rangle,x)), \text{ i.e.}
\end{aligned}
$$

(4.5) $\lambda(x,y)\lambda(u,v) = \lambda(x,\langle yuv\rangle) = \lambda(\langle xyu\rangle,u)$

consequently

$\mathcal{L}_0$ is a subalgebra of $\mathcal{E}$. Let E denote the unit element of $\mathcal{E}$, then

$$\mathcal{L} := \Phi E + \mathcal{L}_0$$

is a subalgebra of $\mathcal{E}$, too.

The Φ-module $\mathcal{T}$ is in a natural way an $\mathcal{E}$ left and an $\mathcal{E}$ right module according to the following definitions. If $A = (A_1, A_2) \in \mathcal{E}$, define

(4.6) $$A \cdot x := A_1 x, \quad x \cdot A := A_2 x$$

and it is obvious that $(A,x) \to A \cdot x$ makes $\mathcal{T}$ a left $\mathcal{E}$ module and $(A,x) \to x \cdot A$ makes $\mathcal{T}$ a right $\mathcal{E}$ module. Since $\mathcal{L}$ is a subalgebra of $\mathcal{E}$ we have the following result:

Lemma 1. *$\mathcal{T}$ together with the maps* $\mathcal{L} \times \mathcal{T} \to \mathcal{T}$, $(A,x) \mapsto A \cdot x$, $\mathcal{T} \times \mathcal{L} \to \mathcal{T}$, $(x,A) \mapsto x \cdot A$ *is a left and a right $\mathcal{E}$ module.* (even an $\mathcal{E}$-bimodule)

Consider the Φ-module

$$\mathfrak{A} := \mathcal{L} \oplus \mathcal{T}$$

and define a product in $\mathfrak{A}$ by the formula

(4.7) $$(A \oplus x)(B \oplus y) := AB + \lambda(x,y) \oplus A \cdot y + x \cdot B$$

Theorem 1. *If $\mathcal{T}$ is an associative triple system of the first kind, then $\mathfrak{A} = \mathcal{L} \oplus \mathcal{T}$ with multiplication as defined in* (4.7) *is an associative algebra with unit element containing $\mathcal{T}$ (isomorphically imbedded) such that* $\langle xyz \rangle = (xy)z$ *for all* $x,y,z \in \mathcal{T}$.

The *proof* is left as an exercise.

4.2. Since for later applications we need a classification of a very similar type of triple systems we do not present a structure theory for associative triple systems of the first kind. We leave it as an exercise to use the methods and arguments we shall develop below to build up parts of a structure theory of associative triple systems of the first kind.

Very similar to the definition in 4.1 is the following: A triple system $\mathfrak{M}$ is called <u>associative</u> (<u>of</u> <u>the</u> <u>second</u> <u>kind</u>), if

(4.8) $\langle\langle xyz\rangle uv\rangle = \langle xy\langle zuv\rangle\rangle = \langle x\langle uzy\rangle v\rangle$

<u>Note:</u> The right hand side equations of (4.1) and (4.8) are different. In the sequel "associative triple system' (= a.t.s) always means "associative triple system of the second kind".

(4.8) is equivalent to either

(4.9) $L(x,y)L(z,u) = L(\langle xyz\rangle,u) = L(x,\langle uzy\rangle)$

(4.10) $R(v,u)R(z,y) = R(\langle zuv\rangle,y) = R(v,\langle uzy\rangle)$

(4.11) $R(v,u)L(x,y) = L(x,y)R(v,u) = P(x,v)P(u,y)$

<u>Example</u>. Let $\mathfrak{A}$ be an associative algebra with involution $x \mapsto \bar{x}$, then $\mathfrak{A}$ together with the map $(x,y,z) \mapsto x\bar{y}z$ is an associative triple system, and so is any submodule of $\mathfrak{A}$ which is closed under $x\bar{y}z$. In particular the Φ-module of all pxq-matrices over Φ together with $(A,B,C) \to AB^tC$ is an a.t.s. (see example 1) in 3.1.).

Let $\mathfrak{M}$ be an a.t.s. We set $\mathcal{E} := \mathrm{End}_\Phi \mathfrak{M} \oplus (\mathrm{End}_\Phi \mathfrak{M})^{op}$ we define

$$l(x,y) := (L(x,y), L(y,x))$$

$$r(x,y) := (R(y,x), R(x,y)).$$

Let $\mathcal{L}_0$ be the submodule of $\mathcal{E}$ spanned by all $l(x,y), x,y \in \mathfrak{M}$ and $\mathcal{R}_0$ be the submodule of $\mathcal{E}^{op}$ spanned by all $r(x,y), x,y \in \mathfrak{M}$.

(4.9) and (4.10) imply (do the computations)

(4.12) $l(x,y)l(u,v) = l(\langle xyu\rangle,v) = l(x,\langle vuy\rangle)$

(4.13) $r(x,y)r(u,v) = r(x,\langle yuv\rangle) = r(\langle uyx\rangle,v)$

(<u>Note:</u> the product on the left hand side of (4.13) is taken in $\mathcal{E}^{op}$.)

The last two equations show that $\mathcal{L}_0$ resp. $\mathcal{R}_0$ are subalgebras of $\mathcal{E}$ resp. $\mathcal{E}^{op}$. The algebras $\mathcal{E}$ and $\mathcal{E}^{op}$ have a natural involution, namely $(A,B) \mapsto \overline{(A,B)} = (B,A)$. Obviously $\overline{l(x,y)} = l(y,x)$, $\overline{r(x,y)} = r(y,x)$. Let E_1 resp. E_2 be the unit element in $\mathcal{E}$ resp. $\mathcal{E}^{op}$. We define

$$\mathcal{L} := \Phi E_1 + \mathcal{L}_0 \quad , \mathcal{R} = \Phi E_2 + \mathcal{R}_0$$

From the preceding discussion it follows

Lemma 2. $\mathcal{L}$ and $\mathcal{R}$ are subalgebras of $\mathcal{E}$ resp. $\mathcal{E}^{op}$ invariant under the canonical involution. $\mathcal{L}_0 (\mathcal{R}_0)$ is an ideal in $\mathcal{L}$ resp. $\mathcal{R}$.

The Φ-module $\mathfrak{M}$ is in a natural way a left $\mathcal{E}$-module and a right $\mathcal{E}^{op}$-module, according to the following compositions. If $A = (A_1,A_2) \in \mathcal{E}$, $B = (B_1,B_2) \in \mathcal{E}^{op}$ and $x \in \mathfrak{M}$ we set

$$(4.14) \qquad A \cdot x := A_1 x \quad , \quad x \cdot B := B_1 x$$

We take an isomorphic copy of $\mathfrak{M}$, denoted by $\overline{\mathfrak{M}}$. By the definitions

$$(4.15) \qquad \bar{x} \cdot A := \overline{A_2 x} \quad , \quad B \cdot \bar{x} := \overline{B_2 x}$$

if $\bar{x} \in \overline{\mathfrak{M}}$, $A = (A_1,A_2) \in \mathcal{E}$ and $B = (B_1,B_2) \in \mathcal{E}^{op}$

it is obvious that $\overline{\mathfrak{M}}$ becomes a right $\mathcal{E}$-module and a left $\mathcal{E}^{op}$-module.

Since $\mathcal{L}$ and $\mathcal{R}$ are subalgebras of $\mathcal{E}$ resp $\mathcal{E}^{op}$ we have the following result

Lemma 3. (i) $\mathfrak{M}$ together with the mappings defined by (4.14) is a left $\mathcal{L}$-module and a right $\mathcal{R}$-module.

(ii) $\overline{\mathfrak{M}}$ together with the mappings defined by (4.15) is a right $\mathcal{L}$-module and a left $\mathcal{R}$-module.

Exercise. Show that $\mathfrak{M}$ (resp. $\overline{\mathfrak{M}}$) is an $(\mathcal{L},\mathcal{R})$-bimodule (resp. $(\mathcal{R},\mathcal{L})$-bimodule), i.e. it is not only a left $\mathcal{L}$-module and a right $\mathcal{R}$-module, but furthermore $(A\cdot x)\cdot B = A\cdot(x\cdot B)$ holds for all $A \in \mathcal{L}$, $B \in \mathcal{R}$, $x \in \mathfrak{M}$.

Now we consider the module

$$\mathfrak{A} := \mathcal{L} \oplus \mathfrak{M} \oplus \overline{\mathfrak{M}} \oplus \mathcal{R}$$

For the convenience of notation we write the elements of $\mathfrak{A}$ in matrix form

$$\begin{pmatrix} A & x \\ \bar{y} & B \end{pmatrix}, \qquad A \in \mathcal{L},\ B \in \mathcal{R},\ x \in \mathfrak{M},\ \bar{y} \in \overline{\mathfrak{M}}.$$

and by means of the module isomorphisms $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, $x \mapsto \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$

etc., we identify $\mathcal{L}, \mathcal{R}, \mathfrak{M}, \overline{\mathfrak{M}}$ with its image. We define a multiplication on $\mathfrak{A}$ by

$$(4.16) \qquad \begin{pmatrix} A & x \\ \bar{y} & B \end{pmatrix} * \begin{pmatrix} A', & x' \\ \overline{y'}, & B' \end{pmatrix} := \begin{pmatrix} AA' + l(x,y'), & A\cdot x' + x\cdot B' \\ \bar{y}\cdot A' + B\cdot \overline{y'}, & r(y,x') + BB' \end{pmatrix}$$

The following result is fundamental:

<u>Theorem 2.</u> <u>If $\mathfrak{M}$ is an a.t.s. then</u>

(i) <u>$\mathfrak{A} = \mathcal{L} \oplus \mathfrak{M} \oplus \overline{\mathfrak{M}} \oplus \mathcal{R}$ together with the product defined by</u> (4.16) <u>is an associative Φ-algebra with unit element</u> $e = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$

(ii) <u>$\mathfrak{A}_0 = \mathcal{L}_0 \oplus \mathfrak{M} \oplus \overline{\mathfrak{M}} \oplus \mathcal{R}_0$ is an ideal in $\mathfrak{A}$</u>

(iii) <u>The map</u> j: $u = \begin{pmatrix} A & x \\ \bar{y} & B \end{pmatrix} \mapsto \bar{u} = \begin{pmatrix} \bar{A}, & y \\ \bar{x}, & \bar{B} \end{pmatrix}$ <u>is an involution of $\mathfrak{A}$</u>.

(iv) <u>If</u> $x,y,z \in \mathfrak{M}$, <u>then</u> $\langle xyz \rangle = x * \bar{y} * z$.

(v) <u>The Peirce component of $\mathfrak{A}$ relative to the idempotent E_1, are</u>

$$\mathfrak{A}_{11} = \mathcal{L},\ \mathfrak{A}_{10} = \mathfrak{M},\ \mathfrak{A}_{01} = \overline{\mathfrak{M}},\ \mathfrak{A}_{00} = \mathcal{R}$$

(For the notations concerning Peirce decomposition see 2.5.).

Proof. The only difficulties related to this theorem are in finding the given construction. The verification of the above statements is done by straightforward computations using (4.9)-(4.16) and the previous 2 lemmas [illegible] and is left as an exercise.

$\mathfrak{A} = \mathfrak{A}(\mathfrak{M})$ is called the standard imbedding of $\mathfrak{M}$.

Lemma 4. Let $\mathfrak{M}$ be an a.t.s. and $\mathfrak{A}$ its standard imbedding. If $\mathcal{J}$ is an ideal in $\mathfrak{A}$ then

(i) $\mathcal{J} = (\mathcal{J}\cap\mathcal{L}) \oplus (\mathcal{J}\cap\mathfrak{M}) \oplus (\mathcal{J}\cap\overline{\mathfrak{M}}) \oplus (\mathcal{J}\cap\mathcal{R})$

(ii) If $\mathcal{J}$ is j-stable then $\mathcal{J}\cap\mathfrak{M}$ is an ideal in $\mathfrak{M}$

(iii) If $\mathcal{J}\subset\mathcal{L}\oplus\mathcal{R}$, then $\mathcal{J} = 0$.

Proof. (i) follows from part (v) of the above theorem and II, Lemma 7.

(ii) If $b \in \mathcal{J}\cap\mathfrak{M}$ and $x,y \in \mathfrak{M}$ then

$$\langle xyb\rangle = x * \bar{y} * b \quad \text{and} \quad \langle bxy\rangle = b * \bar{x} * y$$

are in $\mathfrak{M}\cap\mathcal{J}$. Since $\mathcal{J}$ is j-invariant $\bar{b}$ is in $\mathcal{J}$ and consequently $\langle xby\rangle = x * \bar{b} * y \in \mathcal{J}\cap\mathfrak{M}$.

(iii) If $\mathcal{J}\subset\mathcal{L}\oplus\mathcal{R}$ then $\mathcal{J} = (\mathcal{L}\cap\mathcal{J}) \oplus (\mathcal{R}\cap\mathcal{J})$ and $\mathcal{J}\cap\mathfrak{M} = \mathcal{J}\cap\overline{\mathfrak{M}} = 0$, by part (i). If $A = (A_1, A_2) \in \mathcal{L}\cap\mathcal{J}$ then $A * \mathfrak{M} = A\cdot\mathfrak{M} = A_1\mathfrak{M} \subset \mathcal{J}\cap\mathfrak{M} = 0$, thus $A_1 = 0$ similarly $A_2 = 0$ and also $\mathcal{L}\cap\mathcal{J} = \mathcal{R}\cap\mathcal{J} = 0$.

4.3. Let $\mathfrak{M}$ be an a.t.s. For fixed $u \in \mathfrak{M}$ we consider the map $(x,y) \mapsto \langle xuy\rangle = x\cdot y$. The resulting algebra is denoted by $\mathfrak{M}_u$. It is immediately seen from the equation (4.8) (put $y = u$) that $\mathfrak{M}_u$ is an associative algebra with left multiplication

$L_u(x) = L(x,u)$. In $\mathfrak{M}_u$ we have the notion of quasi invertibility (see 2.2.-2.3.). x is quasi invertible in $\mathfrak{M}_u$, iff there exists an element $y \in \mathfrak{M}$ such that

$$y - x = \langle yux \rangle = \langle xuy \rangle$$

But thinking of $\mathfrak{M}$ as being a submodule of its standard imbedding this is equivalent to:

$x \in \mathfrak{M}$ is quasi invertible in $\mathfrak{A}_{\bar{u}}$ and the quasi inverse y is in $\mathfrak{M}$.

Lemma 5. If $x, u \in \mathfrak{M}$ then the following statements are equivalent,

(i) x is quasi invertible in $\mathfrak{M}_u$

(ii) x is quasi invertible in $\mathfrak{A}_{\bar{u}}$

Proof. We need only prove (ii) $\rightarrow$ (i). This follows from a result on Peirce decomposition. If $x \in \mathfrak{M} = \mathfrak{A}_{10}$ is quasi-invertible then by II, lemma 3 (ii) its quasi-inverse is also in $\mathfrak{A}_{10} = \mathfrak{M}$.

$(\mathfrak{A}\mathfrak{A}_{10} \subset \mathfrak{A}_{10} + \mathfrak{A}_{00}$ and $\mathfrak{A}_{10}\mathfrak{A} \subset \mathfrak{A}_{11} + \mathfrak{A}_{10})$

We define

$\operatorname{Rad}\mathfrak{M} := \{ x \in \mathfrak{M},\ x$ is q.i. in $\mathfrak{M}_u$ for all $u \in \mathfrak{M} \}$

$\operatorname{Rad}\mathfrak{M}$ is called the Jacobson radical of $\mathfrak{M}$. $\mathfrak{M}$ is called semi simple, if $\operatorname{Rad}\mathfrak{M} = 0$.

Theorem 3. If $\mathfrak{M}$ is an a.t.s. and $\mathfrak{A}$ its standard imbedding, then

(i) $\operatorname{Rad}\mathfrak{A} = \operatorname{Rad}\mathcal{L} \oplus \operatorname{Rad}\mathfrak{M} \oplus \operatorname{Rad}\overline{\mathfrak{M}} \oplus \operatorname{Rad}\mathcal{R}$

(ii) $\operatorname{Rad}\mathfrak{M}$ is an ideal in $\mathfrak{M}$

(iii) $\mathfrak{A}$ is semi simple, iff $\mathfrak{M}$ is semi simple.

Proof. According to lemma 4(i), $\operatorname{Rad}\mathfrak{A}$ is a direct sum of its components in $\mathcal{L}, \mathcal{R}, \overline{\mathfrak{M}}, \mathfrak{M}$. But these are the Peirce spaces relative to E_1 and II, theorem 5(ii) shows

$(\text{Rad }\mathfrak{A})\cap\mathcal{L} = \text{Rad }\mathcal{L}$, $(\text{Rad }\mathfrak{A})\cap\mathcal{R} = \text{Rad }\mathcal{R}$.

It remains to show $\text{Rad }\mathfrak{M} = (\text{Rad }\mathfrak{A})\cap\mathfrak{M}$. If $x\in(\text{Rad }\mathfrak{A})\cap\mathfrak{M}$ then in particular x q.i. in $\mathfrak{A}_{\bar{u}}$ for all $\bar{u}\in\overline{\mathfrak{M}}$ but then $x\in\text{Rad }\mathfrak{M}$ by lemma 5. If conversely $x\in\text{Rad }\mathfrak{M}$, then in $\mathfrak{A}$, $q(x,\bar{y})$ exists for all $\bar{y}\in\overline{\mathfrak{M}}$ (again lemma 5.). By theorem 2(v), we have $\overline{\mathfrak{M}} = (1 - E_1)\mathfrak{A}E_1$, hence $q(x,(1 - E_1)aE_1)$ exists for every $a\in\mathfrak{A}$. From the shifting principle (see 2.3.) it follows that $q(x,a) = q(E_1x(1 - E_1),a)$ exists for every $a\in\mathfrak{A}$ (note $x\in\mathfrak{A}_{10}$), this is $x\in\text{Rad }\mathfrak{A}$ by definition. Hence $\text{Rad }\mathfrak{M} = (\text{Rad }\mathfrak{A})\cap\mathfrak{M}$. Since $\text{Rad }\mathfrak{A}$ is j-invariant it follows firstly, that $(\text{Rad }\mathfrak{A})\cap\overline{\mathfrak{M}} = \overline{\text{Rad }\mathfrak{M}}$ (this completes the proof of (i)) and secondly that $\text{Rad }\mathfrak{M} = \text{Rad }\mathfrak{A}\cap\mathfrak{M}$ is an ideal (by lemma 4(ii)). Now, if $\text{Rad }\mathfrak{A} = 0$, obviously $\text{Rad }\mathfrak{M} = 0$. If conversely $\text{Rad }\mathfrak{M} = 0$ then $\text{Rad }\mathfrak{A}\subset\mathcal{L}\oplus\mathcal{R}$, hence $\text{Rad }\mathfrak{A} = 0$ by lemma 4(iii).

<u>Exercise</u>. 1) $\text{Rad}(\mathfrak{M}/\text{Rad }\mathfrak{M}) = 0$

2) If $\mathfrak{N}$ is an ideal in $\mathfrak{M}$ then $\text{Rad }\mathfrak{N} = \mathfrak{N}\cap\text{Rad }\mathfrak{M}$.

3) Carry over the above definitions to a.t.s.'s of the first kind and prove corresponding results.

4) Show that the direct sum of two semi simple a.t.s.'s is semi simple.

The following lemma will be useful:

<u>Lemma 6.</u> <u>If $\mathfrak{M}$ is a simple a.t.s. and $\mathfrak{L}\neq 0$ a j-stable ideal of $\mathfrak{A}$, then $\mathfrak{A}_0\subset\mathfrak{L}$.</u>

<u>Proof.</u> Since $\mathfrak{L}\cap\mathfrak{M}$ is an ideal in $\mathfrak{M}$ (lemma 4(ii)) and $\mathfrak{M}$ is simple we have $\mathfrak{L}\cap\mathfrak{M} = 0$ or $\mathfrak{L}\cap\mathfrak{M} = \mathfrak{M}$. Clearly $\mathfrak{L}\cap\mathfrak{M} = 0$ is equivalent to $\mathfrak{L}\cap\overline{\mathfrak{M}} = 0$ since $\mathfrak{L}$ is j-stable. If $\mathfrak{L}\cap\mathfrak{M} = 0$

then $\mathcal{L} \cap \overline{\mathcal{M}} = 0$ and $\mathcal{L} \subset \mathcal{L} \oplus \mathcal{R}$ by lemma 4(i). But then $\mathcal{L} = 0$ by lemma 4(iii). Since $\mathcal{L} \neq 0$ we must have $\mathcal{L} \cap \mathcal{M} = \mathcal{M}$ and $\mathcal{L} \cap \overline{\mathcal{M}} = \overline{\mathcal{M}}$. Again lemma 4 implies $\mathcal{M} \oplus \overline{\mathcal{M}} \subset \mathcal{L}$. Since $\mathcal{M} * \overline{\mathcal{M}} = \mathcal{L}_0$ and $\overline{\mathcal{M}} * \mathcal{M} = \mathcal{R}_0$ we get $\mathfrak{A}_0 \subset \mathcal{L}$.

4.4. Let $\mathcal{M}$ be an a.t.s. The powers of a submodule $\mathcal{U} \subset \mathcal{M}$ are defined recursively

$$\mathcal{U}^1 := \mathcal{U}\ ,\ \mathcal{U}^{k+2} = \langle \mathcal{U}^k\, \mathcal{U}\, \mathcal{U} \rangle$$

(Note: Only odd powers are defined.) $\mathcal{U}$ is <u>nilpotent</u>, if $\mathcal{U}^n = 0$ for some n. If $\mathfrak{A}$ is the standard imbedding and $\mathcal{U}$ is contained in a j-invariant subalgebra $\mathcal{L}$ of $\mathfrak{A}$, then

$$\mathcal{U}^n \subset \mathcal{L}^n. \tag{4.17}$$

The proof of (4.17) is by an easy induction. By assumption $\mathcal{U} \subset \mathcal{L}$, i.e. $\mathcal{U}^1 \subset \mathcal{L}^1$. Assume (4.17). By theorem 2(iv) and the j-invariance of $\mathcal{L}$ we get

$$\mathcal{U}^{n+2} = \langle \mathcal{U}^n\, \mathcal{U}\, \mathcal{U} \rangle \subset \mathcal{L}^n * \overline{\mathcal{L}} * \mathcal{L} = \mathcal{L}^{n+2}.$$

<u>Theorem 4. If $\mathcal{M}$ is an a.t.s. and the standard imbedding $\mathfrak{A}$ of $\mathcal{M}$ is Artinian, then Rad $\mathcal{M}$ is nilpotent.</u>

<u>Proof.</u> Rad $\mathcal{M} \subset$ Rad $\mathfrak{A}$ by theorem 3, therefore by (4.17)

$$(\text{Rad}\, \mathcal{M})^k \subset (\text{Rad}\, \mathfrak{A})^k$$

since Rad $\mathcal{M}$ is j-invariant. Since $\mathfrak{A}$ is Artinian we get $(\text{Rad}\, \mathfrak{A})^n = 0$ for some n by II, theorem 6.

<u>Corollary. If $\mathcal{U}$ is a simple ideal of $\mathcal{M}$ ($\mathfrak{A}$ Artinian) then $\mathcal{U}$ is semi simple.</u>

<u>Proof.</u> Since Rad $\mathcal{U} \subset$ Rad $\mathcal{M}$ (see exercise 2 above), Rad $\mathcal{U}$ is nilpotent by theorem 4. Now see the proof of the corollary to II, theorem 6.

Exercise: $\mathfrak{U} \subset \mathfrak{M}$ is a left ideal of $\mathfrak{M}$, if $\langle \mathfrak{M}\mathfrak{M}\mathfrak{U} \rangle \subset \mathfrak{U}$. $\mathfrak{M}$ is called Artinian, if $\mathfrak{M}$ has the descending chain condition on left ideals. Show: $\mathfrak{A}$ Artinian implies $\mathfrak{M}$ Artinian. (It is most likely that the converse statement is true if $\mathfrak{M}$ is semi simple; see Lister: Ternary rings.).

Theorem 5. If $\mathfrak{M}$ is an a.t.s. such that the standard imbedding $\mathfrak{A}$ is Artinian, then

(i) $\mathfrak{M}$ is semi simple, iff $\mathfrak{M}$ is the direct sum of a finite number of ideals which are as a.t.s.'s simple and Artinian.

(ii) $\mathfrak{M}$ is simple, iff $(\mathfrak{A}, j)$ is simple.

Proof. The following observation makes the things go: If $\mathfrak{U}$ is an ideal of $\mathfrak{M}$, then

(4.18) $\mathfrak{L} = \mathfrak{L}(\mathfrak{U}) = \mathfrak{A} * \mathfrak{U} * \mathfrak{A} + \mathfrak{A} * \overline{\mathfrak{U}} * \mathfrak{A}$

is a j-invariant ideal in $\mathfrak{A}$ such that $\mathfrak{L} \cap \mathfrak{M} = \mathfrak{U}$.

(The verification of this statement is an easy exercise.).

Let $\mathfrak{M}$ be semi simple; then $\mathfrak{A}$ is semi simple (theorem 3). If $\mathfrak{U}$ is a minimal ideal in $\mathfrak{M}$ ($\mathfrak{U}$ exists, since $\mathfrak{M}$ is Artinian, see exercise above) then the j-invariant ideal $\mathfrak{L} = \mathfrak{L}(\mathfrak{U})$ is complemented in $\mathfrak{A}$, i.e., there exists a j-invariant ideal $\mathfrak{L}'$ in $\mathfrak{A}$ such that $\mathfrak{A} = \mathfrak{L} \oplus \mathfrak{L}'$ (see 2.7.) Now $\mathfrak{M} = (\mathfrak{L} \cap \mathfrak{M}) \oplus (\mathfrak{L}' \cap \mathfrak{M}) = \mathfrak{U} \oplus \mathfrak{U}'$ where $\mathfrak{U}' = \mathfrak{L}' \cap \mathfrak{M}$ is an ideal. Continue this construction with $\mathfrak{U}'$ (take a minimal ideal in $\mathfrak{U}'$, which is an ideal in $\mathfrak{A}$) etc. After a finite number of steps we must have come to an end since any proper chain of ideals of $\mathfrak{M}$ has finite length. Furthermore these ideals are simple (by construction they don't have a proper ideal and $\langle \mathfrak{U}\mathfrak{U}\mathfrak{U} \rangle = 0$ leads to $\mathfrak{U}$ nilpotent in $\mathfrak{A}$ and then $\mathfrak{U} \subset \operatorname{Rad} \mathfrak{A}$, by II, Corollary to theorem 2). Conversely let $\mathfrak{M}$ be the direct sum of simple ideals.

Since the simple ideals of $\mathfrak{M}$ are semi simple (Corollary to theorem 4) and a finite direct sum of semi simple a.t.s.'s is semi simple (see exercise 4) we get that $\mathfrak{M}$ is semi simple. This proves part (i) of the theorem. ad(ii): If $(\mathfrak{A},j)$ is simple, and $\mathfrak{U}$ an ideal in $\mathfrak{M}$, then $\mathcal{L} = \mathcal{L}(\mathfrak{U})$ (see (4.18)) is either 0 or $\mathfrak{A}$, consequently $\mathfrak{U} = \mathcal{L} \cap \mathfrak{M}$ either 0 or $\mathfrak{M}$. If conversely $\mathfrak{M}$ is simple and if $\mathcal{L} \neq 0$ is a j-stable ideal in $\mathfrak{A}$ then by lemma 6 $\mathfrak{A}_0 \subset \mathcal{L}$. Since $\mathfrak{A}$ is semi simple, $\mathcal{L}$ has a direct complement $\mathcal{L}'$ and if $\mathcal{L}' \neq 0$ then $\mathfrak{A}_0 \subset \mathcal{L}'$ by the same lemma, but then $\mathfrak{A}_0 \subset \mathcal{L} \cap \mathcal{L}' = 0$. This is a contradiction, thus $\mathcal{L} = \mathfrak{A}$.

Corollary: *$\mathfrak{M}$ is a simple a.t.s. with Artinian standard imbedding, iff $\mathfrak{M}$ is equal to the Peirce space $\mathfrak{A}_{10}$ relative to some idempotent c in a simple pair $(\mathfrak{A},j)$, where $\mathfrak{A}$ is an Artinian algebra, j an involution of $\mathfrak{A}$ such that j(c) = c, and the ternary composition $(x,y,z) \to \langle xyz \rangle$ in $\mathfrak{M}$ is given by $\langle xyz \rangle = xj(y)z$.* That $\mathfrak{A}_{10}$ is a simple a.t.s. and its standard imbedding is isomorphic to $\mathfrak{A}$ is an easy exercise.

4.5. The above corollary shows that whenever one knows all simple pairs $(\mathfrak{A},j)$, $\mathfrak{A}$ an Artinian associative algebra with involution j, one gets all simple a.t.s.'s with Artinian standard imbedding by computing the $\mathfrak{A}_{10}$ spaces relative to idempotents which are fixed under the involution. A classification of the simple pairs $(\mathfrak{A},j)$ can be found in Jacobson's "Lectures in Abstract Algebra". The result is as follows:

If $(\mathfrak{A},j)$ is a simple pair, $\mathfrak{A}$ an Artinian associative algebra over a field F, then either

(i) $\mathfrak{A} \cong \mathcal{L} \oplus \mathcal{L}^{op}$ where $\mathcal{L} = End_{\Delta}(V)$ is the F-algebra of endomorphisms of a finite dimensional right vector space V over an (associative) division algebra Δ over F, the involution j is given by $j(a,b) = (b,a)$.

(ii) $\mathfrak{A} \cong \mathcal{L}$, $\mathcal{L}$ as in (i) and for j one has the following possibilities:

α) $\Delta = \Gamma$ a (commutative) field extension of F, $\dim_\Gamma V = 2n$, V has a non degenerate alternating bilinear form λ (i.e., $\lambda(x,x) = 0$ for all $x \in V$) and $j(a) = a^*$, where a^* is the adjoint of $a \in \mathrm{End}_\Gamma V$ relative to λ (i.e., $\lambda(ax,y) = \lambda(x,a^*y)$).

β) Δ has an involution $\alpha \to \bar{\alpha}$, V has a nonalternating Hermitian sesquilinear form λ (i.e., $\overline{\lambda(x,y)} = \lambda(y,x)$ and $\lambda(x,y\alpha) = \bar{\alpha}\lambda(x,y)$) and $j(a) = a^*$, where a^* is the adjoint of a relative to λ. (Note $\bar{\alpha} = \alpha$ is not excluded.)

Let $\mathfrak{A}$ be as in (i). $e = (c_1,c_2)$ an idempotent, $j(e) = e$ shows $c_1 = c_2 = c$ and obviously c is an idempotent in $\mathcal{L}$. From the definitions we get

$$\mathfrak{A}_{10} = \mathcal{L}_{10} \oplus (\mathcal{L}^{op})_{10}, \text{ where } \mathcal{L}_{ij} \text{ are the}$$

Peirce spaces of $\mathcal{L}$ relative to c. Since $c^2 = c$ we have $V = \mathrm{Im}\, c \oplus \text{kernel } c$. (Im c: = image c). Let $x_1,\ldots,x_p,x_{p+1},\ldots x_{p+q}$ be a Δ-basis of V consisting of a basis $x_1,\ldots,x_p$ of Im c and a basis $x_{p+1},\ldots,x_{p+q}$ of kernel c, then we identify relative to this basis $\mathrm{End}_\Delta V$ with the F-algebra of all $(p + q)\times(p + q)$ matrices over Δ. c is then of the form $c = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}$, where E is the pxp unit matrix. Now (see the example in 2.5.) we have that $\mathcal{L}_{10}$ is the space of all pxq matrices over Δ. $\mathcal{L}^{op}$, of course, will be identified with the matrices over Δ^{op}.

Putting this together we can conclude that

(i) $\mathfrak{M} = \mathfrak{A}_{10}$ is the space of all pxq matrices over $\Delta\oplus\Delta^{op}$ and $\langle xyz\rangle = x\bar{y}^t z$, where $\bar{y}^t$ is the transposed conjugate of $y = (\alpha_{ij}), \alpha_{ij} \in \Delta\oplus\Delta^{op}$, i.e. $\bar{y} = (\bar{\alpha}_{ij})$ where $\alpha \to \bar{\alpha}$ is the canonical involution in $\Delta\oplus\Delta^{op}$.

(ii) If $\mathfrak{A} = \mathrm{End}_\Delta V$, $\dim [V:\Delta] < \infty$, Δ division algebra over F and λ either skew symmetric or hermitian. A j-invariant idempotent c is in either case selfadjoint and consequently Im c and kernel c are orthogonal with respect to λ (i.e., $\lambda(\mathrm{Im}\ c, \mathrm{kernel}\ c) = 0$) and the restrictions of λ to these two subspaces are non degenerate.

α) If λ is skew symmetric, we choose a basis of Im c, $x_1,\ldots,x_p$ and a basis of kernel c, $x_{p+1},\ldots,x_{p+q}$, such that the matrix of λ, i.e. $(\lambda(x_i,x_j))$, is of the form

$$Q = S_{p+q} = \mathrm{diag}(\overbrace{S, S, \ldots, S}^{p+q}) \quad , \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

The same argument as in (i) shows that we can identify $\mathfrak{M} = \mathfrak{A}_{10}$ with the space of 2px2q matrices over Δ, the adjoint of a is then $a^* = Qa^tQ^{-1}$, consequently we get for $x,y,z \in \mathfrak{A}_{10}$

$$\langle xyz\rangle = xj(y)z = xS_q y^t S_p^{-1} z.$$

β) If λ is hermitian and non alternating we can choose a basis of Im c and kernel c such that the matrix of λ is of the form $D = \mathrm{diag}(\alpha_1,\ldots,\alpha_{p+q})$, $a \neq \alpha_i = \overline{\alpha_i} \in \Delta$. Again we can identify $\mathfrak{M} = \mathfrak{A}_{10}$ with the space of pxq matrices over Δ. The involution is then

$a \to a^* = D\bar{a}^t D^{-1}$ consequently if $x,y,z \in \mathfrak{A}_{10}$

$\langle xyz \rangle = xy^* z = x a_q \bar{y}^t b_p z$, where

$a_q = \mathrm{diag}(\alpha_{p+1},\ldots,\alpha_{p+q})$, $b_p = \mathrm{diag}(\alpha^{-1},\ldots,\alpha_p^{-1})$.

We proved

Theorem 6. If $\mathfrak{M}$ is a simple associative triple system over a field F such that its standard embedding is Artinian, then $\mathfrak{M}$ is isomorphic to one of the following types:

(i) The pxq matrices over $\Delta \oplus \Delta^{op}$, Δ a division algebra over F, together with triple product $\langle xyz \rangle = x(\bar{y}^{\,t})z$.

(ii) The 2px2q matrices over a field extension Γ of F with triple product $\langle xyz \rangle = xS_q y^t S_p^{-1} z$ where

$$S_n = \mathrm{diag}(\underbrace{S,\ldots,S}_{n}), S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(iii) The pxq matrices over a division algebra Δ (over F) with involution $\alpha \to \bar{\alpha}$, the triple product is given by $\langle xyz \rangle = xa(\bar{y}^t)bz$, where $a = \mathrm{diag}\,(\alpha_1,\ldots,\alpha_q)$
$b = \mathrm{diag}(\beta_1,\ldots,\beta_p)$, $0 \neq \alpha_i = \bar{\alpha}_i$, $0 \neq \beta_i = \bar{\beta}_i$.

V. Lie Algebras

5.1. We recall that an algebra $\mathcal{L}$ over Φ (with multiplication $(x,y) \mapsto [xy]$) is called a Lie algebra, if

(5.1) $$[xx] = 0$$

(5.2) $$[[xy]z] + [[yz]x] + [[zx]y] = 0 \quad \text{(Jacobi identity)}$$

for all $x,y,z \in \mathcal{L}$.

In Lie algebras (and only in Lie algebras) one denotes the left-multiplications by adx, $(\mathrm{ad}x)y = [xy]$. (5.1) implies

(5.3) $$[xy] = -[yx],$$

and the Jacobi identity then may be written as

$$\mathrm{ad}[xy] = [\mathrm{ad}x, \mathrm{ad}y].$$

Let $\mathcal{L}$ be a Lie algebra.

<u>Lemma 1. If $\mathfrak{U}, \mathfrak{W}$ are ideals of $\mathcal{L}$, then $[\mathfrak{U}\mathfrak{W}]$ is an ideal of $\mathcal{L}$.</u>

<u>Proof</u>. We need only prove $[a[uv]] \in [\mathfrak{U}\mathfrak{W}]$ for $a \in \mathcal{L}$, $u \in \mathfrak{U}$, $v \in \mathfrak{W}$. But this is immediately seen from the Jacobi identity.

<u>Corollary. If $\mathcal{K}$ is an ideal of $\mathcal{L}$, then the "derived modules" $\mathcal{K}^{(o)} = \mathcal{K}$, $\mathcal{K}^{(n+1)} = [\mathcal{K}^{(n)}\mathcal{K}^{(n)}]$ and the powers of $\mathcal{K}^1 = \mathcal{K}$, $\mathcal{K}^{n+1} = [\mathcal{K}^n\mathcal{K}]$ are ideals of $\mathcal{L}$.</u> If $\mathcal{L}$ is Noetherian (i.e. has a.c.c. on ideals) then there exists a unique maximal solvable ideal of $\mathcal{L}$, the <u>radical</u> of $\mathcal{L}$ (see 1.5). Also there exists a unique maximal nilpotent

ideal in $\mathcal{L}$, the <u>nilradical</u>.

<u>Exercise</u>. a) $[\mathcal{L}^m, \mathcal{L}^n] \subset \mathcal{L}^{m+n}$

b) $\mathcal{L}^{(n)} \subset \mathcal{L}^{2^n}$

c) The nilradical is contained in the radical.

We assume now, that $\mathcal{L}$ is a <u>finite dimensional</u> Lie algebra over a field F. In this case, there is a canonical bilinear form λ on $\mathcal{L}$, the so-called <u>Killing form</u>, defined by

$$\lambda(x,y) = \text{trace } (\text{ad}x)(\text{ad}y).$$

<u>Lemma 2</u>. (i) λ is <u>symmetric and associative</u>

(ii) $\lambda(\alpha x,y) = \lambda(x,\alpha^{-1}y)$ <u>for any</u> $\alpha \in \text{Aut}\,\mathcal{L}$.

<u>Proof</u>. (i) the symmetry of λ is obvious. By definition and Jacobi identity $\lambda([xy],z) = \text{tr ad}[xy]\text{ad}z = \text{tr }(\text{ad}x\,\text{ad}y\,\text{ad}z - \text{ad}y\,\text{ad}x\,\text{ad}z)$

$$= \text{tr }(\text{ad}x\,[\text{ad}y, \text{ad}z]) = \text{tr ad}x\,\text{ad}[yz] = \lambda(x,[yz]).$$

(ii) $\alpha \in \text{Aut}\,\mathcal{L}$ is equivalent to $\alpha\,\text{ad}x\,\alpha^{-1} = \text{ad}(\alpha x)$. Therefore $\lambda(\alpha x,y) = \text{tr ad}\,\alpha x\,\text{ad}y = \text{tr }\alpha\,\text{ad}x\,\alpha^{-1}\,\text{ad}y$

$$= \text{tr ad}x\,\alpha^{-1}(\text{ad}y)\alpha = \lambda(x, \alpha^{-1}y).$$

There is a fundamental result.

<u>Theorem 1</u>. <u>(CARTAN Criterion)</u>. <u>Let $\mathcal{L}$ be a finite dimensional Lie algebra over a field of characteristic</u> 0. <u>Then $\mathcal{L}$ is semi simple, iff the Killing form is non degenerate</u>.

<u>Exercise</u>: Read the proof of the Cartan Criterion in any book on Lie algebras.

An immediate application of theorem 1 and Dieudonné's theorem (1.9) is the following:

Theorem 2. If $\mathcal{L}$ is a finite dimensional semi simple Lie algebra over a field of char. 0, then $\mathcal{L}$ is a direct sum of simple ideals.

Note: Condition (ii) in Dieudonné's theorem holds since $\mathcal{L}$ has no solvable ideal.

We shall give another application of the Cartan Criterion.

Theorem 3. (Zassenhaus). If $\mathcal{L}$ is as in theorem 2, then any derivation D of $\mathcal{L}$ is of the form $D = \text{ad } d$, $d \in \mathcal{L}$.

Proof. Since the Killing form λ is non degenerate, there exists $d \in \mathcal{L}$ such that

$$\text{trace } D \text{ ad}x = \lambda(d,x).$$

Let $E: = D - \text{ad } d$, then E is a derivation and

(5.4) $$\text{trace } E \text{ ad } x = \text{trace } D \text{ ad } x - \text{trace ad } d \text{ ad } x = 0.$$

Then $\lambda(Ex,y) = \text{tr ad}(Ex)\text{ad}y = \text{tr } [E, \text{ad}x]\text{ad}y$

$$= \text{tr } E[\text{ad}x, \text{ad}y] = \text{tr } E \text{ ad}[xy] = 0, \text{ by (5.4)}$$

Since λ is non degenerate, we get $Ex = 0$ for all x or $D = \text{ad } d$.

VI. Lie Triple Systems

6.1. Let $\mathcal{T}$ be a unital Φ-module. $\mathcal{T}$ together with a trilinear map $(x,y,z) \longmapsto [xyz]$, is called a Lie triple system (= L.t.s.), if

$$\text{(i)} \quad [xxz] = 0$$

(6.1) $$\text{(ii)} \quad [xyz] + [yzx] + [zxy] = 0 \quad \text{(Jacobi identity)}$$

$$\text{(iii)} \quad [uv[xyz]] = [[uvx]yz] + [x[uvy]z] + [xy[uvz]]$$

for all $u, v, x, y, z \in \mathcal{T}$.

Examples. 1) Let $\mathcal{L}$ be a Lie algebra with product $(x,y) \longmapsto [xy]$, then $\mathcal{L}$ together with $(x,y,z) \longmapsto [[xy]z]$ is a L.t.s.

2) Any submodule of a Lie algebra closed under $[[xy]z]$ is a L.t.s.; the most important submodules of this type which are not subalgebras are the modules $\mathcal{L}_- = \{x,\ \alpha x = -x\}$ where $\alpha \in \mathrm{Aut}\,\mathcal{L}$, $\alpha^2 = \mathrm{id}$.

3) If $\mathcal{T}$ together with $(x,y,z) \mapsto \langle xyz \rangle$ is an associative triple system, then $\mathcal{T}$ together with

$$[xyz] := \langle xyz \rangle - \langle yxz \rangle - \langle zxy \rangle + \langle zyx \rangle$$

is a L.t.s. An important example of this type is F_n the vector space of column vectors over a field F;

4) Let $\mathfrak{A}$ be a commutative algebra over Φ with multiplication $(x,y) \mapsto xy = L(x)y$. Set $D(x,y) = [L(x),L(y)]$. Assume

(6.2) $$[D(x,y),\ D(u,v)] = D(D(x,y)u,v) + D(u,\ D(x,y)v) \quad \text{for all } x,y,u,v \in \mathfrak{A}.$$

If $\mathcal{T}$ is a submodule of $\mathfrak{A}$ closed under $[xyz] = D(x,y)z$ then $\mathcal{T}$ together with $(x,y,z) \longmapsto [xyz]$ is a Lie triple system. The most important examples for this type of algebras are the Jordan algebras.

Exercise. Verify that the given examples really are L.t.s.'s.

(6.1 i) implies (replace x by x+y)

(6.2) $$[xyz] = -[yxz].$$

Define $L(x,y)$, $R(z,y)$, $P(x,z) \in \mathrm{End}_\Phi \mathfrak{T}$ (see Chapter III) by $[xyz] = L(x,y)z = R(z,y)x = P(x,z)y$. We see that (6.1) is equivalent to

(6.3)
(i) $L(x,x) = 0 \quad (\Rightarrow L(x,y) = -L(y,x))$
(ii) $L(x,y) = R(x,y) - R(y,x)$
(iii) $[L(x,y), L(u,v)] = L([xyu],v) + L(u,[xyv])$.

Lemma 1. A submodule $\mathfrak{U}$ of $\mathfrak{T}$ is an ideal of $\mathfrak{T}$, iff $[\mathfrak{U}\mathfrak{T}\mathfrak{T}] \subset \mathfrak{U}$.

Proof. Clearly the condition is necessary. Since $[\mathfrak{U}\mathfrak{T}\mathfrak{T}] \subset \mathfrak{U}$ imples $[\mathfrak{T}\mathfrak{U}\mathfrak{T}] \subset \mathfrak{U}$ (by (6.2)) and then $[\mathfrak{T}\mathfrak{T}\mathfrak{U}] \subset \mathfrak{U}$ by the Jacobi identity, we see that the given condition is also sufficient.

6.2. Let $\mathfrak{T}$ be a Lie triple system. We recall that $D \in \mathrm{End}_\Phi \mathfrak{T}$ is a derivation, if

(6.4) $$[D, L(x,y)] = L(Dx,y) + L(x,Dy).$$

(6.3iii) shows, that all $L(x,y)$, $x,y \in \mathfrak{T}$ are derivations. Let $\mathfrak{g}$ be the submodule of $\mathfrak{D}(\mathfrak{T})$ (derivation algebra of $\mathfrak{T}$) generated by all $L(x,y)$, $x,y \in \mathfrak{T}$. Another interpretation of (6.4) gives:

Lemma 2. $\mathfrak{g}$ is an ideal of $\mathfrak{D}(\mathfrak{T})$.

Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{D}(\mathfrak{T})$ containing $\mathfrak{h}$. We consider

$$\mathcal{L}(\mathfrak{g},\mathfrak{T}) = \mathfrak{g} \oplus \mathfrak{T}$$

and define for elements $X_i = H_i \oplus x_i$, $H_i \in \mathfrak{g}$, $x_i \in \mathfrak{T}$ $(i = 1,2)$ a product.

(6.5) $$[X_1, X_2]: = [H_1, H_2] + L(x_1, x_2) \oplus (H_1x_2 - H_2x_1).$$

The following result is fundamental.

Theorem 1. If $\mathfrak{T}$ is a Lie triple system, $\mathfrak{g}$ a subalgebra of $\mathfrak{D}(\mathfrak{T})$ containing $\mathfrak{h}$, then

(i) $\mathcal{L}(\mathfrak{g},\mathfrak{T}) = \mathfrak{g} \oplus \mathfrak{T}$ together with the product (6.5) is a Lie algebra,

(ii) $\Theta: H \oplus x \longmapsto (-H) \oplus x$ defines an involution of $\mathcal{L}$,

(iii) $\mathcal{L}(\mathfrak{h},\mathfrak{T}) = \mathfrak{h} \oplus \mathfrak{T}$ is an ideal of $\mathcal{L}(\mathfrak{g},\mathfrak{T})$,

(iv) if $x,y,z \in \mathfrak{T}$, then $[xyz] = [[x,y],z]$,

(v) if $1/2 \in \Phi$ then $\mathfrak{T} = \{X \in \mathcal{L}(\mathfrak{g},\mathfrak{T});\ \Theta X = X\}$.

Proof. Clearly $[X,X] = 0$ for all $X \in \mathcal{L}$. We have to show $\mathfrak{J}(X_1,X_2,X_3) = [[X_1,X_2], X_3] + [[X_2,X_3], X_1] + [[X_3,X_1], X_2] = 0$ for all $X_i \in \mathcal{L}$. It is sufficient to show this equality only for $(X_1,X_2,X_3) \in \mathfrak{U} \times \mathfrak{V} \times \mathfrak{W}$, where $\mathfrak{U}, \mathfrak{V}, \mathfrak{W}$ is either $\mathfrak{g}$ or $\mathfrak{T}$. Since $\mathfrak{g}$ is a subalgebra of $\mathfrak{D}(\mathfrak{T})$ we get $\mathfrak{J}(\mathfrak{g},\mathfrak{g},\mathfrak{g}) = 0$. If $H_i \in \mathfrak{g}$, $x \in \mathfrak{T}$ we get $[[H_1,H_2],x] = [H_1,H_2]x = H_1(H_2x) - H_2(H_1x) = [H_1,[H_2,x]] - [H_2[H_1,x]]$. This shows $\mathfrak{J}(\mathfrak{g},\mathfrak{g},\mathfrak{T}) = 0$, then by cyclic permutation $\mathfrak{J}(\mathfrak{g},\mathfrak{T},\mathfrak{g}) = \mathfrak{J}(\mathfrak{T},\mathfrak{g},\mathfrak{g}) = 0$. Using (6.5) and $\mathfrak{g} \subset \mathfrak{D}(\mathfrak{T})$ we get

$$[[H,x],y] + [[x,y],H] + [[y,H],x] = L(Hx,y) + [L(x,y),H] + L(x,Hy) = 0.$$

Hence $\mathcal{J}(\mathfrak{g}, \mathcal{T}, \mathcal{T}) = 0$ and also $\mathcal{J}(\mathcal{T}, \mathfrak{g}, \mathcal{T}) = \mathcal{J}(\mathcal{T}, \mathcal{T}, \mathfrak{g})$ $= 0$. Finally $\mathcal{J}(x,y,z) = [[x,y],z] + [[y,z],x] + [[z,x],y]$

$$= L(x,y)z + L(y,z)x + L(z,x)y = 0, \text{ by (6.1 ii)}.$$

The other statements are easily verified (using definitions and lemma 2).

The Lie algebra $\mathcal{L} = \mathcal{L}(\mathfrak{H}, \mathcal{T}) = \mathfrak{H} \oplus \mathcal{T}$ is called the <u>standard imbedding</u> of $\mathcal{T}$, Θ is called the <u>main involution</u> of $\mathcal{L}$.

<u>Examples</u>. 1) Let F be a field, $\mathcal{T} = F_n$ the L.t.s. of column vectors over F (see ex. 3, p. 43) We take as triple product $[xyz] = yx^tz - xy^tz$ and get $L(x,y)z = (yx^t - xy^t)z$. Consequently we can identify $L(x,y)$ with the $n \times n$ matrix $yx^t - xy^t$. The space spanned by these matrices is the space of all $n \times n$ skew symmetric matrices. We define a mapping of the standard imbedding $\mathfrak{H} \oplus \mathcal{T}$ onto the Lie algebra of all $(n+1) \times (n+1)$ skew symmetric matrices by

$$A \oplus x \longmapsto \begin{pmatrix} A & x \\ -x^t & 0 \end{pmatrix}$$

This is a (well defined) 1-1 linear map onto. It is an easy computation (and is left as an exercise) that the given map is a Lie algebra homomorphism, hence an isomorphism.

2) The above example may be generalized as follows. We define on $F^{(p,q)}$, the space of all $p \times q$ matrices the triple composition by (see ex. 3, p. 43)

$$[ABC] = BA^tC - AB^tC - CB^tA + CA^tB.$$

Use the same kind of argument to show that the standard imbedding of $K^{(p,q)}$ is the Lie algebra of $(p+q)\times(p+q)$ skew symmetric matrices over F.

6.3. Let $\mathcal{T}_i$ $(i = 1,2)$ be Lie triple systems and $\mathcal{L}_i = \mathcal{H}_i \oplus \mathcal{T}_i$ the corresponding standard imbedding. If $\phi : \mathcal{H}_1 \to \mathcal{H}_2$ and $\eta : \mathcal{T}_1 \to \mathcal{T}_2$ are Φ-linear maps, then $\lambda : \mathcal{L}_1 \to \mathcal{L}_2$ defined by $\lambda(H \oplus x) = \phi(H) \oplus \eta(x)$ is obviously Φ linear.

<u>Lemma 3.</u> $\lambda : \mathcal{L}_1 \to \mathcal{L}_2$ <u>is a Lie algebra homomorphism, if</u> $\phi : \mathcal{H}_1 \to \mathcal{H}_2$ <u>is a Lie algebra homomorphism and</u>

(i) $\phi\, L_1(x,y) = L_2(\eta x, \eta y)$

(ii) $\eta\, H = \phi(H)\eta$

(L_i is the left multiplication of $\mathcal{T}_i$.)

<u>Proof.</u> Easy exercise.

A linear map $\eta : \mathcal{T}_1 \to \mathcal{T}_2$ is an L.t.-homomorphisms, if

$$\eta[xyz] = [(\eta x)(\eta y)(\eta z)] \quad \text{, or equivalently}$$

$$\eta L_1(x,y) = L_2(\eta x, \eta y)\eta$$

If $\eta : \mathcal{T}_1 \to \mathcal{T}_2$ is an L.t.-isomorphisms, then according to lemma 3, the map

$$\Lambda \; : \; H \oplus x \mapsto \eta H \eta^{-1} \oplus \eta x$$

is an isomorphism of $\mathcal{L}_1$ onto $\mathcal{L}_2$. Obviously Λ commutes with the main involutions, i.e. $\Lambda\theta_1 = \theta_2\Lambda$. If conversely $\Lambda : \mathcal{L}_1 \to \mathcal{L}_2$ is an isomorphismus such that $\Lambda\theta_1 = \theta_2\Lambda$ and if $1/2 \in \Phi$ then we get that the restriction of Λ to $\mathcal{T}_1$ maps onto $\mathcal{T}_2$, hence is an L.t.-isomorphism.

The following trivial observation is quite useful for applications. Assume η is an automorphism of $\mathcal{T}$, $\eta^2 = \mathrm{id}$. Then $\Lambda : H + x \mapsto \eta H \eta + \eta x$

is an automorphism of $\mathfrak{L}$ and $\Lambda^2 = \mathrm{id}$. Hence the (-1)-eigenspace of $\mathfrak{L}$, i.e. $\mathfrak{L}_- = \{X \in \mathfrak{L}, \Lambda X = -X\}$ is a L.t.s. (6.1 ex. 2), which is (obviously) in most cases quite different from $\mathfrak{T}$, but which has, in certain cases, the same (isomorphic) standard imbedding as $\mathfrak{T}$, namely $\mathfrak{L}$.

6.4. From now on we assume $1/2 \in \Phi$. In this case $x \in \mathfrak{T}$, iff $\Theta x = x$ (see theorem 1). We shall derive a rather strong connection between ideals in a L.t.s. $\mathfrak{T}$ and ideals in its standard imbedding $\mathfrak{L}$.

Since $\mathfrak{L}$ is a Lie algebra with involution Θ we are mainly interested in Θ-invariant ideals.

If $\mathfrak{K}$ is any Θ-invariant submodule of $\mathfrak{L}$, then $H \oplus x \in \mathfrak{K}$ implies $-H \oplus x \in \mathfrak{K}$, consequently $\mathfrak{K} = \mathfrak{H} \cap \mathfrak{K} \oplus \mathfrak{T} \cap \mathfrak{K}$. Conversely, any submodule of this type is Θ-invariant. Let $\mathfrak{K} = \mathfrak{M} \oplus \mathfrak{U}$ be a Θ-invariant submodule of $\mathfrak{L}$ ($\mathfrak{M} \subset \mathfrak{H}$, $\mathfrak{U} \subset \mathfrak{T}$). $\mathfrak{K}$ is an ideal of $\mathfrak{L}$, iff for any $K = M \oplus u \in \mathfrak{K}$ and any $X = H \oplus x \in \mathfrak{L}$ we have $[X,K] = [H,M] + L(x,u) \oplus Hu - Mx \in \mathfrak{K}$. This is equivalent to

(i) $[H,M]$, $L(x,u) \in \mathfrak{M}$ for all $H \in \mathfrak{H}$, $x \in \mathfrak{T}$, $M \in \mathfrak{M}$, $u \in \mathfrak{U}$

(ii) Hu, $Mx \in \mathfrak{U}$ for all $H \in \mathfrak{H}$, $x \in \mathfrak{T}$, $M \in \mathfrak{M}$, $u \in \mathfrak{U}$

We define $i(\mathfrak{U}) = L(\mathfrak{T}, \mathfrak{U})$

$$j(\mathfrak{U}) = \{A \in \mathfrak{H},\ A\mathfrak{T} \subset \mathfrak{U}\}$$

and get immediately from the above considerations,

<u>Lemma 4.</u> $\mathfrak{K} \subset \mathfrak{L}$ <u>is a Θ-invariant ideal of</u> $\mathfrak{L}$, <u>iff</u> $\mathfrak{K} = \mathfrak{M} \oplus \mathfrak{U}$, <u>where</u> $\mathfrak{U}$ <u>is an ideal of</u> $\mathfrak{T}$, $\mathfrak{M}$ <u>an ideal of</u> $\mathfrak{H}$ <u>such that</u> $i(\mathfrak{U}) \subset \mathfrak{M} \subset$ $\subset j(\mathfrak{U})$.

<u>Corollary 1.</u> $\mathfrak{I}(\mathfrak{U}) = i(\mathfrak{U}) \oplus \mathfrak{U}$. $\mathfrak{J}(\mathfrak{U}) = j(\mathfrak{U}) \oplus \mathfrak{U}$ <u>are</u> <u>(Θ-invariant) ideals of</u> $\mathfrak{L}$ <u>iff</u> $\mathfrak{U}$ <u>is an ideal of</u> $\mathfrak{T}$.

We need only prove that $i(\mathfrak{U}) \subset j(\mathfrak{U})$ and that both are ideals of $\mathfrak{H}$ if $\mathfrak{U}$ is an ideal. Assume $\mathfrak{U}$ ideal, then $L(\mathfrak{T}, \mathfrak{U})\mathfrak{T} \subset \mathfrak{U}$, this already shows $i(\mathfrak{U}) \subset j(\mathfrak{U})$. Let $x \in \mathfrak{T}$, $u \in \mathfrak{U}$, $H \in \mathfrak{H}$, $A \in j(\mathfrak{U})$ then the equations

$$[H, L(x,u)] = L(Hx,u) + L(x,Hu)$$

$$[H,A]x = HAx - AHx$$

show that $i(\mathfrak{U})$ and $j(\mathfrak{U})$ are idealf of $\mathfrak{H}$.

<u>Theorem 2.</u> $\mathfrak{T}$ <u>is simple, iff</u> $(\mathfrak{L}, \theta)$ <u>is simple.</u>

The proof is an immediate application of the above corollary and lemma 4.

If $\mathfrak{T}$ is simple, then either $\mathfrak{L}$ is simple, or $\mathfrak{L} = \mathfrak{L} \oplus \mathfrak{L}^{op}$ (1.8, theorem 3) $\theta(b_1, b_2) = (b_2, b_1)$. In the second case we get some more informations. Since we assume $1/2 \in \Phi$ we get (up to an identification) $(b_1, b_2) \in \mathfrak{L}$ is in $\mathfrak{T}$ iff $\theta(b_1, b_2) = (b_1, b_2)$, this is the case, iff $b_1 = b_2$, consequently

$$\mathfrak{T} = \{(b, b), \ b \in \mathfrak{L}\}.$$

The map $(b, b) \longmapsto (b, 0)$ is obviously a module isomorphism, but furthermore, since

$$([b_1b_2b_3], [b_1b_2b_3]) = ([[b_1b_2], b_3], [[b_1b_2], b_3])$$

this map is an Lt - homomorphism of $\mathfrak{T}$ onto the Lts $\mathfrak{L}$, $(b_1,b_2,b_3) \longmapsto [[b_1b_2], b_3]$. We proved:

<u>Theorem 3. If</u> $\mathfrak{T}$ <u>is a simple Lie triple system, then either</u> $\mathfrak{T}$ <u>is a simple Lie algebra and the triple product is</u> $[xyz] = [[xy]z]$ <u>or</u> $\mathfrak{T}$ <u>is the +1 - space of an involution of a simple Lie algebra.</u>

If $\mathcal{T}$ is simple and every ideal of $\mathcal{L}$ is Θ invariant, then $\mathcal{L}$ is simple, too. If, for example, $E \in \mathcal{G}$ such that $[E, [E, H + x]] = x$ (there are important examples of this type; see Chapter XI) then any ideal $\mathcal{K}$ of $\mathcal{L}$ is Θ-invariant. Let $K = M + u \in \mathcal{K}$, since $[E, [E,K]] = u \in \mathcal{K}$, we see that M and u are both in $\mathcal{K}$. This shows

Theorem 4. *If $\mathcal{T}$ is simple and there is an element $E \in \mathcal{G}$ such that $[E, [E, H + x]] = x$, then $\mathcal{L}$ is simple.*

It is far beyond the scope of these lectures to give a classification of all simple L.t.s.'s (finite dimensional over fields). We restrict ourselves to the presentation of those parts in the theory of L.t.s. which are useful for a better understanding of certain constructions we shall perform later in connection with Jordan algebras and Jordan triple systems.

6.5. Let $\mathcal{T}$ be an arbitrary L.t.s. and $\mathcal{L} = \mathcal{G} \oplus \mathcal{T}$ its standard imbedding. We have seen that for an ideal $\mathcal{U}$ of $\mathcal{T}$

$$\mathcal{J}(\mathcal{U}) = i(\mathcal{U}) \oplus \mathcal{U}, \quad \bar{\mathcal{J}}(\mathcal{U}) = j(\mathcal{U}) \oplus \mathcal{U}$$

are ideals of $\mathcal{L}$, and if $\mathcal{K} = \mathcal{M} \oplus \mathcal{U}$ is a Θ-invariant ideal of $\mathcal{L}$ then $\mathcal{J}(\mathcal{U}) \subset \mathcal{K} \subset \bar{\mathcal{J}}(\mathcal{U})$.

For submodules $\mathcal{U}_1, \mathcal{U}_2$ we set

$$\mathcal{U}_1 * \mathcal{U}_2 := i(\mathcal{U}_1)\mathcal{U}_2 + i(\mathcal{U}_2)\mathcal{U}_1,$$

this is in terms of triple product $\mathcal{U}_1 * \mathcal{U}_2 = [\mathcal{U}_1 \mathcal{T} \mathcal{U}_2] + [\mathcal{U}_2 \mathcal{T} \mathcal{U}_1]$.

Lemma 5. *If $\mathcal{U}_1, \mathcal{U}_2$ are ideals of $\mathcal{T}$, then $\mathcal{U}_1 * \mathcal{U}_2$ is an ideal and*

$$i(\mathfrak{U}_1 * \mathfrak{U}_2) \subset [i(\mathfrak{U}_1), i(\mathfrak{U}_2)] + L(\mathfrak{U}_1, \mathfrak{U}_2) \subset j(\mathfrak{U}_1 * \mathfrak{U}_2).$$

Proof. In $\mathfrak{L}$ we have

$[\mathfrak{J}(\mathfrak{U}_1), \mathfrak{J}(\mathfrak{U}_2)] = [i(\mathfrak{U}_1), i(\mathfrak{U}_2)] + L(\mathfrak{U}_1, \mathfrak{U}_2) + i(\mathfrak{U}_1)\mathfrak{U}_2 + i(\mathfrak{U}_2)\mathfrak{U}_1)$ since the product of two ideals in a Lie algebra is an ideal ($\bar{V}$, lemma 1) and if both ideals are θ-invariant, then, of course, the product is θ-invariant. The result now follows from lemma 4.

(6.6) Corollary. $\mathfrak{J}(\mathfrak{U}_1 * \mathfrak{U}_2) \subset [\mathfrak{J}(\mathfrak{U}_1), \mathfrak{J}(\mathfrak{U}_2)] \subset \mathfrak{J}(\mathfrak{U}_1 * \mathfrak{U}_2)$ ($\mathfrak{U}_i \subset \mathfrak{F}$ ideals.)

We define $\mathfrak{U}^{<o>} := \mathfrak{U}$, $\mathfrak{U}^{<k+1>} := \mathfrak{U}^{<k>} * \mathfrak{U}^{<k>} = [\mathfrak{U}^{<k>} \mathfrak{F}\, \mathfrak{U}^{<k>}]$.
$\mathfrak{U}$ is called L-solvable, if $\mathfrak{U}^{<k>} = 0$ for some k.

Exercise. $\mathfrak{U}$ L-solvable $\Longrightarrow \mathfrak{U}$ solvable (for def. see ch. III)

Lemma 6. If $\mathfrak{U}_1$, $\mathfrak{U}_2$ are L-solvable ideals of $\mathfrak{F}$, then $\mathfrak{U}_1 + \mathfrak{U}_2$ is L-solvable.

Proof. We show by induction

(6.7) $$(\mathfrak{U}_1 + \mathfrak{U}_2)^{<k>} \subset \mathfrak{U}_1^{<k>} + \mathfrak{U}_2^{<k>} + \mathfrak{U}_1 \cap \mathfrak{U}_2.$$

k = 0 is trivial. Assume (6.7) then

$$(\mathfrak{U}_1 + \mathfrak{U}_2)^{<k+1>} = [(\mathfrak{U}_1 + \mathfrak{U}_2)^{<k>} \mathfrak{F} (\mathfrak{U}_1 + \mathfrak{U}_2)^{<k>}]$$

$$[(\mathfrak{U}_1^{<k>} + \mathfrak{U}_2^{<k>} + \mathfrak{U}_1 \cap \mathfrak{U}_2) \mathfrak{F} (\mathfrak{U}_1^{<k>} + \mathfrak{U}_2^{<k>} + \mathfrak{U}_1 \cap \mathfrak{U}_2)] \subset$$

$$\subset \mathfrak{U}_1^{<k+1>} + \mathfrak{U}_2^{<k+1>} + \mathfrak{U}_1 \cap \mathfrak{U}_2.$$

Since obviously $(\mathfrak{U}^{<k>})^{<\ell>} = \mathfrak{U}^{<k+\ell>}$, the lemma is an immediate consequence of (6.7).

Assume $\mathfrak{F}$ Noetherian, then there is a unique maximal L-solvable ideal $\mathcal{R}(\mathfrak{F})$, the <u>L-radical</u> of $\mathfrak{F}$. $\mathfrak{F}$ is called L-semi-simple, if $\mathcal{R}(\mathfrak{F}) = 0$.

<u>Theorem 5.</u> $\mathcal{R}(\mathfrak{F}/\mathcal{R}(\mathfrak{F})) = 0$ <u>and if</u> $\mathcal{R}(\mathfrak{F}/\mathfrak{U}) = 0$ <u>then</u> $\mathcal{R}(\mathfrak{F}) \subset \mathfrak{U}$.

<u>Proof.</u> The ideals of $\overline{\mathfrak{F}} = \mathfrak{F}/\mathcal{R}(\mathfrak{F})$ have the form $\mathfrak{U}/\mathcal{R}(\mathfrak{F})$, where $\mathfrak{U}$ is an ideal of $\mathfrak{F}$ containing $\mathcal{R}(\mathfrak{F})$. Since

$$\left(\frac{\mathfrak{U}}{\mathcal{R}(\mathfrak{F})}\right)^{<k>} = \frac{\mathfrak{U}^{<k>} + \mathcal{R}(\mathfrak{F})}{\mathcal{R}(\mathfrak{F})},$$

if $\mathfrak{U}/\mathcal{R}(\mathfrak{F})$ is L-solvable, then $\mathfrak{U}^{<k>} \subset \mathcal{R}(\mathfrak{F})$ for some k. This implies that $\mathfrak{U}$ is L-solvable in $\mathfrak{F}$, hence $\mathfrak{U} = \mathcal{R}(\mathfrak{F})$, and $\mathfrak{F}/\mathcal{R}(\mathfrak{F})$ is L-semi-simple. In $\mathfrak{F}/\mathfrak{U}$ the ideal $\overline{\mathcal{R}} = \{r + \mathfrak{U},\ r \in \mathcal{R}(\mathfrak{F})\}$ is L-solvable. Consequently $\mathfrak{F}/\mathfrak{U}$ L-semi-simple implies $\overline{\mathcal{R}} = 0$ and $\mathcal{R}(\mathfrak{F}) \subset \mathfrak{U}$.

<u>Lemma 6.</u> <u>If</u> $\mathfrak{U} \subset \mathfrak{F}$ <u>is an ideal, then</u>

(i) $[\mathfrak{j}(\mathfrak{U}), \mathfrak{g}] \subset \mathfrak{i}(\mathfrak{U})$

(ii) $\mathfrak{j}(\mathfrak{U})^{(k+1)} \subset \mathfrak{i}(\mathfrak{U})^{(k)} \subset \mathfrak{j}(\mathfrak{U})^{(k)}$

(Note: in (ii) we have Lie algebra notation).

<u>Proof.</u> If $H \in \mathfrak{j}(\mathfrak{U})$ then $H \in \mathfrak{g}$ and $H\mathfrak{F} \subset \mathfrak{U}$. Since

$[H, L(x,y)] = L(Hx,y) - L(Hy,x)$ and $Hx, Hy \in \mathfrak{U}$, we get $[H,L(x,y)] \in$ $\in i(\mathfrak{U})$. This implies (i).

In particular, since $j(\mathfrak{U}) \subset \mathfrak{G}$, we get $[j(\mathfrak{U}), j(\mathfrak{U})] \subset i(\mathfrak{U})$ and $j(\mathfrak{U})^{(k+1)} \subset i(\mathfrak{U})^{(k)}$ by induction. The other inclusion is trivial, since for an ideal $\mathfrak{U}$ we have $i(\mathfrak{U}) \subset j(\mathfrak{U})$.

<u>Corollary</u>. $i(\mathfrak{U})$ <u>solvable in</u> $\mathfrak{G}$, <u>iff</u> $j(\mathfrak{U})$ <u>solvable in</u> $\mathfrak{G}$.

<u>Lemma 7</u>. <u>If</u> $\mathfrak{U}$ <u>is an ideal of</u> $\mathfrak{F}$, then

(i) $[\mathfrak{J}(\mathfrak{U}), \mathfrak{L}] \subset \mathfrak{I}(\mathfrak{U})$

(ii) $\mathfrak{J}(\mathfrak{U})^{(k+1)} \subset \mathfrak{I}(\mathfrak{U})^{(k)} \subset \mathfrak{J}(\mathfrak{U})^{(k)}$

<u>Proof</u>. $[j(\mathfrak{U}) \oplus \mathfrak{U}, \mathfrak{G} \oplus \mathfrak{F}] = [j(\mathfrak{U}), \mathfrak{G}] + L(\mathfrak{U}, \mathfrak{F}) +$ $j(\mathfrak{U})\mathfrak{F} + \mathfrak{G}\mathfrak{U} \subset \mathfrak{I}(\mathfrak{U})$, by lemma 6.i. (ii) follows from (i) by induction.

<u>Corollary</u>. $\mathfrak{I}(\mathfrak{U})$ <u>is solvable in</u> $\mathfrak{L}$, <u>iff</u> $\mathfrak{J}(\mathfrak{U})$ <u>is solvable</u>.

<u>Theorem 6</u>. <u>If</u> $\mathfrak{U} \subset \mathfrak{F}$ <u>is an ideal, then the following statements are equivalent</u>,

(i) $\mathfrak{U}$ <u>is L-solvable in</u> $\mathfrak{F}$

(ii) $\mathfrak{I}(\mathfrak{U})$ <u>is solvable in</u> $\mathfrak{L}$

(iii) $\mathfrak{J}(\mathfrak{U})$ <u>is solvable in</u> $\mathfrak{L}$.

<u>Proof</u>. ((ii) $\Leftrightarrow$ (iii) is the above Corollary) We show by induction

(6.8) $$\mathfrak{I}(\mathfrak{U}^{\langle k \rangle}) \subset \mathfrak{I}(\mathfrak{U})^{(k)}.$$

$k = 0$ is trivial. Assume (6.8), then by (6.6)

$$\mathfrak{J}(\mathfrak{U}^{<k+1>}) = \mathfrak{J}(\mathfrak{U}^{<k>} * \mathfrak{U}^{<k>}) \subset [\mathfrak{J}(\mathfrak{U}^{<k>}), \mathfrak{J}(\mathfrak{U}^{<k>})]$$
$$[\mathfrak{J}(\mathfrak{U})^{(k)}, \mathfrak{J}(\mathfrak{U})^{(k)}] = \mathfrak{J}(\mathfrak{U})^{(k+1)}.$$

As consequence of (6.8) we see (ii) $\Rightarrow$ (i).

Next we show by induction

a) $\mathfrak{J}(\mathfrak{U})^{(2k-1)} \subset \mathfrak{F}(\mathfrak{U}^{<k>})$, $(k \geq 1)$

b) $\mathfrak{J}(\mathfrak{U})^{(2k)} \subset \mathfrak{J}(\mathfrak{U}^{<k>}) \subset \mathfrak{F}(\mathfrak{U})^{<k>})$ $(k \geq 0)$

If k = 1 then a) follows from (6.6) and b) is trivial for k = 0.

Assume a), b) for k. Then

$$\mathfrak{J}(\mathfrak{U})^{(2(k+1)-1)} = [\mathfrak{J}(\mathfrak{U})^{(2k)}, \mathfrak{J}(\mathfrak{U})^{(2k)}] \subset [\mathfrak{J}(\mathfrak{U}^{<k>}), \mathfrak{J}(\mathfrak{U}^{<k>})]$$
$$\subset (\text{Lemma 7i})\ \mathfrak{F}(\mathfrak{U}^{<k+1>})$$

and $\mathfrak{J}(\mathfrak{U})^{(2k+2)} = [\mathfrak{J}(\mathfrak{U})^{(2k+1)}, \mathfrak{J}(\mathfrak{U})^{(2k+1)}] \subset [\mathfrak{F}(\mathfrak{U}^{<k+1>}),$
$\mathfrak{F}(\mathfrak{U}^{<k+1>})]$ (we just proved a) for k+1) $\subset \mathfrak{J}(\mathfrak{U}^{<k+1>})$ by lemma 7i.

Now (i) $\Rightarrow$ (ii), by a) and b).

Now we assume $\mathfrak{L}$ Noetherian. Then the radical of $\mathfrak{L}$ exists.

Theorem 7. Rad $\mathcal{L} = \mathfrak{J}(\mathcal{R}(\mathcal{T})) = J(\mathcal{R}(\mathcal{T}))$.

Corollary (i) $\mathcal{R}(\mathcal{T}) = \mathcal{T} \cap \text{Rad}\,\mathcal{L}$

(ii) $\mathcal{L}$ semi-simple, iff $\mathcal{T}$ L-semi-simple.

Proof (Of Thm. 7). Let $\mathcal{S} = \text{Rad}\,\mathcal{L}$. Obviously the isomorphic (or antiisomorphic) image of a solvable ideal is solvable and since every solvable ideal is contained in the radical, it is clear that $\mathcal{S}$ is θ-invariant, then $\mathcal{S} = \mathcal{M} \oplus \mathcal{U}$, by lemma 4, and $J(\mathcal{U}) \subset \mathcal{S} \subset \mathfrak{J}(\mathcal{U})$. Since $J(\mathcal{U})$ as a submodule of a solvable ideal is solvable, we get $\mathfrak{J}(\mathcal{U})$ solvable by theorem 6. But $\mathcal{S}$ is maximal solvable, thus $\mathcal{S} = \mathfrak{J}(\mathcal{U})$. Theorem 6 also states that $\mathcal{U}$ is L-solvable, hence $\mathcal{U} \subset \mathcal{R}(\mathcal{T})$ and $\mathcal{S} = \mathfrak{J}(\mathcal{U}) \subset \mathfrak{J}(\mathcal{R}(\mathcal{T}))$. Conversely $\mathcal{R}(\mathcal{T})$ is L-solvable, then $\mathfrak{J}(\mathcal{R}(\mathcal{T}))$ solvable and $\mathfrak{J}(\mathcal{R}(\mathcal{T})) \subset \mathcal{S}$. Hence $\mathcal{S} = \mathfrak{J}(\mathcal{R}(\mathcal{T}))$. From this the corollary is already an immediate consequence. We now consider

$\overline{\mathcal{T}} = \dfrac{\mathcal{T} + J(\mathcal{R})}{J(\mathcal{R})}$, $\mathcal{R} = \mathcal{R}(\mathcal{T})$, with canonical induced triple product. $\overline{\mathcal{T}}$ is a L.t.s. It is easily checked, that the natural module isomorphisms

$$\overline{\mathcal{T}} = \frac{\mathcal{T} + J(\mathcal{R})}{J(\quad)} \cong \frac{\mathcal{T}}{J(\mathcal{R}) \cap \mathcal{T}} \cong \frac{\mathcal{T}}{\mathcal{R}(\mathcal{T})}$$ are L.t.-isomorphisms,

hence $\overline{\mathcal{T}}$ is L-semi-simple. Obviously

$\overline{\mathcal{L}} = \dfrac{\mathcal{L}}{J(\mathcal{R})} = \overline{\mathcal{T}} + [\overline{\mathcal{T}}, \overline{\mathcal{T}}]$ and it is clear that the results so far proved apply to $\overline{\mathcal{T}}$ and $\overline{\mathcal{L}}$, in particular $\overline{\mathcal{L}}$ semi-simple iff $\overline{\mathcal{T}}$ L-semi-simple. Hence $\overline{\mathcal{L}}$ is semi-simple and Rad $\mathcal{L} \subset J(\mathcal{R})$.

This implies $\mathcal{J}(\mathcal{R}) = \mathrm{Rad}\,\mathcal{L}$.

Open question: We know $\mathcal{R}(\mathcal{T}) \subset \mathrm{Rad}\,\mathcal{T}$ (see exercise). Is the converse true? Or more generally, is a solvable ideal of $\mathcal{T}$ L-solvable? This is true for special cases (see theorem 9).

<u>Exercise.</u> Complete the proof of thm. 7 and show $\overline{\mathcal{T}}$ is L-semi-simple, iff $\overline{\mathcal{L}}$ semi-simple.

6.6. Since we want to apply the results from Lie theory we indicated in chapter V, we make for the rest of this chapter the assumption, that $\mathcal{T}$ is a finite dimensional Lie triple system over a field F. Our first result is an application of V, theorem 2.

<u>Theorem 8. If $\mathcal{T}$ is a finite dimensional L.t.s. over F of Char 0, then $\mathcal{T}$ is L-semi-simple, iff $\mathcal{T}$ is the direct sum of simple ideals.</u>

<u>Proof.</u> Let $\mathcal{T}_1$ be a minimal ideal of $\mathcal{T}$. $\mathcal{J}(\mathcal{T}_1)$ is an ideal of $\mathcal{L}$, which is semi-simple by theorem 7, and then a finite sum of simple ideals (or equivalently every ideal of $\mathcal{L}$ is complemented). Consequently $\mathcal{L} = \mathcal{J}(\mathcal{T}_1) \oplus \mathcal{L}'$. Since $\mathcal{J}(\mathcal{T}_1)$ is Θ-invariant, the complement $\mathcal{L}'$ (which is the orthogonal complement of $\mathcal{J}(\mathcal{T}_1)$ relative to the Killing form) is Θ-invariant, too (see V, lemma 2). Then $\mathcal{L}' = \mathcal{M} \oplus \mathcal{U}$, $\mathcal{U}$ ideal of $\mathcal{T}$ and $\mathcal{T} = \mathcal{L} \cap \mathcal{T} = \mathcal{T}_1 \oplus \mathcal{U}$. Standard arguments show that $\mathcal{T}_1$ is simple. The result now follows by induction on the dimension. $\mathcal{T}$ simple implies $\mathcal{L}$ semi-simple and consequently $\mathcal{T}$ semi-simple.

<u>Theorem 9. If $\mathcal{T}$ is finite dimensional over F of char. 0, then an ideal $\mathcal{U}$ of $\mathcal{T}$ is L solvable, iff $\mathcal{U}$ is solvable.</u>

<u>Proof.</u> If $\mathcal{T}$ is L-semi-simple and $\mathcal{U} \neq 0$ a solvable ideal then theorem 8 implies that $\mathcal{U}$ is a direct sum of simple ideals $\mathcal{T}_i$. This leads to a contradiction to $[\mathcal{T}_i \mathcal{T}_i \mathcal{T}_i] = \mathcal{T}_i$. Hence $\mathcal{U} = 0$ and

Rad $\mathcal{T} = 0$. Assume $\mathcal{R}(\mathcal{T}) \neq 0$, then $\mathcal{T}/_{\mathcal{R}(\mathcal{T})}$ is L-semi-simple, consequently $\text{Rad}(\mathcal{T}/_{\mathcal{R}(\mathcal{T})}) = 0$. Then Rad $\mathcal{T} \subset \mathcal{R}(\mathcal{T})$, by III, 3.2. (see also theorem 5). Since $\mathcal{R}(\mathcal{T}) \subset \text{Rad}\,\mathcal{T}$ (see exercise p.51) we get Rad $\mathcal{T} = \mathcal{R}(\mathcal{T})$. Consequently, if $\mathcal{U}$ is solvable it is contained in $\mathcal{R}(\mathcal{T})$ and then L-solvable.

Theorem 10. If $\mathcal{T}$ is a semi-simple L.t.s. over F of char. 0, then any derivation D of $\mathcal{T}$ is of the form $D = \Sigma\, L(u_i, v_i)$.

Proof. Verify that $\delta : \mathcal{L} \to \mathcal{L}$,

$$\delta(X) := [D,H] \oplus Da \quad (\text{if } X = H \oplus a)$$

is a derivation of $\mathcal{L}$. Since $\mathcal{T}$ is L-semi-simple, iff $\mathcal{T}$ is semi-simple, we get $\mathcal{L}$ semi-simple and therefore we can apply V, theorem 3. Consequently $\delta(X) = [U,X]$, $U = H_1 \oplus a_1 \in \mathcal{L}$. Then $Da = \delta a = H_1 a + L(a_1, a)$. But $Da \in \mathcal{T}$, implies $L(a_1, a) = 0$. We end up with $Da = H_1 a$ for all a or $D = H_1 \in \mathcal{G}$.

6.7. Since the Killing form in Lie algebras is of fundamental importance (in the finite dimensional case) we shall compute the Killing form of the standard imbedding $\mathcal{L}$ for finite dimensional L.t.s. $\mathcal{T}$ over F of characteristic $\neq 2$. Let be $\mathcal{L} = \mathcal{G} \oplus \mathcal{T}$, $X = H \oplus a$, $H \in \mathcal{G}$, $a \in \mathcal{T}$, and λ the Killing form of $\mathcal{L}$. Since $-\Theta$ is an automorphism of $\mathcal{L}$, V, lemma 2, implies $-\lambda(H,a) = \lambda(\Theta H, a) = \lambda(H, \Theta a) = \lambda(H,a)$. Consequently $\lambda(\mathcal{G}, \mathcal{T}) = 0$. Hence

$$(6.9) \qquad \lambda(X,X) = \lambda(H,H) + \lambda(a,a).$$

We define

$$\text{ad}_+ X : \mathcal{L} \to \mathcal{L} \text{ by}$$

$$(\text{ad}_+ X)H' = [X,H'], \; (\text{ad}_+ X)u = 0, \; H' \in \mathcal{G}, \; u \in \mathcal{T}.$$

and $ad_-X: \mathcal{L} \to \mathcal{L}$ by

$(ad_-X)H' = 0$, $(ad_-X)u = [X,u]$.

Clearly

(6.10) $adX = ad_+X + ad_-X$.

Next we show

$$\begin{aligned} &\text{(i)} \quad ad_+H\,ad_-H = ad_-H\,ad_+H = 0 \\ (6.11) \quad &\text{(ii)} \quad (ad_+a)^2 = (ad_-a)^2 = 0. \qquad \text{for all } H \in \mathfrak{g}, \ a \in \mathfrak{f}. \end{aligned}$$

Proof. For example,

$(ad_+H\,ad_-H)\mathfrak{g} = o$, by definition of ad_-X.

$(ad_+H\,ad_-H)a = (ad_+H)Ha = 0$, by definition of ad_+X,

then $ad_+H\,ad_-H = 0$.

The rest is left as an exercise.

(6.10) and (6.11) imply

$$\begin{aligned} (6.12) \quad &\text{(i)} \quad (adH)^2 = (ad_+H)^2 + (ad_-H)^2 \text{ and} \\ &\text{(ii)} \quad (ada)^2 = ad_+a\,ad_-a + ad_-a\,ad_+a. \end{aligned}$$

Now we compute traces. Since ad_+H is zero on $\mathfrak{f}$ and equals $ad_{\mathfrak{g}}H$ on $\mathfrak{g}$, we get

$$\text{trace } (ad_+H)^2 = \lambda_{\mathfrak{g}}(H,H)$$

where $\lambda_{\mathfrak{g}}$ denotes the Killing form of $\mathfrak{g}$. $(ad_-H)^2$ is zero on $\mathfrak{g}$, and on $\mathfrak{f}$ we have $(ad_-H)^2u = H^2u$, consequently

$$\text{trace } (ad_-H)^2 = \text{trace } H^2$$

Hence by (6.12) and these two equations,

$$(6.13) \qquad \lambda(H,H) = \lambda_{\mathfrak{g}}(H,H) + \text{trace } H^2$$

By (6.12i) we get

$$\lambda(a,a) = 2 \text{ trace } ad_+a\,ad_-a.$$

Again: $(ad_+a\,ad_-a)$ is zero on $\mathfrak{g}$ and

$$\begin{aligned} (ad_+a\,ad_-a)u &= ad_+a\,[a,u] \\ &= [a,[a,u]] = [uaa] = R(a,a)u. \end{aligned}$$

Hence

(6.14) $$\lambda(a,a) = 2 \text{ trace } R(a,a).$$

Putting these results together ((6.9), (6.13), (6.14)) we end up with

Theorem 11. *If $\mathcal{T}$ is a finite dimensional L.t.s. over F of char $\neq 2$, λ resp. $\lambda_{\mathcal{G}}$ the Killing form on $\mathcal{L}$ resp. $\mathcal{G}$ and $X = H \oplus a \in \mathcal{L}$, then*

$$\lambda(X,X) = \lambda_{\mathcal{G}}(H,H) + \text{trace } H^2 + 2 \text{ trace } R(a,a).$$

Corollary 1: *λ is non degenerate, iff*

$$(H,H') \mapsto \lambda_{\mathcal{G}}(H,H') + \text{trace } (HH') \text{ and}$$
$$(a,b) \mapsto \text{trace}\left[R(a,b) + R(b,a)\right]$$

are non degenerate bilinear forms on $\mathcal{G}$ resp. $\mathcal{T}$.

Corollary 2: *If $\mathcal{T}$ is finite dimensional, semi-simple over F of char 0, then $(a,b) \mapsto \text{trace } [R(a,b) + R(b,a)]$ is non degenerate.*

We define $\varrho(a,b) := \frac{1}{2} \text{ trace}\left[R(a,b) + R(b,a)\right]$ and assume ϱ is non degenerate. ($\mathcal{T}$ fin. dim., char F $\neq 2$). Since ϱ is (up to a scalar β) the restriction of the Killing form, which is associative, we get

$$\varrho(R(a,b)x,z) = \varrho([xba],z) = \beta\lambda([[x,b],a],z) =$$
$$\beta\lambda(x,[b,[a,z]]) = \varrho(x,[zab]) = \varrho(x,R(b,a)z)$$

This shows

(6.14) $$\varrho([xba],z) = \varrho(x,[zab])$$

Using (6.14), the symmetry of ϱ and $[abc] = -[bac]$, we get a chain

(6.15) $$\varrho([xba],z) = \varrho(x,[zab]) = -\varrho([azb],x) = \varrho(a,[bxz]) =$$
$$\varrho([bxz],a) = \varrho(b,[azx])$$

If A^* denotes the adjoint of $A \in \operatorname{End} \mathfrak{Z}$ relative to ρ then we get from (6.15)

(6.16) (i) $R(a,b)^* = R(b,a)$, (ii) $L(x,b)^* = L(b,x) = - L(x,b)$.

Since $\operatorname{tr}A = \operatorname{tr}A^*$ (6.16) implies trace $R(a,b)$ = trace $R(b,a)$ and trace $L(x,b) = 0$, in particular

$$\rho(a,b) = \operatorname{trace} R(a,b).$$

We define $xy^* \in \operatorname{End} \mathfrak{Z}$ by $xy^*z = \rho(z,y)x$ and consider the map $S: \operatorname{End} \mathfrak{Z} \to \operatorname{End} \mathfrak{Z}$ defined by

trace $AL(x,y) = \rho(S(A)x,y)$. By III, 3.4. we have

$$S(xy^*) = L(x,y).$$

(6.16ii) implies $S(xb^*)^* = S((xb^*)^*) = - S(xb^*)$, consequently

(6.17) $$S(A)^* = S(A^*) = - S(A).$$

The defining identity (6.3ii) now becomes (using 6.17)

$$\left[S(xy^*),\, S(uv^*)\right] = S(S(xy^*)uv^* - uv^*S(xy^*)),$$

or

(6.18) $$\left[S(A),S(B)\right] = S(\left[S(A),B\right])$$

Furthermore (6.15) implies

$$\operatorname{tr} uv^*L(x,y) = \rho([xyu],v) = \rho(x,[vuy]) = \rho([uvx],y) = \operatorname{tr} L(u,v)xy^*.$$

Hence

(6.19) $\operatorname{tr} AS(B) = \operatorname{tr} S(A)B$

which means that S is selfadjoint with respect to the trace form on $\operatorname{End} \mathfrak{Z}$.

We shall compute another expression for the Killing form. The Lie algebra $\mathfrak{G}$ is spanned by all $L(x,y)$ or in terms of the Lie algebra $\mathfrak{L}$ by $[x,y]$. To compute the Killing form it is sufficient to compute its value on the generators.

$$\begin{aligned}\text{tr ad}[x,y]\text{ad}[u,v] &= \lambda([x,y],[u,v]), \text{ by (6.9)},\\ &= \lambda([[x,y],u],v), \text{ by V, lemma 2i},\\ &= \lambda([xyu],v), \text{ by theorem 1 (iv)},\\ &= 2\rho(L(x,y)u,v), \text{ by (6.14)}\\ &= \text{trace } S(xy^*)uv^*,\end{aligned}$$

Consequently

$$\lambda(S(A),S(B)) = \text{trace } S(A)B. \tag{6.20}$$

From this expression it is obvious that λ restricted to $\mathfrak{g}$ is non degenerate, consequently

<u>Corollary 3.</u> λ <u>is non degenerate, iff ρ is non degenerate</u>/ and

<u>Corollary 4.</u> (Cartan's Criterion for L.t.s.). <u>If $\mathfrak{T}$ is a finite dimensional L.t.s. over a field of char</u> 0, <u>then $\mathfrak{T}$ is semi-simple, iff ρ is non degenerate</u>.

<u>Proof.</u> ρ non degenerate, iff λ nondegen. (by cor. 3); this is the case, iff $\mathfrak{L}$ semi-simple (Cartan Criterion, V, theorem 1) and by the corollary to theorem 7 this is equivalent to $\mathfrak{T}$ semi-simple.

As another application of our computations of traces we find an expression of the Killing form of $\mathfrak{g}$. Comparing (6.13) (in linearized form) and (6.20), putting $H = S(A)$, $H' = S(B)$ and observing tr $S(A)S(B) = $ tr $S^2(A)B$ we get

$$\lambda_{\mathfrak{g}}(S(A),S(B)) = \text{trace } [2S(A) - S(S(A))]B. \tag{6.21}$$

This equation shows that $\lambda_{\mathfrak{g}}$ is non degenerate, iff $S^2(A) = 2S(A)$ implies $S(A) = 0$. In the case of char $F = 0$ this gives an interesting criterion about semi simplicity of $\mathfrak{g}$.

<u>Problem:</u> (the answer is not known to me). Determine the minimum polynomial of S for simple L.t.s.

<u>Exercise.</u> Define S': End $\mathfrak{T} \to$ End $\mathfrak{T}$ by

$$\text{trace } AR(x,y) = \rho(S'(A)x,y).$$

Show. i) $S'(uv^*) = R(u,v)$

ii) $\operatorname{tr} S'(A)B = \operatorname{tr} AS'(B)$

iii) $S'(Id) = Id$

6.8. Let $\mathcal{T}$ be a set of endomorphisms of a vector space V. A subspace $U \subset V$ is called $\underline{\mathcal{T}\text{-invariant}}$, if $\sigma U \subset U$ for all $\sigma \in \mathcal{T}$. One says that V is irreducible (rel. $\mathcal{T}$) or $\mathcal{T}$ acts irreducibly on V, if V has no proper ($\neq 0, \neq V$) $\mathcal{T}$ - invariant subspace. $\mathcal{T}$ is called completely reducible (on V), if $V = \sum U_\alpha$, and $\mathcal{T}$ acts irreducibly on all subspaces U_α. From linear algebra it should be known: $\mathcal{T}$ is completely reducible, iff every $\mathcal{T}$ - invariant subspace of V has a direct $\mathcal{T}$ - invariant complement, or equivalently $V = \oplus U_i$ where the subspaces U_i are irreducible (rel. $\mathcal{T}$). (see p. 46 of N. Jacobson's Lie algebras).

It can be shown (the proof is non trivial and needs some techniques from Lie theory).

Theorem 12. If $\mathcal{F}$ is a simple L.t.s., finite dimensional over F of char 0, then $\mathcal{D}(\mathcal{F})$ is completely reducible (on $\mathcal{F}$).

Note: By theorem 10 we have $\mathcal{D}(\mathcal{F}) = \mathcal{G}$.

Starting with this result we shall prove

Theorem 13. If $\mathcal{F}$ is as in thm. 12, then either $\mathcal{F}$ is irreducible relative $\mathcal{D}(\mathcal{F})$ or $\mathcal{F} = \mathcal{U}_1 \oplus \mathcal{U}_2$ with invariant irreducible subsystems and $L(\mathcal{U}_i, \mathcal{U}_i) = 0$ $(i = 1,2)$.

Proof. Suppose $\mathcal{F}$ not irreducible and $\mathcal{U}_1$ a proper $\mathcal{D}$ - invariant subspace. Since $\mathcal{D} = \mathcal{G}$, $\mathcal{D}$ - invariance means $[\mathcal{F}\mathcal{F}\mathcal{U}_1] \subset \mathcal{U}_1$.

Then 1) $L(\mathcal{U}_1, \mathcal{U}_1)$ is an ideal in $\mathcal{G}$, (obvious by (6.4))

2) If $\mathcal{M}$ is an ideal of $\mathcal{G}$, then $\mathcal{M}\mathcal{F}$ is $\mathcal{G}$ invariant.

3) If $\mathcal{U}_2$ is a direct $\mathcal{D}$ - invariant complement of $\mathcal{U}_1$

$\mathfrak{T} = \mathfrak{U}_1 \oplus \mathfrak{U}_2$ then $[\mathfrak{U}_1\mathfrak{U}_1\mathfrak{U}_2] = 0.$

For $[\mathfrak{U}_1\mathfrak{U}_1\mathfrak{U}_2] \subset [\mathfrak{T}\mathfrak{T}\mathfrak{U}_2] \subset \mathfrak{U}_2$ and on the other hand $[\mathfrak{U}_1\mathfrak{U}_1\mathfrak{U}_2] \subset [\mathfrak{T}\mathfrak{T}\mathfrak{U}_1] \subset \mathfrak{U}_1$ by Jacobi's identity

4) $[\mathfrak{U}_1\mathfrak{U}_1\mathfrak{U}_1]$ is an ideal in $\mathfrak{T}$. If $u,v,w \in \mathfrak{U}_1$, $x,y \in \mathfrak{T}$ then

$$\begin{aligned}[[uvw]xy] &= [uv[wxy]] - [w[uvx]y] - [wx[uvy]] \in \\ &\in [\mathfrak{U}_1\mathfrak{U}_1\mathfrak{T}] + [[\mathfrak{U}_1\mathfrak{U}_1\mathfrak{T}]\mathfrak{U}_1\mathfrak{T}] + [\mathfrak{U}_1\mathfrak{T}[\mathfrak{U}_1\mathfrak{U}_1\mathfrak{T}]] \\ &\subset [\mathfrak{U}_1\mathfrak{U}_1\mathfrak{U}_1] \text{ by 2) and 3).}\end{aligned}$$

Since $\mathfrak{T}$ is simple and $[\mathfrak{U}_1\mathfrak{U}_1\mathfrak{U}_1] \subset \mathfrak{U}_1 \neq \mathfrak{T}$ we have $[\mathfrak{U}_1\mathfrak{U}_1\mathfrak{U}_1] = 0.$

This together with 3) shows

$$L(\mathfrak{U}_1, \mathfrak{U}_1) = 0.$$

Suppose $\mathfrak{T} = \mathfrak{W}_1 \oplus \mathfrak{W}_2 \oplus \ldots \oplus \mathfrak{W}_r$, $r > 2$, $\mathfrak{W}_i$ invariant irred., then $\mathfrak{W}_1 \oplus \mathfrak{W}_i$ is a proper invariant subspace and the above consideration apply. Consequently, $0 = L(\mathfrak{W}_1 \oplus \mathfrak{W}_i, \mathfrak{W}_1 \oplus \mathfrak{W}_i) = L(\mathfrak{W}_1, \mathfrak{W}_i)$, hence $[\mathfrak{W}_1\mathfrak{T}\mathfrak{T}] = 0$ and $\mathfrak{W}_1$ is a proper ideal. This is a contradiction. We conclude that $r \leq 2$ and $\mathfrak{U}_1, \mathfrak{U}_2$ irreducible. ∎

We keep the assumptions on $\mathfrak{T}$. We consider the case, when $\mathfrak{T}$ is not irreducible. $\mathfrak{T} = \mathfrak{U}_1 \oplus \mathfrak{U}_2$. We consider the map $D: \mathfrak{T} \to \mathfrak{T}$, $Dx = x_1 - x_2$, if $x = x_1 + x_2$, $x_i \in \mathfrak{U}_i$. An easy verification (using $L(\mathfrak{U}_i, \mathfrak{U}_i) = 0$) shows $0 = [D, L(u,v)] = L(Du,v) + L(u,Dv)$. D is a derivation of $\mathfrak{T}$, therefore contained in $\mathfrak{H}$ (see theorem 10). We proved

Lemma 8. *If $\mathfrak{T}$ is not irreducible, then there is an element D in the center of $\mathfrak{H}$ such that $Dx = x_1 - x_2$ (if $x = x_1 + x_2$, $x_i \in \mathfrak{U}_i$)*

(The center of $\mathfrak{H}$ is the set of all $H \in \mathfrak{H}$, such that $[H, \mathfrak{H}] = 0$.)

We come to the conclusive result of this chapter which will (hopefully) give us a better understanding for certain results and constructions we shall obtain later.

Theorem 14. If $\mathcal{J}$ is a simple, finite dimensional L.t.s. over an algebraically closed field of characteristic zero, then either

(i) $\mathcal{J}$ is irreducible relative $\mathcal{D}(\mathcal{J})$ and $\mathcal{D}(\mathcal{J})$ is semi-simple,

(ii) $\mathcal{J} = \mathcal{U}_1 \oplus \mathcal{U}_2$, $\mathcal{U}_i$ are irreducible (rel. $\mathcal{D}(\mathcal{J})$) and isomorphic subsystems such that $L(\mathcal{U}_i, \mathcal{U}_i) = 0$, furthermore $\mathcal{D}(\mathcal{J})$ has a one dimensional center.

Proof. (i) If $\mathcal{J}$ is not irreducible, then by lemma 8, $\mathcal{D}(\mathcal{J})$ has a nontrivial center (which is, of course, a solvable ideal). Hence $\mathcal{D}(\mathcal{J})$ semi-simple implies $\mathcal{J}$ irreducible. If conversely $\mathcal{D}$ is not semi-simple, then the center of $\mathcal{D}$ has to be non trivial. This follows from an important result in Lie theory, that a completely reducible Lie algebra of linear transformations can be decomposed as a direct sum of its center and a semi-simple subalgebra. H is in the center of $\mathcal{D}$, iff H commutes with every element of $\mathcal{D}$. If $\mathcal{J}$ is irreducible then by Schur's lemma $H = \alpha \mathrm{Id}, \alpha \in F$ (F is algebraically closed). But by (6.17) the elements in $\mathcal{G}-\mathcal{D}$ are skew symmetric; therefore $0 = \operatorname{trace} H = \alpha \dim \mathcal{J}$. Consequently $\alpha = 0$. Thus, if $\mathcal{D}$ has nontrivial center, $\mathcal{J}$ must be reducible. So we are in case (ii). Since $[\mathcal{U}_i \mathcal{U}_i \mathcal{U}_i] = 0$, they are isomorphic if they have the same dimension. Consider $Dx = x_1 - x_2$, by lemma 8 it is in $\mathcal{D}$ and therefore trace $D = 0$. But $D|_{\mathcal{U}_1} = \mathrm{id}|_{\mathcal{U}_1}$, and $D|_{\mathcal{U}_2} = -\mathrm{Id}|_{\mathcal{U}_2}$. Consequently

$0 = \operatorname{tr} D = \operatorname{tr} D|_{\mathcal{U}_1} + \operatorname{tr} D|_{\mathcal{U}_2} = \dim \mathcal{U}_1 - \dim \mathcal{U}_2$. Finally let C be any element in the center of $\mathcal{D}$. Then by Schur's lemma, the restriction of C to $\mathcal{U}_i$ is $\alpha_i \mathrm{id}|_{\mathcal{U}_i}$ since $\mathcal{D}$ acts irreducibly on $\mathcal{U}_i$.

Thus $C(x_1 + x_2) = \alpha_1 x_1 + \alpha_2 x_2$. Then $0 = \text{trace } C = \alpha_1 \dim \mathcal{U}_1 + \alpha_2 \dim \mathcal{U}_2$. Since $\dim \mathcal{U}_1 = \dim \mathcal{U}_2$ we get $\alpha_1 = -\alpha_2$ and $C = \alpha_1 D$. This shows that D is a basis of the center of $\mathfrak{J}$.

VII. Linear Jordan Algebras.

7.1. Let Φ be a commutative ring with unit element 1 containing $\frac{1}{2}$. An algebra $\mathfrak{J}$ over Φ with product $(x,y) \mapsto xy$ is called a <u>linear Jordan algebra</u>, if

(J.1) $\quad xy = yx \quad$ "commutativity"

(J.2) $\quad x(x^2y) = x^2(xy) \quad$ "Jordan identity"

for all $x,y \in \mathfrak{J}$.

In terms of the left and right multiplication $L(x), R(x)$, the above definition is obviously equivalent to

(J.1') $\quad L(x) = R(x)$

(J.2') $L(x)L(x^2) = L(x^2)L(x) \quad$ for all $x \in \mathfrak{J}$.

<u>Example.</u> If $\mathfrak{A}$ is an associative algebra over Φ with product $(x,y) \mapsto xy$, then $\mathfrak{A}^+$, i.e., the module $\mathfrak{A}$ together with multiplication $(x,y) \mapsto x \circ y = \frac{1}{2}(xy + yx)$ is a Jordan algebra (see I, 1.1. , ex. 5). The powers of an element in $\mathfrak{A}^+$ are the same as in $\mathfrak{A}$. Furthermore, if $\mathfrak{A}$ has a unit element e, then e is also the unit element of $\mathfrak{A}^+$.

<u>Exercise.</u> If $\mathfrak{J}$ is a Jordan algebra over Φ, then the unital algebra $\hat{\mathfrak{J}} = \Phi \cdot 1 \oplus \mathfrak{J}$ is again a Jordan algebra (see 1.7.).

7.2. A linearized form of the Jordan identity is (replace x by $x + \alpha z$, $\alpha = 1, \frac{1}{2}$).

(7.1) $\quad z(x^2y) + 2x((xz)y) = x^2(zy) + 2(xz)(xy)$

Linearizing again leads to (since we assume $\frac{1}{2} \in \Phi$)

(7.2) $\quad z((xu)y) + u((xz)y) + x((uz)y) = (xu)(zy) + (uz)(xy) + (xz)(uy)$.

This is in operator form (acting on z)

(7.3) $L(y(xu)) + L(u)L(y)L(x) + L(x)L(y)L(u) = L(xu)L(y) + L(uy)L(x) + L(xy)L(u)$.

Since the right hand side of this equation is symmetric in x and y we get

$L(y(xu)) + L(u)L(y)L(x) + L(x)L(y)L(u) = L(x(yu)) + L(u)L(x)L(y) + L(y)L(x)L(u)$,

or equivalently

(7.4) $L(x(yu) - y(xu)) = [[L(x),L(y)], L(u)]$

This equation has the following two interpretations

Lemma 1. The mappings $[L(x),L(y)]$, $x,y \in \mathfrak{J}$, are derivations of $\mathfrak{J}$.

Lemma 2. $L(\mathfrak{J})$ together with $(L(x),L(y),L(z)) \mapsto [[L(x),L(y)] ,L(z)]$ is a Lie triple system.

We denote by $\mathfrak{J}'$ the submodule of $\mathfrak{J}$ spanned by all associators $(xy)z - x(yz)$, $x,y,z \in \mathfrak{J}$. Equation (7.4) shows that any Lie triple product of elements in $L(\mathfrak{J})$ is in $L(\mathfrak{J}')$, consequently, $L(\mathfrak{J}')$ is an ideal of $L(\mathfrak{J})$.

7.3. An important role in the theory of Jordan algebras plays the so-called quadratic representation P of a Jordan algebra $\mathfrak{J}$. This is a map $P : \mathfrak{J} \to \operatorname{End}\mathfrak{J}$, $x \mapsto P(x)$, defined by

(7.5) $P(x) = 2L(x)^2 - L(x^2)$, $x \in \mathfrak{J}$.

Note: $[L(x),L(x^2)] = 0$ implies $[L(x),P(x)] = 0$.

Example. If $\mathfrak{A}$ is associative, then the quadratic representation of $\mathfrak{A}^+$ is given by $P(x)y = xyx$.

The map P is quadratic in the sense that

$P(\alpha x) = \alpha^2 P(x)$ for all $\alpha \in \Phi$, $x \in \mathfrak{J}$, and

$P(x,y) := P(x+y) - P(x) - P(y)$ is bilinear (in x and y).

From the definition (7.5) we obtain easily

(7.6) $P(x,y) = 2\left[L(x)L(y) + L(y)L(x) - L(xy)\right]$, $P(x,x) = 2P(x)$,

Using (7.6) and (7.3) we compute

$P(xy,x) - L(y)P(x) - P(x)L(y) = 2L(xy)L(x) + 2L(x)L(xy) -$
$2L(x(xy)) - 2L(y)L(x)^2 + L(y)L(x^2) - 2L(x)^2L(y) + L(x^2)L(y)$
$= 2\left[L(x), L(xy)\right] + \left[L(y), L(x^2)\right] = 0$, since the last term is the linearized form of $\left[L(x), L(x^2)\right] = 0$, (J.2').

Consequently,

(7.7) $L(y)P(x) + P(x)L(y) = P(xy,x)$.

Furthermore we note that the linearization of $\left[L(x), P(x)\right] = 0$ is

(7.8) $\left[P(x,u), L(x)\right] = \left[L(u), P(x)\right]$.

An important composition in (linear) Jordan algebras is $(x,y,z) \to \{xyz\} := P(x,z)y$.

This is obviously a trilinear composition, i.e., $\mathfrak{J}$ together with this composition is a triple system (see <u>III</u>.). The "left multiplications" of this triple system are $L(x,y) \in \operatorname{End}\mathfrak{J}$, defined by

$$L(x,y)z = \{xyz\} = P(x,z)y$$

Using (7.6) we observe

$$L(x,y) = 2\left[L(x), L(y)\right] + 2L(xy).$$

Applying (7.7) repeatedly (and using $L(x)P(x) = P(x)L(x)$) we derive $\frac{1}{2}P(x)L(y,x) = P(x)L(y)L(x) - L(x)P(x)L(y) + P(x)L(xy) =$

$= [P(xy,x) - L(y)P(x)]\, L(x) - L(x)[P(xy,x) - L(y)P(x)] + P(x)L(xy)$

$= [L(x),L(y)]\, P(x) + [P(xy,x),L(x)] + P(x)L(xy)$

$= [L(x),L(y)]\, P(x) + L(xy)P(x)$ (by (7.8) with $u = xy$)

$= \frac{1}{2}L(x,y)P(x)$.

We proved $P(x)L(y,x) = L(x,y)P(x)$. Both sides of this equation acting on u shows $P(x)\{yxu\} = \{xyP(x)u\}$. Since the left hand side of the last equation is symmetric in y and u, we conclude $\{xyP(x)u\} = \{xuP(x)y\}$. This is in operator form $L(x,y)P(x) = P(P(x)y,x)$. We proved

(7.9) $L(x,y)P(x) = P(x)L(y,x) = P(P(x)y,x)$ "Homotopy formula".

The linearization of (7.7) acting on $v \in \mathcal{J}$ shows (after appropriate change of notation),

(7.10) $y\cdot\{uvw\} = \{(yu)vw\} - \{u(yu)w\} + \{uv(yw)\}$.

It is obvious from the definition, that for any derivation D of $\mathcal{J}$

$$D\{uvw\} = \{(Du)vw\} + \{u(Dv)w\} + \{uv(Dw)\}$$

holds. Then, in particular, this equation holds for $D = [L(x),L(y)]$, by lemma 1. Using this and (7.10) $(y \to xy)$, we derive

$L(x,y)\{uvw\} = 2[L(x),L(y)]\{uvw\} + 2L(xy)\{uvw\}$

$= \{(L(x,y)u)vw\} - \{u(L(y,x)v)w\} + \{uvL(x,y)w\}$.

This is

(7.11) $\{xy\{uvw\}\} - \{uv\{xyw\}\} = \{\{xyu\}vw\} - \{u\{yxv\}w\}$,

or in operator form

(7.11') $[L(x,y),L(u,v)] = L(\{xyu\},v) - L(u,\{yxv\})$.

A particular case of this equation is (setting $u = x$, $y = v$)

(7.12) $\qquad L(P(x)y,y) = L(x,P(y)x)$.

Furthermore we observe that the left hand side of (7.11) is skew symmetric in the pairs (x,y), (u,v), hence

$$(7.13) \quad \{\{xyu\}vw\} - \{u\{yxv\}w\} = \{x\{vuy\}w\} - \{\{uvx\}yw\} .$$

In order to prove the <u>fundamental formula</u>

$$(7.14) \quad P(P(u)v) = P(u)P(v)P(u) \quad \text{for all } u,v \in \mathfrak{J} ,$$

we substitute $x \to \{uvu\}, w \to u$ in (7.11) and obtain (note: $\{xyx\} = 2P(x)y$)

$$(7.15) \quad 8P(P(u)v)y = 2\{uv\{uy\{uvu\}\}\} - \{u\{y\{uvu\}\}u\} .$$

Replacing $u \to y$, $y \to u$, $x \to v$, $v \to u$, $w \to v$ in (7.13) gives

$$\{y\{uvu\}v\} = 2\{\{vuy\}uv\} - \{v\{uyu\}v\} .$$

Substituting this in (7.15) implies

$$8P(P(u)v)y = 2\{uv\{uy\{uvu\}\}\} - 2\{u\{\{uyv\}uv\}u\} + 8P(u)P(v)P(u)y.$$

Since the homotopy formula (7.9) has as consequence

$$\{uv\{uy\{uvu\}\}\} = \{uv\{u\{yuv\}u\}\} = \{u\{vu\{yuv\}\}u\} ,$$

the foregoing reduces to (7.14).

We have seen that the deduction from the axioms (J.1), (J.2) of all the important formulas in Jordan theory (in particular (7.9), (7.12) and (7.14)) depends heavily on the fact that we were able to cancel by 2. On the other hand, a theory of linear Jordan algebras over fields of characteristic 2 does not lead to results, which are "compatible" with results in the case of char $\neq 2$. So one has to think of something else, which would permit a "nice" theory for arbitrary rings. The best approach so far is via "quadratic Jordan algebras", which where "invented" by K. McCrimmon. Before presenting some fundamentals of his theory we shall study some examples of linear Jordan algebras.

VIII. Examples of Linear Jordan Algebras.

Throughout this chapter we assume $\frac{1}{2} \in \Phi$.

8.1. We already know, that for an associative algebra $\mathfrak{A}$ with multiplication $(x,y) \to xy$, the algebra $\mathfrak{A}^+$, i.e., $\mathfrak{A}$ together with $x \circ y := \frac{1}{2}(xy + yx)$, is a Jordan algebra. But then any submodule $\mathfrak{L}$ of $\mathfrak{A}$, closed under $(x,y) \mapsto x \circ y$, is also a Jordan algebra. A (linear) Jordan algebra is called special, if it is isomorphic to a (Jordan -) subalgebra of some $\mathfrak{A}^+$, $\mathfrak{A}$ associative. Which makes the theory more complicated, but more interesting, is the fact, that there are Jordan algebras which are not special. These are called exceptional Jordan algebras.

For the most interesting applications of Jordan algebras one needs simple algebras. Therefore we shall look for conditions on $\mathfrak{A}$ which force $\mathfrak{A}^+$ to be simple. Obviously any associative ideal of $\mathfrak{A}$ is an ideal of $\mathfrak{A}^+$. We shall show the converse. We start with:

Lemma 1. If $\mathfrak{L}$ is an ideal in $\mathfrak{A}^+$, then for all $a,b \in \mathfrak{L}$ and $x \in \mathfrak{A}$, $(ab + ba)x - x(ab + ba) \in \mathfrak{L}$.

Proof. An immediate verification shows

$x(ab + ba) - (ab + ba)x = a(xb - bx) + (xb - bx)a + (xa - ax)b + b(xa - ax)$.

Since $a,b \in \mathfrak{L}$, we have that $ya + ay$ and $yb + by$ are elements in $\mathfrak{L}$ for all $y \in \mathfrak{A}$; so the right hand side of the above equation is in $\mathfrak{L}$, for all $x \in \mathfrak{A}$. This already proves the lemma. An element

$x \in \mathfrak{A}$ is called trivial, if $x\mathfrak{A}x = 0$.

Theorem 1. If $\mathfrak{A}$ has no trivial elements $\neq 0$, then any non-zero ideal $\mathfrak{L}$ of $\mathfrak{A}^+$ contains a non-zero ideal of $\mathfrak{A}$.

Proof. Let $\mathfrak{L} \neq 0$ be an ideal of $\mathfrak{A}^+$. By lemma 1 we get for any $x \in \mathfrak{A}$, $xc - cx \in \mathfrak{L}$, where $c = ab + ba$, $a,b \in \mathfrak{L}$. Since $c \in \mathfrak{L}$, we have $xc + cx \in \mathfrak{L}$, consequently $xc \in \mathfrak{L}$ ($\frac{1}{2} \in \Phi$!) for all $x \in \mathfrak{A}$. But then again $(xc)y + y(xc) \in \mathfrak{L}$ for all y and therefore $xcy \in \mathfrak{L}$ for all $x,y \in \mathfrak{A}$ since we already showed $y(xc) = (yx)c \in \mathfrak{L}$. Then we have $\mathfrak{A}c\mathfrak{A} \subset \mathfrak{L}$. Since $\mathfrak{A}c\mathfrak{A}$ is an ideal in $\mathfrak{A}$, we are done, unless $\mathfrak{A}c\mathfrak{A} = 0$. In this case $c\mathfrak{A}c\mathfrak{A}c\mathfrak{A}c = 0$, which forces $c\mathfrak{A}c = 0$ and then $c = 0$, since $\mathfrak{A}$ has no trivial elements. If we can show, that for some $a,b \in \mathfrak{L}$ the element $c := ab + ba \neq 0$, then by the foregoing $\mathfrak{A}c\mathfrak{A} \neq 0$. Therefore assume $ab + ba = 0$ for all $a,b \in \mathfrak{L}$. Then in particular $a^2 = 0$ and $2\,axa = a(ax + xa) + (ax + xa)a = 0$ since $ax + xa \in \mathfrak{L}$. This shows $a\mathfrak{A}a = 0$. Again our assumption implies $a = 0$, which contradicts $\mathfrak{L} \neq 0$.

Corollary: If $\mathfrak{A}$ is a simple associative algebra then $\mathfrak{A}^+$ is a simple Jordan algebra.

Proof. Firstly we note that $x\mathfrak{A} = 0$ implies that $\mathfrak{A}x$ is an ideal of $\mathfrak{A}$. Since $\mathfrak{A}x = \mathfrak{A}$ would imply $\mathfrak{A}^2 = \mathfrak{A}x\mathfrak{A} = 0$ we have $\mathfrak{A}x = 0$. Then Φx is an ideal and $x \neq 0$ leads to $\mathfrak{A} = \Phi x$, $\mathfrak{A}^2 = 0$. Thus $x = 0$. Also $\mathfrak{A}x = 0$ implies $x = 0$, by the same argument. Next, let c be a trivial element in $\mathfrak{A}$, $c\mathfrak{A}c = 0$. We consider the ideal $\mathfrak{A}c\mathfrak{A}$. Since $\mathfrak{A}c\mathfrak{A} = \mathfrak{A}$ leads to $\mathfrak{A}^2 = \mathfrak{A}c\mathfrak{A}\mathfrak{A}c\mathfrak{A} \subset \mathfrak{A}c\mathfrak{A}c\mathfrak{A} = 0$

we get $\mathfrak{A} c \mathfrak{A} = 0$. Then $\mathfrak{A}c = 0$ and $c = 0$, by the foregoing remark. Therefore $\mathfrak{A}$ has no trivial elements $\neq 0$ and the theorem applies.

8.2. Let V be a vectorspace over $\Phi = F$, F being a field, and $q : V \to F$ a quadratic form on V, i.e.,

$$q(\alpha x) = \alpha^2 q(x) \text{ for all } \alpha \in F,\ x \in V, \text{ and}$$

$$q(x,y) = \tfrac{1}{2}\left[q(x+y) - q(x) - q(y)\right] \text{ is bilinear (in x and y).}$$

We wish to associate with (V,q) a Jordan algebra. The most obvious attempt will do it. We define

$$xy = q(x,y)1.$$

This, of course, is not a composition on V, but it leads to a composition on

$$\mathfrak{Q} = F\cdot 1 \oplus V$$

if we define $(\alpha 1 + x)(\beta 1 + y) := (\alpha\beta + q(x,y))1 + \alpha y + \beta x$.

In particular, for $z = \alpha 1 + x$ we get

$z^2 = 2\alpha z + (\alpha^2 + q(x,x))1$, and furthermore 1 is unit element of $\mathfrak{Q}$. This shows that the left multiplication $L(z^2)$ is a linear combination of $L(z)$ and $L(1) = \mathrm{id}$, which trivially implies $L(z)L(z^2) = L(z^2)L(z)$. Thus $\mathfrak{Q}$ is a Jordan algebra.

Exercise: Show that $\mathfrak{Q}$ is a quadratic extension of F or $\mathfrak{Q} \cong F \oplus F$ if $\dim V = 1$ and q non degenerate. Now assume $\dim V > 2$.

Let $\mathfrak{U}$ be an ideal of $\mathfrak{Q}$. If $\mathfrak{U} \cap V \neq 0$ and $z \neq 0$ is in this intersection, then by the nondegeneracy of q we can find a vector x such that $xu = q(x,u) = 1$. Since $xu \in \mathfrak{U}$, this shows $1 \in \mathfrak{U}$ and consequently $\mathfrak{U} = \mathfrak{Q}$. Let $\rho 1 + v$ be a non zero element in $\mathfrak{U}$ and $\rho \neq 0$. Then for any vector $y \neq 0$, orthogonal to v ($\dim V \geq 2$), we get $(\varrho 1 + v)y = \varrho y \in \mathcal{U} \cap V$ and we are back in the case, from which we derived $\mathcal{U} = \mathcal{Q}$. We proved:

<u>If</u> dim V ≥ 2 <u>and</u> $q(x,y)$ is <u>a</u> <u>non</u> <u>degenerate</u> <u>bilinear</u> <u>form</u> <u>then</u> <u>the</u> <u>Jordan</u> <u>algebra</u> $\mathcal{Q} = F1 \oplus V$ <u>is</u> <u>simple</u>.

Next we show that the Jordan algebra $F1\oplus V$, we considered above, is special. For this purpose we have to introduce the Clifford algebra $\mathcal{C}(V,q)$.

Let $\mathcal{T}(V)$ be the tensor algebra over V, that is

$\mathcal{T}(V) = \bigoplus_{i\geq 0} V^i$, where $V^0 := F1$ and $V^i = \bigotimes_{j=1}^{i} V$, the multiplication in $\mathcal{T}(V)$ is defined for the generators $\alpha \cdot x = \alpha x$,

$(a_1 \otimes \ldots \otimes a_s)\cdot(a_{s+1}\otimes \ldots \otimes a_r) = a_1 \otimes \ldots \otimes a_{s+1} \otimes \ldots \otimes a_r$. (then linearly extended). It is obvious that $\mathcal{T}(V)$ is an associative algebra with unit element 1. Let $\mathcal{K}$ be the ideal generated by $\{x\otimes x - q(x)1; x \in V\}$. The quotient algebra

$$\mathcal{C}(V,q) = \mathcal{T}(V)/\mathcal{K}$$

is called the Clifford algebra of q.

Let $\pi: F1\oplus V \to \mathcal{C}$ the canonical map $x \mapsto x + \mathcal{K}$, then by the definition we have

$$\pi(\alpha 1 + x)^2 = (\alpha 1 + x)\cdot(\alpha 1 + x) + \mathcal{K} = \alpha^2 1 + 2\alpha x + x\otimes x + \mathcal{K}$$
$$= \left[\alpha^2 + q(x,x)\right] 1 + 2\alpha x + \mathcal{K} = \pi((\alpha 1 + x)^2)$$

which implies, that $\pi(F1\oplus V)$ is a (Jordan) subalgebra of $\mathcal{C}^+$, and $\pi: F1\oplus V \to \mathcal{C}^+$ a homomorphism. One can show that π (restricted to $F1\oplus V$) is 1 - 1. This shows that $F1\oplus V$ is isomorphic to the subalgebra $\pi(F1\oplus V)$ in $\mathcal{C}^+$, hence it is special.

8.3. Let $\mathcal{D}$ be an arbitrary algebra over F with involution

$j : x \to \bar{x}$. By $\mathcal{D}_n$ we denote the algebra of $n \times n$ matrices with entries in $\mathcal{D}$. In $\mathcal{D}_n$ we have the <u>standard involution</u> $X \to \bar{X}^t$, where $\bar{X} = (\bar{\alpha}_{ij})$ if $X = (\alpha_{ij})$ and Y^t is the transposed of $Y \in \mathcal{D}_n$. (Verify that $X \to \bar{X}^t$ is an involution.) The space of symmetric elements relative to this involution is denoted by $\mathcal{H}(\mathcal{D}_n)$.

$\mathcal{H}(\mathcal{D}_n) = \{X \in \mathcal{D}_n; \quad X = \bar{X}^t\}$.

Clearly $X \circ Y = \frac{1}{2}(XY + YX) \in \mathcal{H}(\mathcal{D}_n)$ if $X, Y \in \mathcal{H}(\mathcal{D}_n)$

(XY denotes the usual matrix product). This shows, that $\mathcal{H}(\mathcal{D}_n)$ together with $(X,Y) \mapsto X \circ Y$ is an algebra. Without proof we state the following important result (see N. Jacobson, Structure and Representations of Jordan Algebras).

<u>Theorem 2.</u> <u>For</u> $n \geq 3$ $(\mathcal{H}(\mathcal{D}_n), \circ)$ <u>is a Jordan algebra, iff either</u> $\mathcal{D}$ <u>is associative or</u> $n = 3$ <u>and</u> $\mathcal{D}$ <u>is alternative and any j-symmetric element</u> α <u>in</u> $\mathcal{D}$, <u>satisfies</u> $(\alpha x)y = \alpha(xy)$ <u>for all</u> $x, y \in \mathcal{D}$.

An algebra $\mathcal{D}$ is called alternative, if

(8.1) $\quad x(xy) = x^2y$ and $(yx)x = yx^2$ for all $x, y \in \mathcal{D}$.

<u>If</u> $(\mathcal{D}, j)$ <u>is a simple pair and</u> $\mathcal{D}$ <u>an associative Artinian algebra, then</u> $\mathcal{H}(\mathcal{D}_n)$ <u>is a simple Jordan algebra.</u>

8.4. In order to present a class of exceptional Jordan algebras we first have to introduce Cayley algebras.

Let $\mathcal{L}$ be an alternative algebra with unit element e and non degenerate quadratic form q such that

$$x^2 - t(x)x + q(x)e = 0$$

for all $x \in \mathcal{L}$, where $t(x) := q(x,e) = q(x+e) - q(x) - q(e)$.

For example F, or $F \oplus F$, or the algebra of 2×2 matrices over F have these properties, relative to $q(\alpha) = \alpha^2, q(\alpha \oplus \beta) = \alpha\beta$ or $q(a) =$ det a.

It is fairly easy to show that

$$x \to \bar{x} := t(x)e - x$$

defines an involution on $\mathcal{L}$. (Compare the following with the construction of the complex numbers from the reals.) Let $\mathcal{L}$ be as described above and $\mathcal{L}1$ an isomorphic copy of $\mathcal{L}$ (identify $e1$ with 1) and $\mu \in F, \mu \neq 0$. In the direct sum

$$(\mathcal{L}, \mu) = \mathcal{L} \oplus \mathcal{L}1$$

we define a product by

$$(x + y1)(u + v1) := (xu + \mu\bar{v}y) + (vx + y\bar{u})1.$$

A simple verification shows

$$(x + y1)^2 - t(x + y1)(x + y1) + q(x + y1)e = 0,$$

where $q(x + y1) := q(x) - \mu q(y)$, which is again non degenerate ($\mu \neq 0$). But it is not clear whether the alternative laws (8.1) hold in $(\mathcal{L}, \mu)$. This is settled by the following result:

a) $(\mathcal{L}, \mu)$ <u>alternative, iff</u> $\mathcal{L}$ <u>associative</u>,

b) $(\mathcal{L}, \mu)$ <u>associative</u>, <u>iff</u> $\mathcal{L}$ <u>associative</u> <u>and</u> <u>commutative</u>,

c) $(\mathcal{L}, \mu)$ <u>commutative</u>, <u>iff</u> $\mathcal{L} = Fe$.

Therefore we can easily construct four classes of alternative algebras with the required properties. Starting with

$\mathcal{L}_0 = Fe$, and $\mu_1 \neq 0$ we get

$\mathcal{L}_1 = (Fe, \mu_1)$, which is commutative; then for $\mu_2 \neq 0$

$\mathcal{L}_2 = (Fe, \mu_1, \mu_2)$ is associative, and

$\mathcal{L}_3 = (Fe, \mu_1, \mu_2, \mu_3)$ is alternative ($\mu_3 \neq 0$).

It can be shown that $\mathcal{L}_3$ is not associative, therefore an algebra $\mathcal{L}_4 = (\mathcal{L}_3, \mu)$, $\mu \neq 0$, would no longer be alternative. The indicated construction is called the Cayley-Dickson construction.

$\mathcal{L}_1$ is either a quadratic extension of Fe, or $\mathcal{L}_1 = \mathrm{Fe} \oplus \mathrm{Fe}$. $\mathcal{L}_2 = (\mathrm{Fe}, \mu_1, \mu_2)$ is called a (generalized) quaternion algebra and $\mathcal{L}_3 = (\mathrm{Fe}, \mu_1, \mu_2, \mu_3)$ is called a Cayley algebra (or octonion algebra).

Exercise: Choose an appropriate basis in $\mathcal{L}_i$ $(i = 1,2,3)$ and determine the multiplication table of this basis. (For more information about these algebras (and, of course, many other topics) see: Braun-Koecher, Jordan-Algebren; N. Jacobson, Structure and Representations of Jordan Algebras; and R.D. Schafer, An Introduction to Nonassociative Algebras.).

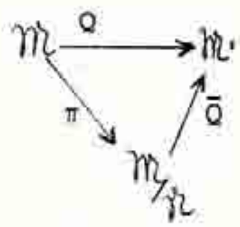
8.5. Now let $\mathcal{C}$ be a Cayley algebra, then $\mathcal{C}$ has an involution $x \to \bar{x} = t(x)e - x$, the symmetric elements then are obviously exactly the elements in Fe. But for $\alpha \in F$ we have trivially $\alpha(xy) = x(\alpha y)$. Therefore theorem 2 applies to show, that

$$\mathcal{G}(\mathcal{C}_3) = \left\{ X = \begin{pmatrix} \alpha_1 x_1 x_2 \\ \bar{x}_1 \alpha_2 x_3 \\ \bar{x}_2 \bar{x}_3 \alpha_3 \end{pmatrix} \; ; \; \alpha_i \in F,\ x_i \in \mathcal{C} \right\}$$

together with $X \circ Y = \frac{1}{2}(XY + YX)$ is a Jordan algebra. This algebra is simple and exceptional.

IX. Quadratic Jordan Algebras.

9.1. Let Φ be a commutative ring with unit element 1. A map of (unital) Φ-modules $Q: \mathfrak{M} \to \mathfrak{M}'$ is called <u>quadratic</u>, if $Q(\alpha x) = \alpha^2 Q(x)$ for all $x \in \mathfrak{M}$, $\alpha \in \Phi$, and $Q(x,y) := Q(x + y) - Q(x) - Q(y)$ is linear in x and y. The kernel of Q, denoted by ker Q, is the set of elements $x \in \mathfrak{M}$ such that $Q(x) = 0$ and $Q(x,\mathfrak{M}) = 0$. Clearly ker Q is a submodule of $\mathfrak{M}$. If $\mathfrak{N}$ is a submodule of $\mathfrak{M}$, contained in ker Q, then we can factorize Q in the usual way. We define $\bar{Q}: \bar{\mathfrak{M}} = \mathfrak{M}/\mathfrak{N} \to \mathfrak{M}'$ by $\bar{Q}(a + \mathfrak{N}) := Q(a)$. Since $\mathfrak{N}$ is in ker Q, this is well defined and it is obvious that $\bar{Q}$ is quadratic. The above factorization is visualized in the following commutative diagram:



$Q = \bar{Q}\pi$, where π is the canonical surjection.

Whenever we have such a commutative diagram with quadratic maps $Q, \bar{Q}$, then $\mathfrak{N} \subset \ker Q$. If $Q: \mathfrak{M} \to \mathfrak{M}'$ is quadratic and $x = \Sigma \alpha_i x_i, \alpha_i \in \Phi, x_i \in \mathfrak{M}$, then $Q(x) = \Sigma \alpha_i^2 Q(x_i) + \sum_{i<j} \alpha_i \alpha_j Q(x_i, x_j)$. This shows that if $X = \{x_i, i \in I\}$ is a set of generators of $\mathfrak{M}$, then Q is uniquely determined by the values $Q(x_i)$ and $Q(x_i, x_j)$, $i, j \in I$.

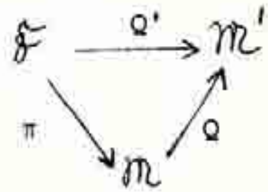
If Ω is a unital commutative associative algebra over Φ (i.e., an <u>extension</u> of Φ) we denote $\mathfrak{M}_\Omega := \Omega \otimes_\Phi \mathfrak{M}$. ($\mathfrak{M}_\Omega$ is an Ω-module, $w'(w \otimes m) = w'w \otimes m$.)

<u>Lemma 1.</u> <u>If</u> $Q: \mathfrak{M} \to \mathfrak{M}'$ <u>is a quadratic map of Φ-modules, and Ω an extension of Φ, then Q has a unique extension</u> $Q_\Omega: \mathfrak{M}_\Omega \to \mathfrak{M}'_\Omega$ <u>such</u>

that

(9.1) $Q_\Omega(\Sigma w_i \otimes m_i) = \Sigma w_i^2 \otimes Q(m_i) + \sum_{i<j} w_i w_j \otimes Q(m_i, m_j)$.

Proof. Clearly if Q has an extension such that $Q_\Omega(1\otimes m) = Q(m)$, it must have the form (9.1). Conversely, if we can show that (8.1) defines a map $\mathfrak{M}_\Omega \to \mathfrak{M}'_\Omega$ then we are done. The quadratic nature of Q_Ω is obvious. We know that any module is the quotient of a free module (up to isomorphism). Let $\mathfrak{M} = \mathfrak{F}/_{\mathfrak{N}}$, $\mathfrak{F}$ free and $\mathfrak{N}$ a submodule of $\mathfrak{F}$. Let $F = \{f_i, i \in I\}$ be a basis of $\mathfrak{F}$ and $i \to b_i$, $\{i,j\} \to b_{ij}$ mappings of I into $\mathfrak{M}'$ resp. of the set of unordered pairs $\{i,j\}$, $i,j \in I$, in $\mathfrak{M}'$. If $x \in \mathfrak{F}$, then x has a unique representation $x = \Sigma \xi_k f_{ik}$. Then $Q'(x) = \sum_k \xi_k^2 b_{i_k} + \sum_{k<l} \xi_k \xi_l b_{i_k i_l}$ is well defined, hence $Q' : \mathfrak{F} \to \mathfrak{M}'$ is a quadratic map. Let $\pi : \mathfrak{F} \to \mathfrak{M}$ be the canonical projection and $\pi f_i = c_i$, $i \in I$. If we set $b_i = Q(c_i)$ and $b_{ij} = Q(c_i, c_j)$, then we get a quadratic map $Q' : \mathfrak{F} \to \mathfrak{M}'$ (defined as above) which makes the following diagram commutative



The projection $\pi : \mathfrak{F} \to \mathfrak{M}$ has a natural extension $\pi_\Omega : \mathfrak{F}_\Omega \to \mathfrak{M}_\Omega$, namely $\pi_\Omega : w\otimes x \mapsto w \otimes \pi(x)$. Obviously, π_Ω is onto with kernel $\mathfrak{N}_\Omega = \Omega \otimes \mathfrak{N}$. Since $\{1\otimes f_i, i \in I\}$ is a basis of the free Ω-module $\mathfrak{F}_\Omega$, we get (as above) a quadratic map $Q'_\Omega : \mathfrak{F}_\Omega \to \mathfrak{M}'_\Omega$, defined by

$$Q'_\Omega(\Sigma w_i \otimes f_i) = \Sigma w_i^2 \otimes Q(c_i) + \sum_{i<j} w_i w_j \otimes Q(c_i, c_j).$$

The kernel of Q'_Ω clearly contains $\Omega \otimes \ker Q'$. Since by our construction $\ker \pi = \mathfrak{N}$ is contained in $\ker Q'$, we see that $\mathfrak{N}_\Omega$ is contained in $\ker Q'_\Omega$. Therefore we have a factorization

$$\begin{array}{ccc} \mathfrak{F}_\Omega & \xrightarrow{\;Q'_\Omega\;} & \mathfrak{M}'_\Omega \\ & {\scriptstyle \pi_\Omega} \searrow \quad \nearrow {\scriptstyle Q_\Omega} & \\ & \mathfrak{M}_\Omega & \end{array}$$

where Q_Ω is the map given in (9.1).

9.2. Let $\mathfrak{M}$ be a unital Φ-module and $P: \mathfrak{M} \to \mathrm{End}_\Phi \mathfrak{M}$ a quadratic map. In this case we call $(\mathfrak{M}, P)$ a <u>quadratic</u> <u>triple</u> <u>system</u>. The reason for this is the following: P induces in a natural way a composition $(x,y) \to P(x)y$ on $\mathfrak{M}$ which is quadratic in x and linear in y; and P induces a trilinear composition $(x,y,x) \mapsto \{xyz\} := P(x,z)y$. Hence P induces on $\mathfrak{M}$ the structure of a triple system in the sense we studied in chapter III. We observe that the induced trilinear composition might be trivial, i.e., $\{\mathfrak{M}\mathfrak{M}\mathfrak{M}\} = 0$, although $P \neq 0$. To refer to the fact that the quadratic map P is the principle object (rather than the induced trilinear composition) we use the adjective "quadratic" in this context.

<u>Example:</u> Let $\mathfrak{A}$ be any algebra. The mapping $x \to L(x^2)$ obviously is quadratic (L denotes the left multiplication). The induced trilinear composition is $\{xyz\} = (xy + yx)z$, which is trivial in case $2\mathfrak{A} = 0$ and $\mathfrak{A}$ commutative.

To a given quadratic map $P: \mathfrak{M} \to \mathrm{End}\,\mathfrak{M}$ we always associate a bilinear map $L: \mathfrak{M} \times \mathfrak{M} \to \mathrm{End}\,\mathfrak{M}$ defined by

$$L(x,y)z := \{xyz\} = P(x,z)y.$$

In some cases the quadratic map and certain other assumptions induce on $\mathfrak{M}$ the structure of an algebra; and if we want to emphasize the algebra structure we refer to $(\mathfrak{M},P)$ as a quadratic algebra (rather than a quadratic triple system).

9.3. A triple $(\mathfrak{J},P,e)$, where $\mathfrak{J}$ is a Φ-module, $P: \mathfrak{J} \to \mathrm{End}\,\mathfrak{J}$ a quadratic map and $e \in \mathfrak{J}$, is called a unital quadratic Jordan algebra, if

(U.Q.F.1) $\qquad P(e) = \mathrm{Id}$

(U.Q.F.2) $\qquad L(x,y)P(x) = P(x)L(y,x)$ "Homotopy formula"

(U.Q.F.3) $\qquad P(P(x)y) = P(x)P(y)P(x)$ "Fundamental formula"

hold in $\mathfrak{J}$ and all extensions $\mathfrak{J}_\Omega$.

We agree to call $\mathfrak{J}$, rather than the triple $(\mathfrak{J},P,e)$, the Jordan algebra.

Examples. 1) It is easily seen that a linear Jordan algebra (over Φ containing $\frac{1}{2}$) remains a linear Jordan algebra under all ring extensions of Φ. Consequently (7.9) and (7.14) hold under all extensions. Taking the quadratic representation $P(x) = 2L(x)^2 - L(x^2)$ (see 7.3) it is obvious that $P(e) = \mathrm{Id}$ for a unit element e of $\mathfrak{J}$. Consequently:

If $\frac{1}{2} \in \Phi$ and $\mathfrak{J}$ is a linear Jordan algebra with unit element e and quadratic representation P, then $(\mathfrak{J},P,e)$ is a unital quadratic

Jordan algebra.

Note: The examples of linear Jordan algebras given in chapter VIII have analogs in the quadratic case. (For more information on quadratic Jordan algebras we refer to N. Jacobson's lecture notes (Tata Institut, Bombay) "Lectures on Quadratic Jordan Algebras").

2) A standard example for arbitrary Φ is derived from an associative algebra $\mathfrak{A}$ with unit element e. We define $P: \mathfrak{A} \to \operatorname{End} \mathfrak{A}$ by $P(x)a = xax$, then it is easily checked, that $(\mathfrak{A}, P, e)$ is a quadratic Jordan algebra. In accordance with the linear notation, we denote this algebra by $\mathfrak{A}^+$, too.

As in 7.3. we observe that the homotopy formula (U.Q.J.2) implies

$$(9.2)\quad L(x,y)P(x) = P(x)L(y,x) = P(P(x)y,x)$$

and this equation also holds in all extensions $\mathfrak{J}_\Omega$.

The fact that (9.2) and the fundamental formula hold in all extensions allows us to linearize these formulas. Let λ be an indeterminate over Φ, and $\Omega = \Phi[\lambda]$ the ring of polynomials in λ (over Φ). Since Ω is a free Φ-module ($\{\lambda^0 = 1, \lambda, \lambda^2, \ldots, \lambda^k, \ldots\}$ is a basis), the canonical map of $\mathfrak{J}$ into $\mathfrak{J}_\Omega$, $x \to 1 \otimes x$, is injective. (Prove this statement.) So we may identify $\mathfrak{J}$ with its image in $\mathfrak{J}_\Omega$. The elements of $\mathfrak{J}_\Omega$ then can be uniquely represented in the form $\Sigma \lambda^i m_i, m_i \in \mathfrak{J}$. The endomorphisms of $\mathfrak{J}_\Omega$ are of the form $A = \Sigma \lambda^i A_i$, where $A_i \in \operatorname{End} \mathfrak{J}$ (extended to $\operatorname{End} \mathfrak{J}_\Omega$, i.e., $A_i(\omega \otimes x) = \omega \otimes A_i x$.) Note $\Sigma \lambda^i a_i = \Sigma \lambda^i b_i, a_i, b_i \in \mathfrak{J}$, then $a_i = b_i$, and if $\Sigma \lambda^i A_i = \Sigma \lambda^i B_i, A_i, B_i \in \operatorname{End} \mathfrak{J}$, then $A_i = B_i$. In other words, we can equate corresponding coefficients.

We apply this method to linearize (9.2) and the fundamental formula. Since by assumption these formulas hold in $\mathfrak{J}_\Omega, \Omega = \Phi[\lambda]$ we can replace x by $x + \lambda u$, $x,u \in \mathfrak{J}$.

$$L(x + \lambda u,y)P(x + \lambda u) = P(x + \lambda u)L(y,x + \lambda u) = P(P(x + \lambda u)y,x + \lambda u)$$

$$P(P(x + \lambda u)y) = P(x + \lambda u)P(y)P(x + \lambda u).$$

Using $P(a + b) = P(a) + P(a,b) + P(b)$ we represent both sides of these equations in the form $\Sigma\lambda^i A_i$, $A_i \in \mathrm{End}\,\mathfrak{J}$, and compare the resulting coefficients of λ^i. We obtain

(9.3) $$L(x,y)P(x,u) + L(u,y)P(x) = P(x)L(y,u) + P(x,u)L(y,x) = P(P(x,u)y,x) + P(P(x)y,u),$$

(9.4) $$P(P(x)y,P(x,u)y) = P(x)P(y)P(x,u) + P(x,u)P(y)P(x),$$

(9.5) $$P(P(x,u)y) + P(P(x)y,P(u)y) = P(x)P(y)P(u) + P(u)P(y)P(x) + P(x,u)P(y)P(x,u).$$

Note that we displayed only "new" identities obtained by this process. It is obvious that equating constant coefficients and the coefficients of the highest power of λ does not yield any new identity. Notice furthermore, in order to linearize an equation $f(x) = 0$, which is quadratic in x, we don't need ring extension arguments since the coefficient of λ in $f(x + \lambda u)$ equals $f(x,u) = f(x + u) - f(x) - f(u)$. Another remark might be useful: If Φ is a field with sufficiently many elements, then one does not need ring extension arguments in order to get the linearizations of a given formula. In this case one replaces the "variable" x in the given formula, say $f(x) = 0$, by $x + \alpha u$, $x,u \in \mathfrak{J}$, $\alpha \in \Phi$. In the expansion $f(x + \alpha u) = \Sigma\alpha^i f_i(x,u) = 0$ we can choose different elements $\alpha_j (1 \leq j \leq s)$ and get a system of s linear equation $\Sigma\alpha_j^i f_i(x,u) = 0$. If the matrix of coefficients has non zero determinant then we get $f_i(x,u) = 0$. The determinant in question is the Vandermonde determinant, which is $\neq 0$ if all α_j are different (and $\neq 0$).

<u>Exercise:</u> Prove, if the fundamental formula and (9.2) - (9.5) hold in $\mathcal{J}$, then they hold in all extensions $\mathcal{J}_\Omega$. This shows that it is equivalent to assume either the homotopy- and fundamental formula hold in all extensions $\mathcal{J}_\Omega$, or these formulas and all their linearizations hold in $\mathcal{J}$.

9.4. A homomorphism ϕ of unital quadratic Jordan algebras $(\mathcal{J},P,e),(\mathcal{J}',P',e')$ is what it ought to be, namely a linear map $\phi\colon \mathcal{J} \to \mathcal{J}'$ such that $\phi(e) = e'$ and $\phi(P(x)y) = P'(\phi(x))\phi(y)$. The class of unital quadratic Jordan algebras together with its homomorphisms is the <u>category of unital quadratic Jordan algebras</u>.

9.5. Unital quadratic Jordan algebras are in particular triple systems. The reason why they are called algebras is the following: If we look at our standard example $\mathfrak{A}^+$, $\mathfrak{A}$ associative with unit element e, we observe that we can recover the multiplication from the quadratic map. Since in this example $P(x)y = xyx$, we obtain $x^2 = P(x)e$. Therefore, of course, we also define for an arbitrary unital quadratic Jordan algebra $(\mathcal{J},P,e)$ the "squaring" $x \to x^2$, by

$$x^2 := P(x)e.$$

The bilinearization of this map defines a multiplication $(x,y) \mapsto x\circ y$, where $x\circ y = (x+y)^2 - x^2 - y^2 = P(x,y)e$. Note: If $2\mathcal{J} = 0$ the multiplication $x\circ y$ might be trivial, whereas the squaring is not.

From the definitions and $P(e) = id$, we obtain

(9.6) i) $e^2 = e$; $x \circ x = 2x^2$; iii) $x \circ y = y \circ x$.

Using the notations

$L(x,y)z = \{xyz\} = P(x,z)y$,

(see 9.2) we get for the leftmultiplication $L(x): y \to x\circ y = P(x,y)e$

$$L(x) = L(x,e).$$

Taking x = e in the homotopy formula gives

$L(y) = L(e,y)$, or equivalently $\{yex\} = \{eyx\}$, or $L(x) = P(x,e)$.

In particular $L(e) = P(e,e) = 2P(e) = 2\,\mathrm{Id}$. Thus

(9.7) $L(x) = L(x,e) = L(e,x) = P(e,x)$; $L(e) = 2\mathrm{Id}$.

Substituting $u \to e$ in (9.3) (left hand side equation), applying the result to e and using (9.7) gives

$4P(x)y + y\circ x^2 = 2P(x)y + x\circ(x\circ y)$, i.e.

(9.8) $$2P(x) = L(x)^2 - L(x^2).$$

The linearization of this is

(9.8') $2P(x,y) = L(x)L(y) + L(y)L(x) - L(x\circ y)$.

<u>Note:</u> Since a special case of the homotopy formula is $L(x)P(x) = P(x)L(x)$ we see immediately from (9.8) that $2L(x)L(x^2) = 2L(x^2)L(x)$ which is $L(x)L(x\circ x) = L(x\circ x)L(x)$. This shows that $\mathcal{J}$ together with $(x,y) \to \frac{1}{2}x\circ y$ is a linear Jordan algebra (if $\frac{1}{2}\in\Phi$) and that the quadratic representation of this linear Jordan algebra is P (see VII,7.3). Therefore if $\frac{1}{2}\in\Phi$, a unital quadratic Jordan algebra may as well be considered as a unital linear Jordan algebra, and conversely.

Replacing x by e in (9.3) an using (9.8') we get

$$L(u,y) + L(y)L(u) = P(y,u) + L(u\circ y)$$
$$= P(y,u) + L(u)L(y) + L(y)L(u) - 2P(u,y),$$

consequently

(9.9) $L(u,y) = L(u)L(y) - P(u,y)$.

Setting $u = e$ in (9.2), substituting $x \to e, y \to x, u \to y$ in (9.4) and $u = e$ in (9.5), we obtain respectively,

(9.10) $L(y)P(x) + L(x,y)L(x) = P(x)L(y) + L(x)L(y,x) = L(P(x)y) + P(xoy,x)$

(9.11) $P(xoy,x) = P(x)L(y) + L(y)P(x)$

(9.12) $P(xoy) + P(P(x)y,y) = P(x)P(y) + P(y)P(x) + L(x)P(y)L(x)$

Replacing the second term on the right hand side of (9.10) by (9.11) shows

(9.13) $L(P(x)y) + P(x)L(y) = L(x,y)L(x)$

(9.13') $L(P(x)y) + L(y)P(x) = L(x)L(y,x)$

Observing that part of (9.12) is symmetric in x and y we conclude (in other words, interchange x and y and substract)

(9.14) $P(P(x)y,y) - P(x,P(y)x) = L(x)P(y)L(x) - L(y)P(x)L(y)$

Now we are ready to prove the important formula

(9.15) $L(P(x)y,y) = L(x,P(y)x)$.

Proof. Using (9.9) we see that (9.15) is equivalent to

$L(P(x)y)L(y) - P(P(x)y,y) = L(x)L(P(y)x) - P(x,P(y)x)$.

But this equation is equivalent to (using (9.14))

$L(P(x)y)L(y) + L(y)P(x)L(y) = L(x)L(P(y)x) + L(x)P(y)L(x)$

Using (9.13) we see that the left hand side of this equation equals $L(x)L(y,x)L(y)$ and using (9.13')(x interchanged with y) the right hand side equals $L(x)L(y,x)L(y)$; consequently the last equation holds and then (9.15) holds, too.

9.6. We introduced unital quadratic Jordan algebras. But in the classical theory of linear Jordan algebras (or any other theory of algebras) generally one does not assume the existence of a unit element. One has two alternatives: to forget all about multiplication (this will be done in the next chapter) or to require (by a set of axioms) the existence of a squaring which induces the multiplication and which has properties compatible with the classical theory. One requirement, which is most reasonable, is that any kind of Jordan algebra should be imbeddable in a unital Jordan algebra (see exercise in 7.1.) These considerations led to the following definition.

A triple $(\mathfrak{J}, P, {}^2)$ where $\mathfrak{J}$ is a unital Φ-module, $P: \mathfrak{J} \to \mathrm{End}\,\mathfrak{J}$ and ${}^2: \mathfrak{J} \to \mathfrak{J}$, $x \to x^2$, are quadratic maps, is called a <u>quadratic Jordan algebra</u>, if

(Q.F.1) $\quad L(x,x) = L(x^2)$

(Q.F.2) $\quad P(x)L(x) = L(x)P(x)$

(Q.F.3) $\quad P(x)x^2 = (x^2)^2$

(Q.F.4) $\quad P(x)P(y)x^2 = (P(x)y)^2$

(Q.F.5) $\quad P(x^2) = P(x)^2$

(Q.F.6) $\quad P(P(x)y) = P(x)P(y)P(x)$

hold in $\mathfrak{J}$ and all extensions $\mathfrak{J}_\Omega$, where, as before, $L(x)y = x \circ y = (x+y)^2 - x^2 - y^2$, $L(x,y)z = \{xyz\} = P(x,z)y$. Without proof we state the following result.

<u>Theorem 1. Any quadratic Jordan algebra $(\mathfrak{J}, P, {}^2)$ can be imbedded as subalgebra in a unital quadratic Jordan algebra $\hat{\mathfrak{J}} = \Phi 1 \oplus \mathfrak{J}$ with unit element 1 and quadratic map $\hat{P}$ defined by</u>

(9.16) $\hat{P}(\alpha 1 + x)(\beta 1 + y) = \alpha^2\beta 1 + \alpha^2 y + 2\alpha\beta x + \alpha x \circ y + \beta x^2 + P(x)y$

<u>Note:</u> 1) Theorem 1 shows that a quadratic Jordan algebra is nothing else than a submodule of a unital quadratic Jordan algebra closed

under cubic operation $(x,y) \mapsto P(x)y$ and squaring $x \mapsto x^2$, but not necessarily containing a unit element.

2) Since unital Jordan algebras are sometimes easier to deal with, one often proves certain results at first for the unital case, i.e., for $\hat{\mathfrak{J}} = \Phi 1 \oplus \mathfrak{J}$, then one gives an interpretation of this result for $\mathfrak{J}$. In particular, in order to prove identities in $\mathfrak{J}$, they certainly hold if they are true in $\hat{\mathfrak{J}}$, for example (9.15) holds in $\mathfrak{J}$. A map $\varphi: \mathfrak{J} \to \mathfrak{J}'$ of quadratic Jordan algebras is called a <u>homomorphism</u>, if $\varphi(x^2) = \varphi(x)^2$ and $\varphi(P(x)y) = P'(\varphi(x)\varphi(y)$. A submodule $\mathfrak{U}$ of $\mathfrak{J}$ is a <u>subalgebra</u>, if $\mathfrak{U}^2 \subset \mathfrak{U}$ and $P(\mathfrak{U})\mathfrak{U} \subset \mathfrak{U}$. $\mathfrak{U}$ is an <u>ideal</u>, if $\mathfrak{U}^2 \subset \mathfrak{U}$, $\mathfrak{U} \circ \mathfrak{J} \subset \mathfrak{U}$, $P(\mathfrak{U})\mathfrak{J} \subset \mathfrak{U}$, $P(\mathfrak{J})\mathfrak{U} \subset \mathfrak{U}$. As usual, $\mathfrak{U}$ is an ideal, iff it is the kernel of some homomorphism. Furthermore, if $\mathfrak{U}$ is an ideal, then we have on $\overline{\mathfrak{J}} = \mathfrak{J}/\mathfrak{U}$ a well defined cubic operation (which gives a quadratic map of $\overline{\mathfrak{J}} \to \operatorname{End}\overline{\mathfrak{J}}$) and a squaring defined by

$$\bar{P}(\bar{u})\bar{x} = \overline{P(u)x}, \quad \bar{x}^2 = \overline{x^2}.$$

Obviously, $(\bar{\mathfrak{J}}, \bar{P}, {}^2)$ is a quadratic Jordan algebra an $x \mapsto \bar{x}$ a homomorphism of $\mathfrak{J}$ onto $\overline{\mathfrak{J}}$.

An <u>inner ideal</u> $\mathfrak{L}$ of $\mathfrak{J}$ is a submodule such that $P(\mathfrak{L})\mathfrak{J} \subset \mathfrak{L}$.

<u>Exercise</u>: Show that if $\mathfrak{L}$ is an inner ideal in $\mathfrak{J}$, so is $P(x)\mathfrak{L}$ for any $x \in \mathfrak{J}$.

9.7. Let $(\mathfrak{J}, P, e)$ be a unital quadratic Jordan algebra. An element $x \in \mathfrak{J}$ is called invertible, if there is an element $y \in \mathfrak{J}$ such that

(9.17) $P(x)y = x$ and $P(x)y^2 = e$.

y is called an <u>inverse</u> of x.

Exercise: Show: x is invertible in $\mathfrak{A}^+$, iff x is invertible in $\mathfrak{A}$.

For later applications we need

Theorem 2. *Let* $(\mathfrak{J}, P, e)$ *be a unital quadratic Jordan algebra and* $x \in \mathfrak{J}$. *The following statements are equivalent.*

i) *x is invertible*

ii) *P(x) is invertible*

iii) $e \in \text{Image } P(x)$

In either case the inverse $y =: x^{-1}$ *of x is uniquely determined by* $x^{-1} = P(x)^{-1}x$

Proof. If x is invertible then by (9.17) (and fundamental formula) there exists an element $y \in \mathfrak{J}$ such that $P(x)P(y^2)P(x) = \text{Id}$; consequently P(x) is invertible. ii) → iii) is trivial. Now assume $P(x)u = e$ for some $u \in \mathfrak{J}$. Then $P(x)P(u)P(x) = \text{Id}$, in particular P(x) is surjective and there is an element y such that $P(x)y = x$. Then $P(x)y^2 = P(x)P(y)e = P(x)P(y)P(x)u = P(P(x)y)u = P(x)u = e$. From (9.17) and ii) the uniqueness and the formula $x^{-1} = P(x)^{-1}x$ are clear.

Exercise: Assume $x, y \in \mathfrak{J}$ are invertible. Show

i) $P(x^{-1}) = P(x)^{-1}$, ii) $(x^{-1})^{-1} = x$

iii) $P(x)y$ is invertible and $(P(x)y)^{-1} = P(x)^{-1}y^{-1}$.

The equivalence of ii) and iii) in theorem 2 suggests that the same might be true in any quadratic Jordan algebra. This is the case. An element $e \in \mathfrak{J}$ ($\mathfrak{J}$ an arbitrary quadratic Jordan algebra) if called a unit element of $\mathfrak{J}$, if

$$P(e) = \text{Id} \quad \text{and} \quad P(x)e = x^2 \quad \text{for all } x \in \mathfrak{J}.$$

If e, e' are unit elements of $\mathfrak{J}$, then

$e = P(e)e = e^2 = P(e)e' = e'$. Hence there is at most one unit element in $\mathfrak{J}$. Clearly the notion of unital quadratic Jordan algebras is equivalent to the notion of quadratic Jordan algebras with unit element; only the homomorphisms are in general not the same (why?)

9.8. Let $(\mathcal{J}, P, {}^2)$ be a quadratic Jordan algebra (not necessarily with unit element) and let $u \in \mathcal{J}$. We define a new quadratic map P_u and squaring $x \to x^{(2,u)}$ by

$$P_u(x) := P(x)P(u) \ ; \ x^{(2,u)} := P(x)u$$

Theorem 4. If $(\mathcal{J}, P, {}^2)$ is a quadratic Jordan algebra, then for all $u \in \mathcal{J}$, $(\mathcal{J}, P_u, {}^{(2,u)})$ is a quadratic Jordan algebra.

Proof: Since $\mathcal{J}_\Omega$ remains a quadratic Jordan algebra for all extensions we only have to verify (Q.$\mathcal{J}$.1) - (Q.$\mathcal{J}$.6). Using the given formulas the verification is straight forward and is left as an exercise. Observe that by the remark following theorem 1 the homotopy formula holds in any Jordan algebra.

Notation: We set $\mathcal{J}_u := (\mathcal{J}, P_u, {}^{(2,u)})$ and call $\mathcal{J}_u$ the u-homotope of $(\mathcal{J}, P, {}^2)$. In case u is invertible, we call $\mathcal{J}_u$ the u-isotope of $\mathcal{J}$.

For unital Jordan algebras the u-homotope need not be unital, for example take $u = 0$. But the following hold:

Theorem 5. If $\mathcal{J}$ is a quadratic Jordan algebra and $u \in \mathcal{J}$. Then $\mathcal{J}_u$ has a unit element, iff $\mathcal{J}$ has a unit element and u is invertible in $\mathcal{J}$. In this case the unit element of $\mathcal{J}_u$ is u^{-1}.

Proof. If g is a unit element of $\mathcal{J}_u$, then by definition

(9.18) $\quad P(g)P(u) = \mathrm{Id}$ and $P(x)P(u)g = P(x)u$. (see 9.7.)

The first equation shows that $P(g)$ is surjective, therefore $\mathcal{J}$ contains a unit element and g is invertible, by theorem 3. Then $P(u) = P(g)^{-1}$

and u is invertible. Using theorem 2 we obtain the inverse of u by $u^{-1} = P(g)u = g^{(2,u)} = g$, since g is the unit element of $\mathfrak{J}_u$.

If conversely $\mathfrak{J}$ has a unit and u is invertible, then from the properties of inverse elements (see 9.7.) it is clear that (9.18) holds for $g = u^{-1}$.

Exercise: 1) $(\mathfrak{J}_u)_v = \mathfrak{J}_{P(u)v}$

2) If $\mathfrak{J}$ has a unit element and u is invertible in $\mathfrak{J}$ then $\mathfrak{J} = (\mathfrak{J}_u)_g$ where $g = (u^{-1})^2$.

9.9. One of the most powerful tools in the study of Jordan algebras is the Peirce decomposition relative to an idempotent. (see 4.5). Let $\mathfrak{J}$ be a quadratic Jordan algebra and $c \in \mathfrak{J}$ an idempotent, i.e., $c^2 = c$. We define

$E_1 := P(c)$; $E_{\frac{1}{2}} := P(c, 1 - c)$; $E_0 := P(1 - c)$.

These are mappings of $\hat{\mathfrak{J}} = \Phi 1 \oplus \mathfrak{J}$, but they make sense in $\mathfrak{J}$, since they leave $\mathfrak{J}$ invariant.

We observe $\quad Id = E_1 + E_{\frac{1}{2}} + E_0$.

We state (without proof) the following fundamental result.

Theorem 6. Let $\mathfrak{J}$ be a quadratic Jordan algebra and $c^2 = c$ an idempotent of $\mathfrak{J}$. If we define $\mathfrak{J}_i := E_i\mathfrak{J}$, E_i as above $(i = 0, \frac{1}{2}, 1)$ then

$$\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_{\frac{1}{2}} \oplus \mathfrak{J}_0$$

and the following relations hold:

(1) $\mathfrak{J}_i^2 \subset \mathfrak{J}_i$; $P(\mathfrak{J}_i)\mathfrak{J}_i \subset \mathfrak{J}_i$ $(i = 0,1)$ and $\mathfrak{J}_1 \circ \mathfrak{J}_0 = 0$.

(2) $\mathcal{J}_i \circ \mathcal{J}_{\frac{1}{2}} \subset \mathcal{J}_{\frac{1}{2}}$ (i = 0,1), $\mathcal{J}_{\frac{1}{2}}^2 \subset \mathcal{J}_1 + \mathcal{J}_0$

(3) $P(\mathcal{J}_{\frac{1}{2}})\mathcal{J}_i \subset \mathcal{J}_{1-i}$ (i = 0,1) $P(\mathcal{J}_{\frac{1}{2}})\mathcal{J}_{\frac{1}{2}} \subset \mathcal{J}_{\frac{1}{2}}$

(4) $\{\mathcal{J}_1 \mathcal{J}_{\frac{1}{2}} \mathcal{J}_0\} \subset \mathcal{J}_{\frac{1}{2}}$

(5) $\{\mathcal{J}_i \mathcal{J}_i \mathcal{J}_{\frac{1}{2}}\} \subset \mathcal{J}_{\frac{1}{2}}$ <u>and</u> $\{\mathcal{J}_{\frac{1}{2}} \mathcal{J}_{\frac{1}{2}} \mathcal{J}_i\} \subset \mathcal{J}_i$ (i = 0,1)

<u>and all other products are zero</u>.

<u>Notation:</u> $\mathcal{J}_i$ are called the Peirce - i-spaces (or components, or modules) of $\mathcal{J}$ relative c.

<u>Note:</u> 1) $\mathcal{J}_1$ and $\mathcal{J}_0$ are inner ideals of $\mathcal{J}$; i.e., $P(\mathcal{J}_i)\mathcal{J} \subset \mathcal{J}_i$ (i = 0,1)

2) $\mathcal{J}_1$, $\mathcal{J}_0$ are closed under P(x)y and squaring, hence they are subalgebras.

3) Although $\mathcal{J}_{\frac{1}{2}}$ is not closed under squaring it is closed under P(x)y.

More properties of Jordan algebras will be introduced when they are needed. Concerning the examples of quadratic Jordan algebras, we may for a while content ourselves with $\mathfrak{A}^+$, $\mathfrak{A}$ associative, and all submodules of $\mathfrak{A}$ which are closed under xyx and x^2, for example $\mathcal{H}(\mathfrak{A},j) = \{x \in \mathfrak{A}, j(x) = x\}$ where j is an involution of $\mathfrak{A}$. More examples are given in chapter VIII, for rings which contain ½.

X. Jordan Triple Systems.

10.1. In 9.2. we already introduced the notion of a quadratic triple system. From now on we shall omit the adjective "quadratic". We recall, if Φ is a commutative ring with 1, $\mathfrak{A}$ a unital Φ-module and $P: \mathfrak{A} \to \operatorname{End}\mathfrak{A}$ a quadratic map, then the pair $(\mathfrak{A}, P)$ is a triple system. (Most often we call $\mathfrak{A}$ rather than $(\mathfrak{A}, P)$ the triple system.) The map P induces a bilinear map $L: \mathfrak{A}\times\mathfrak{A} \to \operatorname{End}\mathfrak{A}$, $(x,y) \to L(x,y)$ given by $L(x,y)z = \{xyz\} = P(x,z)y$.

The pair $(\mathfrak{A}, P)$ is called a <u>Jordan triple system</u> (Jts), if

(J.T.1) $L(x,y)P(x) = P(x)L(y,x)$ "homotopy formula"

(J.T.2) $L(P(x)y,y) = L(x,P(y)x)$

(J.T.3) $P(P(x)y) = P(x)P(y)P(x)$ "fundamental formula"

hold in $\mathfrak{A}$ and all extensions $\mathfrak{A}_\Omega$.

It is immediately clear from the definition that there must be a strong connection between Jordan algebras and Jordan triple systems.

If $(\mathfrak{J}, P, {}^2)$ or $(\mathfrak{J}, P, e)$ <u>is a quadratic (or unital quadratic) Jordan algebra, then</u> $(\mathfrak{J}, P)$ <u>is a Jordan triple system</u>, since the fundamental formula is in each case among the axioms, the homotopy formula is either an axiom or follows from Theorem 9.1, and (J.T.2) was shown in (9.15)(see the remark following theorem 9.1). But then any submodule of a Jordan algebra $\mathfrak{J}$ closed under $P(x)y$ is a Jts; this includes subalgebras, but also other submodules, for example Peirce -½- spaces of $\mathfrak{J}$. (see 9.9.).

The homotopy formula has as immediate consequence (see 7.3)

(10.1) $L(x,y)P(x) = P(x)L(y,x) = P(P(x)y,x)$.

Exercise: Let $\mathcal{T}$ be a Lie triple system over a field of char. $\neq 2,3$ such that $\mathcal{T} = \mathcal{U} \oplus \overline{\mathcal{U}}$, where $\overline{\mathcal{U}}$ is isomorphic to $\mathcal{U}$ and $[\mathcal{U}\,\mathcal{U}\,\mathcal{T}] = [\overline{\mathcal{U}}\,\overline{\mathcal{U}}\,\mathcal{T}] = 0$ (see 6.8.) Define $P: \mathcal{U} \to \operatorname{End}\mathcal{U}$ by $P(x)y = [x\bar{y}x]$ and show that $(\mathcal{U},P)$ is a Jordan triple system. (Hint: Set $\{xyz\} = P(x,z)y$. The Jacobi identity implies $[x\bar{y}z] = [z\bar{y}x]$ and the Lie triple identity (6.1iii) yields (*) $\{xy\{uvw\}\} - \{uv\{xyw\}\} = \{\{xyu\}vw\} - \{u\{yxv\}w\}$. A special case of (*) shows that (J.T.2) holds. Next put $x = w = u$ in (*) and use the fact that part of the resulting equation is symmetric in y and v to derive the homotopy formula. For the fundamental formula compare the results in 7.3.

10.2. There is no additional multiplicative structure (or distinguished elements) required in defining a Jordan triple system. But as we have seen in several instances before, the quadratic map P induces multiplications. Guided by the corresponding definitions in Jordan algebras we define for $u \in \mathfrak{A}$, $(\mathfrak{A},P)$ arbitrary Jts, a quadratic map P_u and a squaring $x \to x^{(2,u)}$ by

$P_u(x) := P(x)P(u)$, $x^{(2,u)} := P(x)u$.

The corresponding leftmultiplications are denoted by $L_u(x)$, resp. $L_u(x,y)$. We have $L_u(x) = x\circ y = P(x,y)u = L(x,u)y$ and $L_u(x,y)z = P(x,z)P(u)y = L(x,P(u)y)z$, thus

(10.2) i) $L_u(x) = L(x,u)$ ii) $L_u(x,y) = L(x,P(u)y)$.

The following result is important

Theorem 1. Let $(\mathfrak{A},P)$ be a Jordan triple system and $u \in \mathfrak{A}$.

<u>Define</u> $P_u(x) := P(x)P(u)$ <u>and</u> $x^{(2,u)} := P(x)u$, <u>then</u> $(\mathfrak{A}, P_u, {}^{(2,u)})$ <u>is a quadratic Jordan algebra</u>.

<u>Proof.</u> Since $\mathfrak{A}_\Omega$ remains a Jts, for any extension Ω, we only have to verify the axioms (Q.F.1) - (Q.F.6) in $\mathfrak{A}$. Using (10.2) we get

$L_u(x,x) = L(x,P(u)x) = L(P(x)u,u)$ (by F.T.2)) $= L_u(x^{(2,u)})$.

This corresponds to (Q.F.1) From the homotopy formula (in $\mathfrak{A}$) it follows $L_u(x)P_u(x) = L(x,u)P(x)P(u) = P(x)L(u,x)P(u) = P(x)P(u)L(x,u)$ $= P_u(x)L_u(x)$, so (Q.F.2) holds. Using the fundamental formula we see $P_u(x)x^{(2,u)} = P(x)P(u)P(x)u = P(P(x)u)u = P(x^{(2,u)})u = (x^{(2,u)})^{(2,u)}$; this is (Q.F.3). Next

$P_u(x)P_u(y)x^{(2,u)} = P(x)P(u)P(y)P(u)P(x)u = P(P(x)P(u)y)u = \left[P_u(x)y\right]^{(2,u)}$

shows that (Q.F.4) holds. Again using the fundamental formula we get $P_u(x^{(2,u)}) = P(P(x)u)P(u) = \left[P(x)P(u)\right]^2 = P_u(x)^2$ and $P_u(P_u(x)y) = P(P(x)P(u)y)P(u) = P_u(x)P_u(y)P_u(x)$. So (Q.J.5) and (Q.F.6) hold and the proof is complete. (compare with 9.8)

We set $\mathfrak{A}_u := (\mathfrak{A}, P_u, {}^{(2,u)})$ and call $\mathfrak{A}_u$ the <u>u-homotope</u> of $\mathfrak{A}$.

<u>Note:</u> If $(\mathfrak{A}, P)$ happens to be the Jts of a Jordan algebra $(\mathfrak{A}, P, {}^2)$ or $(\mathfrak{A}, P, e)$, then the u-homotopes of the triple system are exactly the u-homotopes of the algebra.

Since the u-homotope of a Jts is a Jordan algebra and thus (in the natural way) a Jts, we can iterate the process of forming u-homotopes. By definition the v-homotope of the u-homotope $\mathfrak{A}_u$ of $\mathfrak{A}$ has quadratic map $(P_u)_v(x) = P(x)P(u)P(v)P(u) =$ $P(P(u)v)$, and squaring $x \to P(x)P(u)v = x^{(2,P(u)v)}$. This shows

(10.3) $\qquad (\mathfrak{A}_u)_v = \mathfrak{A}_{P(u)v}$.

10.3. There is an obvious (and rather important) generalization of forming homotopes. To the Jordan triple system $(\mathfrak{A}_u, P_u)$ (we neglect the squaring) corresponds the quadratic map given by $P_u(x) = P(x)P(u)$. An interesting question arises now: Given Jts $(\mathfrak{A}, P)$, for which $V \in \operatorname{End}\mathfrak{A}$ does $P_V(x) = P(x)V$ define on $\mathfrak{A}$ the structure of a Jts. As we have seen, this is the case for $V = P(u)$, $u \in \mathfrak{A}$.

We denote the triple system $(\mathfrak{A}, P_V)$, where $P_V(x) = P(x)V$, by $\mathfrak{A}_V$ and call it, of course, the V-homotope of $\mathfrak{A}$ and V-isotope in case that V is invertible. (It is, of course, convenient to denote the P(u)-homotope as u-homotope as we did in 10.2.) The corresponding trilinear composition in $\mathfrak{A}_V$ is given by

$$L_V(x,y)z = P_V(x,z)y = P(x,z)Vy = L(x,Vy)z.$$

Therefore $\mathfrak{A}_V = (\mathfrak{A}, P_V)$ is again a Jts, iff

i) $L(x,Vy)P(x)V = P(x)VL(y,Vx)$

ii) $L(P(x)Vy,Vy) = L(x,VP(y)Vx)$

iii) $P(x)VP(y)VP(x)V = P(P(x)Vy)V$

It is easy to see, that whenever $P(Vx) = VP(x)V$ holds for all $x \in \mathfrak{A}$, then these equations hold. For in this case iii) holds, using the fundamental formula and also ii) holds using (J.T.2). That under the given assumptions i) holds is seen from the following chain of equalities $P(x)V\{y(Vx)z\} = P(x)VP(y,z)Vx = P(x)P(Vy,Vz)x = P(x)L(Vy,x)Vz = L(x,Vy)P(x)Vz$. We have proved

Theorem 2. *If $(\mathfrak{A}, P)$ is a Jordan triple system and $V \in \operatorname{End}\mathfrak{A}$ such that $P(Vx) = VP(x)V$ for all $x \in \mathfrak{A}$, then the V-homotope (resp. V-isotope) $\mathfrak{A}_V = (\mathfrak{A}, P_V)$, $P_V(x) = P(x)V$, is again a Jordan triple system.*

As an application of this concept we shall look at the following situation. Let $(\mathfrak{A},P)$ be a Jordan triple system and $u\in\mathfrak{A}$ such that $P(u)$ is surjective. The same argument as used in the proof of theorem 9.3 shows that $P(u)$ is bijective. More precisely, there exists $v\in\mathfrak{A}$ such that $P(u)v = u$ and $P(u)P(v) = P(v)P(u) = \mathrm{Id}$. This gives

$P_u(v) = \mathrm{Id}$ and $P_u(x)v = P(x)P(u)v = P(x)u = x^{(2,u)}$

which shows that v is the unit element in the Jordan algebra $\mathfrak{A}_u$. (see 9.7. and 9.8.) Since $P(u)P(v) = \mathrm{Id}$ we observe $P(x)y = P(x)P(u)P(v)y = P_u(x)P(v)y$ which allows the following interpretation.

Lemma 1. If the Jts $(\mathfrak{A},P)$ contains an element u for which $P(u)$ is surjective, then $(\mathfrak{A},P)$ is the $P(v)$-isotope of the unital Jordan algebra $(\mathfrak{A}_u,P_u,v)$ (considered as Jts.)

Note: Since $P(u)P(v) = P(v)P(u)$ we obtain $P_u(P(v)x) = P(v)P(x)P(v)P(u) = P(v)P_u(x)P(v)$, this shows that we can apply theorem 2 to show that $(\mathfrak{A}_u)_{P(v)}$ is a Jts. Note, however, $P(v)$ need not have the form $P_u(y)$ for any y.

10.4. Let $(\mathfrak{A},P)$, $(\mathfrak{A}',P')$ be Jordan triple systems over Φ. A linear map $\phi\colon \mathfrak{A}\to\mathfrak{A}'$ is called a homomorphism, if $\phi(P(x)y) = P'(\phi(x))\phi(y)$.

Note: If $(\mathfrak{J},P,e)$ or $(\mathfrak{J},P,{}^2)$ are Jordan algebras then clearly any algebra homomorphism is a Jts-homomorphism, but the converse is not true (compare the definitions). For example $x\to -x$ is a Jts homomorphism, but (in general) not an algebra homomorphism.

By $\mathrm{Aut}\,\mathfrak{A}$ we denote the group of automorphisms of $\mathfrak{A}$.

$D: \mathfrak{A} \to \mathfrak{A}$ is a <u>derivation</u>, if

$$[D, P(x)] = P(Dx, x) \quad \text{for all } x \in \mathfrak{A}. \; (D \in \operatorname{End} \mathfrak{A})$$

(For Jordan algebras one also requires $D(x^2) = x \circ Dx$ or $De = 0$.). It is immediately checked, that for derivations D, D' of $\mathfrak{A}$, $DD' - D'D$ is again a derivation. Hence $\mathfrak{D}(\mathfrak{A})$, the module of all derivations of $\mathfrak{A}$, is a sub-(Lie)-algebra of $(\operatorname{End} \mathfrak{A})^-$. The defining identity for derivations implies for $D \in \mathfrak{D}$ (linearize and apply to y)

$$D\{xyz\} - \{xDyz\} = \{Dxyz\} + \{xyDz\}, \text{ or}$$

$$[D, L(x,y)] = L(Dx,y) + L(x,Dy).$$

A submodule $\mathfrak{L}$ of $\mathfrak{A}$ is a <u>Jt-subsystem</u>, if $P(\mathfrak{L})\mathfrak{L} \subset \mathfrak{L}$. $\mathfrak{L}$ is an ideal, if $P(\mathfrak{L})\mathfrak{A} \subset \mathfrak{L}$, $P(\mathfrak{A})\mathfrak{L} \subset \mathfrak{L}$, $\{\mathfrak{A}\mathfrak{A}\mathfrak{L}\} \subset \mathfrak{L}$. As usual $\mathfrak{L}$ is an ideal, iff it is the kernel of some homomorphism, and the quotient $\bar{\mathfrak{A}} = \mathfrak{A}/\mathfrak{L}$ ($\mathfrak{L}$ ideal) together with the induced map $\bar{P}$, $\bar{P}(\bar{x})\bar{y} = \overline{P(x)y}$, is a Jts. The usual homomorphism and isomorphism theorems hold.

10.5. Of particular interest in the theory of Jordan triple systems (and in Jordan algebras) are the <u>inner ideals</u>, which are submodules $\mathfrak{L}$ of $\mathfrak{A}$ such that $P(\mathfrak{L})\mathfrak{A} \subset \mathfrak{L}$, i.e., $P(b)\mathfrak{A} \subset \mathfrak{L}$ for any $b \in \mathfrak{L}$. (An inner ideal in a Jordan algebra $\mathfrak{J}$ is the same as an inner ideal in the corresponding Jts $(\mathfrak{J}, P)$.) Clearly the intersection of inner ideals in $\mathfrak{A}$ is an inner ideal. If S is any subset of $\mathfrak{A}$ we denote by $\langle S \rangle$ the inner ideal generated by S, or equivalently, the intersection of all inner ideals containing S.

Lemma 2. i) *The inner ideal generated by a* $a \in \mathfrak{A}$ *is*

$$\langle a\rangle = \Phi a + P(a)\mathfrak{A}$$

ii) $P(\langle a\rangle)\mathfrak{A} \subset P(a)\mathfrak{A}$.

Proof. Obviously $\Phi a + P(a)\mathfrak{A}$ is contained in any inner ideal containing a. We have to prove ii) which in particular shows that $\Phi a + P(a)\mathfrak{A}$ is itself an inner ideal. Using (10.1) and the fundamental formula we derive

$$P(\alpha a + P(a)b)x = P(a)\alpha^2 x + P(a,P(a)b)\alpha x + P(P(a)b)x$$
$$= P(a)\left[\alpha^2 \mathrm{Id} + \alpha L(b,a) + P(b)P(a)\right] x \in P(a)\mathfrak{A} ,$$

this completes the proof.

A particular case of the above equation is of importance. Take $\alpha = 1$ and replace $b \to -b$. Using $P(a)L(b,a) = L(a,b)P(a)$ we obtain

(10.4) $P(a - P(a)b) = P(a)B(b,a) = B(a,b)P(a)$

where $B(a,b) := \mathrm{Id} - L(a,b) + P(a)P(b)$.

Since $P(a)\mathfrak{A} \subset \langle a\rangle$, part ii) of the above lemma shows that $P(a)\mathfrak{A}$ is an inner ideal, for every $a \in \mathfrak{A}$. We call $P(a)\mathfrak{A}$ the *principal inner ideal* generated by a.

Lemma 3. *If* $\mathfrak{B}$ *is an inner ideal of* $\mathfrak{A}$ *and* $(V,U) \in \mathrm{End}\,\mathfrak{A} \times \mathrm{End}\,\mathfrak{A}$ *such that* $P(Vb) = VP(b)U$ *for all* $b \in \mathfrak{B}$ then $V\mathfrak{B}$ *is an inner ideal, in particular* $P(x)\mathfrak{B}$ *is an inner ideal for all* $x \in \mathfrak{A}$.

Proof. Let $b \in \mathfrak{B}$.

$P(Vb)\mathfrak{A} = VP(b)U\mathfrak{A} \subset VP(b)\mathfrak{A} \subset V\mathfrak{B}$ since $P(b)\mathfrak{B} \subset \mathfrak{B}$.

(This lemma, too, implies that $P(x)\mathfrak{A}$ is an inner ideal.)

An element $u \in \mathfrak{A}$ is called *trivial* (or an *absolute zero division*), if $P(u) = 0$. In this case $P(u)\mathfrak{A} = 0$ and $\langle u\rangle = \Phi u$.

Consequently

Lemma 4. *If $u \in \mathfrak{A}$ is trivial, then Φu is an inner ideal.*

We state some more properties, which will be needed.

Lemma 5. i) *The sum of an inner ideal and an ideal is an inner ideal.*

ii) *The image of an inner ideal under a surjective homomorphism is an inner ideal.*

iii) *If $f: \mathfrak{A} \to \mathfrak{A}'$ is a homomorphism, then the complete inverse image $f^{-1}(\mathfrak{L}')$ of an inner ideal in $\mathfrak{A}'$ is an inner ideal in $\mathfrak{A}$.*

The proof is left as an exercise.

Let $\mathfrak{L}$ be an inner ideal of $\mathfrak{A}$ and $V \in \operatorname{End} \mathfrak{A}$ with $P(Vx) = VP(x)V$ then trivially for $b \in \mathfrak{L}$, $P_V(b)\mathfrak{A} = P(b)V\mathfrak{A} \subset P(b)\mathfrak{A} \subset \mathfrak{L}$ which shows that $\mathfrak{L}$ is an inner ideal in the V-homotope $\mathfrak{A}_V$. If V is invertible, then obviously the converse is true. In general, if $\mathfrak{A}_V$ is a V-homotope and $\mathfrak{L}$ an inner ideal in $\mathfrak{A}_V$, $b \in \mathfrak{L}$, then $P(Vb)\mathfrak{A} = VP(b)V\mathfrak{A} = VP_V(b)\mathfrak{A} \subset Vb$, hence $V\mathfrak{L}$ is an inner ideal in $\mathfrak{A}$. We proved:

Lemma 6. i) *If $\mathfrak{L}$ is an inner ideal in $\mathfrak{A}$, then it is an inner ideal in every homotope.*

ii) *$\mathfrak{A}$ and its isotopes have the same inner ideals*

iii) *If $\mathfrak{L}$ is an inner ideal in the V-homotope $\mathfrak{A}_V$, then $V\mathfrak{L}$ is an inner ideal in $\mathfrak{A}$.*

We define $K_u :=$ kernel $P(u)$ and

K_v = kernel V, if $P(Vx) = VP(x)V$.

Lemma 7. i) *K_V is an ideal in $\mathfrak{A}_V$*

ii) *If $\mathfrak{L}$ is an inner ideal without trivial elements $\neq 0$ then K_b is an ideal in the Jordan algebra $\mathfrak{A}_b$, for every*

$b \in \mathcal{L}$.

iii) <u>If $\frac{1}{2} \in \Phi$ then K_b is an ideal in $\mathfrak{A}_b$ for every</u> $b \in \mathfrak{A}$.

<u>Proof.</u> i) We have to show (see 10.4)

a) $P_V(\mathfrak{A})K_V \subset K_V$; b) $P_V(K_V)\mathfrak{A} \subset K_V$; c) $\{\mathfrak{A}(V\mathfrak{A})K_V\} \subset K_V$

Since $P_V(x) = P(x)V$, a) is trivial; so is b), since $VP_V(K_V)\mathfrak{A} = VP(K_V)V\mathfrak{A} = P(VK_V) = 0$. The same argument applies to c): $VP(\mathfrak{A},K_V)V\mathfrak{A} = P(V\mathfrak{A},VK_V)\mathfrak{A} = 0$

ii) and iii): To show that K_b is an ideal in $\mathfrak{A}_b$, we have to show moreover (see 9.6.) a) $K_b^{(2,b)} \subset K_b$ and b) $L_b(\mathfrak{A})K_b \subset K_b$.

We first show b). Using the homotopy formula we get $P(b)\{\mathfrak{A}bK_b\} = \{b\mathfrak{A}P(b)K_b\} = 0$.

Finally take $x \in K_b$, then $P(b)x^{(2,b)} = P(b)P(x)b$.

Assume $\frac{1}{2} \in \Phi$. Then $2P(b)x^{(2,b)} = P(b)\{xbx\} = L(b,x)P(b)x = 0$ shows $x^{(2,b)} \in K_b$ (for $b \in \mathfrak{A}$). This proves iii). In the other case we consider $P(P(b)x^{(2,b)}) = P(b)P(x)P(b)P(x)P(b) = P(P(b)x)P(x)P(b) = 0$. Since $P(b)x^{(2,b)} \in \mathcal{L}$ and $\mathcal{L}$ is without trivial elements $\neq 0$ we conclude $P(b)x^{(2,b)} = 0$, consequently $x^{(2,b)} \in K_b$.

<u>Lemma 8.</u> <u>There is a 1 - 1 correspondence between the inner ideals of</u> $\mathfrak{A}_V/K_V$ <u>and the inner ideals of $\mathfrak{A}$ contained in</u> $V\mathfrak{A}$.

<u>Proof.</u> K_V is an ideal in $\mathfrak{A}_V$ and therefore $\bar{\mathfrak{A}}_V = \mathfrak{A}_V/K_V$ is (in a natural way) a $\mathfrak{J}$ts.

Let $\bar{\mathcal{L}}$ be an inner ideal in $\bar{\mathfrak{A}}_V$, then $\mathcal{L}$ is an inner ideal in $\mathfrak{A}_V$, by lemma 5 iii). Then $V\mathcal{L}$ is an inner ideal in $\mathfrak{A}$ (lemma 6iii) which is contained in $V\mathfrak{A}$. If conversely $\mathfrak{U} \subset V\mathfrak{A}$ is an inner ideal, $\mathfrak{U} = V\mathcal{L}'$, then $\bar{\mathcal{L}}' = \dfrac{\mathcal{L}' + K_V}{K_V}$ is an inner ideal in $\bar{\mathfrak{A}}_V$. This is clear since $VP(b')V\mathfrak{A} = P(Vb')\mathfrak{A} \subset V\mathcal{L}'$ for $b' \in \mathcal{L}'$, equivalently $P_V(b')\mathfrak{A} \subset \mathcal{L}'$ (mod K_V). Furthermore it is clear that the indicated correspondence ($V\mathcal{L} \leftrightarrow \bar{\mathcal{L}}$) is 1 - 1.

There is a nice characterization of Jordan triple systems without (nontrivial) inner ideals.

<u>Theorem 3.</u> <u>$(\mathfrak{A},P)$ is the isotope of a Jordan division algebra, iff $\mathfrak{A}$ is not trivial and has no proper inner ideals.</u>

<u>Proof.</u> If $\mathfrak{J}$ is a division algebra, let $\mathcal{L} \subset \mathfrak{J}$, $\mathcal{L} \neq 0$ be an inner ideal. Since $b \neq 0$ is invertible we have $\mathfrak{J} = P(b)\mathfrak{J} \subset \mathcal{L}$ if $b \in \mathcal{L}$, $b \neq 0$. Any isotope $\mathfrak{J}_V$ has the same inner ideals as $\mathfrak{J}$ (lemma 6.2).

Conversely, since $P(x)\mathfrak{A}$ is an inner ideal in $\mathfrak{A}$ we have $P(x)\mathfrak{A} = \mathfrak{A}$ or $P(x)\mathfrak{A} = 0$. If $P(x)\mathfrak{A} = 0$ then x is a trivial element and Φx an inner ideal. If $x \neq 0$, then $\mathfrak{A} = \Phi x$ and $P(\mathfrak{A})\mathfrak{A} = 0$ which contradicts our assumption that $\mathfrak{A}$ is non-trivial. Consequently, for every $x \neq 0$ in $\mathfrak{A}$, $P(x)$ is invertible. But then for any fixed $u \in \mathfrak{A}$, $u \neq 0$, $P(x)P(u)$ is invertible for all $x \neq 0$. This shows that $\mathfrak{A}_u$ is a Jordan division algebra. The rest follows from lemma 1.

10.6. An inner ideal $\mathcal{L}$ of $\mathfrak{A}$ is called <u>mimimal</u>, if $\mathcal{L} \neq 0$ and for any quadratic ideal $\mathcal{K} \neq 0$ with $\mathcal{K} \subset \mathcal{L}$, we have $\mathcal{L} = \mathcal{K}$.

Let $\mathcal{L}$ be a mimimal inner ideal of $\mathfrak{A}$. If $\mathcal{L}$ contains a trivial element $u \neq 0$ then Φu is a non zero inner ideal contained in $\mathcal{L}$, hence $\mathcal{L} = \Phi u$, by the minimality of $\mathcal{L}$. Therefore we assume from now on that $\mathcal{L}$ does not contain trivial elements $\neq 0$, i.e., $P(b)\mathfrak{A} \neq 0$ for all $0 \neq b \in \mathcal{L}$. But for $b \in \mathcal{L}$, $P(b)\mathfrak{A} \subset \mathcal{L}$ (by definition of inner ideals) and $P(b)\mathfrak{A}$ is an inner ideal (lemma 1 or 3). By the minimality of $\mathcal{L}$ we get

$$(10.5) \qquad P(b)\mathfrak{A} = \mathcal{L} \qquad \text{for all } b \in \mathcal{L},\ b \neq 0.$$

If $b \in \mathcal{L}$, then by lemma 3, $P(b)\mathcal{L}$ is an inner ideal contained in $\mathcal{L}$. Thus we have either $P(b)\mathcal{L} = 0$ or $P(b)\mathcal{L} = \mathcal{L}$. Assume $P(b)\mathcal{L} = 0$ for some $b \neq 0$, $b \in \mathcal{L}$. If b' is an arbitrary element in $\mathcal{L}$, (10.5) implies $b' = P(b)x$ for some $x \in \mathfrak{A}$. Then $P(b')\mathcal{L} = P(P(b)x)\mathcal{L} = P(b)P(x)P(b)\mathcal{L} = 0$, by the fundamental formula and the assumption $P(b)\mathcal{L} = 0$. So far we have proved

Lemma 9. If $\mathcal{L} \neq 0$ is a minimal inner ideal of $\mathfrak{A}$, then we have the possibilities

(I) $\mathcal{L} = \Phi u$, where u is a trivial element of $\mathfrak{A}$.

(II) $P(b)\mathfrak{A} = \mathcal{L}$ for all $b \in \mathcal{L}, b \neq 0$, but $P(b)\mathcal{L} = 0$ for all $b \in \mathcal{L}$.

(III) $P(b)\mathfrak{A} = \mathcal{L}$ and $P(b)\mathcal{L} = \mathcal{L}$ for all $b \in \mathcal{L}$, $b \neq 0$.

In case II and III, we observe, that for $b \neq 0$, $b \in \mathcal{L}$, the inner ideal $P(b)\mathfrak{A}$ is minimal (also, $\mathcal{L}$ has no trivial elements). An immediate application of lemma 7, lemma 8 and theorem 3 is

Corollary. A minimal inner ideal $\mathcal{L}$ of $\mathfrak{A}$ is of type II or III, iff for every $0 \neq b \in \mathcal{L}$ the Jordan algebra $\mathfrak{A}^{b}/K_b$ is a division algebra.

In the cases II and III, we have seen, that for any $b \in \mathcal{L}$, there exists an element $x \in \mathfrak{A}$ such that $b = P(b)x$. We define: An element $u \in \mathfrak{A}$ is regular (or von Neumann regular), if $u \in P(u)\mathfrak{A}$, i.e., $u = P(u)x$ for some x. A pair $(u,v) \in \mathfrak{A} \times \mathfrak{A}$ is called a regular pair, if $P(u)v = u$ and $P(v)u = v$.

Lemma 10. If $x \in \mathfrak{A}$ is regular, then there exists $y \in \mathfrak{A}$ such that (x,y) is a regular pair, i.e., $P(x)y = x$, $P(y)x = y$.

Proof. Assume $x = P(x)u$. We set $y := P(u)x$ and obtain $P(x)y = P(x)P(u)x = P(x)P(u)P(x)u = P(P(x)u)u = P(x)u = x$, and $P(y)x = P(P(u)x)x = P(u)P(x)P(u)x = P(u)P(x)y = P(u)x = y$, thus (x,y) is a regular pair.

We continue the study of minimal inner ideals. Let $\mathcal{L}$ be minimal of type II or III, $\mathcal{L} = P(b)\mathfrak{A}$, $b \neq 0$. What we said before, $b \in \mathcal{L}$ is regular and ther is $d \in \mathfrak{A}$ such that (b,d) is a regular pair; $P(b)d = b$, $P(d)b = d$. We set $\vartheta := P(d)\mathfrak{A}$. ϑ is an inner ideal. Since $P(d)P(b)P(d)a = P(P(d)b)a = P(d)a$, and $P(b)P(d)P(b)a = P(P(b)d)a = P(b)a$, we observe, that $P(d): \mathcal{L} \to \vartheta$ and $P(b): \vartheta \to \mathcal{L}$ are 1 - 1, since $P(b)P(d)|_{\mathcal{L}} = Id|_{\mathcal{L}}$ and $P(d)P(b)|_{\vartheta} = Id|_{\vartheta}$. Obviously $P(b)$, $P(d)$ respects the inner ideal structure of ϑ, resp. $\mathcal{L}$, thus ϑ is minimal, since $\mathcal{L}$ is minimal. Moreover

(10.6) $\mathcal{L} = P(b)\vartheta = P(b)P(d)\mathfrak{A}$ and $\mathcal{L} = P(b)\vartheta = P(b)P(d)\mathcal{L}$.

By lemma 6 we have, that $\mathcal{L}$ is also an inner ideal in the homotope $\mathfrak{A}_d$. We claim, it is also minimal in $\mathfrak{A}_d$. Otherwise there is an inner ideal $\mathcal{L}'$ of $\mathfrak{A}_d$, $\mathcal{L}' \subset \mathcal{L}$. Then, again by lemma 6, $P(d)\mathcal{L}' \subset \vartheta$ is an inner ideal in $\mathfrak{A}$, but ϑ is minimal and $P(d): \mathcal{L} \to \vartheta$ one-to-one. It still might be the case, that $\mathcal{L}$ contains a trivial element in $\mathfrak{A}_d$. Assume $\mathcal{L} = \Phi u$, $P(u)P(d) = 0$. But then $b = \alpha u$ and $P(b) = P(b)P(d)P(b) = \alpha^2 P(u)P(d)P(b) = 0$, which is a contradiction. Therefore

$\mathcal{L} = P_d(b')\mathfrak{A}$ and $\mathcal{L} = P_d(b')\mathcal{L}$ for all $b' \in \mathcal{L}$, $b' \neq 0$. (Using 10.6 and obvious arguments used to prove lemma 9.)

We proved

Lemma 11. If $\mathcal{L}$ is a minimal inner ideal of $\mathfrak{A}$, which contains no trivial element $\neq 0$, then $\mathcal{L}$ is minimal of type III in $\mathfrak{A}$ or in a homotope $\mathfrak{A}_d$.

Note: 1) If $\mathcal{L}$ is of type III, $P(b)\mathcal{L} = \mathcal{L}$, for all $b \in \mathcal{L}$, $b \neq 0$. Consider $(\mathcal{L}, P)$ as a Jts. (P restricted to $\mathcal{L}$). Then every $P(b)$, $b \neq 0$, is invertible and $\mathcal{L}$ is the isotope of a Jordan division algebra. (see theorem 3 and lemma 1).

2) For Jordan algebras there is a complete classification of the minimal inner ideals (see Jacobson: Lectures on quadratic Jordan algebras). This is due to the additional multiplicative structure. We wish to outline some of the additional properties in this case. Assume $\mathcal{L}$ is a minimal inner ideal of type II or III in a Jordan algebra $\mathfrak{J}$. Since $b \in \mathcal{L}$ is regular we have $P(b)a = b$ for some $a \in \mathfrak{J}$. Then by (Q.F.4) we get $b^2 = [P(b)a]^2 = P(b)P(a)b^2 \in P(b)\mathfrak{A} \subset \mathcal{L}$. This shows that $\mathcal{L}$ is actually a subalgebra. If $\mathcal{L}$ is of type III, then $\mathcal{L}$ is a Jordan division algebra which has a unit element e, and then $\mathcal{L} = P(e)\mathfrak{A}$, $e^2 = e$. In the other case ($\mathcal{L}$ of type II) we get $P(b^2)\mathfrak{A} = P(b)P(b)\mathfrak{A} \subset P(b)\mathcal{L} = 0$, since $b^2 \in \mathcal{L}$ and $\mathcal{L}$ has no trivial elements $\neq 0$ we get $\mathcal{L}^2 = 0$. Take $b \neq 0$ in $\mathcal{L}$ and $d \in \mathfrak{A}$ such that (b,d) is a regular pair. $\mathcal{V} = P(d)\mathfrak{A}$ is minimal of type II or III. If it is of type III, then as above there is an idempotent $c = c^2$, $c \neq 0$ in $\mathcal{V}$ such that $\mathcal{V} = P(c)\mathfrak{A}$. Otherwise, as we have just seen $\mathcal{V}^2 = 0$, in particular $d^2 = 0$. $(d = P(d)b \Rightarrow d \in \mathcal{V})$. We set $f := b \circ d$ and show $f^2 = f$, $P(f)b = b$, $P(f)d = d$. We apply (see (9.12))

$P(f) = P(e \circ d) = P(b)P(d) + P(d)P(b) + L(b)P(d)L(b) - P(b,d)$ to

1,b,d and obtain the stated results (We may think of $\mathcal{J}$ as being imbedded in $\hat{\mathcal{J}}$) (the details are left as an exercise). We proved also the following important result: If $\mathcal{J}$ contains a minimal inner ideal of type II or III then $\mathcal{J}$ contains an idempotent $\neq 0$. A result like this does not hold in general for Jordan triple system. For example, consider a (fin. dim.) vector space over $\mathbb{R}$ equipped with a negative definite bilinear form σ, V together with $P(x)y = \sigma(x,y)x$ has no idempotents, since $P(e)e = e$ is (in this case) equivalent with $\sigma(e,e) = 1$, which is not possible.

10.7. We want to make some further remarks on the regularity in Jordan triple systems.

Lemma 12. (McCoy) If $P(a)b - a$ is regular for some $b \in \mathfrak{A}$ then a is regular.

Proof. Let $P(a)b - a = P(P(a)b - a)u$. Using (10.4) we obtain $P(a)b - a = P(a)B(b,a)u$, equivalently $a = P(a)[b - B(b,a)u]$, which shows that a is regular.

Lemma 13. If $\mathfrak{A}$ is a Jts and $\mathcal{L}$ an ideal of $\mathfrak{A}$, then $\mathfrak{A}$ is regular, iff $\mathcal{L}$ and $\mathfrak{A}/\mathcal{L}$ are regular.

Proof. It is obvious that a homomorphic image of a regular triple system is regular, hence $\mathfrak{A}/\mathcal{L}$ is regular if $\mathfrak{A}$ is. If $\mathfrak{A}$ is regular an $b \in \mathcal{L}$, then by lemma 10 we can find $y \in \mathfrak{A}$ such that $b = P(b)y$ and $y = P(y)b$, the second equation shows $y \in \mathcal{L}$ and then b is regular in $\mathcal{L}$. If conversely $\mathcal{L}$ and $\mathfrak{A}/\mathcal{L}$ are regular and $a \in \mathfrak{A}$, then there is $\bar{u} \in \mathfrak{A}/\mathcal{L}$ such that $P(\bar{a})\bar{u} - \bar{a} = 0$, i.e., $P(a)u - a \in \mathcal{L}$, hence regular. Consequently a is regular by

McCoy's lemma.

Now the usual techniques using a homomorphism theorem apply to prove

Lemma 14. The sum of two regular ideals of $\mathfrak{A}$ is regular.

The proof is left as an exercise.

Theorem 4. Any Jordan triple system $\mathfrak{A}$ has a unique maximal regular ideal $\mathfrak{I}$. $\mathfrak{A}/\mathfrak{I}$ has no regular ideal $\neq 0$.

Proof. Since the property of an ideal to be regular is defined in terms of its elements, the set of all regular ideals of $\mathfrak{A}$ is inductively ordered and consequently has a maximal element $\mathfrak{I}$, by Zorn's lemma. By lemma 14 $\mathfrak{I}$ is unique, it contains all regular ideals of $\mathfrak{A}$. If $\bar{\mathfrak{L}} = \mathfrak{L}/\mathfrak{I}$ is regular then $\mathfrak{L}$ is regular by lemma 13 and therefore $\mathfrak{L} \subset \mathfrak{I}$, i.e., $\bar{\mathfrak{L}} = 0$.

XI. Some connections between Jordan triple systems, Lie triple systems and Lie algebras.

11.1. There is a very strong relation between Jordan triple systems and Lie triple systems (resp. Lie algebras). The material presented in chapter VI will be needed throughout this chapter. Let $(\mathfrak{A},P)$ be a Jordan triple system over Φ. Then we have (among others) the equations

(11.1) $L(x,y)P(x) = P(x)L(y,x) = P(P(x)y,x)$

(11.2) $L(x,y)P(x,u) + L(u,y)P(x) = P(x,u)L(y,x) + P(x)L(y,u) = P(P(x,u)y,x) + P(P(x)y,u)$

(11.3) $L(P(x)y,u) + L(P(x)u,y) = L(x,P(y,u)x)$

((11.2) is the linearization of (11.1) and (11.3) the linearization of (J.T.2). (11.3) is equivalent to

$\{(P(x)y)uz\} + \{(P(x)u)yz\} = \{x\{yxu\}z\}$.

Considering this as an operation on u we get

$$L(z,y)P(x) + P(P(x)y,z) = P(x,z)L(y,x).$$

Replacing $P(P(x)y,z)$ in this equation by the corresponding expression obtained from the right hand side equation of (11.2) we obtain

$$L(z,y)P(x) + P(x)L(y,z) = P(\{zyz\},x) \tag{11.4}$$

Remark: In the case that $\mathfrak{A}$ has no 2 and 3 torsion, all equations in $\mathfrak{A}$ are consequences of (11.4).

We recall: D is a derivation of $\mathfrak{A}$, if

$[D,P(x)] = P(Dx,x)$; $\mathcal{D}(\mathfrak{A})$ is the Lie algebra of derivations of $\mathfrak{A}$.

Setting $D(x,y) := L(x,y) - L(y,x)$

We observe immediately from (11.4)

Lemma 1. The mappings $D(x,y)$, $x,y \in \mathfrak{A}$, are derivations of $\mathfrak{A}$.

The following result, although almost trivial, is very important.

Theorem 1. If $(\mathfrak{A},P)$ is a Jordan triple system, then $\mathfrak{A}$ together with $[xyz] := D(x,y)z = \{xyz\} - \{yxz\}$ is a Lie triple system.

Proof. $[xxz] = 0$ is obvious. Also the Jacobi identity $[xyz] + [yzx] + [zxy] = 0$ is immediately verified from the definition and using $\{uvw\} = \{wvu\}$. In order to prove the Lie triple identity (6.1iii) which is equivalent to (6.3iii)(operator form) we use $[D,L(x,y)] = L(Dx,y) + L(x,Dy)$ for $D \in \mathcal{D}(\mathfrak{A})$ (see 10.4) and lemma 1. We obtain

$$\begin{aligned}[D(x,y),D(u,v)] &= [D(x,y),L(u,v) - L(v,u)] \\ &= L(D(x,y)u,v) + L(u,D(x,y)v) - L(D(x,yv,u) - L(v,D(x,y)u) \\ &= D(D(x,y)u,v) + D(u,D(x,y)v),\end{aligned}$$

which corresponds to (6.3iii).

<u>Corollary 1.</u> <u>If</u> $V \in \operatorname{End}\mathfrak{A}$ <u>such that</u> $P(Vx) = VP(x)V$ <u>for all</u> x, <u>then</u> $\mathfrak{A}$ <u>together with</u> $[xyz] = \{xVyz\} - \{yVxz\}$ <u>is a Lie triple system.</u>

<u>Proof.</u> We apply the theorem to the V-homotope $\mathfrak{A}_V$ (see Theorem 10.2).

<u>Corollary 2.</u> <u>If</u> $j \in \operatorname{Aut}\mathfrak{A}$, $j^2 = \mathrm{Id}$, <u>then</u> $\mathfrak{A}$ <u>together with</u> $[xyz] = \{xj(y)z\} - \{yj(x)z\}$ <u>is a Lie triple system.</u>

<u>Proof.</u> $P(j(x)) = jP(x)j$.

<u>Corollary 3.</u> <u>If</u> $\mathfrak{A}$ <u>is a</u> Jts, <u>then</u> $\mathfrak{F} := \mathfrak{A} \oplus \tilde{\mathfrak{A}} = \{(x_1,x_2),\ x_i \in \mathfrak{A}\}$ <u>together with</u> $[(x_1,x_2)(y_1,y_2)(z_1,z_2)] := (\{x_1y_2z_1\} - \{y_1x_2z_1\},$

$\{x_2y_1z_2\} - \{y_2x_1z_2\})$

<u>is a Lie triple system.</u>

<u>Proof.</u> $\mathfrak{F}$ is, as a direct sum of Jts, together with $\{xyz\} = (\{x_1y_1z_1\}, \{x_2y_2z_2\})$ a Jts ($x = (x_1,x_2)$ etc). The exchange map $j: \mathfrak{F} \to \mathfrak{F}$, $j(x_1,x_2) = (x_2,x_1)$ clearly is in $\operatorname{Aut}\mathfrak{F}$ and $j^2 = \mathrm{Id}$. Then apply corollary 2.

Starting with any Jts $\mathfrak{A}$ and V such that $P(Vx) = VP(x)V$, we can construct lots of Lie algebras, according to theorem 6.1. Before discussing these constructions we have to introduce the structure (algebra) of a Jts.

11.2. Now we consider the following generalization of (10.5). Let $\mathfrak{L} := \operatorname{End}\mathfrak{A} \oplus \operatorname{End}\mathfrak{A}$ (the ordinary direct sum of associative algebras, i.e., componentwise multiplication). $\mathfrak{L}^-$ the associated Lie algebra (the product is given by $[(A,B),(A',B')] = ([A,A'],[B,B'])$). We define

$\mathcal{T}(\mathfrak{A}) := \{(U,V) \in \mathcal{L}$; $UP(x) - P(x)V = P(Ux,x)$ and

$VP(x) - P(x)U = P(Vx,x)$ for all $x \in \mathfrak{A}\}$.

Obviously

(11.5) i) $E := (Id, -Id) \in \mathcal{T}(\mathfrak{A})$; ii) $(D,D) \in \mathcal{T}(\mathfrak{A})$ for all $D \in \vartheta(\mathfrak{A})$ iii) $(L(x,y), -L(y,x)) \in \mathcal{T}(\mathfrak{A})$ (follows from (11.4)).

One more definition:

$l(x,y) := (L(x,y), -L(y,x))$ and $\mathcal{G}(\mathfrak{A})$ denotes the submodule in $\mathcal{T}(\mathfrak{A})$ spanned by all $l(x,y)$, $x,y \in \mathfrak{A}$.

<u>Lemma 2.</u> <u>$\mathcal{T}(\mathfrak{A})$ is a subalgebra of $\mathcal{L}^-$ and $\mathcal{G}(\mathfrak{A})$ an ideal in</u> $\mathcal{T}(\mathfrak{A})$.

<u>Proof</u>. $\mathcal{T}(\mathfrak{A})$ is a submodule of $\mathcal{L}$. If $(U,V),(U',V') \in \mathcal{T}(\mathfrak{A})$, then using the definition we compute:

$$UU'P(x) = UP(U'x,x) + UP(x)V'$$

$$= P(UU'x,x) + P(U'x,Ux) + P(U'x,x)V + P(Ux,x)V' + P(x)VV'$$

(the defining relations of $\mathcal{T}(\mathfrak{A})$ can be linearized!)

Interchanging $(U,V),(U',V')$ and substracting gives

$$[U,U']P(x) = P([U,U']x,x) + P(x)[V,V'] \text{ ; similarly}$$

$$[V,V']P(x) = P([V,V']x,x) + P(x)[U,U'] .$$

This shows

$$[(U,V),(U',V')] = ([U,U'],[V,V']) \in \mathcal{T}(\mathfrak{A}).$$

$(U,V) \in \mathcal{T}(\mathfrak{A})$ implies

$$UP(x,y)z - P(x,y)Vz = P(Ux,y)z + P(x,Uy)z,$$

which is, considered as action on y

$$(11.6) \quad [U,L(x,z)] = L(Ux,z) + L(x,Vz), \text{ similarly}$$

$$[V,L(x,z)] = L(Vx,z) + L(x,Uz)$$

Since $l(u,v) = (L(u,v), -L(v,u))$ (which is in $\mathfrak{T}(\mathfrak{A})$, see (11.5)) we get

$$[(U,V),l(u,v)] = ([U,L(u,v)], -[V,L(v,u)])$$
$$= (L(Uu,v) + L(u,Vv), -L(Vv,u) - L(v,Uu))$$
$$= l(Uu,v) + l(u,Vv),$$ which shows that

$\mathfrak{G}(\mathfrak{A})$ is an ideal. This completes the proof. A particular case of (11.6) is

(11.8) $[L(u,v),L(x,y)] = L(L(u,v)x,y) - L(x,L(v,u)y)$.

We call $\mathfrak{T}(\mathfrak{A})$ the <u>structure algebra</u> of $\mathfrak{A}$, and the ideal $\mathfrak{G}(\mathfrak{A})$ generated by the $l(x,y)$ is called the <u>inner structure algebra</u>. $\mathfrak{T}(\mathfrak{A})$ has a canonical involutorial automorphism

$\theta: \mathfrak{T}(\mathfrak{A}) \to \mathfrak{T}(\mathfrak{A})$, $\theta:(U,V) \mapsto (V,U)$.

(This is clearly the restriction of the corresponding automorphism in $\mathfrak{L}$). The derivation algebra $\mathfrak{D}(\mathfrak{A})$ may be identified via $D \mapsto (D,D)$ with the fixed point set of θ, i.e.,

$\mathfrak{D}(\mathfrak{A}) \cong \{x, \theta x = x\} = \mathfrak{T}_+(\mathfrak{A})$ $(\frac{1}{2} \in \Phi)$

<u>Lemma 3.</u> <u>$\mathfrak{T}(\mathfrak{A})$ is a subalgebra of the derivation algebra of the Lts $\mathfrak{A} \oplus \tilde{\mathfrak{A}}$.</u>

<u>Proof.</u> Let $x = (x_1,x_2)$, $y = (y_1,y_2)$, $z = (z_1,z_2)$ etc.

$$S(x,y)z := [xyz] = [l(x_1,y_2) - l(y_1,x_2)](z_1,z_2)$$

(see Corollary 3; $l(u,v) = (L(u,v), -L(v,u))$.). i.e.,

(11.9) $$S(x,y) = l(x_1,y_2) - l(y_1,x_2).$$

Using (11.7) we derive for $(U,V) \in \mathfrak{T}(\mathfrak{A})$

$$[(U,V),S(x,y)] = l(Ux_1,y_2) + l(x_1,Vy_2) - l(Uy_1,x_2) - l(y_1,Vx_2)$$
$$= S((U,V)x,y) + S(x,(U,V)y),$$ by (11.9) and

$(U,V)z = (Uz_1,Vz_2)$. This is the defining identity for derivations in Lie triple systems.

This lemma allows us to apply theorem 6.1 for the Lts $\mathfrak{A} \oplus \tilde{\mathfrak{A}}$, $[xyz] = S(x,y)z$ and a subalgebra $\mathfrak{g}$ between $\mathfrak{h}$ and $\mathfrak{T}(\mathfrak{A})$. (Note: $\mathfrak{h}(\mathfrak{A})$ is spanned by all $S(x,y)$, by (11.9).)

Translating the construction performed in Ch. VI to this particular case, using (11.9) and other obvious definitions we get the following Lie algebra:

$$\mathcal{L}(\mathfrak{g},\mathfrak{A}) = \mathfrak{g} \oplus \mathfrak{A} \oplus \tilde{\mathfrak{A}}$$

with product given by the rules

i) the given Lie product in $\mathfrak{g}$, i.e., $\mathfrak{g}$ is subalgebra of $\mathcal{L}$.

(11.10) ii) $[x_1 \oplus \tilde{x}_2, y_1 \oplus \tilde{y}_2] = S(x,y) = l(x_1,y_2) - l(y_1,x_2)$

iii) $[(U,V), x_1 \oplus \tilde{x}_2] = Ux_1 \oplus \widetilde{Vx_2}$.

We proved:

<u>Theorem 2. If $\mathfrak{A}$ is a Jts and $\mathfrak{g}$ is a subalgebra in $\mathfrak{T}(\mathfrak{A})$ containing $\mathfrak{h}(\mathfrak{A})$, then</u>

$$\mathcal{L}(\mathfrak{g},\mathfrak{A}) = \mathfrak{g} \oplus \mathfrak{A} \oplus \tilde{\mathfrak{A}}$$

<u>with product given in</u> (11.10) <u>is a Lie algebra</u>.

For convenience we shall assume from now on that $\mathfrak{g}$ is invariant under $\Theta: (U,V) \to (V,U)$.

We collect some properties of $\mathcal{L}(\mathfrak{g},\mathfrak{A})$. As usual we identify $\mathfrak{g}, \mathfrak{A}, \tilde{\mathfrak{A}}$ with its canonical images in $\mathcal{L}$. $a \leftrightarrow (0,a,0)$, etc.

The multiplication rules (11.10) give the following

i) $[\mathfrak{A},\mathfrak{A}] = [\tilde{\mathfrak{A}},\tilde{\mathfrak{A}}] = 0$

ii) $[(U,V),a] = Ua$, $[(U,V),\tilde{b}] = \widetilde{Vb}$

iii) $[a,\tilde{b}] = l(a,b)$, which implies in particular

iv) $[[a,\tilde{b}],c] = L(a,b)c = \{abc\}$

Extending Θ to $\mathfrak{L}$ it is obvious

(11.11) v) $\Theta : G \oplus a \oplus \tilde{b} \to \Theta G \oplus b \oplus \tilde{a}$ is an involutorial automorphism of $\mathfrak{L}(\mathfrak{g},\mathfrak{A})$.

vi) $j : G \oplus a \oplus \tilde{b} \to G \oplus -a \oplus -\tilde{b}$ is an involutorial automorphism of $\mathfrak{L}$ (with (-1)-space $\mathfrak{A} \oplus \tilde{\mathfrak{A}}$ if $\frac{1}{2} \in \Phi$, see theorem 6.1)

vii) If $E := (Id,-Id) \in \mathfrak{g}$, then $(adE)^3 = adE$.

If $\frac{1}{2} \in \Phi$, then $\mathfrak{g}, \mathfrak{A}, \tilde{\mathfrak{A}}$ are the eigenspaces of adE belonging to the eigenvalues 0, +1, -1, respectively.

The "biggest" algebra, obtained from theorem 2 is $\mathfrak{L}(\mathfrak{T}(\mathfrak{A}),\mathfrak{A})$, observe $E \in \mathfrak{T}(\mathfrak{A})$, by (11.5), the smallest algebra is $\mathfrak{L}(\mathfrak{h}(\mathfrak{A}),\mathfrak{A})$, E is not neccessarily in $\mathfrak{h}$, but in the most important examples it is in $\mathfrak{h}$, as we shall see later.

viii) $\mathfrak{L}(\mathfrak{h},\mathfrak{A})$ is an ideal in $\mathfrak{L}(\mathfrak{g},\mathfrak{A})$, by lemma 2, and is the standard imbedding of $\mathfrak{J} = \mathfrak{A} \oplus \tilde{\mathfrak{A}}$.

The Lie algebra $\mathfrak{L}(\mathfrak{g},\mathfrak{A})$ is called the <u>KOECHER-TITS-algebra of</u> $(\mathfrak{g},\mathfrak{A})$.

11.3. It is clear that we are going to apply the results obtained in chapter VI, therefore we assume for the rest of this chapter

$$\tfrac{1}{2} \in \Phi.$$

Due to the indicated construction many properties of $\mathfrak{J} = \mathfrak{A} \oplus \tilde{\mathfrak{A}}$ (and then properties of $\mathfrak{L}(\mathfrak{h},\mathfrak{A})$, which is the standard imbedding of $\mathfrak{J}$) can be expressed in terms of $\mathfrak{A}$. Our first concern is getting some informations about the ideal structure of $\mathfrak{J}$.

We recall: a submodule $\mathfrak{U}$ of $\mathfrak{T}$ is a Lie triple ideal of $\mathfrak{T}$, iff $[\mathfrak{U}\mathfrak{T}\mathfrak{T}] \subset \mathfrak{U}$. Using (11.9) in our case this is equivalent to

(11.12) $\left[1(u_1,y_2) - 1(y_1,u_2)\right](z_1,z_2) \in \mathfrak{U}$ for all $u = (u_1,u_2) \in \mathfrak{U}$, $y = (y_1,y_2)$, $z = (z_1,z_2)$. Different selections of y_i, z_i (for example $y_1 = z_2 = 0$ or $y_2 = z_1 = 0$) show

<u>Lemma 4.</u> *$\mathfrak{U} \subset \mathfrak{T} = \mathfrak{A} \oplus \tilde{\mathfrak{A}}$ is a Lt ideal, iff*
$\{\mathfrak{U}_i \mathfrak{A}\mathfrak{A}\} \subset \mathfrak{U}_i$, $\{\mathfrak{A}\,\mathfrak{U}_i\,\mathfrak{A}\} \subset \mathfrak{U}_j$ $(i,j = 1,2;\ i \neq j)$
where $\mathfrak{U}_1$, $\tilde{\mathfrak{U}}_2$ are the projections of $\mathfrak{U}$ into $\mathfrak{A}$, $\tilde{\mathfrak{A}}$ respectively.

A submodule $\mathfrak{W}$ of $\mathfrak{A}$ is $\mathfrak{G}$-<u>invariant</u>, if $\{\mathfrak{A}\mathfrak{A}\mathfrak{W}\} \subset \mathfrak{W}$. A Jts $\mathfrak{A}$ is called $\mathfrak{G}$-<u>irreducible</u> (or an irreducible $\mathfrak{G}$-module) if there is no $\mathfrak{G}$-invariant submodule in $\mathfrak{A}$ other than 0 and $\mathfrak{A}$.
For a submodule $\mathfrak{W}$ of $\mathfrak{A}$ we define

$$\bar{\mathfrak{W}} := P(\mathfrak{A})\mathfrak{W} = \{\mathfrak{A}\mathfrak{W}\mathfrak{A}\}.$$

Lemma 4 then says that $\mathfrak{U} \subset \mathfrak{T}$ is an ideal, iff $\mathfrak{U}_i$ are $\mathfrak{G}$-invariant and $\bar{\mathfrak{U}}_1 \subset \mathfrak{U}_2$, $\bar{\mathfrak{U}}_2 \subset \mathfrak{U}_1$. $\mathfrak{W}$ in $\mathfrak{A}$ is an ideal, iff $\{\mathfrak{A}\mathfrak{W}\mathfrak{A}\} \subset \mathfrak{W}$ and $\{\mathfrak{W}\mathfrak{A}\mathfrak{A}\} \subset \mathfrak{W}$. ($P(\mathfrak{W})\mathfrak{A} \subset \mathfrak{W}$ follows from $\{\mathfrak{W}\mathfrak{A}\mathfrak{A}\} \subset \mathfrak{W}$ since $\frac{1}{2} \in \Phi$).
This shows

<u>Corollary.</u> *$\mathfrak{W} \subset \mathfrak{A}$ if a Jt ideal, iff $\mathfrak{W} \oplus \tilde{\bar{\mathfrak{W}}}$ is a Lt ideal in $\mathfrak{A} \oplus \tilde{\mathfrak{A}}$.*

<u>Lemma 5.</u> *If $\mathfrak{L}$ is a $\mathfrak{G}$-invariant submodule of $\mathfrak{A}$, then $\bar{\mathfrak{L}}$ is $\mathfrak{G}$-invariant and $\bar{\bar{\mathfrak{L}}} \subset \mathfrak{L}$.*

<u>Proof.</u> Using (11.4) we obtain for $\mathfrak{G}$-invariant $\mathfrak{L}$

$L(z,y)P(x)\mathfrak{L} \subset P(\{zyx\},x)\mathfrak{L} + P(x)L(y,z)\mathfrak{L} \subset P(\mathfrak{A})\mathfrak{L} = \bar{\mathfrak{L}}$,

which shows that $\bar{\mathfrak{L}}$ is $\mathfrak{G}$-invariant. In order to show $\bar{\bar{\mathfrak{L}}} \subset \mathfrak{L}$ it suffices to show $P(x)P(y)\mathfrak{L} \subset \mathfrak{L}$ for all $x,y \in \mathfrak{A}$.

We apply (11.4) to y and consider the result as operation on z. We find

$$2P(x)P(y) = L(x,y)L(x,y) - L(P(x)y,y).$$

Since $\{\mathfrak{A}\mathfrak{A}\mathfrak{L}\}\subset\mathfrak{L}$ this shows $P(x)P(y)\mathfrak{L}\subset\mathfrak{L}$.

Lemma 5 and lemma 4 together show

Corollary 1: *If $\mathfrak{L}$ is an $\mathfrak{G}$-invariant submodule of $\mathfrak{A}$, then*

$$\mathfrak{L}\oplus\tilde{\bar{\mathfrak{L}}}, \qquad \bar{\mathfrak{L}}\oplus\tilde{\mathfrak{L}}$$

are Lt-ideals in $\mathfrak{A}\oplus\tilde{\mathfrak{A}}$.

Corollary 2: *If the Lts $\mathfrak{A}\oplus\tilde{\mathfrak{A}}$ is simple, then $\mathfrak{A}$ is $\mathfrak{G}$-irreducible.*

Next we assume $E = (\mathrm{Id}, -\mathrm{Id})\in\mathfrak{G}$ and $\mathfrak{U}$ an ideal of $\mathfrak{T} = \mathfrak{A}\oplus\tilde{\mathfrak{A}}$. Let $u = (u_1,u_2)\in\mathfrak{U}$, $y = (0,y)$, $z = (z,0)\in\mathfrak{A}\oplus\tilde{\mathfrak{A}}$, then by (11.12) $\{u_1yz\}\in\mathfrak{U}$, similarly $\{u_2yz\}\in\mathfrak{U}$ for all $y,z\in\mathfrak{A}$. If $E = \Sigma l(z,y)$, then $\mathrm{Id} = \Sigma L(z,y)$ and by what we just said $u_1 = \mathrm{Id}u_1 = \Sigma\{zyu_1\}\in\mathfrak{U}$, similarly $u_2\in\mathfrak{U}$. I.e., $\mathfrak{U}$ is split, or $\mathfrak{U} = \mathfrak{U}_1\oplus\mathfrak{U}_2$. Assume $\mathfrak{A}$ is $\mathfrak{G}$-irreducible then $\mathfrak{U}_1$, $\mathfrak{U}_2$ are either 0 or $\mathfrak{A}$ (since they are $\mathfrak{G}$-invariant). If $\mathfrak{U}_1 = 0$, $\mathfrak{U}_2 = \mathfrak{A}$ then (see lemma 4) $P(\mathfrak{A})\mathfrak{U}_2 = P(\mathfrak{A})\mathfrak{A} = 0$. But if $\mathfrak{A}\neq 0$, our assumption $E\in\mathfrak{G}$ implies $\{\mathfrak{A}\mathfrak{A}\mathfrak{A}\}\neq 0$. Therefore $\mathfrak{U}_1 = 0$ implies $\mathfrak{U}_2 = 0$ and of course $\mathfrak{U}_2 = 0$ implies $\mathfrak{U}_1 = 0$. This shows $\mathfrak{U} = 0$ or $\mathfrak{U} = \mathfrak{A}\oplus\tilde{\mathfrak{A}}$. We proved

Lemma 6. *If $\mathfrak{A}\neq 0$ is $\mathfrak{G}$-irreducible and $E\in\mathfrak{G}$, then $\mathfrak{A}\oplus\tilde{\mathfrak{A}}$ is a simple Lie triple system.*

Using the fact that the Koecher-Tits-algebra $\mathcal{L}(\mathfrak{G},\mathfrak{A})$ together with the involution J: $H\oplus a\oplus b\mapsto -H\oplus a\oplus b$ (see (11.11vi), $J = -j$) is the standard imbedding of $\mathfrak{T} = \mathfrak{A}\oplus\tilde{\mathfrak{A}}$ and using theorem 6.2, we get that $(\mathcal{L}(\mathfrak{G},\mathfrak{A}),J)$ is a simple pair, iff $\mathcal{F}$ is simple. Moreover we know, if $\mathcal{F}$ is simple and any ideal in $\mathcal{L}(\mathfrak{G},\mathfrak{A})$ is J-invariant (see 6.4.) then $\mathcal{L}(\mathfrak{G},\mathfrak{A})$ is simple.

<u>Theorem 3.</u> <u>If the Jordan triple system $\mathfrak{A}$ is $\mathfrak{G}$-irreducible and if $E\in\mathfrak{G}$, then the Koecher-Tits-algebra $\mathcal{L}(\mathfrak{G},\mathfrak{A})$ is a simple Lie algebra.</u>

<u>Proof.</u> Using lemma 6 and the above remarks we must still show that any ideal in $\mathcal{L}$ splits. Let $\mathfrak{K}$ be an ideal in $\mathcal{L}$ and $X = H\oplus a\oplus\tilde{b}$ in $\mathfrak{K}$, then $a\oplus\tilde{b} = [E[E,X]]\in\mathfrak{K}$, which already completes the proof.

We still have to investigate relations between simplicity and $\mathfrak{G}$-irreducibility of $\mathfrak{A}$.
Since any ideal of $\mathfrak{A}$ is in particular $\mathfrak{G}$-invariant, we see if $\mathfrak{A}$ is $\mathfrak{G}$-irreducible and $\{\mathfrak{A}\mathfrak{A}\mathfrak{A}\}\neq 0$ then $\mathfrak{A}$ is simple. If conversely $\mathfrak{A}$ is simple and $\mathfrak{L}$ an $\mathfrak{G}$-invariant submodule, then by lemma 5 we get that $\mathfrak{L}\cap\overline{\mathfrak{L}}$ and $\mathfrak{L}+\overline{\mathfrak{L}}$ are ideals in $\mathfrak{A}$. Assuming $0\neq\mathfrak{L}\neq\mathfrak{A}$ we conclude $\mathfrak{L}\cap\overline{\mathfrak{L}} = 0$ and $\mathfrak{A} = \mathfrak{L}\oplus\overline{\mathfrak{L}}$ (as direct sum of triple systems). Thus we proved
<u>Theorem 4.</u> <u>A simple Jts $\mathfrak{A}$ is either $\mathfrak{G}$-irreducible or decomposes into the direct sum of two $\mathfrak{G}$-invariant and $\mathfrak{G}$ irreducible subsystems $\mathfrak{A} = \mathfrak{L}\oplus\overline{\mathfrak{L}}$.</u>
<u>Remark:</u> 1) If $\mathfrak{A}$ is finite dimensional over a field of characteristic zero we shall see later that under the given assumptions the trace form $\sigma(x,y) = \mathrm{tr}L(x,y)$ is non degenerate on $\mathfrak{A}$. Since $\{\mathfrak{L}\mathfrak{L}\mathfrak{A}\} = \{\overline{\mathfrak{L}}\,\overline{\mathfrak{L}}\mathfrak{A}\} = 0$, $\mathfrak{L}$ and $\overline{\mathfrak{L}}$ are totally isotropic subspaces

which implies $\dim \mathcal{L} = \dim \bar{\mathcal{L}}$.

2) An example in which a simple Jts is not irreducible is the following:

$$\mathfrak{A} = \left\{ X = \begin{pmatrix} 0 & x \\ y^t & 0 \end{pmatrix};\ x,y \text{ all } 1\times n \text{ matrices over a field } F \right\}$$

together with $P(X)Y = XY^{t}X$.

Exercise. Let $\mathfrak{A} = \mathcal{L} \oplus \bar{\mathcal{L}}$ as in theorem 4. Consider the subalgebras $\mathcal{L}_1$, $\mathcal{L}_2$ of $\mathcal{L}(\mathcal{G}, \mathfrak{A})$

$$\mathcal{L}_1 := l(\mathcal{L}, \bar{\mathcal{L}}) \oplus \mathcal{L} \oplus \tilde{\bar{\mathcal{L}}}$$
$$\mathcal{L}_2 := l(\bar{\mathcal{L}}, \mathcal{L}) \oplus \bar{\mathcal{L}} \oplus \tilde{\mathcal{L}}$$

and prove: i) $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$

ii) If $E \in \mathcal{G}$ then $\mathcal{L}_i$ is simple.

11.4. We are going to apply the results about the radical in Lie triple systems (see 6.5.) to $\mathfrak{A} \oplus \tilde{\mathfrak{A}}$. We give a short review of these results. If $\mathcal{T}$ is a Lts with composition $S(x,y)z = [xyz]$ and $\mathcal{U}$ an ideal of $\mathcal{T}$, then $i(\mathcal{U}) \oplus \mathcal{U}$, $i(\mathcal{U}) = S(\mathcal{T}, \mathcal{U})$, is an ideal in the standard imbedding $\mathcal{L}(\mathcal{T})$. For ideals $\mathcal{U}_1$, $\mathcal{U}_2$ of $\mathcal{T}$, the "product"

$$\mathcal{U}_1 * \mathcal{U}_2 = i(\mathcal{U}_1)\mathcal{U}_2 + i(\mathcal{U}_2)\mathcal{U}_1$$

is an ideal of $\mathcal{T}$. Certain powers of an ideal $\mathcal{U} \subset \mathcal{T}$ are defined by

$$\mathcal{U}^{<o>} = \mathcal{U},\ \mathcal{U}^{<k+1>} = \mathcal{U}^{<k>} * \mathcal{U}^{<k>} = [\mathcal{U}^{<k>}\ \mathcal{T}\ \mathcal{U}^{<k>}] .$$

$\mathcal{U}$ is L-solvable, if $\mathcal{U}^{<k>} = 0$ for some k. Under appropriate finiteness condition we proved $\operatorname{Rad} \mathcal{L}(\mathcal{T}) = i(\mathcal{R}(\mathcal{T})) \oplus \mathcal{R}(\mathcal{T})$, where $\mathcal{R}(\mathcal{F})$ is the maximal L-solvable ideal in $\mathcal{F}$ (or L-radical). For finite dimensional $\mathcal{F}$ over a field of char. 0 L-solvability and (ordinary) solvability is the same.

Now let $\mathcal{F} = \mathfrak{A} \oplus \tilde{\mathfrak{A}}$, $\mathfrak{A}$ a Jts. $x = (x_1, x_2)$, $y = (y_1, y_2)$ etc. $S(x,y) = l(x_1,y_2) - l(y_1,x_2)$, (see (11.9)). If $\mathfrak{B}$ is an ideal of $\mathfrak{A}$, then $\mathfrak{U} = \mathfrak{B} \oplus \tilde{\mathfrak{B}}$ is an ideal in $\mathcal{F}$ (corollary to lemma 4). By definition $S(\mathcal{F}, \mathfrak{U}) = l(\mathfrak{A}, \mathfrak{B}) + l(\mathfrak{B}, \mathfrak{A})$. Let $\mathfrak{B}$, $\mathfrak{B}'$ be ideals in $\mathfrak{A}$, $\mathfrak{U} = \mathfrak{B} \oplus \tilde{\mathfrak{B}}$, $\mathfrak{U}' = \mathfrak{B}' \oplus \tilde{\mathfrak{B}}'$ then a simple computation shows

$$\mathfrak{U} * \mathfrak{U}' = \mathfrak{B} \circ \mathfrak{B}' \oplus \widetilde{\mathfrak{B} \circ \mathfrak{B}'}, \text{ where}$$

$$\mathfrak{B} \circ \mathfrak{B}' := \{\mathfrak{A}\mathfrak{B}\mathfrak{B}'\} + \{\mathfrak{A}\mathfrak{B}'\mathfrak{B}\} + \{\mathfrak{B}\mathfrak{A}\mathfrak{B}'\}.$$

Since $\mathfrak{U} * \mathfrak{U}'$ is a Lt-ideal again the corollary to lemma 4 shows that $\mathfrak{B} \circ \mathfrak{B}'$ is a Jt ideal in $\mathfrak{A}$. These considerations lead us (in a natural way) to define for an ideal $\mathfrak{B} \subset \mathfrak{A}$

$$\mathfrak{B}^{<o>} = \mathfrak{B}, \quad \mathfrak{B}^{<k+1>} = \mathfrak{B}^{<k>} \circ \mathfrak{B}^{<k>},$$

and call $\mathfrak{B}$ L-<u>solvable</u>, if $\mathfrak{B}^{<k>} = 0$ for some k.

<u>Lemma 7.</u> a) $(\mathfrak{B} \oplus \tilde{\mathfrak{B}})^{<k>} = \mathfrak{B}^{<k>} \oplus \widetilde{\mathfrak{B}^{<k>}}$

b) $\mathfrak{B} \oplus \tilde{\mathfrak{B}}$ <u>is L-solvable, iff</u> $\mathfrak{B}$ <u>is L-solvable</u>.

<u>Proof:</u> a) by induction, b) immediate consequence of a).

<u>Lemma 8. If</u> $\mathfrak{B}$, $\mathfrak{B}'$ <u>are L-solvable ideals of</u> $\mathfrak{A}$, <u>then</u> $\mathfrak{B} + \mathfrak{B}'$ <u>is L-solvable</u>.

<u>Proof.</u> This follows from lemma 7 and the fact that $(\mathfrak{B} \oplus \tilde{\mathfrak{B}}) + (\mathfrak{B}' \oplus \tilde{\mathfrak{B}}')$ is L-solvable in $\mathcal{F}$ (lemma 6. 6.).

Assume the appropriate finiteness conditions then there is a unique maximal L-solvable ideal in $\mathfrak{A}$, denoted by $\mathcal{R}(\mathfrak{A})$, the L-radical of $\mathfrak{A}$.

Theorem 5. $\mathcal{R}(\mathfrak{A} \oplus \tilde{\mathfrak{A}}) = \mathcal{R}(\mathfrak{A}) \oplus \widetilde{\mathcal{R}(\mathfrak{A})}$

Proof. Since $\mathcal{R}(\mathfrak{A})$ is L-solvable, so is $\mathcal{R}(\mathfrak{A}) \oplus \widetilde{\mathcal{R}(\mathfrak{A})}$ in $\mathcal{F}$, by lemma 7. Then $\mathcal{R}(\mathfrak{A}) \oplus \widetilde{\mathcal{R}(\mathfrak{A})} \subset \mathcal{R}(\mathfrak{A} \oplus \tilde{\mathfrak{A}})$. Conversely, if $\mathcal{R}_1$, $\mathcal{R}_2$ are the projections of $\mathcal{R}(\mathcal{F})$ into $\mathfrak{A}$ resp. $\tilde{\mathfrak{A}}$ then our first observation is $\mathcal{R}_1 = \mathcal{R}_2$, since the radical is invariant under all automorphisms, in particular under the exchange automorphism $(x_1, x_2) \to (x_2, x_1)$. Then $\mathcal{R}_1$ is an ideal of $\mathfrak{A}$, by lemma 4. Since $\mathcal{R}_1^{\langle k\rangle}$ clearly is contained in the projection of $\mathcal{R}(\mathcal{F})^{\langle k\rangle}$ to $\mathfrak{A}$, we obtain that $\mathcal{R}_1 (= \mathcal{R}_2)$ is L-solvable. Consequently $\mathcal{R}(\mathcal{F}) \subset \mathcal{R}_1 \oplus \tilde{\mathcal{R}}_1 \subset \mathcal{R}(\mathfrak{A}) \oplus \widetilde{\mathcal{R}(\mathfrak{A})}$.

Corollary 1. $\operatorname{Rad} \mathcal{L}(\mathfrak{g}, \mathfrak{A}) = 1(\mathfrak{A}, \mathcal{R}(\mathfrak{A})) + 1(\mathcal{R}(\mathfrak{A}), \mathfrak{A}) \oplus \mathcal{R}(\mathfrak{A}) \oplus \widetilde{\mathcal{R}(\mathfrak{A})}$.

Calling $\mathfrak{A}$ L-semi-simple, if $\mathcal{R}(\mathfrak{A}) = 0$ we have

Corollary 2. $\mathcal{L}(\mathfrak{g}, \mathfrak{A})$ is semi-simple, iff $\mathfrak{A}$ is L-semi-simple.

Besides L-solvability we have the notion of (ordinary) solvability. If $\mathfrak{Q}$ is an ideal in $\mathfrak{A}$ we set

$$\mathfrak{Q}^{(o)} = \mathfrak{Q}, \quad \mathfrak{Q}^{(k+1)} = \{\mathfrak{Q}^{(k)} \mathfrak{Q}^{(k)} \mathfrak{Q}^{(k)}\}.$$

$\mathfrak{Q}$ is solvable, if $\mathfrak{Q}^{(k)} = 0$ for some k. If $\mathfrak{U} = \mathfrak{Q} \oplus \tilde{\mathfrak{Q}}$, clearly $[\mathfrak{U}\mathfrak{U}\mathfrak{U}] = \{\mathfrak{Q}\mathfrak{Q}\mathfrak{Q}\} \oplus \widetilde{\{\mathfrak{Q}\mathfrak{Q}\mathfrak{Q}\}}$ and by induction $\mathfrak{U}^{(k)} = \mathfrak{Q}^{(k)} \oplus \tilde{\mathfrak{Q}}^{(k)}$ which shows, that $\mathfrak{U}$ is solvable if $\mathfrak{Q}$ is solvable. We apply theorem 6.9 and obtain

Theorem 6. a) If $\mathfrak{A}$ is a finite dimensional Jts over F of char. 0, then an ideal $\mathfrak{Q}$ in $\mathfrak{A}$ is solvable, iff $\mathfrak{Q}$ is L-solvable.

b) $\mathcal{R}(\mathfrak{A}) = \operatorname{Rad} \mathfrak{A}$.

(Rad $\mathfrak{A}$ denotes the maximal solvable ideal of $\mathfrak{A}$, see 3.2)

11.5. In Lie theory the Killing form of a Lie algebra is an important tool. We assume $\mathfrak{A}$ is a finite dimensional Jts over a field F. Using theorem 6.11 and (6.7.) we compute the Killing form λ of the Koecher-Tits-algebra $\mathcal{L}(\tilde{\mathcal{G}}, \mathfrak{A})$.

We already know (ch. VI), if $X = H \oplus a \oplus \tilde{b} \in \mathcal{L}$, then

$$\lambda(X,X) = \lambda(H,H) + \lambda(a\oplus\tilde{b}, a\oplus\tilde{b}), \text{ and}$$

(11.13) $$\lambda(H,H) = \lambda_{\mathcal{G}}(H,H) + \mathrm{tr}H^2$$

$$\lambda(a\oplus\tilde{b}, a\oplus\tilde{b}) = 2\mathrm{tr}R(a\oplus\tilde{b}, a\oplus\tilde{b}), \text{ where}$$

$S(x,y)z = [xyz] = R(z,y)x$ (in $\mathfrak{A} \oplus \tilde{\mathfrak{A}}$).

Since $$[xyz] = (\{x_1y_2z_1\} - \{y_1x_2z_1\}, \{x_2y_1z_2\} - \{y_2x_1z_2\})$$

$$= (x_1,x_2)\begin{pmatrix} L(z_1,y_2) & -P(y_2^*,z_1) \\ -P(z_1,y_1^*) & L(z_2,y_1) \end{pmatrix}$$

we obtain

(11.14) $$\mathrm{tr}R(z,y) = \mathrm{tr}\ L(z_1,y_2) + L(z_2,y_1)$$

This leads us to consider

$$\sigma(a,b) := 1/2\ \mathrm{tr}[L(a,b) + L(b,a)] \qquad , a,b \in \mathfrak{A},$$

then (11.14) implies

(11.15) $$\rho(z,y) = 1/2\ \mathrm{tr}[R(z,y) + R(y,z)]$$

$$= \sigma(z_1,y_2) + \sigma(z_2,y_1)$$

This, together with theorem 6.11 and the results in 11.4. show

Theorem 7. a) λ is non degenerate, iff σ is non degenerate.

If Char F = 0, then

b) $\mathcal{L}(\mathcal{G},\mathfrak{A})$ <u>is</u> <u>semi-simple</u>, <u>iff</u> σ <u>is</u> <u>non</u> <u>degenerate</u>.

c) $\mathfrak{A}$ <u>is</u> <u>semi-simple</u>, <u>iff</u> σ <u>is</u> <u>non</u> <u>degenerate</u>.

This result, of course, makes it interesting to have a closer look at the bilinear form σ on $\mathfrak{A}$. $\sigma(a,b) := 1/2 \operatorname{trace}\left[L(a,b) + L(b,a)\right]$. Our first observation is, taking traces in (11.8):

$$\sigma(\{xyz\},u) = \sigma(z,\{yxu\})$$

i.e., σ is associative. Since the left hand side of this equation is symmetric in x and z, we also get

$$\sigma(z,\{yxu\}) = \sigma(x,\{yzu\})$$

(compare with chapter III.)

Now assume σ is non degenerate.

We denote by A^* the adjoint of $A \in \operatorname{End}\mathfrak{A}$ relative to σ, and $(xy^*)z = \sigma(z,y)x$. (Ch. III.)

(11.15) shows

$$L(x,y)^* = L(y,x),$$

hence, $\qquad \sigma(x,y) = \operatorname{tr} L(x,y).$

Since the maps xy^* generate $\operatorname{End}\mathfrak{A}$, we have in particular $\mathrm{Id} = \sum u_i v_i^*$. Then by the associativity of σ,

$$\sigma(x,y) = \operatorname{tr} L(x,y)\,\mathrm{Id} = \sum \operatorname{tr}\{xyu_i\}v_i^* = \sum \sigma(\{xyu_i\},v_i)$$

$$= \sigma(\sum\{u_i v_i x\},y).$$

Since σ is non degenerate, this implies $x = \sum L(u_i,v_i)x$ for all x, consequently $\mathrm{Id} = \sum L(u_i,v_i)$. We proved

<u>Lemma 9.</u> <u>If</u> σ <u>is</u> <u>non</u> <u>degenerate</u>, <u>then</u> $E = (\mathrm{Id}, -\mathrm{Id}) \in \mathcal{G}$.

We next consider the map $T: \operatorname{End}\mathfrak{A} \to \operatorname{End}\mathfrak{A}$, defined by $T(xy^*) = L(x,y)$. Then $L(x,y)^* = L(y,x)$ shows $T(A)^* = T(A^*)$, and in terms of the map T, we get

$$\mathcal{G} = \{(T(A), -T(A)^*),\ \ A \in \mathrm{End}\,\mathfrak{A}\}.$$

We still have to determine $\lambda(H,H')$, $H,H' \in \mathcal{G}$. As usual we compute this expression at first for the generators of $\mathcal{G}$.

$$\begin{aligned}
\lambda(l(x,y), l(u,v)) &= \lambda([x,\tilde{y}],[u,\tilde{v}]) \\
&= \lambda([[x,\tilde{y}],u],\tilde{v}) \quad \text{(associativity of } \lambda) \\
&= \lambda(\{xyu\},\tilde{v}) \quad (11.11\text{iv}) \\
&= 4\sigma(\{xyu\},v) \quad ((11.13),(11.15)).
\end{aligned}$$

If $H = (T(A), -T(A)^*)$, $H' = (T(B), T(B)^*)$, $A = \Sigma xy^*$, $B = \Sigma uv^*$, then this equation shows

$$\lambda(H,H') = 4 \text{ trace } T(A)B.$$

This, together with (11.13),(11.15), gives the Killing form on $\mathcal{L}$. If $X = (T(A), -T(A)^*)\oplus a \oplus \tilde{b}$, $X' = (T(A'), -T(A')^*)\oplus a' \oplus \tilde{b}'$, then

$$\lambda(X,X') = 4 \text{ tr } T(A)A' + 2\sigma(a,b') + 2\sigma(a',b).$$

11.6. Next we wish to determine the derivations of $\mathcal{L}(\mathcal{G},\mathfrak{A})$ and $\mathcal{L}(\gamma(\mathfrak{A}),\mathfrak{A})$. (We assume $E \in \mathcal{G}$.) We can do this simultaneously by setting $\mathfrak{g} = \mathcal{G}$ or $\gamma(\mathfrak{A})$ and looking at derivations

$$D: \mathcal{L}(\mathfrak{g},\mathfrak{A}) \longrightarrow \mathcal{L}(\gamma(\mathfrak{A}),\mathfrak{A}) \quad \text{i.e.,}$$
$$D([X,Y]) = [DX,Y] + [X,DY].$$

We set $D(E) := S \oplus p \oplus \tilde{q}$.

Since $\mathcal{L}(\mathfrak{g},\mathfrak{A})$ is an ideal in $\mathcal{L}(\gamma(\mathfrak{A}),\mathfrak{A})$ the leftmultiplications ad$Y$ of $\mathcal{L}(\gamma(\mathfrak{A}),\mathfrak{A})$ are derivations of $\mathcal{L}(\mathfrak{g},\mathfrak{A})$, by the Jacobi identity. Therefore $D' := D + \mathrm{ad}(p\oplus - \tilde{q})$ is a derivation. We obtain $D'(E) = D(E) + [p\oplus - \tilde{q}, E] = S\oplus p \oplus \tilde{q} - p\oplus - \tilde{q} = S$

Consequently $D'a = D'([E,a]) = [D'(E),a] + [E,D'a]$

$$= S_1 a + (D'a)_1 - (D'a)_2$$

where $S = (S_1,S_2)$ and $D'(X)_1, D'(X)_2$ are the components of $D'(x)$ in $\mathfrak{a}$, $\tilde{\mathfrak{a}}$ respectively. Comparing the corresponding components on both sides of this equation we get

$S_1 a = 0$, $(D'a)_2 = 0$ and then $D'a \in \mathfrak{a}$, similarly

$S_2 b = 0$ $(D'\tilde{b})_1 = 0$, and $D'\tilde{b} \in \tilde{\mathfrak{a}}$.

If we denote by U the restriction of D' to $\mathfrak{a}$ and by V the map defined by $D'\tilde{b} = \widetilde{Vb}$, then these results show

$D'(E) = 0$, $D'(a \oplus \tilde{b}) = Ua \oplus \widetilde{Vb}$. This immediately implies

$$D'l(x,y) = D'([x,\tilde{y}]) = [Ux,\tilde{y}] + [x,\tilde{V}y]$$

(11.16?)

$$= l(Ux,y) + l(x,Vy).$$

And finally $D'[T,a \oplus \tilde{b}] = UT_1 a \oplus \widetilde{VT_2 b} = [D'(T),a \oplus \tilde{b}] + [T,Ua \oplus \widetilde{Vb}]$ implies (comparing terms)

$$D'T = [(U,V),T].$$

But then (11.16) shows $(U,V) \in \mathfrak{r}(\mathfrak{a})$ and we proved

$$D' = \operatorname{ad}H, \quad H \in \mathfrak{r}(\mathfrak{a}).$$

Since $D' = D + \operatorname{ad}(p \oplus - \tilde{q})$ we obtain that a derivation $D: \mathcal{L}(\mathfrak{g},\mathfrak{a}) \to \mathcal{L}(\mathfrak{r}(\mathfrak{a}),\mathfrak{a})$ is of the form $\operatorname{ad}X, X \in \mathcal{L}(\mathfrak{r}(\mathfrak{a}),\mathfrak{a})$. And conversely these maps are derivations of $\mathcal{L}(\mathfrak{g},\mathfrak{a}) \to \mathcal{L}(\mathfrak{g},\mathfrak{a})$.

<u>Theorem 8.</u> $\mathfrak{D}(\mathcal{L}(\mathfrak{g},\mathfrak{a})) = \mathfrak{D}(\mathcal{L}(\mathfrak{r}(\mathfrak{a}),\mathfrak{a})) \cong \mathcal{L}(\mathfrak{r}(\mathfrak{a}),\mathfrak{a})$.

<u>Proof.</u> We just proved the first equality and moreover $\mathfrak{D}(\mathcal{L}) = \operatorname{ad}\mathcal{L}$. By the Jacobi identity the map $X \to \operatorname{ad}X$ is a homomorphism of Lie algebras. We show, that in our case it is an isomorphism. Assume $\operatorname{ad}X = 0$, $X \in \mathcal{L}(\mathfrak{g})$; i.e., $[X,Y] = 0$

for all $Y \in \mathcal{L}(\mathfrak{g})$, in particular $[X,E] = 0$ which implies $X \in \mathfrak{g}$. Since we have also $[X, a\oplus \tilde{b}] = 0$ we end up with $X = 0$. This completes the proof.

If we assume that the bilinear form σ on $\mathfrak{A}$ $\sigma(x,y) = 1/2\ \mathrm{tr}[L(x,y) + L(y,x)]$ is non degenerate, then by Theorem 7, we know that the Killing form λ on $\mathcal{L}(\mathcal{G},\mathfrak{A})$ is non degenerate, and $E \in \mathcal{G}$, (lemma 9). Using theorem 5.3 we get that any derivation of $\mathcal{L}(\mathcal{G},\mathfrak{A})$ is inner, i.e., is of the form adX, $X \in \mathcal{L}(\mathcal{G},\mathfrak{A})$. From theorem 8 we then derive easily (some details left as ecercise).

<u>Corollary 1.</u> <u>If</u> σ <u>is non degenerate on</u> $\mathfrak{A}$ <u>then</u>

$$\mathcal{G}(\mathfrak{A}) = \mathcal{T}(\mathfrak{A}).$$

This corollary in turn has strong implications for the derivations of $\mathfrak{A}$. If $D \in \mathcal{D}(\mathfrak{A})$, then $(D,D) \in \mathcal{T}(\mathfrak{A})$, by (11.5). The above corollary yields

$$(D,D) = \sum_{(u,v)} l(u,v), \text{ componentwise:}$$

$$D = \Sigma L(u,v),\ D = -\Sigma L(v,u). \quad \text{Then}$$

$$D = 1/2(D + D) = \sum_{(u,v)} 1/2\left[L(u,v) - L(v,u)\right] = \Sigma 1/2 D(u,v).$$

By lemma 1, the maps $D(u,v)$ are derivations, this proves

<u>Corollary 2.</u> <u>If</u> σ <u>is non degenerate, then</u> $\mathcal{D}(\mathfrak{A}) = D(\mathfrak{A},\mathfrak{A})$.

<u>Exercise:</u> Use a trace argument to show $\sigma(Dx,y) + \sigma(x,Dy) = 0$ for $D \in \mathcal{D}(\mathfrak{A})$.

11.7. It is quite natural to determine the automorphisms of the

Lts $\mathcal{T} = \mathfrak{A} \oplus \tilde{\mathfrak{A}}$ which are induced by mappings on $\mathfrak{A}$. By this we mean automorphisms of the form $\varphi = (\alpha,\beta)$, $\varphi(a \oplus \tilde{b}) = \alpha a \oplus \widetilde{\beta b}$. We proceed more generally and consider linear maps $\alpha,\beta : \mathfrak{A}' \to \mathfrak{A}$ of Jordan triple systems $\mathfrak{A}, \mathfrak{A}'$. Then by (11.9) the map $\varphi = (\alpha,\beta)$ is a homomorphism of the corresponding Lts $\mathcal{T}' = \mathfrak{A}' \oplus \tilde{\mathfrak{A}}'$ and $\mathcal{T} = \mathfrak{A} \oplus \tilde{\mathfrak{A}}$, iff

$$(\alpha\{x_1 y_2 z_1\} - \alpha\{y_1 x_2 z_1\}, \beta\{x_2 y_1 z_2\} - \beta\{y_2 x_1 z_2\})$$
$$= (\{\alpha x_1 \beta y_2 \alpha z_1\} - \{\alpha y_1 \beta x_2 \alpha z_1\}, \{\beta x_2 \alpha y_1 \beta z_2\} - \{\beta y_2 \alpha x_1 \beta z_2\})$$

This is easily to be seen equivalent to

$$\begin{aligned} &(11.17) \qquad \alpha P'(x,z) = P(\alpha x, \alpha z)\beta \\ &\qquad\qquad\quad \beta P'(x,z) = P(\beta x, \beta z)\alpha \qquad \text{for all } x,z \in \mathfrak{A}' \end{aligned}$$

where P', resp. P denotes the quadratic map on $\mathfrak{A}'$, resp. $\mathfrak{A}$.

(11.7) has an immediate application to isotopes. Let $V \in Gl(\mathfrak{A})$ such that $P(Vx) = VP(x)V$, $\mathfrak{A}_V$ the V-isotope of $\mathfrak{A}$. Taking $\mathfrak{A}' = \mathfrak{A}_V$, $P'(x) = P(x)V$, $\alpha = Id$, $\beta = V$, then (11.7) reduces to $P'(x,z) = P(x,z)V$ and $P(Vx,Vz) = VP(x,z)V$, these relations are fulfilled by definition of $\mathfrak{A}_V$ or assumption on V, respectively. We proved

<u>Theorem 9. If $\mathfrak{A}$ is a Jts and $V \in Gl(\mathfrak{A})$ such that $P(Vx) = VP(x)V$, $\mathfrak{A}_V$ the V-isotope of $\mathfrak{A}$, then the Lie triple systems $\mathfrak{A} \oplus \tilde{\mathfrak{A}}$ and $\mathfrak{A}_V \oplus \tilde{\mathfrak{A}}_V$ are isomorphic.</u>

Since isomorphic Lts have isomorphic standard imbeddings (see 6.3.) we get

<u>Corollary</u>. $\mathcal{L}(\mathfrak{g}(\mathfrak{A}_V), \mathfrak{A}_V) \cong \mathcal{L}(\mathfrak{g}(\mathfrak{A}), \mathfrak{A})$.

Exercise. Apply the corollary to the u-isotope $\mathcal{J}_u$, where u is an invertible element in the unital Jordan algebra $\mathcal{J}$.

The fundamental formula and (11.7) might serve as motivation for the following definitions.[1]

Let $(\mathfrak{A},P)$ be a Jts (over arbitrary Φ). We define $\Gamma(\mathfrak{A}) := \{(U,V) \in \mathcal{E}$; $P(Ux) = UP(x)V$ and $P(Vx) = VP(x)U$ for all $x \in \mathfrak{A}\}$

Exercise: $\Gamma(\mathfrak{A})$ is a multiplicative submonoid of $\mathrm{End}\,\mathfrak{A} \times \mathrm{End}\,\mathfrak{A}^{op}$.

$\Gamma(\mathfrak{A})$ is called the structure monoid of $\mathfrak{A}$. Clearly

i) $(Id,Id) \in \Gamma(\mathfrak{A})$; ii) $(P(x),P(x)) \in \Gamma(\mathfrak{A})$ for all $x \in \mathfrak{A}$ (fundamental formula).

iii) $(U,V) \in \Gamma(\mathfrak{A}), \alpha \in \Phi \rightarrow (\alpha U, \alpha V) \in \Gamma(\mathfrak{A})$.

iv) $(\phi,\phi^{-1}) \in \Gamma(\mathfrak{A})$ for all $\phi \in \mathrm{Aut}\,\mathfrak{A}$.

We apply standard techniques (linearize the defining identities in "direction" of y, apply to z and consider the result as operation on y), to obtain

(11.18) $L(Ux,z)U = UL(x,Vz)$

$L(Vx,z)V = VL(x,Uz)$ for all $(U,V) \in \Gamma(\mathfrak{A}), x,z \in \mathfrak{A}$.

Note: if $1/2 \in \Phi$ then (11.18) can serve as a definition for $\Gamma(\mathfrak{A})$.

Next we define a group $H(\mathfrak{A})$.

$H(\mathfrak{A}) := \{(U,V) \in Gl(\mathfrak{A}) \times Gl(\mathfrak{A})$; $(U,V^{-1}) \in \Gamma(\mathfrak{A})$.

Exercise: $H(\mathfrak{A})$ is a subgroup of $Gl(\mathfrak{A}) \times Gl(\mathfrak{A})$. (The product is componentwise multiplication.).

$H(\mathfrak{A})$ is called the structure group of $\mathfrak{A}$. The following result gives a nice characterization of the structure group.

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[1] In a more systematical approach these definitions should occur elsewhere. But in order to have different concepts come up in a natural way we preferred to introduce the structure group in this context.

Theorem 10. *If $\mathfrak{A}$ is a Jts over Φ, $1/2 \in \Phi$, then* $(U,V) \in Gl(\mathfrak{A}) \times Gl(\mathfrak{A})$ *is in the structure group of $\mathfrak{A}$, iff* (U,V) *is an automorphism of the Lie triple system* $\mathcal{T} = \mathfrak{A} \oplus \tilde{\mathfrak{A}}$.

Proof. $(U,V) \in H(\mathfrak{A}) \leftrightarrow (U,V^{-1}) \in \Gamma(\mathfrak{A}) \leftrightarrow P(Ux) = UP(x)V^{-1}$ and $P(V^{-1}x) = V^{-1}P(x)U \leftrightarrow UP(x) = P(Ux)V$ and $VP(x) = P(Vx)U \Longleftrightarrow$ $UP(x,z) = P(Ux,Uz)V$, $VP(x,z) = P(Vx,Vz)U$ $(1/2 \in \Phi)$. Now compare with (11.17).

Remark: In connection with Lie triple systems the assumption $1/2 \in \Phi$ always comes in since we did not make an attempt to define "quadratic Lie triple systems".

Looking at the standard imbedding $\mathcal{L}(\mathfrak{g},\mathfrak{A})$ of $\mathfrak{A} \oplus \tilde{\mathfrak{A}}$ (and therefore back to the assumption $1/2 \in \Phi$) and using the fact that an automorphism ϕ of $\mathfrak{A} \oplus \tilde{\mathfrak{A}}$ induces an automorphism $\Lambda_\phi: T \oplus a \oplus \tilde{b} \to \phi T \phi^{-1} \oplus \phi(a \oplus \tilde{b})$ (6.3.) We have the following application of theorem 10.

Corollary. *If* $(U,V) \in H(\mathfrak{A})$, *then*

$\Lambda_{(U,V)}: T \oplus a \oplus \tilde{b} \to (U,V)T(U^{-1},V^{-1}) \oplus Ua \oplus \widetilde{Vb}$ *is an automorphism of* $\mathcal{L}(\mathfrak{g},\mathfrak{A})$.

Exercise: For $a \in \mathfrak{A}$ we define

$$\Lambda_a: T \oplus x \oplus \tilde{y} \to T + l(a,y) \oplus (x - T \cdot a - P(a)y) \oplus \tilde{y}$$

prove:

1) $\Lambda_a \in \operatorname{Aut} \mathcal{L}(\mathfrak{g},\mathfrak{A})$ for all $a \in \mathfrak{A}$.
2) $\Lambda_a \Lambda_b = \Lambda_{a+b}$
3) $\Theta\, \Lambda_{(U,V)} = \Lambda_{(V,U)} \Theta$ for $(U,V) \in H(\mathfrak{A})$ (for Θ see (11.11v))
4) $\Lambda_{(U,V)} \Lambda_a = \Lambda_{Ua} \Lambda_{(U,V)}$ for $(U,V) \in H(\mathfrak{A})$.

Note: $\Lambda_a = \exp(\operatorname{ad}(a)) = \operatorname{Id} - \operatorname{ad}(a) + 1/2(\operatorname{ad}(a))^2$; $(\operatorname{ad}(a))^3 = 0$.

(see N. Jacobson: Lie algebras, p. 8).

XII. Examples

12.1. As a first example we shall determine the Koecher-Tits algebra of a special Jordan algebra. We assume $1/2 \in \Phi$. Let $\mathfrak{A}$ be a unital subalgebra of $\mathcal{L}^+$, where $\mathcal{L}$ is an associative algebra. Let e be the unit element of $\mathfrak{A}$. We may assume that $\mathcal{L}$ is generated by $\mathfrak{A}$. Then e is also unit element of $\mathcal{L}$.

The Jordan triple structure of $\mathfrak{A}$ is given by $P(x)y = xyx$, resp. $L(x,y)z = xyz + zyx$. We denote $x \circ y = xy + yx$, $[x,y] = xy - yx$ and get trivially $xy = \frac{1}{2} x \circ y + \frac{1}{2}[x,y]$. Thus if $\Sigma L(x_i,y_i) = 0$ then $\Sigma L(x_i,y_i)e = \Sigma x_i \circ y_i = 0$ and $0 = \Sigma L(x_i,y_i)z = \frac{1}{2} \Sigma [x_i,y_i]z - \frac{1}{2} z \Sigma [x_i,y_i]$. We shall assume moreover

$$[\mathfrak{A},\mathfrak{A}] \cap \text{Center}\,\mathcal{L} = 0\,,$$

and then $\Sigma[x_i y_i] = 0$. Consequently $\Sigma\, x_i y_i = 0$. These considerations show that $\rho(L(x,y)) := xy$ defines a mapping $\rho\colon L(\mathfrak{A},\mathfrak{A}) \to \mathfrak{A} + [\mathfrak{A},\mathfrak{A}]$. It is obvious from the definition

$$\rho(L(x,y))z + z\rho(L(y,x)) = L(x,y)z$$

$$\rho(U)x + x\rho(V) = (U,V)\cdot x\,,\quad (U,V) \in \mathfrak{H}\,.$$

Next we define a map $\omega\colon \mathfrak{L}(\mathfrak{A}) \to \mathcal{L}^-_2$ by

$$\omega((U,V) + x + \tilde{y}) := \begin{pmatrix} \rho(U) & x \\ y & -\rho(V) \end{pmatrix}$$

Theorem 1. ω is an injective homomorphism of Lie algebras.

Proof. Exercise.

Due to the theorem we have

$$(12.1)\qquad \omega(\mathcal{L}(\mathfrak{A})) = \left\{ \begin{pmatrix} u + \Sigma[v_i,w_i] & x \\ y & -u + \Sigma[v_i,w_i] \end{pmatrix} , \begin{matrix} u,\ v_i,\ w_i, \\ x,\ y \in \mathfrak{A} \end{matrix} \right\} .$$

We shall obtain more information on $\omega(\mathcal{L}(\mathfrak{A}))$ for special choices of $\mathfrak{A}$.

Let $\Phi = F$ be a field, $\mathcal{L} = F^{(n,n)}$ the algebra of $n \times n$ matrices over F and $\mathfrak{A} = \mathcal{L}^+$. Assume char $F \nmid n$.

We denote by $\mathfrak{M}(n,F)$ the subalgebra of $\mathcal{L}^-$ of elements of trace zero.

$$\mathfrak{M}(n,F) = \{A \in \mathcal{L} \text{ , trace } A = 0\} \ .$$

Lemma 1. $[\mathfrak{A},\mathfrak{A}] = \mathfrak{M}(n,F)$.

Proof. Let E_{ij} denote the matrix with 1 at the intersection of the i-th row and j-th column. An easy verification shows $[E_{ii},E_{ij}] = E_{ij}$ $(i \neq j)$, and $[E_{i1},E_{1i}] = E_{ii} - E_{11}$ $(i > 1)$, which implies that the linear generators E_{ij} $(i \neq j)$ and $E_{ii} - E_{11}$ $(i > 1)$ of $\mathfrak{M}(n,F)$ are in $[\mathfrak{A},\mathfrak{A}]$. Since trace $[A,B] = 0$ we also have $[\mathfrak{A},\mathfrak{A}] \subset \mathfrak{M}(n,F)$.

Theorem 2. If $\mathfrak{A} = \mathcal{L}^+$, $\mathcal{L} = F^{(n,n)}$ and char $F \nmid n$, then $\mathcal{L}(\mathfrak{A}) \cong \mathfrak{M}(2n,F)$.

Proof. By (12.1) $\omega(\mathcal{L}(\mathfrak{A})) \subset \mathfrak{M}(2n,F)$. If conversely $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{M}(2n,F)$ $A,B,C,D \in \mathcal{L}$, then trace $A = -$ trace D. We set

$$C' = \frac{1}{2}(A + D)\ ,\ B' = \frac{1}{2}(A - D)$$

and get

$$A = B' + C'\ ,\ D = -B' + C'$$

where $C' \in \mathfrak{K}(n,F) = [\mathfrak{A},\mathfrak{A}]$, by Lemma 1. Again (12.1) shows that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} B' + C' & B \\ C & -B' + C' \end{pmatrix}$$

is in $\omega(\mathfrak{L}(\mathfrak{A}))$.

<u>Exercise</u>: Char $F \nmid n \Rightarrow [\mathfrak{A},\mathfrak{A}] \cap$ Center $\mathfrak{L} = 0$.

As another application of Theorem 1 we look at $\mathfrak{L} = F^{(n,n)}$ and $\mathfrak{A} = \{A \in \mathfrak{L},\ A^t = A\}$. If $\mathfrak{T}$ is the set of skewsymmetric matrices then $\mathfrak{T}$ is linearly generated by the matrices $E_{ij} - E_{ji}$ $(1 \leq i < j \leq n)$. Since $[E_{ii}, E_{ij}+E_{ji}] = E_{ij} - E_{ji}$ $i \neq j$, we get $[\mathfrak{A},\mathfrak{A}] = \mathfrak{T}$. Thus $\mathfrak{L} = \mathfrak{A} \oplus [\mathfrak{A},\mathfrak{A}]$. Let

$$S := \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$$

where E is the $n \times n$ unit matrix and

$$\mathfrak{F}(2n,S) = \{A \in \mathfrak{L}\ ,\ S^{-1}A^tS = -A\}$$

the set of S-skewsymmetric matrices. An easy computation shows

$$\mathcal{T}(2n,S) = \left\{ \begin{pmatrix} T & A \\ B & -T^t \end{pmatrix}, \ A,B \in \mathfrak{A}, \ T \in \mathcal{L} \right\}$$

$$= \left\{ \begin{pmatrix} U + V & A \\ B & -U + V \end{pmatrix} \ A,B,U \in \mathfrak{A}, \ V \in [\mathfrak{A},\mathfrak{A}] \right\} .$$

The last equality follows from $\mathcal{L} = \mathfrak{A} \oplus [\mathfrak{A},\mathfrak{A}]$. We proved

<u>Theorem 3.</u> <u>If</u> $\mathcal{L} = F^{(n,n)}$, $\mathfrak{A} = \{A;\ A^t = A\}$ <u>and</u> char $F \nmid n$, <u>then</u> $\mathcal{L}(\mathfrak{A}) \cong \mathcal{T}(2n,S)$.

12.2. Let $\mathfrak{A}$ be the Jts defined by a non degenerate symmetric bilinear form μ .

$$\{xyz\} = \mu(y,z)x - \mu(x,z)y + \mu(x,y)z$$

Let $(xy^*)z = \mu(z,y)x$; then

$$L(x,y) = xy^* - yx^* + (\text{trace } xy^*)\text{Id} .$$

Since $L(y,x) = L(x,y)^*$ we may identify $L(x,y)$ with $(L(x,y),L(y,x))$ resp. $\mathfrak{H}$ with $L(\mathfrak{A},\mathfrak{A})$. Then $\mathfrak{H} = \mathcal{T} \oplus F\cdot\text{Id}$, if char $F \nmid n$, where $\mathcal{T}$ is the set of skewsymmetric linear mappings (relative to μ).

We choose a basis $a_1, \cdots, a_n$ of $\mathfrak{A}$. Let $M = (\mu(a_i,a_j))$ and $\mathfrak{x} = (\xi_1, \cdots, \xi_n)$, if $x = \Sigma\xi_i a_i$. Then $\mu(x,y) = \mathfrak{x}^t M \mathfrak{y}$. Define

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & M \end{pmatrix}$$

and $\mathfrak{x}^* = \mathfrak{x}^t M$; an easy computation shows that

$$U \in \mathcal{J}(n+2,\ T) \quad \text{(for def. see 12.1.)}$$

iff

$$U = \begin{pmatrix} -\alpha & 0 & * \\ 0 & \alpha & - \quad * \\ & - & D \end{pmatrix} \quad \begin{matrix} \alpha \in F, \quad , \xi, \eta \in F_n \text{ and} \\ MD = -D^t M. \end{matrix}$$

The map

$$\sigma\,(uv^* - vu^* + \mu(u,v)\,\mathrm{Id} \oplus x \oplus \tilde{y}) :$$

$$= \begin{pmatrix} -\mu(u,v) & 0 & \eta^* \\ 0 & \mu(u,v) & -\tau^* \\ \tau & -\eta & \tilde{u}v^* - \omega\tilde{u}^* \end{pmatrix}$$

obviously is 1 - 1 and onto $\mathcal{J}(n+2,\ T)$. It is left as an exercise to verify that it is a Lie algebra homomorphism.

Theorem 4. If $\mathfrak{A}$ is the Jts of dimension n defined by a non degenerate bilinearform, then $\mathcal{L}(\mathfrak{A}) \cong \mathcal{J}(n+2,\ T)$.

12.3. For deeper results concerning the determination of the Koecher-Tits algebra for the (-1)-eigenspace $\mathfrak{A}_-$ of an involutorial automorphism of a Jordan algebra $\mathfrak{A}$ we refer to the author's paper in Manuscripta Math. 3 (1970), 115-132. Without proof (and further comments) we mention the following result.

Theorem 5. If $\mathfrak{A}$ is a finite dimensional central simple Jordan algebra of degree s over F, char $F \nmid s$ and $\mathfrak{A}_{1/2}$ the Peirce-1/2-space relative to

<u>an idempotent</u> ($\neq$ <u>unit element</u>) <u>then</u> $\mathcal{L}(\mathfrak{A}_{1/2})$ <u>is isomorphic to the quotient of the structure algebra</u> $[L(\mathfrak{A}), L(\mathfrak{A})] \oplus L(\mathfrak{A})$ <u>of</u> $\mathfrak{A}$ <u>modulo its center.</u>

<u>Remark.</u> If $\mathfrak{A}$ is central simple, finite dimensional exceptional over F of char 0 then $\mathcal{L}(\mathfrak{A})$ is of type E_7 and $\mathcal{L}(\mathfrak{A}_{1/2})$ of type E_6.

XIII. Identities and the radical in Jordan triple systems.

13.1. Let Φ be a commutative ring containing 1 and $(\mathfrak{A},P)$ a Jordan triple system over Φ, that is, $\mathfrak{A}$ is a unital Φ-module, $P:\mathfrak{A} \to \operatorname{End}\mathfrak{A}$ a quadratic map such that

(13.1) $L(x,y)P(x) = P(x)L(y,x) = P(P(x)y,x)$

(13.2) $L(P(x)y,y) = L(x,P(y)x)$

(13.3) $P(P(x)y) = P(x)P(y)P(x)$

hold in $\mathfrak{A}$ and all scalarextensions of $\mathfrak{A}$. We use the following notations: $P(x,y) = P(x+y) - P(x) - P(y)$, $L(x,y)z = \{xyz\} = P(x,z)y$. For the convenience of the reader we restate the linearizations of these formulas and some formulas which we already proved in chapter XI. All of these formulas will be needed in the sequel. The linearizations are

(13.4) $L(x,y)P(x,u) + L(u,y)P(x) = P(x,u)L(y,x) + P(x)L(y,u) =$
$$P(P(x,u)y,x) + P(P(x)y,u) .$$

(13.5) $L(P(x)y,u) + L(P(x)u,y) = L(x,P(y,u)x)$.

(13.6) $L(P(x,u)y,y) = L(x,P(y)u) + L(u,P(y)x)$.

(13.7) $P(P(x)y,P(x,u)y) = P(x)P(y)P(x,u) + P(x,u)P(y)P(x)$.

(13.8) $P(P(x,u)y) + P(P(x)y,P(u)y) = P(x)P(y)P(u) + P(u)P(y)P(x) +$
$$P(x,u)P(y)P(x,u).$$

And the formulas from chapter XI are

(13.9) $L(z,y)P(x) + P(P(x)y,z) = P(x,z)L(y,x)$

(13.10) $L(z,y)P(x) + P(x)L(y,z) = P(\{zyx\},x)$

(13.11) $[L(x,y),L(u,v)] = L(\{xyu\},v) - L(u,\{yxv\})$
$$= L(x,\{vuy\}) - L(\{uvx\},y).$$

(The last equality follows from the fact that the left hand side of (13.11) is skewsymmetric in the pairs $(x,y),(u,v)$.) We shall need some more identities. We apply (13.6) to a and consider the result as operation on u to obtain

(13.12) $L(a,y)L(x,y) - P(x,a)P(y) = L(a,P(y)x)$.

The identity which is in a certain sense dual to (13.12) is obtained from (13.10) by the same technique; after appropriate change of notation we get

(13.13) $L(y,x)L(y,a) - P(y)P(x,a) = L(P(y)x,a)$.

Since (13.10) is linear in two variables we also derive

(13.14) $P(x)L(y,z) + P(z,P(x)y) = L(x,y)P(x,z)$,

which is the "dual" of (13.9).

If F is a formula in L's and P's, we call the formula $F^{\#}$ which is obtained from F by replacing $L(x,y)$ by $L(y,x)$ and reversing the order of the L's and P's the <u>dual</u> of F. Inspecting the formulas (13.1) - (13.14) we observe that to any of these formulas its dual is among (13.1) - (13.14). As a consequence of this observation we have the following result:

<u>Duality Principle</u>. <u>If F is any identity derived from</u> (13.1) - (13.14) <u>then its dual</u> $F^{\#}$ <u>is also an identity</u>.

<u>Proof.</u> We define $l(x,y) := (L(x,y),L(y,x)), p(x) = (P(x),P(x))$. Then it is easily checked that in the associative algebra $\mathcal{E} = \operatorname{End}\mathfrak{A} \oplus (\operatorname{End}\mathfrak{A})^{op}$ the following set S of identities holds:

$l(x,y)p(x) = p(x)l(y,x) = p(P(x)y,x)$

$l(P(x)y,y) = l(x,P(y)x)$

$p(x)p(y)p(x) = p(P(x)y)$

$l(z,y)p(x) + p(P(x)y,z) = p(x,z)l(y,x)$

$$l(z,y)p(x) + p(x)l(y,z) = p(\{zyx\},x)$$

$$[l(x,y),l(u,v)] = l(\{xyu\},v) - l(u,\{yxv\}) = l(x,\{vuy\}) - l(\{uvx\},y)$$

$$l(a,y)l(x,y) - p(x,a)p(y) = l(a,P(y)x)$$

$$l(y,x)l(y,a) - p(y)p(x,a) = l(P(y)x,a)$$

$$p(x)l(y,z) + p(P(x)y,z) = l(x,y)p(x,z)$$

for all $x,y,a,u,v,z \in \mathfrak{A}$.

($p(x,y)$ is the linearization of $p(x)$).

In $\mathfrak{L}$ we have a canonical involution $^{\#}$: $(A,B) \mapsto (B,A)$. Obviously $l(x,y)^{\#} = l(y,x)$ and $p(x)^{\#} = p(x)$. Furthermore it is immediately seen that the set S is invariant under this involution. Consequently, to any identity derived from elements in S the dual formula is an identity, too. Taking the projection $\tau : \mathfrak{L} \to \mathrm{End}\,\mathfrak{A}$ onto the first component we get the desired result.

13.2. We recall that for fixed $u \in \mathfrak{A}$ we get a quadratic Jordan algebra $\mathfrak{A}_u = (\mathfrak{A}, P_u, {}^{(2,u)})$ by setting $P_u(x) = P(x)P(u)$ and $x^{(2,u)} = P(x)u$ (see 10.2.). Let $\hat{\mathfrak{A}}_u$ be the unital Jordan algebra obtained from $\mathfrak{A}_u$ by adjoining a unit element.

Since the notion of quasi-invertibility has proved very useful in the theory of associative algebras (chapter II) it is reasonable to try the same concept in Jordan algebras or Jordan triple systems. An element x in a Jordan algebra $\mathfrak{J}$ is called <u>quasi invertible</u>, with <u>quasi inverse</u> y, if $1 - x$ is invertible in the unital Jordan algebra $\hat{\mathfrak{J}}$ and has inverse $1 + y$. Let $\mathfrak{A}$ be a Jordan triple system over Φ. We shall study the notion of quasi invertibility in the homotopes $\mathfrak{A}_u$. Our first result is a translation of the inverse theorem (theorem 9.2) in Jordan algebras. We recall:

If $(\mathfrak{J}, U, e)$ is a unital quadratic Jordan algebra, then $a \in \mathfrak{J}$ is invertible with inverse b, if one of the following (equivalent) conditions hold:

i) $U(a)b = a$ and $U(a)b^2 = e$

ii) $U(a)$ is invertible

iii) $U(a)$ is surjective

iv) e is in the image of $U(a)$.

In either case b is uniquely determined by $b = U(a)^{-1}a$.

We define

$$B(x,y) := Id - L(x,y) + P(x)P(y)$$

(see (10.4)) and are ready to prove

Theorem 1. Let $\mathfrak{A}$ be a Jordan triple system, $x,u \in \mathfrak{A}$. The following statements are equivalent:

i) x is quasi invertible in $\mathfrak{A}_u$ (with quasi inverse y),

ii) there exists $y \in \mathfrak{A}$ such that $B(x,u)y = x - P(x)u$ and

$$B(x,u)P(y)u = P(x)u,$$

iii) $B(x,u)$ is invertible,

iv) $B(x,u)$ is surjective,

v) $2x - P(x)u$ is in the image of $B(x,u)$.

In either case y is uniquely determined by

$$y = B(x,u)^{-1}(x - P(x)u). \tag{13.15}$$

Proof. Using the definition of $\mathfrak{A}_u$ and the formula for $\hat{P}_u$ (see thm. 9.1.) we firstly observe

$$\hat{P}_u(1 - x)1 = 1 - 2x + P(x)u$$
$$\hat{P}_u(1 - x)z = B(x,u)z \quad \text{for all } x,u,z \in \mathfrak{A}.$$

In particular, we note

$$B(x,u) = \hat{P}_u(1 - x)|_{\mathfrak{A}} .$$

Using these relations we translate the inverse conditions:

$\hat{P}_u(1 - x)(1 + y) = 1 - x$ is equivalent to

$1 - 2x + P(x)u + B(x,u)y = 1 - x$, or

(a) $$B(x,u)y = x - P(x)u.$$

$\hat{P}_u(1 - x)(1 + y)^{(2,u)} = 1$ is equivalent to

$1 - 2x + P(x)u + 2B(x,u)y + B(x,u)P(y)u = 1.$

(a) together with this equation is equivalent to (a) and $B(x,u)P(y)u = P(x)u$. This shows i) $\leftrightarrow$ ii). Since $\hat{P}_u(1 - x)$ is invertible and maps $\mathfrak{A}$ into $\mathfrak{A}$ (and also its inverse maps $\mathfrak{A}$ into $\mathfrak{A}$) its restriction $B(x,u)$ to $\mathfrak{A}$ is invertible, then $B(x,u)$ is surjective and in particular $2x - P(x)u$ is in the image of $B(x,u)$. But this last condition is immediately seen to be equivalent to $1 \in$ image $\hat{P}_u(1 - x)$, which in turn give the invertibility of $1 - x$ in $\hat{\mathfrak{A}}_u$. The last statement in our theorem can be read off from iii) and ii). ∎

<u>Remark.</u> In a unital Jordan algebra $(\mathfrak{J}, U, e)$ the conditions $U(a)b = a$ and $U(a)b^2 = e$ imply

$$a \circ b = U(a,e)b = U(a,U(a)b^2)b = V(a,b^2)U(a)b$$
$$= V(a,b^2)a = 2U(a)b^2 = 2e. \quad (V(x,y)z = U(x,z)y)$$

The translation of this to the situation considered above yields: If x is quasi invertible in $\mathfrak{A}_u$ with quasi inverse y, then

$$2(x - y) = \{xuy\} .$$

If x is quasi invertible in $\mathfrak{A}_u$ with quasi inverse y then we set

$$y =: q(x,u)$$

Using this notation, (13.15) becomes

(13.15') $\qquad q(x,u) = B(x,u)^{-1}(x - P(x)u).$

We shall proceed along the lines presented in chapter II, although the proof won't be that easy.

Theorem 2. (Symmetry Principle)

a) $B(x,u)$ *is invertible, iff* $B(u,x)$ *is invertible,*

b) $q(x,u)$ *exists, iff* $q(u,x)$ *exists.*

In this case $q(x,u) = x + P(x)q(u,x)$.

Note: In particular $q(x,u) \in \Phi x + P(x)\mathfrak{A}$.

Proof. The fundamental formula

$$\hat{P}_u(\hat{P}_u(1 - x)(1 - z)) = \hat{P}_u(1 - x)\hat{P}_u(1 - z)\hat{P}_u(1 - x)$$

restricted to $\mathfrak{A}$ shows

(13.16) $B(x,u)B(z,u)B(x,u) = B(2x - P(x)u + B(x,u)z,u).$

Since the fundamental formula in $\hat{\mathfrak{A}}_u$ can be derived by using (3.1) - (3.14) the Duality Principle applies to give

(13.16') $B(u,x)B(u,z)B(u,x) = B(u,2x - P(x)u + B(x,u)z).$

If $B(x,u)$ is invertible then there exists $z' \in \mathfrak{A}$ such that $2x - P(x)u + B(x,u)z' = 0$ and since $B(u,0) = Id$ we get $B(u,x)B(u,z')B(u,x) = Id$ from (13.16'), which shows that $B(u,x)$ is invertible. This proves part a). Part b) is an application of a) and theorem 1. For the last statement it is sufficient to show

(*) $\qquad B(x,u)q(x,u) = B(x,u)x + B(x,u)P(x)q(u,x),$

since $B(x,u)$ is invertible. Using (13.1) and the defintion of

$B(x,u)$ we get $B(x,u)P(x) = P(x) - L(x,u)P(x) + P(x)P(u)P(x) = P(x)B(u,x)$, and using this and (13.15'), (*) is equivalent to

$$x - P(x)u = B(x,u)x + P(x)(u - P(u)x).$$

But this is a trivial identity, by definition of $B(x,u)$. ($L(x,u)x = 2P(x)u$).

We note particular cases of (13.16) and (13.16') $(z = 0)$:

(13.17) $\quad B(x,u)^2 = B(2x - P(x)u,u)$

(13.17') $\quad B(u,x)^2 = B(u,2x - P(x)u)$

We recall, the structure monoid $\Gamma(\mathfrak{A})$ of $\mathfrak{A}$ is defined by

$\Gamma(\mathfrak{A}) = \{(U,V) \in \operatorname{End}\mathfrak{A} \times (\operatorname{End}\mathfrak{A})^{op};\ P(Ux) = UP(x)V, P(Vx) = VP(x)U$ for all $x \in \mathfrak{A}\}$.

(see 11.7.) If $(U,V) \in \Gamma(\mathfrak{A})$ then

$L(Ux,z)U = UL(x,Vz)$; $L(Vx,z)V = VL(x,Uz)$ for all $x,z \in \mathfrak{A}$.

And furthermore

$UB(x,Vu) = U - UL(x,Vu) + UP(x)P(Vu)$

$\qquad = U - L(Ux,u) + P(Ux)P(u)U = B(Ux,u)U.$

By a similar computation we get for $(U,V) \in \Gamma(\mathfrak{A})$

(13.18) $\quad UB(x,Vu) = B(Ux,u)U; \quad VB(x,Uu) = B(Vx,u)V.$

<u>Theorem 3.</u> <u>(Shifting Principle)</u>

<u>Let</u> $(U,V) \in \Gamma(\mathfrak{A})$; <u>then</u> $q(x,Vu)$ <u>exists</u>, <u>iff</u> $q(Ux,u)$ <u>exists</u>.

<u>In</u> <u>this</u> <u>case</u> $Uq(x,Vu) = q(Ux,u)$.

<u>Proof.</u> Assume $q(x,Vu)$ exists. Then by theorem 1

$Ux - UP(x)Vu = UB(x,Vu)q(x,Vu)$ and

$UB(x,Vu)P(q(x,Vu))Vu = UP(x)Vu.$

Using the definition of $\Gamma(\mathfrak{A})$ and (13.18) we then get

$Ux - P(Ux)u = B(Ux,u)Uq(x,Vu)$ and

$B(Ux,u)P(Uq(x,Vu))u = P(Ux)u$. This already shows that $q(Ux,u)$ exists and equals $Uq(x,Vu)$, by theorem 1. Conversely, assume $q(Ux,u)$ exists, then by the Symmetry Principle $q(u,Ux)$ exists and then $q(Vu,x)$ exists (by what we just proved) and again by symmetry $q(x,Vu)$ exists. We already know that $(P(x),P(x)) \in \Gamma(\mathfrak{A})$ and shall show below that $(B(x,y),B(y,x)) \in \Gamma(\mathfrak{A})$, therefore we have

<u>Corollary.</u> a) $q(x,P(y)u)$ <u>exists</u>, <u>iff</u> $q(P(y)x,u)$ <u>exists</u>,

b) $q(x,B(a,b)u)$ <u>exists</u>, <u>iff</u> $q(B(b,a)x,u)$ <u>exists</u>.

<u>for all</u> $x,y,a,b,u \in \mathfrak{A}$.

In order to prove $(B(x,y),B(y,x)) \in \Gamma$, we cannot avoid some computations, since we don't have (not yet) nice formula proving principles at hand.

To show that $(B(x,y),B(y,x)) \in \Gamma(\mathfrak{A})$ we have to prove

$$P(B(x,y)z) = B(x,y)P(z)B(y,x). \tag{13.19}$$

<u>Proof:</u> Using the definition of $B(x,y)$ (resp. $B(y,x)$) we compute

$$\begin{aligned} B(x,y)P(z)B(y,x) &= [P(z) - L(x,y)P(z) + P(x)P(y)P(z)]B(y,x) \\ &= P(z) - L(x,y)P(z) + P(x)P(y)P(z) - P(z)L(y,x) + L(x,y)P(z)L(y,x) \\ &\quad - P(x)P(y)P(z)L(y,x) + P(z)P(y)P(x) - L(x,y)P(z)P(y)P(x) + \\ &\quad P(x)P(y)P(z)P(y)P(x) \end{aligned}$$

and

$$\begin{aligned} P(B(x,y)z) = P(z) &- P(L(x,y)z,z) + P(P(x)P(y)z,z) - \\ &- P(P(x)P(y)z,L(x,y)z) + P(L(x,y)z) + P(P(x)P(y)z) \end{aligned}$$

Using the fundamental formula and (13.10) we see that we are done, if we can prove

(13.20) $P(x)P(y)P(z) + P(z)P(y)P(x) + L(x,y)P(z)L(y,x) =$

$$P(\{xyz\}) + P(P(x)P(y)z,z)$$

and

(13.21) $P(P(x)P(y)z,L(x,y)z) = P(x)P(y)P(z)L(y,x) + L(x,y)P(z)P(y)P(x).$

Proof of (13.20):

Comparing (13.20) with (13.8) we observe that we have to prove

$$L(x,y)P(z)L(y,x) = P(x,z)P(y)P(x,z) + P(P(x)P(y)z,z) - P(P(x)y,P(z)y).$$

Using (13.9),(13.13) and (13.2) we get

$$\begin{aligned} L(x,y)P(z)L(y,x) &= P(z,x)L(y,z)L(y,x) - P(P(z)y,x)L(y,x) \\ &\overset{(9)}{=} P(z,x)P(y)P(z,x) + P(z,x)L(P(y)z,x) - P(P(z)y,x)L(y,x) \\ &\overset{(13)}{=} P(z,x)P(y)P(z,x) + P(P(x)P(y)z,z) + L(z,P(y)z)P(x) \\ &\overset{(9)}{-} P(P(x)y,P(z)y) - L(P(z)y,y)P(x) \\ &\underset{(3)}{=} P(z,x)P(y)P(z,x) + P(P(x)P(y)z,z) - P(P(x)y,P(z)y). \end{aligned}$$

which is the desired formula.

Proof of (13.21):

We have to use (13.9) and its dual (13.14), (13.1) and (13.2).

We start with a linearized form of the fundamental formula.

(*) $P(P(x)y,P(x,z)y) = P(x)P(y)P(x,z) + P(x,z)P(y)P(x).$

Linearizing again yields

$$P(P(x)u,P(x,z)y) + P(P(x)y,P(x,z)u) = P(x)P(y,u)P(x,z) + P(x,z)P(y,u)P(x).$$

In this formula we replace u by $P(y)z$, together with (13.1) and (13.2) we obtain

$$\begin{aligned} P(P(x)P(y)z,L(x,y)z) &= P(x)P(y)L(z,y)P(x,z) + P(x,z)L(y,z)P(y)P(x) \\ &\quad - P(P(x)y,\{xy(P(z)y)\}) \Rightarrow :A \end{aligned}$$

For the first term of A we use (13.9) and for the second term we use (13.14) to transform A into

$$A = P(x)P(y)P(z)L(y,x) + P(x)P(y)P(P(z)y,x) + L(x,y)P(z)P(y)P(x)$$
$$+ P(P(z)y,x)P(y)P(x) - P(P(x)y,\{xy(P(z)y)\})$$
$$= P(x)P(y)P(z)L(y,x) + L(x,y)P(z)P(y)P(x),$$

since all other terms have sum zero; this can be seen by replacing z by $P(z)y$ in (*).

This completes the proof of (13.19). and the Corollary on p. 141

For the proof of the next theorem we need two more identities. We already proved (see (10.4)).

(13.22) $P(x - P(x)u) = P(x)B(u,x) = B(x,u)P(x)$

The other formula is

(13.23) $P(B(x,u)z, P(x)u - x) = B(x,u)\left[L(z,u)P(x) - P(x,z)\right]$
$$= \left[P(x)L(u,z) - P(x,z)\right]B(u,x).$$

<u>Proof.</u> Using the definition of $B(x,u)$ and expanding we obtain

$$P(B(x,u)z,P(x)u - x) = P(P(x)u,z) - P(P(x)u,\{xuz\}) + P(P(x)u,P(x)P(u)z)$$
$$- P(x,z) + P(\{xuz\},x) - P(P(x)P(u)z,x) = :A$$

For the first term of A we use (13.14), for the second and third term a linearized form of the fundamental formula, for the fifth term (13.10) and for the last term (13.1) to transform A into

$$A = L(x,u)P(x,z) - P(x)L(u,z) - P(x)P(u)P(x,z) - P(x,z)P(u)P(x)$$
$$+ P(x)P(u,P(u)z)P(x) - P(x,z) + L(z,u)P(x) + P(x)L(u,z)$$
$$- L(x,P(u)z)P(x)$$
$$= - B(x,u)P(x,z) + \left[L(z,u) - P(x,z)P(u) + P(x)P(u)L(z,u) - L(x,P(u)z)\right]P(x)$$
$$= B(x,u)\left[L(z,u)P(x) - P(x,z)\right] \text{, by (13.12).}$$

The other equality follows from the Duality Principle.

Theorem 4. (Addition formula)

If x is quasi invertible in $\mathfrak{A}_u$, then

a) $B(x,u)B(q(x,u),z) = B(x,u + z)$

b) $B(z,q(x,u))B(u,x) = B(z + u,x)$

for all $z \in \mathfrak{A}$.

Proof. Let x be quasi invertible in $\mathfrak{A}_u$ and $y = q(x,u)$. Thus $B(x,u)y = x - P(x)u$, by theorem 1. Then $B(x,u)P(y)B(u,x) = P(B(x,u)y) = P(x - P(x)u) = P(x)B(u,x) = B(x,u)P(x)$ by (13.19) and (13.22). According to theorem 1 and the Symmetry Principle, $B(u,x)$ is invertible, thus

(13.24) $B(x,u)P(y) = P(x)$ and $P(y)B(u,x) = P(x)$.

$$\begin{aligned}\text{Now } B(x,u)B(y,z) &= B(x,u)\left[Id - L(y,z) + P(y)P(z)\right]\\ &= B(x,u) - B(x,u)L(y,z) + P(x)P(z), \text{ by (13.24).}\end{aligned}$$

While $B(x,u + z) = Id - L(x,u) - L(x,z) + P(x)P(u) + P(x)P(u,z) + P(x)P(z)$.

This shows, that all we have to prove is

$B(x,u)L(y,z) = L(x,z) - P(x)P(u,z)$.

From a linearization of (13.19) and (13.23) we derive

$$\begin{aligned}B(x,u)P(y,a)B(u,x) &= P(B(x,u)y,B(x,u)a)\\ &= P(x - P(x)u,B(x,u)a)\\ &= \left[P(x,a) - P(x)L(u,a)\right]B(u,x)\end{aligned}$$

And again, since $B(u,x)$ is invertible, we get

$B(x,u)P(y,a) = P(x,a) - P(x)L(u,a)$.

Applying this formula to z and taking the result as operation on a we get the desired result. Since the proof we gave for the identity a) does not allow us to apply the Duality Principle to conclude b), we must give a direct proof of b). This is left as an exercise (see below).

Exercise:

1) $B(x,u)[L(z,P(u)x) - L(z,u)] = [L(x,P(u)z) - L(z,u)]B(x,u)$

for all $x,z,u \in \mathfrak{A}$. (Hint: Multiply the right hand side identity of (13.23) by $P(u)$, use (13.22), (13.1), replace $P(x)P(u) = B(x,u) + L(x,u) - \mathrm{Id}$ and use (13.12).)

2) If $y = q(x,u)$, then

$$L(B(u,x)z,y) = - L(u,P(x)z) + L(z,x) \quad \text{(Hint: use 1))}$$

3) Prove part b) of theorem 4.

Corollary 1. If x is quasi invertible in $\mathfrak{A}_u$, $z \in \mathfrak{A}$, then $q(x,u + z)$ exists, iff $q(q(x,u),z)$ exists. In either case $q(x,u + z) = q(q(x,u),z)$.

Proof. The first part of the corollary is an immediate consequence of the addition formula in connection with theorem 1. Using (13.15) we derive

$$\begin{aligned} q(x,u + z) &= B(x,u + z)^{-1}[x - P(x)(u + z)] \\ &= B(q(x,u),z)^{-1}B(x,u)^{-1}[x - P(x)(u + z)], \text{ by thm. 4} \\ &= B(q(x,u),z)^{-1}[q(x,u) - B(x,u)^{-1}P(x)z], \text{ by (13.5)} \\ &= B(q(x,u),z)^{-1}[q(x,u) - P(q(x,u))z], \text{ by (13.24)} \\ &= q(q(x,u),z). \end{aligned}$$

Corollary 2. The set $R \subset \mathfrak{A} \times \mathfrak{A}$, defined by $R: = \{(x,y) \in \mathfrak{A} \times \mathfrak{A}, \quad x = q(y,u)$ for some $u \in \mathfrak{A}$ is an equivalence relation.

Proof. $(x,x) \in R$ since $x = q(x,0)$. If $(y,x) \in R$, say $y = q(x,u)$ then $x = q(x,0) = q(x,u - u) = q(y, - u)$, by corollary 1. Thus $(x,y) \in R$. In order to show the transitivity of R we use corollary 1 again.

Assume $(x,y),(y,z) \in R$, by definition of $R, x = q(y,u)$ and $y = q(z,w)$ for some $u,w \in \mathfrak{A}$. Then

$x = q(y,u) = q(q(z,w),u) = q(z,w+u)$, in particular $(x,z) \in R$.

13.3 Of course, results like the foregoing should have some strong applications. That this is the case will be seen very soon when we introduce the radical (Jacobson radical) for Jordan triple systems. Let again $\mathfrak{A}$ be a Jordan triple system over Φ.

We define $x \in \mathfrak{A}$ to be properly quasi invertible, (p.q.i.) if x is quasi invertible in all Jordan algebras $\mathfrak{A}_u$, $u \in \mathfrak{A}$. In other words:

x is p.q.i. $\leftrightarrow$ $q(x,u)$ exists for all $u \in \mathfrak{A}$ $\leftrightarrow$ $B(x,u)$ is invertible for all $u \in \mathfrak{A}$.

We define

$$\text{Rad}\,\mathfrak{A} := \{ x \in \mathfrak{A} \text{ , x is p.q.i.} \}$$

<u>Theorem 5:</u> <u>Rad $\mathfrak{A}$ is an ideal of $\mathfrak{A}$.</u>

<u>proof</u>: We have to show:

a) Rad $\mathfrak{A}$ is a submodule

b) $P(\text{Rad}\,\mathfrak{A})\mathfrak{A} \subset \text{Rad}\,\mathfrak{A}$

c) $\{\mathfrak{A}\mathfrak{A}(\text{Rad}\,\mathfrak{A})\} \subset \text{Rad}\,\mathfrak{A}$

d) $P(\mathfrak{A})\,\text{Rad}\,\mathfrak{A} \subset \text{Rad}\,\mathfrak{A}$

(see 10.4.)

proof of a): Clearly $B(\alpha x,u) = B(x,\alpha u)$, $\alpha \in \Phi$ which already shows that $\alpha x \in \text{Rad}\,\mathfrak{A}$ if $x \in \text{Rad}\,\mathfrak{A}$ and $\alpha \in \Phi$. Let $u, z \in \text{Rad}\,\mathfrak{A}$, then by the Symmetry Principle and corollary 1 of thm. 4 we conclude $u + z \in \text{Rad}\,\mathfrak{A}$. In order to prove the other properties we firstly observe, that the Shifting Principle has as an immediate consequence the following result

Lemma 1. If $(U,V) \in \Gamma(\mathfrak{A})$, then $U(\mathrm{Rad}\,\mathfrak{A}) \subset \mathrm{Rad}\,\mathfrak{A}$.

Since $(P(z),P(z))$ and $(B(a,b),B(b,a)) \in \Gamma(\mathfrak{A})$ for all $z,a,b \in \mathfrak{A}$ (fundamental formula and (13.19)) we have $P(\mathfrak{A})\,\mathrm{Rad}\,\mathfrak{A} \subset \mathrm{Rad}\,\mathfrak{A}$ (this is d)) and $B(a,b)x = x - \{abx\} + P(a)P(b)x \in \mathrm{Rad}\,\mathfrak{A}$ if $x \in \mathrm{Rad}\,\mathfrak{A}$. By what we already proved we conclude

$$\{\mathfrak{A}\mathfrak{A}\ \mathrm{Rad}\,\mathfrak{A}\} \subset \mathrm{Rad}\,\mathfrak{A}\text{, this is c).}$$

The addition formula allows us also to conclude that for[1)*] $x \in \mathrm{Rad}\,\mathfrak{A}$ the quasi inverse $q(x,u)$ in any $\mathfrak{A}_u$ is in $\mathrm{Rad}\,\mathfrak{A}$. Now let $x \in \mathrm{Rad}\,\mathfrak{A}$ and $y \in \mathfrak{A}$. Then x is q.i. in $\mathfrak{A}_y$ and the quasi-inverse $q(x,y)$ is in $\mathrm{Rad}\,\mathfrak{A}$. By (13.15) $B(x,y)q(x,y) = x - P(x)y$ and consequently $P(x)y \in \mathrm{Rad}\,\mathfrak{A}$ since $B(x,y)\,\mathrm{Rad}\,\mathfrak{A} \subset \mathrm{Rad}\,\mathfrak{A}$. This completes the proof. ∎

$\mathrm{Rad}\,\mathfrak{A}$ is called the Jacobson radical of $\mathfrak{A}$.

We shall show that $\mathrm{Rad}\,\mathfrak{A}$ has all the properties one generally requires from a nice radical.

Theorem 6. a) $\mathrm{Rad}\,(\mathfrak{A}/_{\mathrm{Rad}\,\mathfrak{A}}) = 0$

b) If $\mathfrak{L}$ is an ideal of $\mathfrak{A}$ such that $\mathrm{Rad}\,(\mathfrak{A}/_{\mathfrak{L}}) = 0$, then $\mathrm{Rad}\,\mathfrak{A} \subset \mathfrak{L}$.

Proof. a) Let $\bar{\mathfrak{A}} = \mathfrak{A}/_{\mathrm{Rad}\,\mathfrak{A}}$, $\bar{x} \in \mathrm{Rad}\,\bar{\mathfrak{A}}$ and $u \in \mathfrak{A}$. Then there exists $\bar{v} \in \bar{\mathfrak{A}}$ such that $B(\bar{x},\bar{u})\bar{v} + 2\bar{x} - P(\bar{x})\bar{u} = 0$, by thm. 1,V. Then $B(x,u)v + 2x - P(x)u =: a \in \mathrm{Rad}\,\mathfrak{A}$ and $B(a,z)$ is invertible for all $z \in \mathfrak{A}$. By (13.16), $B(a,u) = B(x,u)B(v,u)B(x,u)$. Thus $B(x,u)$ is invertible and $x \in \mathrm{Rad}\,\mathfrak{A}$.

b) Let $\phi: \mathfrak{A} \to \mathfrak{A}'$ be a surjective homomorphism of Jts's, then by thm. 1, ii) $\phi(\mathrm{Rad}\,\mathfrak{A}) \subset \mathrm{Rad}\,\mathfrak{A}'$. In particular

*1) See also the remark following the Symmetry Principle.

we have $\pi(\mathrm{Rad}\,\mathfrak{A}) \subset \mathrm{Rad}\,(\mathfrak{A}/\mathfrak{L}) = 0$ for the natural map $\pi : \mathfrak{A} \to \mathfrak{A}/\mathfrak{L}$. Consequently $\pi(\mathrm{Rad}\,\mathfrak{A}) = 0$, that is $\mathrm{Rad}\,\mathfrak{A} \subset \mathfrak{L}$.

Theorem 7. If $\mathfrak{L}$ is an ideal of $\mathfrak{A}$, then

$$\mathrm{Rad}\,\mathfrak{L} = \mathfrak{L} \cap \mathrm{Rad}\,\mathfrak{A}.$$

Proof. If $x \in \mathfrak{L} \cap \mathrm{Rad}\,\mathfrak{A}$, then theorem 1 ii) implies $x \in \mathrm{Rad}\,\mathfrak{L}$ since the quasi inverse of an element of $\mathfrak{L}$ is in $\mathfrak{L}$. For the converse we shall firstly establish

(13.25) $B(x,y)B(-x,y) = B(x,P(y)x) = B(P(x)y,y)$ for all $x,y \in \mathfrak{A}$.

Proof: Expand the left hand side and use the fundamental formula, (13.1), (13.2) and (13.12). Now, if $x \in \mathrm{Rad}\,\mathfrak{L}$ then again by theorem 1 ii), x is q.i. in $\mathfrak{A}_u$ for all $u \in \mathfrak{L}$, equivalently $B(x,u)$ is invertible for all $u \in \mathfrak{L}$. If $y \in \mathfrak{A}$, then $P(y)x \in \mathfrak{L}$ and $B(x,P(y)x)$ is invertible, then $B(x,y)$ is invertible, by (13.25). Thus $x \in \mathfrak{L} \cap \mathrm{Rad}\,\mathfrak{A}$.

13.4. The Jacobson radical of a Jordan algebra $\mathfrak{J}$ is defined as $\mathrm{Rad}\,\mathfrak{J}$, where $\mathfrak{J}$ is considered as a Jordan triple system. The connection of the radical of $\mathfrak{A}$ and the radicals of the Jordan algebras $\mathfrak{A}_u$ is given in the next result. We recall, that $\mathfrak{A}_u$ viewed as a Jts is the $P(u)$ - homotope of $\mathfrak{A}$ (see Chapter 10). More generally, if $P(Vx) = VP(x)V$ for all $x \in \mathfrak{A}$, then $P_V(y) = P(y)V$ defines the V-homotope $\mathfrak{A}_V = (\mathfrak{A}, P_V)$ of $\mathfrak{A}$. $\mathfrak{A}_V$ is a Jts. Since $(P_V)_u(x) = P(x)VP(u)V = P(x)P(Vu) = P_{Vu}(x)$ we have

$$(\mathfrak{A}_V)_u = \mathfrak{A}_{Vu}.$$

Theorem 8. a) $\mathrm{Rad}\,\mathfrak{A}_V = \{x \in \mathfrak{A},\ Vx \in \mathrm{Rad}\,\mathfrak{A}\}$

$$\mathrm{Rad}\,\mathfrak{A}_u = \{x \in \mathfrak{A},\ P(u)x \in \mathrm{Rad}\,\mathfrak{A}\}$$

b) $\operatorname{Rad} \mathfrak{A} = \bigcap \{\operatorname{Rad} \mathfrak{A}_u;\ u \in \mathfrak{A}\}$.

<u>Proof.</u> a): $x \in \operatorname{Rad} \mathfrak{A}_V \leftrightarrow x$ q.i. in $(\mathfrak{A}_V)_u = \mathfrak{A}_{V_u}$ for all $u \in \mathfrak{A} \leftrightarrow$ $q(x, V_u)$ exists for all $u \in \mathfrak{A} \leftrightarrow$ (Shifting) $q(Vx, u)$ exists for all $u \in \mathfrak{A} \leftrightarrow Vx \in \operatorname{Rad} \mathfrak{A}$.

b): Clarly $\operatorname{Rad} \mathfrak{A} \subset \bigcap \{\operatorname{Rad} \mathfrak{A}_u, u \in \mathfrak{A}\}$, since $P(u) \operatorname{Rad} \mathfrak{A} \subset \operatorname{Rad} \mathfrak{A}$ for all $u \in \mathfrak{A}$. Let $x \in \operatorname{Rad} \mathfrak{A}_u$ for all $u \in \mathfrak{A}$, then $P(u)x \in \operatorname{Rad} \mathfrak{A}$ and then in particular $B(P(u)x, x) = B(u,x)B(-u,x)$ is invertible. Then by symmetry $B(x,u)$ is invertible, thus $x \in \operatorname{Rad} \mathfrak{A}$.

<u>Corollary 1.</u> $u \in \operatorname{Rad} \mathfrak{A}$, <u>iff</u> $\operatorname{Rad} \mathfrak{A}_u = \mathfrak{A}$.

<u>Proof.</u> Immediate consequence of a) and (13.25).

<u>Corollary 2.</u> <u>If</u> $\mathfrak{A}_V$ <u>is</u> ~~the~~ a <u>V-isotope</u> of $\mathfrak{A}$, <u>then</u>

$$\operatorname{Rad} \mathfrak{A}_V = \operatorname{Rad} \mathfrak{A}$$

<u>Proof.</u> If $(U,V) \in \Gamma(\mathfrak{A})$, then $U(\operatorname{Rad} \mathfrak{A}) \subset \operatorname{Rad} \mathfrak{A}$, by 13.3, thus $\operatorname{Rad} \mathfrak{A} \subset \operatorname{Rad} \mathfrak{A}_V$. If V is invertible, then $\mathfrak{A} = (\mathfrak{A}_V)_{V^{-1}}$, by (10.3), which shows that $\mathfrak{A}$ is a homotope of $\mathfrak{A}_V$ and then $\operatorname{Rad} \mathfrak{A}_V \subset \operatorname{Rad} \mathfrak{A}$.

We call $\mathfrak{A}$ <u>semi simple</u>, if $\operatorname{Rad} \mathfrak{A} = 0$.

<u>Corollary 3.</u> <u>If</u> $\mathfrak{A}$ <u>is semi simple and</u> $u \in \mathfrak{A}$, <u>then</u>

$$\operatorname{Rad} \mathfrak{A}_u = \operatorname{kernel} P(u).$$

A certain information what kind of elements we may expect in $\operatorname{Rad} \mathfrak{A}$ is given by our next (almost trivial) result.

<u>Theorem 9.</u> a) <u>If x is a trivial element in</u> $\mathfrak{A}$, <u>i.e.</u>, $P(x) = 0$, <u>then</u> $x \in \operatorname{Rad} \mathfrak{A}$.

b) <u>If x is von Neumann regular and</u> $x \in \operatorname{Rad} \mathfrak{A}$, <u>then</u> $x = 0$.

Proof. a) $P(x) = 0$ implies $B(x,y)B(-x,y) = \mathrm{Id}$ for all $y \in \mathfrak{A}$, by (13.25). Thus $x \in \mathrm{Rad}\,\mathfrak{A}$.

b) $x = P(x)u$ and x q.i. in $\mathfrak{A}_u$ implies $q(x,u) = 0$, by (13.15'). Then 1 is the inverse of $1 - x$ in $\hat{\mathfrak{A}}_u$, this can only be the case if $x = 0$.

Thus, if $\mathfrak{A}$ is semi simple ($\mathrm{Rad}\,\mathfrak{A} = 0$) then $\mathfrak{A}$ has no trivial elements $\neq 0$, by a). We can also prove the converse, if $\mathfrak{A}$ has d.c.c. on inner ideals (see 10.6).

Theorem 10. If $\mathfrak{A}$ has d.c.c on inner ideals, then $\mathfrak{A}$ is semi simple, iff $\mathfrak{A}$ has no trivial elements $\neq 0$.

Proof. Let $\mathfrak{A}$ have d.c.c., then $\mathrm{Rad}\,\mathfrak{A}$ contains a minimal inner ideal. Then $\mathcal{L} = \Phi u$ where u is trivial, or the elements of $\mathcal{L}$ are regular, by lemma 10.9. Since the only regular element in $\mathrm{Rad}\,\mathfrak{A}$ is 0, by theorem 9b) this shows that $\mathrm{Rad}\,\mathfrak{A}$ contains a trivial element $\neq 0$ if $\mathrm{Rad}\,\mathfrak{A} \neq 0$. Equivalently, if $\mathfrak{A}$ has no trivial elements, then $\mathrm{Rad}\,\mathfrak{A} = 0$.

There is another proof of theorem 10 which is of some interest. Let $\mathrm{Rad}\,\mathfrak{A} \neq 0$. Consider the set of inner ideals $\{P(x)\mathfrak{A},\ 0 \neq x \in \mathrm{Rad}\,\mathfrak{A}\}$. Let $P(u)\mathfrak{A}$ be a minimal element in this set, then either u is trivial or $P(u)\mathfrak{A} \neq 0$. For any $a \in \mathfrak{A}$ we have $P(P(u)a)\mathfrak{A} = P(u)P(a)P(u)\mathfrak{A} \subset P(u)\mathfrak{A}$. But if $P(u)a \neq 0$ then $P(u)a$ (which is in $\mathrm{Rad}\,\mathfrak{A}$) is not contained in $P(P(u)a)\mathfrak{A}$, otherwise it would be regular, therefore $P(P(u)a)\mathfrak{A} \neq P(u)\mathfrak{A}$ and consequently $P(P(u)a)\mathfrak{A} = 0$ by the minimality of $P(u)\mathfrak{A}$. Thus $P(u)\mathfrak{A}$ consists entirely of trivial elements.

13.5. We have to mention some facts concerning the powers of an element in a quadratic Jordan algebra. Let $(\mathfrak{J}, U, {}^2)$ be a quadratic Jordan algebra. Then the given squaring $x \to x^2$ induces the bilinear composition $(x,y) \to x\circ y = (x+y)^2 - x^2 - y^2$. The powers of an element $x \in \mathcal{J}$ are defined by

$x^1 := x$, x^2 (is given) and $x^{n+2} := U(x)x^n$, $n \geq 1$.

(If $\mathcal{J}$ is unital we set $x^0 = e$)

By an easy induction, using the fundamental formula and (Q.J.5) we get

(13.26) $\qquad U(x^n) = U(x)^n.$

An application of this formula is

13.27 $\qquad U(x^n)x^m = x^{2n+m}.$

And also by induction on n resp. n + m one proves

(13.28) $\quad (x^m)^n = x^{mn}$ and $x^n \circ x^m = 2x^{n+m}$

Exercise: Prove (13.28).

Also without proof we state the following results

(13.29) $\quad V(a^n, a^m) = V(a^{n+m})$ and $U(f(a))U(g(a)) = U((fg)(a))$ for all polynomials $f,g \in \lambda\Phi[\lambda]$ (in the unital case for all $f,g \in \Phi[\lambda]$), where, as usual $V(x,y)z = U(x,z)y$, $V(x)y = x\circ y$, and also $[V(a^n), V(a^m)] = [V(a^n), U(a^m)] = 0$. $x \in \mathcal{J}$ is nilpotent, if $x^n = 0$ for some n. Although this does not imply $x^m = 0$ for all $m \geq n$, we have for all $k \geq 1$: $x^{2n+k} = U(x^n)x^k = 0$, by (13.27) and $x^n = 0$. Since $x^{2n} = (x^n)^2 = 0$ we therefore have: if $x^n = 0$, then $x^m = 0$ for all $m \geq 2n$.

Lemma 2. a) $x \in \mathcal{J}$ is nilpotent, iff $U(x)$ is nilpotent.

b) if $x \in \mathcal{J}$ is nilpotent, then $V(x)$ is nilpotent.

Proof. If $x^n = 0$, then $0 = U(x^n) = U(x)^n$. Conversely, if $U(x)^m = 0$ then $x^{2m+1} = U(x)^m x = 0$.

b) is by induction on m: $V(y)$ is nilpotent for all $y \in \mathcal{J}$ such that $y^m = 0$. This is clearly true for $m = 1$. Assume $x^{m+1} = 0$, then $(x^4)^m = x^{4m} = 0$, since $4m \geq 2(m+1)$. Thus $V(x^4)$ is nilpotent by induction hypothesis. Now we use

$$V(a)^2 = 2U(a) + V(a^2) \qquad \text{(see (9.8))}$$

with $a = x^2$. Since $U(x^2) = U(x)^2$ and $V(x^4)$ commute and both are nilpotent we conclude that $V(x^2)^2$ is nilpotent, but this is the case, iff $V(x^2)$ is nilpotent, then, by the same formula, putting $a = x$, we get that $V(x)^2$, hence $V(x)$, is nilpotent.

Now let $(\mathfrak{A}, P)$ be a Jordan triple system. The odd powers of $x \in \mathfrak{A}$ are defined inductively

$$x^1 := x \quad , \; x^{n+2} := P(x)x^n.$$

We denote by $x^{(n,u)}$ the n-th power of x in the Jordan algebra $\mathfrak{A}_u$. Our first observation is

(13.30) $\qquad x^{(n,x)} = x^{2n-1} \quad , \; n \geq 1.$

This is easily proved by induction. $n = 1,2$: $x^1 = x = x^{(1,x)}$. $x^{(2,x)} = P(x)x = x^3 = x^{2\cdot 2-1}$, by definition of the squaring in $\mathfrak{A}_x$. Then $x^{(n+2,x)} = P_x(x)x^{(n,x)} = P(x)P(x)x^{2n-1} = x^{2n+3}$.

Using the fundamental formula (and trivial induction) we obtain

(13.31) $\quad P(x^n) = P(x)^n;\; P(x^n)x^m = x^{2n+m};\; (x^m)^n = x^{mn} \qquad m, n$ both odd

The next result is also proved by induction.

(13.32) a) $\quad L(x^n, x^m) = L(x^m, x^n) = L(x^{n+m-1}, x)$

b) $\quad \{x^n x^m x^k\} = 2x^{m+n+k}$.

Proof. We firstly prove the second equality in a). The case $n = 1$ is trivial. Let $m = 2m' - 1$, $n = 2n' + 1$. Then

$$L(x^m, x^n) = L(x^{(m',x)}, P(x)x^{(n',x)}) \quad , \text{ by } (13.30)$$
$$= L(x^{(m'+n',x)}, x) \quad , \text{ by } (13.29)$$
$$= L(x^{2m'+2n'-1}, x)$$
$$= L(x^{m+n-1}, x) \; ,$$

Since $L(a, P(x)b)$ corresponds to $V(a,b)$ in $\mathfrak{A}_x$ and

$L(a,x)$ to $V(a)$ (in $\mathfrak{A}_x$), see ($\mathcal{A}.\overline{\underline{X}}$).

But the right hand side of a) is symmetric in m and n, thus $L(x^m, x^n) = L(x^n, x^m)$.

Applying a) to x^k and using (13.28) we get

$$\{x^m x^n x^k\} = \{x^{m+n-1} x x^k\} = x^{m+n-1} \circ x^k \quad (\text{in } \mathfrak{A}_x)$$
$$= 2x^{m+n+k}.$$

Let $\Phi[x]$ denote the subsystem generated by x. The foregoing equations show $\Phi[x] = \sum_{k \geq 0} \Phi x^{2k+1}$. And as a consequence of (13.32) we have

$$L(u,v) = L(v,u) \qquad \text{for all } u,v \in \Phi[x].$$

13.6 We wish to apply lemma 2 to the Jordan triple system $\mathfrak{A}$. Since the quadratic map of the Jordan algebra $\mathfrak{A}_u$ is given by $P_u(x) = P(x)P(u)$ and left multiplication by $L_u(x) = L(x,u)$ an immediate application of lemma 2 is

<u>Lemma 3.</u> <u>$x \in \mathfrak{A}$ is nilpotent in $\mathfrak{A}_u$, iff $P(x)P(u)$ is nilpotent. In this case $L(x,u)$ is nilpotent.</u>

Since $P(x)P(u)$ is nilpotent, iff $P(u)P(x)$ is nilpotent, we have

<u>Corollary 1.</u> <u>x is nilpotent in $\mathfrak{A}_u$, iff u is nilpotent in $\mathfrak{A}_x$.</u>

It is well known from elementary linear algebra, that for a nilpotent linear map N, Id - N is invertible. Since $L(x,u)$ and $P(x)P(u)$ commute (by (13.1)) we get as another application of lemma 3 and theorem 1:

Corollary 2: a) *If x is nilpotent in $\mathfrak{A}_u$, then $B(x,u)$ is invertible.*

b) *If x is nilpotent in $\mathfrak{A}_u$, then x is q.i. in $\mathfrak{A}_u$.*

And one more straightforward application is a shifting principle for nilpotent elements. Let $(U,V) \in \Gamma(\mathfrak{A})$ and x be nilpotent in $\mathfrak{A}_{Vu}$. This is the case, iff $P(x)P(Vu) = P(x)VP(u)U$ is nilpotent. And again, since AB is nilpotent iff BA is nilpotent we see that x is nilpotent in $\mathfrak{A}_{Vu}$, iff $UP(x)VP(u) = P(Ux)P(u)$ is nilpotent, equivalently, iff Ux is nilpotent in $\mathfrak{A}_u$.

Corollary 3: *If $(U,V) \in \Gamma(\mathfrak{A})$, then x is nilpotent in $\mathfrak{A}_{Vu}$ iff Ux is nilpotent in $\mathfrak{A}_u$.*

We define an element $x \in \mathfrak{A}$ to be properly nilpotent (p.n) if x is nilpotent in every Jordan algebra $\mathfrak{A}_u$, $u \in \mathfrak{A}$. A subalgebra (or an ideal) $\mathfrak{L}$ of $\mathfrak{A}$ is called *properly nil*, if every element of $\mathfrak{L}$ is properly nilpotent.

We note that all p.n. elements of $\mathfrak{A}$ are in Rad$\mathfrak{A}$, by Cor. 2.

Lemma 4. a) *Subsystems and homomorphic images of p.n. Jts's are p.n.*

b) *If $\mathfrak{L}$ is an ideal of $\mathfrak{A}$, then $\mathfrak{A}$ is p.n., iff $\mathfrak{L}$ and $\mathfrak{A}/\mathfrak{L}$ are p.n.*

Proof. a) is obvious from the definition.

b) One direction follows from a). Let $x,u \in \mathfrak{A}$, then $z := x^{(n,u)} \in \mathfrak{L}$ for some n and z is nilpotent in $\mathfrak{L}_z$; then z is also nilpotent in $\mathfrak{A}_z$. Thus $P(z)P(z)$ is nilpotent, equivalently

$P(z)$ is nilpotent. $P(z) = [P(x)P(u)]^n$ shows that $P(x)P(u)$ is nilpotent, which means that x is nilpotent in $\mathfrak{A}_u$.

A standard argument using an isomorphism theorem shows that finite sumes of p.n. ideals are p.n. And since the property of an ideal to be p.n. is defined elementwise, ZORN'S lemma applies to give a maximal p.n. ideal of $\mathfrak{A}$ which is unique by the foregoing remark. We call this unique maximal p.n. ideal $N(\mathfrak{A})$ the p.n. - radical of $\mathfrak{A}$. Clearly

$$N(\mathfrak{A}/_N) = 0,$$

since $N(\mathfrak{A}/_N) = \mathfrak{L}/_N$, $\mathfrak{L}$ ideal of $\mathfrak{A}$, lemma 4 shows that $\mathfrak{L}$ is p.n., hence $\mathfrak{L} \subset N$.

Since by a previous remark all p.n. elements are in Rad $\mathfrak{A}$ we have

$$N(\mathfrak{A}) \subset \text{Rad}\,\mathfrak{A} .$$

Next we consider the module $Z(\mathfrak{A})$ consisting of all finite sums of trivial elements. Using $P(P(x)z) = P(x)P(z)P(x)$ and $P(B(z,b)z) = B(a,b)P(z)B(b,a)$ we firstly observe that for a trivial element z, $P(x)z$ and $B(a,b)z$ are also trivial, which shows $P(x)Z \subset Z$ and $B(a,b)Z \subset Z$ for all $x,a,b \in \mathfrak{A}$, consequently $P(\mathfrak{A})Z \subset Z$ and $\{\mathfrak{A}\mathfrak{A}\,Z\} \subset Z$. In order to show $P(z)\mathfrak{A} \subset Z$, we set $a = \Sigma z_i, z_i$ trivial. Then $P(a) = \sum_{i<j} P(z_i,z_j)$ and we only have to show that for trivial elements z_1, z_2, and $x \in \mathfrak{A}$, $P(z_1,z_2)x$ is trivial. But this can be seen from (13.20). These considerations show that $Z(\mathfrak{A}) = \{\Sigma z_i, z_i \text{ trivial}\}$ is an ideal of $\mathfrak{A}$.

Without proof we mention the following result

<u>Theorem 11.</u> $Z(\mathfrak{A})$ <u>is properly nil</u>.

Thus we have the following chain of ideals

$$Z(\mathfrak{A}) \subset N(\mathfrak{A}) \subset \mathrm{Rad}\,\mathfrak{A} .$$

Theorem 12. If $\mathfrak{A}$ has d.c.c. on inner ideals, then

$\mathrm{Rad}\,\mathfrak{A} = N(\mathfrak{A}) = \{x, x \text{ is p.n.}\} = \{x \in \mathfrak{A}, \mathfrak{A}_x \text{ is a nil algebra}\}$.

Proof. Let $x \in \mathrm{Rad}\,\mathfrak{A}$ and $u \in \mathfrak{A}$. Consider the descending chain of inner ideals

$$P(x)P(u)\mathfrak{A} > \ldots \quad [P(x)P(u)]^k\mathfrak{A} > \ldots$$

This chain becomes stationary, say $[P(x)P(u)]^\ell \mathfrak{A} = [P(x)P(u)]^{\ell+j}\mathfrak{A}$. then $x^{(2\ell+1,u)} = [P(x)P(u)]^\ell x = [P(x)P(u)]^{2\ell+1}v$, for some v. Thus $x^{(2\ell+1,u)} = P(x^{(2\ell+1,u)})P(u)v$ which means that $x^{(2\ell+1,u)} \in \mathrm{Rad}\,\mathfrak{A}$ is regular, this can only be the case if $x^{(2\ell+1,u)} = 0$, by theorem 9. Since u is arbitrary we get that x is p.n. and $\mathrm{Rad}\,\mathfrak{A} = \{x, x \text{ p.n.}\}$. Since all p.n. elements are in $\mathrm{Rad}\,\mathfrak{A}$ we have equality and $\{x, x \text{ p.n.}\}$ is a p.n. ideal. The remaining equality follows from Cor. 1 to lemma 3.

13.7. We shall call $\mathfrak{A}$ strongly semi prime, if $\mathfrak{A}$ has no trivial elements, and an ideal $\mathfrak{K}$ of $\mathfrak{A}$ is called strongly semi prime, if $\mathfrak{A}/\mathfrak{K}$ is a strongly semi prime Jts. Equivalently, the ideal $\mathfrak{K}$ of $\mathfrak{A}$ is strongly semi prime, if $P(a)\mathfrak{A} \subset \mathfrak{K}$ implies $a \in \mathfrak{K}$.

Example: 1) $\mathrm{Rad}\,\mathfrak{A}$ is strongly semi prime. Let $P(a)\mathfrak{A} \subset \mathrm{Rad}\,\mathfrak{A}$, then $\mathfrak{A} = \mathrm{Rad}\,\mathfrak{A}_a$, by thm. 8, and then $a \in \mathrm{Rad}\,\mathfrak{A}$, by Cor. 1 of thm 8.

2) $N(\mathfrak{A})$ is strongly semi prime.

Since $N(\mathfrak{A}/N) = 0$ and $Z \subset N$ we see that $\mathfrak{A}/N$ has no nontrivial trivial elements.

There is an interesting result concerning strongly semi prime ideals.

<u>Theorem 13.</u> <u>If $\mathcal{B}$ is an ideal of $\mathfrak{A}$ and $\mathcal{C}$ a strongly semi prime ideal of $\mathcal{B}$ then $\mathcal{C}$ is an ideal of $\mathfrak{A}$.</u>

<u>Proof.</u> We have to show:

a) $P(\mathcal{C})\mathfrak{A} \subset \mathcal{C}$, b) $P(\mathfrak{A})\mathcal{C} \subset \mathcal{C}$ and c) $\{\mathfrak{A}\mathfrak{A}\mathcal{C}\} \subset \mathcal{C}$

ad a): Let $c \in \mathcal{C}$, $a \in \mathfrak{A}$, then $P(c)a \in \mathcal{B}$ and $P(P(c)a)\mathcal{B} = P(c)P(a)P(c)\mathcal{B} \subset P(c)\mathcal{B} \subset \mathcal{C}$, consequently $P(c)a \in \mathcal{C}$, by assumption on $\mathcal{C}$.

ad b): Let $b_1 := P(a)c$, $c \in \mathcal{C}$, $a \in \mathfrak{A}$.

$b_2 := P(b_1)b$, $b_3 := P(b_2)b'$, $b, b' \in \mathcal{B}$. In order to show $b_1 \in \mathcal{C}$ it suffices to prove $P(b_3)\mathcal{B} \subset \mathcal{C}$, because then $b_3 \in \mathcal{C}$ and since b, b' are arbitrary we get b_2 and $b_1 \in \mathcal{C}$.

$$
\begin{aligned}
P(b_3)\mathcal{B} &= P(b_2)P(b')P(b_2)\mathcal{B} \\
&= P(b_1)\big(P(b)P(a)P(c)\big)\big(P(a)P(b')P(a)\big)\big(P(c)P(a)P(b)P(b_1)\big)\mathcal{B} \\
&\subset P(b_1)P(b)P(a)P(c)P(P(a)b')\mathcal{C} \\
&\subset P(\mathcal{B})P(b)P(a)P(c)\mathcal{C}, \text{ since } P(a)b' \in \mathcal{B}.
\end{aligned}
$$

Using

$$P(b)P(a)P(c) = P(\{bac\}) + P(P(b)a, P(c)a) - P(c)P(a)P(b) - P(b,c)P(a)P(b,c)$$

we observe, that the right hand side applied to an element of $\mathcal{C}$ is an element of $\mathcal{C}$ again, thus $P(b_3)\mathcal{B} \subset P(\mathcal{B})\mathcal{C} \subset \mathcal{C}$, which we wanted to prove.

ad c): The identity

$$P(\{axc\}) = P(a)P(x)P(c) + P(c)P(x)P(a) + P(a,c)P(x)P(a,c) - P(P(a)x, P(c)x)$$

shows that in order to prove $P(\{axc\})\mathcal{B} \subset \mathcal{C}$, $a, x \in \mathfrak{A}$, $c \in \mathcal{C}$, it suffices to show $P(a,c)b \in \mathcal{C}$, $a \in \mathfrak{A}$, $b \in \mathcal{B}$, $c \in \mathcal{C}$. (using a) and b).) Instead of showing $\{abc\} \in \mathcal{C}$ we show $B(a,b)c \in \mathcal{C}$. Now we proceed as in b).

$b_1 := B(a,b)c$, $b_2 := P(b_1)b'$, $b_3: P(b_2)b"$.

$$P(b_3)\mathcal{L} = P(b_2)P(b")P(b_2)\mathcal{L}$$
$$= P(b_1)P(b')B(a,b)P(c)B(b,a)P(b")B(a,b)P(c)B(b,a)P(b')P(b_1)\mathcal{L}$$
$$\subset P(b_1)P(b')B(a,b)P(c)P(B(b,a)b")\mathcal{L}$$
$$\subset P(\mathcal{L})P(\mathcal{L})B(a,b)P(c)\mathcal{L} \ ,$$

this is contained in $\mathcal{L}$ if we can show that $L(a,b)P(c)\mathcal{L} \subset \mathcal{L}$.
But this is clear from $L(a,b)P(c)c' = P(\{abc\},c)c' - P(c)L(b,a)c'$.
(The first term at the right hand side is in $\mathcal{L}$ since $\{abc\} \in \mathcal{L}$ and $\mathcal{L}$ ideal in $\mathcal{L}$ and the second term is in $\mathcal{L}$ by b)).

Using the fact that $N(\mathfrak{A})$ is strongly semi prime we have as an immediate application

$N(\mathcal{L}) = \mathcal{L} \cap N(\mathfrak{A})$ for any ideal $\mathcal{L}$ of $\mathfrak{A}$.

XIV Regularity.

14.1. In 10.7 we already made some remarks on the regularity of Jts. We recall, $x \in \mathfrak{A}$ ($\mathfrak{A}$ is a Jts) is <u>regular</u> (= von Neumann regular), if $x = P(x)u$ for some $u \in \mathfrak{A}$. Since we derived more formulas and properties in $\mathfrak{A}$ we can expect more results.

<u>Lemma 1. If $\mathcal{L}$ is an ideal of $\mathfrak{A}$ and $\mathcal{L}$ a regular ideal in $\mathcal{L}$, then $\mathcal{L}$ is an ideal of $\mathfrak{A}$.</u>

<u>Proof.</u> We have to show

a) $P(\mathcal{L})\mathfrak{A} \subset \mathcal{L}$, b) $P(\mathfrak{A})\mathcal{L} \subset \mathcal{L}$ c) $\{\mathfrak{A}\mathfrak{A}\mathcal{L}\} \subset \mathcal{L}$.

We recall that $\mathcal{L}$ is already regular as a subsystem (see lemma 10.13).

Let $c \in \mathcal{C}$, $x, y \in \mathfrak{A}$. Since $\mathcal{C}$ is regular we have c', c'' in $\mathcal{C}$ such that $c = P(c)c'$, $c' = P(c')c''$, thus $c = P(c)P(c')c''$. Then $P(c)\mathfrak{A} = P(c)P(c')P(c)\mathfrak{A} \subset P(c)P(c')\mathcal{L} \subset \mathcal{C}$, since $\mathcal{L}$ is ideal in $\mathfrak{A}$ and $\mathcal{C}$ ideal in $\mathcal{L}$. This is a).

ad c): $\{xyc\} = L(x,y)P(c)c' = P(\{xyc\},c)c' - P(c)\{yxc'\}$, by (13.10). The right hand side is contained in $P(\mathcal{L},\mathcal{C})\mathcal{C} + P(\mathcal{C})\mathfrak{A} \subset \mathcal{C}$, by a) and assumption.

ad b): $P(x)c = P(x)P(c)P(c')c''$

$$= P(\{xcc'\})c'' + P(P(x)c, P(c')c)c'' - P(c')P(c)P(x)c'' - P(x,c')P(c)P(x,c')c''$$

and the right hand side is in $\mathcal{C}$, by a) and b).

In 10.7. we proved the existence of a unique maximal regular ideal $\mathfrak{r}(\mathfrak{A})$ of any Jts $\mathfrak{A}$.

<u>Corollary</u>. *$\mathfrak{r}(\mathcal{L}) = \mathcal{L} \cap \mathfrak{r}(\mathfrak{A})$ for any ideal $\mathcal{L}$ of $\mathfrak{A}$.*

<u>Proof.</u> $\mathcal{L} \cap \mathfrak{r}(\mathfrak{A})$ is regular in $\mathfrak{A}$ and then regular as a subsystem by lemma 10.13. Thus $\mathcal{L} \cap \mathfrak{r}(\mathfrak{A}) \subset \mathfrak{r}(\mathcal{L})$. Conversely $\mathfrak{r}(\mathcal{L})$ is an ideal of $\mathfrak{A}$, by the above lemma, and therefore $\mathfrak{r}(\mathcal{L}) \subset \mathfrak{r}(\mathfrak{A})$.

14.2. The next result shows how useful it can be to know whether a given Jordan algebra is unital.

<u>Theorem 1.</u> *Let $\mathfrak{A}$ be a strongly semi prime Jordan triple system and $x \in \mathfrak{A}$. Then x is regular, iff the Jordan algebra $\mathfrak{A}_x/_{\text{kernel } P(x)}$ is unital.*

<u>Proof.</u> We note that kernel $P(x)$ is an ideal in $\mathfrak{A}_x$ by lemma 10.7. We set $\overline{\mathfrak{A}}_x = \mathfrak{A}_x/_{\text{kernel } P(x)}$. The quadratic map and squaring in $\overline{\mathfrak{A}}_x$ are given by

$\overline{P}_x(\overline{u})\overline{v} = \overline{P(u)P(x)v}$ and $\overline{u}^2 = \overline{P(u)x}$.

If $\overline{\mathfrak{A}}_x$ is unital with unit element $\overline{e}$, then $\overline{e}\circ\overline{v} = \overline{P(e,v)x} = 2\overline{v}$ (see (9.7)) for all $v \in \mathfrak{A}$ and $\overline{P}_x(\overline{e}) = \mathrm{Id}\,\big|\,\overline{\mathfrak{A}}_x$. This implies $\overline{v} - \{\overline{exv}\} + P(e)P(x)v = 0$, equivalently $B(e,x)\mathfrak{A} \subset$ kernel $P(x)$, resp. $P(x)B(e,x) = P(x - P(x)e) = 0$, by (13.22). $\mathfrak{A}$ is strongly semi prime and therefore we have $x = P(x)e$; i.e., x is regular. Conversely, if $x = P(x)c$ for some $c \in \mathfrak{A}$, then $P(x) = P(x)P(c)P(x)$, equivalently $\overline{P}_x(\overline{c}) = \mathrm{Id}$ on $\overline{\mathfrak{A}}_x$ and therefore $\overline{\mathfrak{A}}_x$ is unital, by theorem 9.3.

The best result we know of concerning the existence of a unit element is the following

<u>Theorem 2. A semi simple Jordan algebra with dcc on inner ideals has a unit element.</u>

We shall not prove this result, although there exist fairly simple proofs of it.

We recall that semi simple and d.c.c. is equivalent to strongly semi prime and dcc, by theorem 13.10. We shall show that these properties carry over from $\mathfrak{A}$ to $\overline{\mathfrak{A}}_x$.

<u>Lemma 2. If $\mathfrak{A}$ is strongly semi prime and has dcc, then all Jordan algebras $\overline{\mathfrak{A}}_x, x \in \mathfrak{A}$, are strongly semi prime and have dcc (on inner ideals).</u>

<u>Proof.</u> First of all, kernel $P(x)$, $x \in \mathfrak{A}$, is an ideal in $\mathfrak{A}_x$ and $(\overline{\mathfrak{A}}_x, \overline{P}_x)$ is well defined. Next, let $\overline{\mathfrak{L}}$ be an inner ideal of $\overline{\mathfrak{A}}_x$, $b \in \mathfrak{L}$, $z \in \mathfrak{A}$, then by definition $P(b)P(x)z \equiv b'$ mod kernel $P(x)$, consequently $P(P(x)b)z = P(x)b' \in P(x)\mathfrak{L}$. This shows that $P(x)\mathfrak{L}$ is an inner ideal of $\mathfrak{A}$.

Now, if $\overline{\mathcal{L}}_1 \supset \ldots \supset \overline{\mathcal{L}}_k \supset \ldots$ is a descending chain of inner ideals, then $P(x)\mathcal{L}_1 \supset \ldots \supset P(x)\mathcal{L}_k \supset \ldots$ is a descending chain of inner ideals which becomes stationary by assumption. Let $P(x)\mathcal{L}_n = P(x)\mathcal{L}_{n+j}, j \geq 0$, and $u \in \mathcal{L}_n$. Then $P(x)u \in P(x)\mathcal{L}_n$ and for any $j \geq 0$ there is an element $u_j \in \mathcal{L}_{n+j}$ such that $P(x)u = P(x)u_j$. This implies $\overline{u} = \overline{u}_j$, consequently $\overline{\mathcal{L}}_n = \overline{\mathcal{L}}_{n+j}$. Since $\mathrm{Rad}\,\mathfrak{A} = 0$ we get kernel $P(x) = \mathrm{Rad}\,\mathfrak{A}_x$ from theorem 13.8. Then $\mathfrak{A}_x/_{\text{kernel } P(x)}$ is semi simple and has no trivial elements $\neq 0$.

A combination of theorem 2 and lemma 2 is the next result

<u>Corollary:</u> <u>If $\mathfrak{A}$ is semi simple and has dcc on inner ideals, then all Jordan algebras $\mathfrak{A}_x/_{\text{kernel } P(x)}$, $x \neq 0$, are unital.</u>

And combining this result with theorem 1, we end up with

<u>Theorem 3.</u> <u>Let $\mathfrak{A}$ be a Jordan triple system with dcc on inner ideals. The following properties are equivalent,</u>

a) <u>$\mathfrak{A}$ is semi simple,</u>

b) <u>$\mathfrak{A}$ is strongly semi prime,</u>

c) <u>$\mathfrak{A}$ is von Neumann regular.</u>

14.3. As an application of the addition theorem we proved that

$$R = \{(x,y) \in \mathfrak{A} \times \mathfrak{A}\ ;\ x = q(y,u) \text{ for some } u \in \mathfrak{A}\}$$

is an equivalence relation. Clearly

$$Q = \{(x,y) \in \mathfrak{A} \times \mathfrak{A}\ ;\ P(x)\mathfrak{A} = P(y)\mathfrak{A}\}$$

is also an equivalence relation.

<u>Lemma 3.</u> $R \subset Q$

<u>Proof.</u> Let $(x,y) \in R$. Then $x = q(y,u)$ and

$P(x)B(u,y) = P(y)$, by (13.24). Since $B(u,y)$ is invertible this implies $P(x)\mathfrak{A} = P(x)B(u,y)\mathfrak{A} = P(y)\mathfrak{A}$, thus $(x,y) \in Q$. Now assume $1/2 \in \Phi$, then $(x,y) \in R$ implies $y - x = \{xuy\}$ for some $u \in \mathfrak{A}$. Assume $R = Q$, then $2x - x = P(x)w'$, since $(x,2x) \in Q$. Consequently, $\mathfrak{A}$ is regular.

Theorem 4. _Let $\mathfrak{A}$ be a Jts over Φ and $1/2 \in \Phi$. Then $R = Q$, iff $\mathfrak{A}$ is regular._

Proof. We have only to prove that $Q \subset R$ if $\mathfrak{A}$ is regular. Assume $P(x)\mathfrak{A} = P(y)\mathfrak{A}$ and $\mathfrak{A}$ regular. Then $x = P(x)v$ for some $v \in \mathfrak{A}$ and the fundamental formula shows $P(x) = P(x)P(v)P(x)$, which means that $P(x)P(v)$ restricted to $P(x)\mathfrak{A} = P(y)\mathfrak{A}$ is the identity. Since $y \in P(y)\mathfrak{A}$ we have $P(x)P(v)y = y$. Furthermore $P(x)\mathfrak{A} = P(x)P(v)P(x)\mathfrak{A} \subset P(x)P(v)\mathfrak{A} \subset P(x)\mathfrak{A}$, thus $P(x)\mathfrak{A} = P(x)P(v)\mathfrak{A}$. This shows that $P(x)\mathfrak{A}$ is the Peirce - 1 - space of the Jordan algebra $\mathfrak{A}_v$ relative to the idempotent x (in $\mathfrak{A}_v$). The chain $P(x)\mathfrak{A} = P(y)\mathfrak{A} = P(P(x)P(v)y)\mathfrak{A} = P(y)P(v)P(x)\mathfrak{A}$ shows that y is invertible in $P(x)\mathfrak{A}$. If a is the inverse and $w: = P(v)(x - a)$ then a verification shows $B(x,w)y = x - P(x)w$ and $B(x,w)P(y)w = P(x)w$. Thus $(x,y) \in R$.

XV. The Peirce Decomposition

15.1. Let $\mathfrak{A}$ be a Jordan triple system. An element $c \in \mathfrak{A}$ is called an idempotent, if $c = P(c)c$; or, equivalently, if c is an idempotent in the Jordan algebra $\mathfrak{A}_c$.

Note: If c is an idempotent, then $-c$ is also an idempotent.

The fact that c is an idempotent in the Jordan algebra $\mathfrak{A}_c$ suggests the study of the Peirce decomposition of $\mathfrak{A}_c$ relative to c. If $c = P(c)c$ is an idempotent, then

(15.1) $$L(c,c)P(c) = P(c)L(c,c) = 2\,P(c)\ ,\quad \text{by (13.1)}$$

(15.2) $$2\,P(c)P(c) = L(c,c)^2 - L(c,c)\ ,\quad \text{by (13.12)}$$

(15.3) $$P(c)B(c,c) = B(c,c)P(c) = 0\ ,\quad \text{by (13.22).}$$

Lemma 1. If c is an idempotent of $\mathfrak{A}$, then $E_1 := P(c)P(c)$; $E_o := B(c,c)$; $E_{1/2} = L(c,c) - 2P(c)P(c)$ are orthogonal projections and $Id = E_1 + E_{1/2} + E_o$.

Proof. $Id = E_1 + E_{1/2} + E_o$ is obvious. We have to show $E_i E_j = \delta_{ij} E_i$.

$E_1^2 = P(c)P(c)P(c)P(c) = P(P(c)c)P(c) = P(c)P(c) = E_1$.

$E_o^2 = B(c,c)^2 = B(c,c) = E_o$, by (13.17).

$E_1 E_o = E_o E_1 = 0$, by (15.3). Then from $Id = \sum E_i$ we get immediately $E_o E_{1/2} = E_1 E_{1/2} = E_{1/2} E_1 = E_{1/2} E_o = 0$ and then $E_{1/2}^2 = E_{1/2}$.

Since the E_i are orthogonal projections, $Id = \sum E_i$, we have the direct sum decomposition

$$\mathfrak{A} = \oplus\, \mathfrak{A}_i(c)\ ;\ \mathfrak{A}_i(c) = E_i \mathfrak{A}\ ,\quad i = 0,\ 1,\ 1/2\ .$$

This is the <u>Peirce decomposition</u> of $\mathfrak{A}$ relative c. Using (15.1), (15.2) we derive easily

(15.4)
$$\text{a)}\quad L(c,c)E_1 = E_1L(c,c) = 2\, E_1$$
$$\text{b)}\quad L(c,c)E_o = E_oL(c,c) = 0$$
$$\text{c)}\quad L(c,c)E_{1/2} = E_{1/2}\, L(c,c) = E_{1/2}$$

Since $\mathfrak{A}_i = E_i\mathfrak{A}$ we therefore have

(15.5)
$$\{ccx_i\} = 2\, i\, x_i \ , \ x_i \in \mathfrak{A}_i \ , \ i = 0,\ 1,\ 1/2\ .$$

If $1/2 \in \Phi$ this equation characterizes completely the elements in $\mathfrak{A}_i$. But in general this is only true for $i = 1/2$.

(15.6)
$$\mathfrak{A}_{1/2}(c) = \{x \in \mathfrak{A}\ ;\ \{ccx\} = x\}\ .$$

We have $\mathfrak{A}_{1/2}(c) \subset \{x \in \mathfrak{A};\ \{ccx\} = x\}$, by (15.5). If conversely $\{ccx\} = x$, then using (15.5) and the decomposition $x = x_o + x_{1/2} + x_1$, $x_i \in \mathfrak{A}_i$, we get $x = \{ccx\} = x_{1/2} + 2x_1$, thus $x_1 = x_o = 0$ and $x \in \mathfrak{A}_{1/2}$. The definition of E_i shows

(15.7)
$$P(E_ix) = E_iP(x)E_i \quad , \quad i = 0,\ 1 \quad (\text{for all } x \in \mathfrak{A})$$

and for $E_{1/2}$ we shall prove

(15.8)
$$P(E_{1/2}x) = E_{1/2}P(x)E_{1/2} + E_1P(x) + P(x)E_1 - P(E_1x,x), \quad (\text{for all } x \in \mathfrak{A}).$$

<u>Proof</u>. Using the definition of $E_{1/2}$, (13.20) and (13.21) we compute

$$P(E_{1/2}x) = P(\{ccx\} - 2\, P(c)P(c)x) = P(\{ccx\}) + 4\, P(P(c)P(c)x) - 2\, P(\{ccx\},$$
$$P(c)P(c)x) = P(c)P(c)P(x) + P(x)P(c)P(c) + L(c,c)P(x)L(c,c) - P(P(c)P(c)x,x)$$
$$- 2\, P(c)P(c)P(x)L(c,c) - 2\, L(c,c)P(x)P(c)P(c) + 4\, P(c)P(c)P(x)P(c)P(c).$$

Theorem 1. (PEIRCE decomposition)

If c *is an idempotent of* $\mathfrak{A}$, *let* $E_1 = P(c)P(c), E_o = B(c,c), E_{1/2} = L(c,c) - 2\,P(c)P(c)$. *Then* $\mathfrak{A} = \oplus\, \mathfrak{A}_i$, $\mathfrak{A}_i = E_i$, $i = 0,\ 1/2,\ 1$ *and*

a) $P(\mathfrak{A}_i)\mathfrak{A}_i \subset \mathfrak{A}_i$, $i = 0,\ 1,\ 1/2$

b) $P(\mathfrak{A}_{1/2})\mathfrak{A}_i \subset \mathfrak{A}_{1-i}$, $i = 0,\ 1$

c) $\{\mathfrak{A}_1 \mathfrak{A}_{1/2} \mathfrak{A}_o\} \subset \mathfrak{A}_{1/2}$

d) $\{\mathfrak{A}_i \mathfrak{A}_i \mathfrak{A}_{1/2}\} \subset \mathfrak{A}_{1/2}$ $\quad i = 0,\ 1$

e) $\{\mathfrak{A}_{1/2} \mathfrak{A}_{1/2} \mathfrak{A}_i\} \subset \mathfrak{A}_i$ $\quad i = 0,\ 1$

while all other compositions are zero.

Proof: If $i = 0,\ 1$, then by (15.7):

$$P(\mathfrak{A}_i)\mathfrak{A}_j = P(E_i\mathfrak{A})E_j\mathfrak{A} = E_iP(\mathfrak{A})E_iE_j\,\mathfrak{A} \quad (j = 0,\ 1/2,\ 1)$$

which shows

$$P(\mathfrak{A}_1)\mathfrak{A}_1 \subset \mathfrak{A}_1 \ , \ P(\mathfrak{A}_1)\mathfrak{A}_o = 0 \ , \ P(\mathfrak{A}_1)\mathfrak{A}_{1/2} = 0$$

$$P(\mathfrak{A}_o)\mathfrak{A}_o \subset \mathfrak{A}_o \ , \ P(\mathfrak{A}_o)\mathfrak{A}_1 = 0 \ , \ P(\mathfrak{A}_o)\mathfrak{A}_{1/2} = 0$$

If $u \in \mathfrak{A}_{1/2}$, then

$E_{1/2}P(u)E_1 = L(c,c)P(u)P(c)P(c)$, since $E_1P(u)E_1 = 0$

$\qquad = P(\{ccu\},u)P(c)P(c) - P(u)L(c,c)P(c)P(c)$, by (13.9)

$\qquad = 0$, by (15.5) and (15.4a). Similarly

$E_1P(u)E_{1/2} = 0$. Then $P(u)E_1 = (E_1 + E_{1/2} + E_o)P(u)E_1 = E_oP(u)E_1$ and $E_1P(u) = E_1P(u)E_o$. Thus (15.8) becomes (for $u \in \mathfrak{A}_{1/2}$)

$$P(u) = E_{1/2}P(u)E_{1/2} + E_1P(u)E_o + E_oP(u)E_1.$$

From this equation we deduce immediately

$$P(\mathfrak{A}_{1/2})\mathfrak{A}_{1/2} \subset \mathfrak{A}_{1/2}, \ P(\mathfrak{A}_{1/2})\mathfrak{A}_1 \subset \mathfrak{A}_o, \ P(\mathfrak{A}_{1/2})\mathfrak{A}_o \subset \mathfrak{A}_1.$$

So far we proved a) and b) (and part of the last statement). We shall use (13.5), (13.6) twice: Let $a_o \in \mathfrak{A}_o$, then

$L(a_o,c) = L(a_o,P(c)c) = L(P(a_o,c)c,c) = 0$, since $P(c)a_o = 0$, $\{cca_o\} = 0$.

For $x \in \mathfrak{A}_1$, $y \in \mathfrak{A}_o$ we then obtain

$L(x,y) = L(P(c)P(c)x,y) = L(c,\{(P(c)x)cy\}) = 0$, since $L(y,c) = 0$.

Similarly (using the dual formula) we obtain $L(y,x) = 0$. Thus

$$\{\mathfrak{A}_1\mathfrak{A}_o\mathfrak{A}\} = \{\mathfrak{A}_o\mathfrak{A}_1\mathfrak{A}\} = 0.$$

Replacing $x = y = c$, $u = u_i \in \mathfrak{A}_i, v = v_j \in \mathfrak{A}_j$ and applying this equation to $z_k \in \mathfrak{A}_k$ shows

$$\{cc\{u_iv_jz_k\}\} = 2(i-j+k)\ \{u_iv_jz_k\}\ , \text{ by } (15.5) \tag{15.9}$$

In particular:

$$\{cc\{u_1v_{1/2}z_o\}\} = \{u_1v_{1/2}z_o\}$$

and

$$\{cc\{u_iv_iz_{1/2}\}\} = \{u_iv_iz_{1/2}\}$$

which already proves c) and d), using the characterization (15.6) of $\mathfrak{A}_{1/2}$.

Next we use a linearization of (15.8) and obtain for $x_i \in \mathfrak{A}_i, x_{1/2} \in \mathfrak{A}_{1/2}$

$$P(E_1 x_i, x_{1/2}) = E_{1/2} P(x_i, x_{1/2}) E_{1/2} + E_1 P(x_i, x_{1/2}) + P(x_i, x_{1/2}) E_1$$

Applying this to $z_{1/2} \in \mathfrak{A}_{1/2}$ leads to

$$P(E_1 x_i, x_{1/2}) z_{1/2} = E_{1/2} P(x_i, x_{1/2}) z_{1/2} + E_1 P(x_i, x_{1/2}) z_{1/2}.$$

Since $E_1 x_o = 0$ this equation shows that the 1/2- and 1- component of $\{x_o z_{1/2} x_{1/2}\}$ is zero, thus $\{\mathfrak{A}_o \mathfrak{A}_{1/2} \mathfrak{A}_{1/2}\} \subset \mathfrak{A}_o$. Taking $i = 1$ the above equation shows that $\{x_1 z_{1/2} x_{1/2}\} \in \mathfrak{A}_{1/2} + \mathfrak{A}_1$. But (15.9) shows $\{cc\{x_1 z_{1/2} x_{1/2}\}\} = 2\{x_1 z_{1/2} x_{1/2}\}$. This implies that the 1/2-component of $\{x_1 z_{1/2} x_{1/2}\}$ is zero, since $u = u_{1/2} + u_1$ and $2u_{1/2} + 2u_1 = 2u = \{ccu\} = u_{1/2} + 2u_1$ obviously give $u_{1/2} = 0$. This completes the proof of e). (The last statement has been established in the course of the proof.)

Part a) of the last theorem shows in particular that the $\mathfrak{A}_i$ are subsystems of $\mathfrak{A}$.

Some useful details concerning the Peirce decomposition are collected in the next lemma.

<u>Lemma 2.</u> <u>If</u> $c = P(c)c$ <u>is an idempotent of</u> $\mathfrak{A}$, $\mathfrak{A} = \oplus \mathfrak{A}_i$ <u>the Peirce decomposition of</u> $\mathfrak{A}$ <u>relative</u> c, <u>then</u>

a) $P(c)\mathfrak{A} = \mathfrak{A}_1$, kernel $P(c) = \mathfrak{A}_{1/2} \oplus \mathfrak{A}_o$

b) $P(c): a \mapsto \bar{a} = P(c)a$ is an involutoric automorphism of $\mathfrak{A}_1$,

(15.10)

c) $P(x)a = P_c(x)\bar{a}$, $x \in \mathfrak{A}$, $a \in \mathfrak{A}_1$,

d) $L(a,c) = L(c,\bar{a})$, $a \in \mathfrak{A}_1$,

e) $\{ac\{acz\}\} = \{P(a)ccz\}$, $a \in \mathfrak{A}_1$, $z \in \mathfrak{A}_o \oplus \mathfrak{A}_{1/2}$,

f) $\overline{\{uvc\}} = \{vuc\}$, $u,v \in \mathfrak{A}_{1/2}$.

g) $\{abz\} = \{ac\{\bar{b}cz\}\}$, $a,b \in \mathfrak{A}_1$, $z \in \mathfrak{A}_o + \mathfrak{A}_{1/2}$

h) $\{xya\} = \{x\{\bar{a}cy\}c\}$, $a \in \mathfrak{A}_1$, x,y $\mathfrak{A}_o + \mathfrak{A}_{1/2}$.

Proof. Since $P(c)\mathfrak{A} = P(P(c)c)\mathfrak{A} = E_1P(c)\mathfrak{A}$ (fundamental formula), we see that $P(c)\mathfrak{A} \subset \mathfrak{A}_1$. But $\mathfrak{A}_1 = P(c)P(c)\mathfrak{A} \subset P(c)\mathfrak{A}$, thus we have equality. Since $P(c)P(c) = E_1$ is the identity on $\mathfrak{A}_1$ we clearly have $\bar{\bar{a}} = a$, $a \in \mathfrak{A}_1$ and $P(x)a = P(x)P(c)P(c)a = P_c(x)\bar{a}$. $x \in$ kernel $P(c) \Leftrightarrow 0 = P(c)x_1 + P(c)x_{1/2} + P(c)x_o = \bar{x}_1$, by (15.1) - (15.3), $\Leftrightarrow x_1 = 0 \Leftrightarrow x \in \mathfrak{A}_o \oplus \mathfrak{A}_{1/2}$. Part c) follows from (13.6) and (15.5) $2L(a,c) = L(\{cca\},c) = L(P(a,c)c,c) = L(c,P(c)a) + L(a,P(c)c)$. Applying $2P(a)P(c) = L(a,c)L(a,c) - L(P(a)c,c)$ (see (13.12)) to $z \in \mathfrak{A}_o \oplus \mathfrak{A}_{1/2}$ and observing $P(c)z = 0$, gives $L(a,c)^2z = L(P(a)c,c)z$; this is e). Using (13.10) we get f):

$$\{vuc\} = L(v,u)P(c)c = P(\{vuc\},c)c - P(c)L(u,v)c$$

$$= 2\{vuc\} - \overline{\{uvc\}}, \text{ since } \{vuc\} \in \mathfrak{A}_1.$$

g): $\{abz\} = P(a,z)P(c)\bar{b} = L(a,c)L(z,c)\bar{b}$, by (13.12) and $P(c)z = 0$.

h): $\{xya\} = L(x,y)P(c)\bar{a} = P(c,x)L(y,c)\bar{a}$, by (13.9) and $P(c)y = 0$.

If $\mathfrak{L}$ is an ideal of $\mathfrak{A}$, then clearly $E_i\mathfrak{L} \subset \mathfrak{L}$ from the definition of E_i ($i = 0, 1, 1/2$). This implies immediately

<u>Lemma 3</u>. <u>If $\mathcal{L}$ is an ideal of $\mathfrak{A}$, then</u>

$$\mathcal{L} = (\mathcal{L} \cap \mathfrak{A}_1) \oplus (\mathcal{L} \cap \mathfrak{A}_{1/2}) \oplus (\mathcal{L} \cap \mathfrak{A}_0)$$

<u>Example</u>. Let F be a field and $\mathfrak{A} = F^{(m,n)}$ $(m \leq n)$ the Jts of all $m \times n$ matrices over F (composition is $P(x)y = xy^t x$). The matrix $c_k = \begin{pmatrix} e_k & 0 \\ 0 & 0 \end{pmatrix}$, where e_k is the $k \times k$ unit matrix, is an idempotent in $F^{(m,n)}$. It is easily checked that $\mathfrak{A}_1$ consists of all matrices of the form $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $a \in F^{(k,k)}$, $\mathfrak{A}_0$ is the space of matrices of the form $\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$, $b \in F^{(m-k,n-k)}$, and

$$\mathfrak{A}_{1/2} = \left\{ \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix} ; u \in F^{(k,n-k)}, v \in F^{(m-k,k)} \right\}.$$

It is clear that c_m is (in a certain sense) maximal and the Peirce decomposition relative to c_m has $\mathfrak{A}_1 \cong F^{(m,m)}$, $\mathfrak{A}_{1/2} \cong F^{(m,n-m)}$ and $\mathfrak{A}_0 = 0$.

15.2. The Peirce components $\mathfrak{A}_i$ are Jts (as subsystems of $\mathfrak{A}$); of importance is a relation between Rad $\mathfrak{A}_i$ and Rad $\mathfrak{A}$.

<u>Theorem 2</u>. <u>If</u> c <u>is an idempotent of</u> $\mathfrak{A}$ <u>and</u> $\mathfrak{A}_i(c)$ $(i = 0, 1/2, 1)$ <u>the Peirce components of</u> $\mathfrak{A}$ <u>relative</u> c, <u>then</u>

$$\operatorname{Rad} \mathfrak{A}_i(c) = \mathfrak{A}_i(c) \cap \operatorname{Rad} \mathfrak{A}.$$

<u>Proof</u>. $i = 0, 1$: We recall that in this case $(E_i, E_i) \in \Gamma(\mathfrak{A})$ and the shifting principle yields

$$(*) \qquad E_i\, q(x, E_i y) = q(E_i x, y)$$

(if one of these expressions exists). If $x \in \mathfrak{A}_i(c) \cap \operatorname{Rad} \mathfrak{A}$, then

$q(E_1x,y) = q(x,y)$ exists for all $y \in \mathfrak{A}$, in particular for all $y \in \mathfrak{A}_1$. Then (*) reads $q(x,y) = E_1q(x,y)$ for all $y \in \mathfrak{A}_1$ which shows that $q(x,y) \in \mathfrak{A}_1$ and then $x \in \text{Rad}\,\mathfrak{A}_1$. If conversely $x \in \text{Rad}\,\mathfrak{A}_1$, then $q(x,E_1y)$ exists for all $y \in \mathfrak{A}$ ($\mathfrak{A}_1 = E_1\mathfrak{A}$) and again by (*), $q(x,y)$ exists for all $y \in \mathfrak{A}$; i.e., $x \in \text{Rad}\,\mathfrak{A}$.

$i = 1/2$: Let $x \in \mathfrak{A}_{1/2} \cap \text{Rad}\,\mathfrak{A}$, then $q(x,u)$ exists for all $u \in \mathfrak{A}$, in particular for all $u \in \mathfrak{A}_{1/2}$. Using the formula $B(x,u)q(x,u) = x - P(x)u$ we show that $q(x,u) \in \mathfrak{A}_{1/2}$, if $x,u \in \mathfrak{A}_{1/2}$. We decompose $q(x,u) = q_o + q_1 + q_{1/2}$ into its Peirce components. From Theorem 1 we easily derive $B(x,u)q_i = q_i - \{xuq_i\} + P(x)P(u)q_i \in \mathfrak{A}_i$. Since $B(x,u)q(x,u) = x - P(x)u \in \mathfrak{A}_{1/2}$ we therefore get $B(x,u)(q_o+q_1) = 0$ and then $q_o + q_1 = 0$ ($B(x,u)$ is invertible). This shows $x \in \text{Rad}\,\mathfrak{A}_{1/2}$. To prove the converse, we first observe $\mathfrak{A} = \mathfrak{A}_{1/2} \oplus (\mathfrak{A}_o \oplus \mathfrak{A}_1)$ and $P(\mathfrak{A}_o \oplus \mathfrak{A}_1)\mathfrak{A}_{1/2} \subset P(\mathfrak{A}_1)\mathfrak{A}_{1/2} + P(\mathfrak{A}_o)\mathfrak{A}_{1/2} + \{\mathfrak{A}_o\mathfrak{A}_{1/2}\mathfrak{A}_1\} \subset \mathfrak{A}_{1/2}$, by Theorem 1. This shows that $\text{Rad}\,\mathfrak{A}_{1/2} \subset \text{Rad}\,\mathfrak{A}$ will follow from the following lemma.

<u>Lemma</u> 4. <u>If</u> $\mathfrak{A} = \mathfrak{U} + \mathfrak{V}$, $\mathfrak{U}$ <u>subsystem and</u> $P(v)\mathfrak{U} \subset \mathfrak{U}$ <u>for all</u> $v \in \mathfrak{V}$, <u>then</u> $\text{Rad}\,\mathfrak{U} \subset \text{Rad}\,\mathfrak{A}$.

<u>Proof</u>. $z \in \text{Rad}\,\mathfrak{U} \Rightarrow q(z,u)$ exists for all $u \in \mathfrak{U}$. Since $P(v)u \in \mathfrak{U}$ (for $v \in \mathfrak{V}$, $u \in \mathfrak{U}$) we have in particular $q(z,P(v)z)$ exists, or equivalently, $B(z,v)B(z,-v) = B(z,P(v)z)$ is invertible. This implies $B(z,v)$ is invertible, or $q(z,v)$ exists for all $v \in \mathfrak{V}$. For arbitrary $x \in \mathfrak{A}$, we decompose $x = u + v$, $u \in \mathfrak{U}$, $v \in \mathfrak{V}$, and $q(z,u)$ exists (by the choice of z) and this is an element of $\text{Rad}\,\mathfrak{U}$ again. We apply the foregoing to conclude that $q(q(z,u),v)$ exists. Then $q(z,x) = q(z,u+v) = q(q(z,u),v)$ exists, by the addition theorem. Thus $z \in \text{Rad}\,\mathfrak{A}$.

Another immediate application of lemma 4 is

<u>Corollary 1. If $\mathfrak{J}$ is a Jordan algebra,</u> α <u>an involutoric automorphism of $\mathfrak{J}$ and $\mathfrak{J} = \mathfrak{J}_+ + \mathfrak{J}_-$, where</u> $\mathfrak{J}_\varepsilon = \{x;\ \alpha x = \varepsilon x\}$, <u>then</u> Rad $\mathfrak{J}_\varepsilon = \mathfrak{J}_\varepsilon \cap$ Rad $\mathfrak{J}$.

Now, let A be an associative algebra over Φ with unit element and involution $\xi \mapsto \bar{\xi}$. The Φ-module $A^{(m,n)}$ of all $m \times n$ matrices over A together with the composition $P(x)y = x\bar{y}^t x$ is a Jts, where $\bar{y}^t$ denotes the conjugate transposed of y. If R(A) denotes the Jacobson radical of A, then we have

<u>Corollary 2</u>: Rad $A^{(m,n)} = R(A)^{(m,n)}$

<u>Proof</u>. The map $u \mapsto \begin{pmatrix} 0 & u \\ \bar{u}^t & 0 \end{pmatrix}$ defines an imbedding (injective homomorphism into) of $A^{(m,n)}$ into the Jordan algebra $\mathfrak{J} = H(A_{m+n})$ of symmetric (relative to the induced involution in A_{m+n}) $(m+n) \times (m+n)$ matrices over A. It is well known (and very easy to verify) that $A^{(m,n)}$ is isomorphic to the Peirce-1/2-component of $\mathfrak{J}$ relative to the idempotent $c = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$, where e is the unit matrix in $A^{(m,m)}$. A well known result for Jordan algebras is

$$\text{Rad } A^{(m,n)} = H(A_{m+n}) \cap R(A)^{(m+n,m+n)}$$

consequently

$$\text{Rad } A^{(m,n)} \cong \text{Rad } \mathfrak{J}_{1/2} = \mathfrak{J}_{1/2} \cap \text{Rad } \mathfrak{J} \cong \mathfrak{J}_{1/2} \cap H(A_{m+n}) \cap R(A)^{(m+n,m+n)}$$
$$\cong R(A)^{(m,n)}.$$

Theorem 3.

$$\operatorname{Rad}\mathfrak{A} = \operatorname{Rad}\mathfrak{A}_1 \oplus \operatorname{Rad}\mathfrak{A}_{1/2} \oplus \operatorname{Rad}\mathfrak{A}_0$$

Proof. $\operatorname{Rad}\mathfrak{A}$ is an ideal in $\mathfrak{A}$ and then $\operatorname{Rad}\mathfrak{A} = \oplus(\mathfrak{A}_i \cap \operatorname{Rad}\mathfrak{A})$, by Lemma 3. Theorem 2 gives the result.

Corollary. $\mathfrak{A}$ is semi simple, iff $\mathfrak{A}_i$ $(i = 0, 1, 1/2)$ is semi simple.

15.3. Of particular importance for the structure theory of Jts we shall present later is the Peirce decomposition of $\mathfrak{A}$ relative to a maximal idempotent. An idempotent $c \in \mathfrak{A}$ is called maximal, if $\mathfrak{A}_0(c) = 0$. (See the example in 15.1.)

Let c be maximal, then

$$\mathfrak{A} = \mathfrak{A}_1(c) \oplus \mathfrak{A}_{1/2}(c)$$

and besides the Peirce rules in Theorem 1 and Lemma 2 we have

(15.11) $$P(\mathfrak{A}_{1/2})\mathfrak{A}_1 = \{\mathfrak{A}_{1/2}\mathfrak{A}_1\mathfrak{A}_{1/2}\} = 0 \quad \text{(see Theorem 1b)}$$

This fact has rather strong implications.

Let $\mathfrak{L}$ be an ideal of $\mathfrak{A}$, then $\mathfrak{L} = \mathfrak{U} \oplus \mathfrak{Q}$, by Lemma 3. Using Theorem 1 and (15.11) we easily verify $\mathfrak{L} = \mathfrak{U} \oplus \mathfrak{Q}$, $\mathfrak{U} \subset \mathfrak{A}_1$, $\mathfrak{Q} \subset \mathfrak{A}_{1/2}$, in $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_{1/2}$, iff $\mathfrak{U}$, resp. $\mathfrak{Q}$, is an ideal in $\mathfrak{A}_1$, resp. $\mathfrak{A}_{1/2}$ and

(15.12)

a) $\{\mathfrak{A}_1\mathfrak{A}_{1/2}\mathfrak{Q}\} \subset \mathfrak{U}$ d) $\{\mathfrak{A}_1\mathfrak{A}_1\mathfrak{Q}\} \subset \mathfrak{Q}$

b) $\{\mathfrak{A}_{1/2}\mathfrak{A}_{1/2}\mathfrak{U}\} \subset \mathfrak{U}$ e) $\{\mathfrak{U}\mathfrak{A}_1\mathfrak{A}_{1/2}\} \subset \mathfrak{Q}$

c) $\{\mathfrak{A}_1\mathfrak{Q}\mathfrak{A}_{1/2}\} \subset \mathfrak{U}$ f) $\{\mathfrak{A}_1\mathfrak{U}\mathfrak{A}_{1/2}\} \subset \mathfrak{Q}$

An ideal $\mathfrak{U}$ in $\mathfrak{A}_1$ (as Jts) is in particular an ideal in the unital Jordan algebra $(\mathfrak{A}_1, P_c, c)$ which is invariant under the involution. A particular case of d) is $\mathfrak{A}_1 \circ \mathfrak{Q} = \{\mathfrak{A}_1 c \mathfrak{Q}\} \subset \mathfrak{Q}$, but this is actually equivalent to d) using (15.10g). A special case of e) is $\mathfrak{U} \circ \mathfrak{A}_{1/2} = \{\mathfrak{U} c \mathfrak{A}_{1/2}\} \subset \mathfrak{Q}$ and again (15.10g) shows $\{\mathfrak{U}\mathfrak{A}_1\mathfrak{A}_{1/2}\} \subset \mathfrak{U}\circ\mathfrak{A}_{1/2}$. Thus e) is equivalent to $\mathfrak{U}\circ\mathfrak{A}_{1/2} \subset \mathfrak{Q}$. b) can be omitted, because it is a consequence of e) and c) (using (15.10h)). Also g) can be omitted, using (15.10g) and $\overline{\mathfrak{U}} = \mathfrak{U}$, $\mathfrak{U}\circ\mathfrak{A}_{1/2} \subset \mathfrak{Q}$, $\mathfrak{A}_1 \circ \mathfrak{Q} \subset \mathfrak{Q}$. a) and c) reduce to $\{c\mathfrak{A}_{1/2}\mathfrak{Q}\} \subset \mathfrak{U}$ (or $\{c\mathfrak{Q}\mathfrak{A}_{1/2}\}$) using 15.10 h) and f). So far we have proved

<u>Lemma 5.</u> $\mathfrak{J} = \mathfrak{U} \oplus \mathfrak{Q}$ <u>is an ideal in</u> $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_{1/2}$ $(\mathfrak{U} \subset \mathfrak{A}_1$, $\mathfrak{Q} \subset \mathfrak{A}_{1/2})$ <u>iff</u> a) $\mathfrak{U}$ <u>is an ideal in</u> $\mathfrak{A}_1$

b) $\mathfrak{Q}$ <u>is an ideal in</u> $\mathfrak{A}_{1/2}$

c) $\{\mathfrak{A}_1 c \mathfrak{Q}\} \subset \mathfrak{Q}$, $\{\mathfrak{A}_{1/2} c \mathfrak{U}\} \subset \mathfrak{Q}$, $\{c\mathfrak{A}_{1/2}\mathfrak{Q}\} \subset \mathfrak{U}$.

Next, let $\mathfrak{U}$ be an ideal in $\mathfrak{A}_1$. Set $\mathfrak{Q} := \{\mathfrak{A}_{1/2} c \mathfrak{U}\}$. We first show that for these submodules $\mathfrak{U}, \mathfrak{Q}$ condition c) of the above lemma is fulfilled.

$$\{\mathfrak{A}_1 c \{\mathfrak{U} c \mathfrak{A}_{1/2}\}\} \subset \{P(\mathfrak{A}_1, \mathfrak{U}) cc\ \mathfrak{A}_{1/2}\} + \{\mathfrak{U} c\ \{\mathfrak{A}_1 c \mathfrak{A}_{1/2}\}\}$$

(by a linearization of (15.10e)) $\subset \mathfrak{Q}$, since $\mathfrak{U}$ is an ideal. For $u \in \mathfrak{U}$, $x, y \in \mathfrak{A}_{1/2}$ we get from (13.11)

$$\{c\bar{u}\{cyx\}\} = \{uyx\} - \{c\{\bar{u}cy\}x\} + \{cy\{c\bar{u}x\}\}$$

$$= \{cy\{ucx\}\}\ \text{, by (15.10h) and d)).}$$

The left hand side of this equation is in $\mathfrak{U}$, thus $\{c\mathfrak{A}_{1/2}\{\mathfrak{A}_{1/2}c\mathfrak{U}\}\}\subset\mathfrak{U}$. Secondly we have to show that $\mathfrak{Q} = \{\mathfrak{A}_{1/2}\, c\,\mathfrak{U}\}$ is an ideal in $\mathfrak{A}_{1/2}$. Let $x,y,z \in \mathfrak{A}_{1/2}$, $u \in \mathfrak{U}$. Then

$$P(x)\{ycu\} = P(x)L(y,c)u + L(c,y)P(x)u \quad (P(x)u = 0)$$

$$= P(\{cyx\},x)u = \{uax\} \quad (a = \{cyx\})$$

$$= \{uc\{\bar{a}cx\}\} \in \mathfrak{Q} .$$

The following theorem is true (McCrimmon), but my original proof given here was incorrect.

<u>Theorem 4</u>. <u>Let</u> $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_{1/2}$ <u>be the Peirce decomposition of</u> $\mathfrak{A}$ <u>relative to a maximal idempotent and</u> $\mathfrak{U}$ <u>an ideal of</u> $\mathfrak{A}_1$, <u>then</u> $\mathfrak{B} := \mathfrak{U} \oplus \{\mathfrak{A}_{1/2}c\mathfrak{U}\}$ <u>is an ideal in</u> $\mathfrak{A}$.

<u>Corollary 1</u>. <u>If</u> $\mathfrak{A}$ <u>is simple</u>, <u>then</u> $\mathfrak{A}_1$ (as Jts) <u>is simple</u>.

<u>Note</u>: We only proved this result for a maximal idempotent. The corresponding result for arbitrary idempotents has not been proved yet.

Since any P(c)-invariant ideal of the Jordan algebra $(\mathfrak{A}_1, P_c)$ is an ideal in the triple system $\mathfrak{A}_1$ we have

<u>Corollary 2</u>. <u>If $\mathfrak{A}$ is simple, then</u> $(\mathfrak{A}_1, P(c))$ <u>(as Jordan algebra) is a simple pair</u>.

We wish to prove a partial converse of corollary 1.

<u>Lemma 6</u>. <u>If $\mathfrak{A}$ is semi simple</u>, $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_{1/2}$, <u>and</u> $\mathfrak{A}_1$ <u>is simple, then $\mathfrak{A}$ is simple</u>.

<u>Proof</u>. If $\mathfrak{L}$ is an ideal of $\mathfrak{A}$, $\mathfrak{L} = \mathfrak{U} \oplus \mathfrak{V}$ (as above) then either $\mathfrak{U} = 0$ or $\mathfrak{U} = \mathfrak{A}_1$. If $\mathfrak{U} = \mathfrak{A}_1$, then $\mathfrak{A}_{1/2} = \{\mathfrak{A}_{1/2} c \mathfrak{A}_1\} \subset \mathfrak{V}$ and $\mathfrak{L} = \mathfrak{A}$. If $\mathfrak{U} = 0$ then $\{c\, \mathfrak{A}_{1/2} \mathfrak{V}\} = 0$, by Lemma 5c). For $v \in \mathfrak{V}$ we get $P(v)\mathfrak{A}_1 = 0$, by (15.11). Also, using (13.8) we get

$$P(v)\mathfrak{A}_{1/2} = P(\{ccv\})\mathfrak{A}_{1/2} = [P(c)P(c)P(v) + P(v)P(c)P(c) + P(v,c)P(c)P(v,c)]\,\mathfrak{A}_{1/2}$$

$$= 0, \text{ since } P(c)\mathfrak{A}_{1/2} = 0,\ P(v)c = 0 \text{ and } \{c\mathfrak{A}_{1/2}v\} = 0.$$

This shows that v is a trivial element and therefore contained in $\mathrm{Rad}\,\mathfrak{A} = 0$. Thus $\mathfrak{L} = 0$.

15.4. Let $c, d \in \mathfrak{A}$ be idempotents. c is <u>orthogonal</u> to d (denoted by $c \perp d$) if $d \in \mathfrak{A}_o(c)$.

<u>Lemma 7</u>. $c \perp d \Longleftrightarrow P(c)d = \{ccd\} = 0$

$$\Longleftrightarrow P(d)c = \{ddc\} = 0.$$

<u>In either case</u> $L(c,d) = L(d,c) = 0$.

<u>Proof</u>. If $d \in \mathfrak{A}_o(c)$, then $P(c)d = \{ccd\} = 0$, by Thm. 1. If conversely $P(c)d = \{ccd\} = 0$ and $d = d_1 + d_{1/2} + d_o$ the Peirce decomposition of d

relative c, then $P(c)d = 0$ shows $\bar{d}_1 = 0$, hence $d_1 = 0$. Then $0 = \{ccd\} = \{cc(d_o + d_{1/2})\} = d_{1/2}$, by (15.5). Thus $d = d_o \in \mathfrak{A}_o(c)$. Since $P(\mathfrak{A}_o)\mathfrak{A}_1 = 0$ and $L(\mathfrak{A}_1, \mathfrak{A}_o) = 0$, by Theorem 1, we get in particular for $d \in \mathfrak{A}_o(c)$, $P(d)c = \{ddc\} = 0$.

Using this lemma it is easily seen that for orthogonal idempotents c, d the sum $c \pm d$ is also an idempotent. Idempotents $c_1, \ldots, c_r$ are called orthogonal, if $c_i \perp c_j$ $(i \neq j)$. If c, d, e are orthogonal idempotents then c is orthogonal to $d \pm e$. This implies that for orthogonal idempotents $c_1, c_2, \ldots, c_r$, the sum $c: = \Sigma\, c_i$ (as well as $\Sigma\, \varepsilon_i c_i$, $\varepsilon_i = \pm 1$) is an idempotent of $\mathfrak{A}$. An idempotent is called primitive, if it has no nontrivial decomposition as sum of orthogonal idempotents.

Lemma 8. Let c, d, e, f be idempotents in $\mathfrak{A}$.

i) If $c \perp d$, then $P(c)$, $P(d)$, $P(c,d)$ are orthogonal.*)

ii) If c, d, e are orthogonal, then $P(c)$, $P(c,d)$, $P(d,e)$ are orthogonal.

iii) If c, d, e, f are orthogonal, then $P(c,d)$, $P(g,h)$ are orthogonal.

iv) $c \perp d$, then $\mathfrak{A}_1(c) \subset \mathfrak{A}_o(d)$.

Proof. $P(c)P(d) = P(c)P(c)P(c)P(d)$

$$= P(c)[P(\{ccd\}) + P(P(c)c, P(d)c) - P(c,d)P(c)P(d,c) - P(d)P(c)P(c)] \qquad \text{(by (13.8))}$$

$$= 0, \text{ since } \{ccd\} = P(d)c = P(c)d = 0.$$

Using (13.13) we get for arbitrary $a \in \mathfrak{A}$

*) Linear maps A, B are orthogonal, if $AB = BA = 0$.

$P(c)P(d,a) = L(c,d)L(c,a) - L(P(c)d,a) = 0$, also $P(d,a)P(c) = 0$, by the dual formula. This completes i) and shows already part of ii). From a linearization of (13.13) (resp. its dual)

$$P(y,z)P(x,a) = L(y,x)L(z,a) + L(z,x)L(y,a) - L(P(y,z)x,a)$$

we obtain for arbitrary $a \in \mathfrak{A}$

$$P(c,d)P(e,a) = 0, \text{ since } L(c,e) - L(d,e) = 0, \text{ by Lemma } 7 .$$

This completes i)-iii). Now let $x = P(c)P(c)x \in \mathfrak{A}_1(c)$. We show that the 1/2- and 1-component of x in the decomposition relative to d are zero. By definition of these components we have to show $P(d)P(d)x = 0$, which is immediate, since $P(d)P(c) = 0$, and $[L(d,d)-2P(d)P(d)]x = 0$. It remains to show $\{ddx\} = 0$. But this is a consequence of $L(d,d)P(c) = P(c,d)L(d,c) - P(P(c)d,d) = 0$, by (13.9) and the orthogonality relations.

15.5. Let $c_1, \cdots, c_r$ be a set of orthogonal idempotents in $\mathfrak{A}$ and $c := \Sigma c_i$. c is an idempotent of $\mathfrak{A}$. We consider the Jordan algebra $\mathfrak{A}_c$. By Lemma 7 we have $P(c_i)c_j = 0$ for $i \neq j$ and therefore $c_i^{(2,c)} = P(c_i)c = \Sigma P(c_i)c_j = P(c_i)c_i = c_i$. Thus c_i is an idempotent in $\mathfrak{A}_c$. Furthermore, since

$$\{c_i c_j c_k\} = 0 \qquad \text{if } j \neq i \text{ or } j \neq k$$

we have for $i \neq j$

$$P_c(c_i)c_j = P(c_i)P(c)c_j = P(c_i)(\Sigma P(c_k) + \sum_{k<\ell} P(c_k,c_\ell))c_j = 0$$

and $c_i \circ c_j = \{c_i c c_j\} = \Sigma\{c_i c_k c_j\} = 0$. This shows that the c_i are orthogonal idempotents in the Jordan algebra $\mathfrak{A}_c$. This allows us to apply the well known results about the Peirce decomposition in Jordan algebras relative to a set of orthogonal idempotents. It is almost trivial, but we have to notice it: c_i is idempotent in $\mathfrak{A}_{c_i}$ and in $\mathfrak{A}_c$. The Peirce-λ-spaces ($\lambda = 0, 1/2, 1$) $\mathfrak{A}_\lambda(c_i)$ rel. c_i (of $\mathfrak{A}_{c_i}$) are (by definition)

$$\mathfrak{A}_1(c_i) = P(c_i)P(c_i)\mathfrak{A},\ \mathfrak{A}_o(c_i) = B(c_i,c_i)\mathfrak{A},\ \mathfrak{A}_{1/2}(c_i) = [L(c_i,c_i) - 2P(c_i)P(c_i)]\mathfrak{A}\ .$$

We also have the Peirce decomposition of $\mathfrak{A}$ relative c, and the decomposition of $\mathfrak{A}_c$ relative c_i, say $\mathfrak{A} = \mathcal{L}_1 \oplus \mathcal{L}_o \oplus \mathcal{L}_{1/2}$, where by definition

$$\mathcal{L}_1 = P_c(c_i)\mathfrak{A} = P(c_i)P(c)\mathfrak{A}$$

$$\mathcal{L}_o = B(c_i,c)\mathfrak{A} \text{ and } \mathcal{L}_{1/2} = [L(c_i,c) - 2P(c_i)P(c)]\mathfrak{A}$$

Fortunately, since $P(c_i)P(c) = P(c_i)P(c_i)$, by Lemma 8, and $L(c_i,c) = L(c_i,c_i)$, we have $\mathcal{L}_\lambda = \mathfrak{A}_\lambda(c_i)$ and we need not distinguish these two decompositions.

Now we define

$$E_{ii} := P(c_i)P(c),\ E_{ij} = E_{ij} : = P(c_i,c_j)P(c) \quad i \neq j,\ i,j \neq 0$$

$$E_{oo} := B(c,c) \qquad E_{io} = E_{oi} : = L(c_i,c) - P(c_i,c)P(c),\ i \neq 0$$

From the definition and Lemma 8 we get that the E_{ij} form a set of orthogonal projections and $\mathrm{Id} = \sum_{o\leq i,j\leq r} E_{ij}$.

The last equation is seen from the following relations $E_1 = P(c)P(c) = \sum_{1\le i,j\le r} E_{ij}$

(15.13) $$E_o = E_{oo}$$

$$E_{1/2} = \sum_{1\le i} E_{oi}$$

The sum $Id = \Sigma E_{ij}$ of orthogonal projections induces a decomposition of $\mathfrak{A}$

$$\mathfrak{A} = \bigoplus_{0\le i,j\le r} \mathfrak{A}_{ij}, \text{ where } \mathfrak{A}_{ij} := E_{ij}\mathfrak{A}.$$

(15.13) clearly implies

(15.14) $$\mathfrak{A}_1(c) = \bigoplus_{1\le i,j\le r} \mathfrak{A}_{ij}, \ \mathfrak{A}_o(c) = \mathfrak{A}_{oo}, \ \mathfrak{A}_{1/2}(c) = \bigoplus_{1\le i} \ {}_{io}$$

Next we wish to relate this decomposition to the Peirce decomposition relative to c_i. We claim

a) $\mathfrak{A}_1(c_i) = \mathfrak{A}_{ii}$ b) $\mathfrak{A}_{1/2}(c_i) = \bigoplus_{k\ne i} \mathfrak{A}_{ik}$

c) $\mathfrak{A}_o(c_i) = \bigoplus_{k,j\ne i} \mathfrak{A}_{kj}$ d) $\mathfrak{A}_{oi} = \mathfrak{A}_{1/2}(c) \cap \mathfrak{A}_{1/2}(c_i)$

(15.15) e) $\mathfrak{A}_{ij} = \mathfrak{A}_{1/2}(c_i) \cap \mathfrak{A}_{1/2}(c_j)$,

$i \ne j$, $i,j \ne 0$

The <u>proof</u> of 15.15 is left as an exercise. Hint: show that the corresponding projections are equal; for d) and e) compare (15.14) and (15.15b).

<u>Theorem 5.</u> <u>If $c_1, \ldots, c_r$ is a set of orthogonal idempotents in $\mathfrak{A}$,</u> $c := \Sigma c_i$, $E_{ii} = P(c_i)P(c)$, $E_{ij} = E_{ji} = P(c_i, c_j)P(c)$ $(i \ne j,\ i,j \ne 0)$, $E_{oo} = B(c,c)$, $E_{io} = E_{oi} = L(c_i,c) - P(c_i,c)P(c)$ $(i \ne 0)$, <u>then the</u> E_{ij} <u>are orthogonal projections of $\mathfrak{A}$ with sum</u> Id <u>and</u>

$$\mathfrak{A} = \bigoplus_{0\le i\le j\le r} \mathfrak{A}_{ij} \quad \underline{\text{where}} \quad \mathfrak{A}_{ij} = E_{ij}\mathfrak{A}$$

<u>and the following composition rules hold</u>: (i,j,k,ℓ <u>distinct</u>)

a) $P(\mathfrak{A}_{ii})\mathfrak{A}_{ii} \subset \mathfrak{A}_{ii}$

b) $P(\mathfrak{A}_{ij})\mathfrak{A}_{jj} \subset \mathfrak{A}_{ii}$

c) $P(\mathfrak{A}_{ij})\mathfrak{A}_{ij} \subset \mathfrak{A}_{ij}$

d) $\{\mathfrak{A}_{ii}\mathfrak{A}_{ij}\mathfrak{A}_{jj}\} \subset \mathfrak{A}_{ij}$

e) $\{\mathfrak{A}_{ii}\mathfrak{A}_{ii}\mathfrak{A}_{ij}\} \subset \mathfrak{A}_{ij}$

f) $\{\mathfrak{A}_{ii}\mathfrak{A}_{ij}\mathfrak{A}_{ji}\} \subset \mathfrak{A}_{ii}$

g) $\{\mathfrak{A}_{ii}\mathfrak{A}_{ij}\mathfrak{A}_{jk}\} \subset \mathfrak{A}_{ik}$

h) $\{\mathfrak{A}_{jk}\mathfrak{A}_{ii}\mathfrak{A}_{ik}\} \subset \mathfrak{A}_{jk}$

i) $\{\mathfrak{A}_{ij}\mathfrak{A}_{ji}\mathfrak{A}_{ik}\} \subset \mathfrak{A}_{ik}$

j) $\{\mathfrak{A}_{ij}\mathfrak{A}_{jk}\mathfrak{A}_{ki}\} \subset \mathfrak{A}_{ii}$

k) $\{\mathfrak{A}_{ij}\mathfrak{A}_{jk}\mathfrak{A}_{k\ell}\} \subset \mathfrak{A}_{i\ell}$

<u>while all other products are zero</u>.

<u>Note</u>: The "product" of 3 elements is zero unless it can be written in the form $\{x_{ij}y_{jk}z_{k\ell}\}$, in which case it is an element in $\mathfrak{A}_{i\ell}$.

<u>Proof</u>. If in $P(\mathfrak{A}_{ij})\mathfrak{A}_{k\ell}$ or $\{\mathfrak{A}_{ij}\mathfrak{A}_{k\ell}\mathfrak{A}_{mn}\}$ the indices k,ℓ are both $\ne 0$, then $\mathfrak{A}_{k\ell} \subset \mathfrak{A}_1(c)$, by (15.14). Since in this case $P(c)\mathfrak{A}_{k\ell} = \mathfrak{A}_{k\ell}$ we can express $P(\mathfrak{A}_{ij})\mathfrak{A}_{k\ell} = P_c(\mathfrak{A}_{ij})\mathfrak{A}_{k\ell}$ and $P(\mathfrak{A}_{ij},\mathfrak{A}_{mn})\mathfrak{A}_{k\ell} = P_c(\mathfrak{A}_{ij},\mathfrak{A}_{mn})\mathfrak{A}_{k\ell}$ as corresponding compositions in the Jordan algebra $\mathfrak{A}_c$, for which the given composition rules are well known. Therefore we have to establish the rules only if k or ℓ (or both) are zero. Since the proof we have for these cases is a case by case checking we leave it as an exercise to find a nice proof for the remaining cases or a nice unified proof of the theorem without refering to Jordan algebras.

15.6. Let $\mathfrak{A}$ be a semi simple Jordan triple system with d.c.c. and c a maximal idempotent in $\mathfrak{A}$.

$\mathfrak{A} = \mathfrak{A}_1(c) \oplus \mathfrak{A}_{1/2}(c)$ (see 15.3).

$\mathfrak{A}_1 = \mathfrak{A}_1(c)$ is a subalgebra of $\mathfrak{A}_c$, since $P(c)|\mathfrak{A}_1$ is invertible the Jt structure of $\mathfrak{A}_1$ comes from an isotope of the algebra $\mathfrak{A}_1$ (Lemma 10.1). Therefore $\mathfrak{A}_1$ as an algebra is semi simple, since $\mathfrak{A}_1$ as a $\mathfrak{J}$ts is semi simple (Theorem 15.2). Also dcc carries over from $\mathfrak{A}$ to $\mathfrak{A}_1$ (easy exercise). We know from Jordan theory that the Jordan algebra $\mathfrak{A}_1$ is the finite direct sum of simple algebras $\mathfrak{J}_i$. Every $\mathfrak{J}_i$ has a unit element e_i. $P(c)$ is an involution of $\mathfrak{A}_1$ (Lemma 15.2) and we have

$$\mathfrak{A}_1 = \bigoplus \mathfrak{L}_i \quad \text{where } \mathfrak{L}_i = \mathfrak{J}_i + P(c)\mathfrak{J}_i.$$

The $\mathfrak{L}_i$ have $P(c)$ invariant unit element $c_i = e_i + P(c)e_i$. Then the c_i are orthogonal idempotents in the Jts $\mathfrak{A}$, $c = \Sigma\, c_i$ and we have the Peirce decomposition $\mathfrak{A} = \bigoplus(\mathfrak{A}_{ii} \oplus \mathfrak{A}_{io})$, $\mathfrak{A}_{ii} = \mathfrak{L}_i$. From Theorem 5 we can easily derive that $\mathfrak{A}_{ii} \oplus \mathfrak{A}_{io}$ is an ideal of $\mathfrak{A}$.

Since $(\mathfrak{L}_i, P(c)|_{\mathfrak{L}_i})$ is a simple pair, the $\mathfrak{J}$ts $\mathfrak{A}_{ii} = \mathfrak{L}_i$ is simple. Also $\mathfrak{A}_{ii} \oplus \mathfrak{A}_{oi}$ is semi simple (since $\mathfrak{A}$ is semi simple). Now we can apply Lemma 15.6 to $\mathfrak{A}_{ii} \oplus \mathfrak{A}_{oi}$ (idempotent c_i) to conclude that this is a simple Jts. We proved

<u>Theorem 6</u>. <u>If $\mathfrak{A}$ is a semi simple Jts with dcc and maximal idempotent, then $\mathfrak{A}$ is the (finite) direct sum of simple Jts with dcc, and maximal idempotent</u>.

XVI. Alternative Triple Systems

16.1 A unital Φ-module $\mathcal{A}$ together with a trilinear map $\mathcal{A}\times\mathcal{A}\times\mathcal{A}\to\mathcal{A}$, $(x,y,z)\mapsto\langle xyz\rangle$ is called an <u>alternative triple system</u>, if for all $x,y,z,u,v\in\mathcal{A}$

(AT 1) $$\langle xy\langle uvz\rangle\rangle+\langle uv\langle xyz\rangle\rangle=\langle\langle uvx\rangle yz\rangle+\langle x\langle vuy\rangle z\rangle$$

(AT 2) $$\langle xy\langle xyz\rangle\rangle=\langle\langle xyx\rangle yz\rangle$$

(AT 3) $$\langle uv\langle xyx\rangle\rangle=\langle\langle uvx\rangle yx\rangle$$

<u>Examples</u>. 1) Any associative triple system (of the second kind) is an alternative triple system.

2) If $\mathcal{L}$ is an alternative algebra with involution $x\mapsto\bar{x}$, then $\mathcal{L}$ together with $\langle xyz\rangle=(x\bar{y})z$ is an alternative triple system.

3) Let $\mathcal{C}$ be a commutative associative Φ-algebra with involution $a\to\bar{a}$, $\mathcal{M}$ a $\mathcal{C}$-module and $\varphi:\mathcal{M}\times\mathcal{M}\to\mathcal{C}$ a hermitian sesquilinear form (that is, $x\mapsto\varphi(y,x)$ is linear and $\overline{\varphi(x,y)}=\varphi(y,x)$). Furthermore let $j:\mathcal{M}\to\mathcal{M}$ be a $\mathcal{C}$-antilinear map such that $j^2=-\,\mathrm{Id}$, $\gamma\in\Phi$, and α an alternating $\mathcal{C}$-bilinearform on $\mathcal{M}$ such that $\varphi(x,y)+\alpha(j(x),y)=0$. Then $\mathcal{M}$ together with

$$\langle xyz\rangle=\varphi(y,z)x+\alpha(x,z)j(y)$$

is an alternative triple system. (The verification of this statement is left as an exercise).

As a first observation we note that the left hand side of (AT 1) is symmetric in the pairs (x,y), (u,v), thus

(16.1) $$\langle\langle xyu\rangle vz\rangle + \langle u\langle yxv\rangle z\rangle = \langle\langle uvx\rangle yz\rangle + \langle x\langle vuy\rangle z\rangle.$$

Setting $z = x$ in (AT 1) we observe that (in the presence of (AT 1)) the equation (AT 3) is equivalent to

(16.2) $$\langle xy\langle uvx\rangle\rangle = \langle x\langle vuy\rangle x\rangle.$$

A linearization of (AT 2) is

$$\langle xy\langle xuz\rangle\rangle + \langle xu\langle xyz\rangle\rangle = \langle\langle xyx\rangle uz\rangle + \langle\langle xux\rangle yz\rangle.$$

And from (AT 1) and (16.1) we get

$$\langle xy\langle xuz\rangle\rangle + \langle xu\langle xyz\rangle\rangle = \langle\langle xyx\rangle uz\rangle + \langle x\langle yxu\rangle z\rangle.$$

Comparing these two equations we obtain

(16.3) $$\langle\langle xux\rangle yz\rangle = \langle x\langle yxu\rangle z\rangle.$$

Another linearization of (AT 2) is

$$\langle uy\langle xyz\rangle\rangle + \langle xy\langle uyz\rangle\rangle = \langle\langle uyx\rangle yz\rangle + \langle\langle xyu\rangle yz\rangle$$

Comparing this equation with (AT 1) ($v = y$) gives

(16.4) $$\langle\langle xyu\rangle yz\rangle = \langle x\langle yuy\rangle z\rangle$$

In particular ($z = u$) we get from this equation

(16.5) $$\langle xy\langle uyu\rangle\rangle = \langle x\langle yuy\rangle u\rangle,$$

by using (AT 3). Again (AT 1) shows

$$\langle xy\langle uyx\rangle\rangle + \langle uy\langle xyx\rangle\rangle = \langle\langle xyu\rangle xy\rangle + \langle u\langle yxy\rangle x\rangle.$$

This equation and (16.5) imply

(16.6) $$\langle xy\langle uyx\rangle\rangle = \langle\langle xyu\rangle yx\rangle.$$

<u>Exercise:</u> Define $x^1 := x$, $x^{n+2} := \langle xx^nx\rangle$ ($n \geq 1$ odd)
and show: $\langle x^k x^\ell x^m\rangle = x^{k+\ell+m}$ (k, ℓ, m odd)

16.2. <u>Theorem</u> 1. <u>If $\mathfrak{A}$ is an alternative triple system over Φ, then $\mathfrak{A}^+ = (\mathfrak{A}, P)$, where P is defined by</u> $P(x)y = \langle xyx\rangle$, <u>is a Jordan triple system.</u>

<u>Proof</u>: $L(x,y)z = P(x,z)y = \langle xyz\rangle + \langle zyx\rangle$. We have to show

a) $L(x,y)P(x) = P(x)L(y,x)$

b) $L(P(x)y,y) = L(x,P(y)x)$

c) $P(P(x)y) = P(x)P(y)P(x)$

ad a): $L(x,y)P(x)z = \langle xy\langle xzx\rangle\rangle + \langle\langle xzx\rangle yx\rangle$

$= \langle x\langle zxy\rangle x\rangle + \langle x\langle yxz\rangle x\rangle$, by (16.2) and (16.3)

$= P(x)L(y,x)z.$

ad b): $L(P(x)y,y)z = \langle\langle xyx\rangle yz\rangle + \langle zy\langle xyx\rangle\rangle$

$= \langle x\langle yxy\rangle z\rangle + \langle z\langle yxy\rangle x\rangle$, by (16.3) and (16.5)

$= L(x,P(y)x)z.$

Next we observe that a particular case of (16.2) is

$$P(y)P(x)u = \langle yx\langle uxy\rangle\rangle.$$

Then $P(x)P(y)P(x)u = \langle x\langle yx\langle uxy\rangle\rangle x\rangle$

$= \langle\langle x\langle uxy\rangle x\rangle yx\rangle$, by (16.3)

$= \langle\langle xyx\rangle ux\rangle yx\rangle$, by (16.3)

$= \langle\langle xyx\rangle u\langle xyx\rangle\rangle$, by (AT 3)

$= P(P(x)y)u.$

This is c) and the proof is complete.

Another important result relating alternative and Jordan triple systems is the following:

<u>Theorem 2</u>. <u>If</u> ($\mathfrak{A}$, P) <u>is a Jordan triple system and</u> c <u>an idempotent such that</u> $\mathfrak{A}_o(c) = 0$. <u>Then</u> $\mathfrak{A}_{1/2}(c)$ <u>together with the map</u> $(x,y,z) \mapsto \langle xyz\rangle$, <u>defined by</u>

$$\langle xyz\rangle = \{\{xyc\}cz\}$$

<u>is an alternative triple system such that</u>

(16.7) $$P(x)y = \langle xyx\rangle.$$

<u>Proof</u>: Let $x, y, u, v, z \in \mathfrak{A}_{1/2}(c)$. Then

$$\begin{aligned} P(x)y &= P(x)L(c,c)y &&, \text{ by } (15.5) \\ &= L(x,c)P(x,c)y, &&, \text{ by } (13.4) \text{ and } P(x)c = 0 \\ &= \{xc\{xyc\}\} = \langle xyx\rangle &&, \text{ by definition.} \end{aligned}$$

Next we have to verify (AT 1) - (AT 3).

By definition of the composition and by (a linearization of) (15,10e) ($\{xyc\} \in \mathfrak{A}_1$) we get

$$\begin{aligned} &\langle xy\langle uvz\rangle\rangle + \langle uv\langle xyz\rangle\rangle \\ &\quad = \{\{xyc\}c\{\{uvc\}cz\}\} + \{\{uvc\}c\{\{xyc\}cz\}\} \\ &\quad = \{\{\{xyc\}c\{uvc\}\}cz\} =: A \end{aligned}$$

Now let $a \in \mathfrak{A}_1 (x,y \in \mathfrak{A}_{1/2})$ then

$$\begin{aligned} \{ac\{cyx\}\} &= \{c\bar{a}\{cyx\}\} &&, \text{ by } (15.10) \\ &= \{cy\{c\bar{a}x\}\} + \{c\{yc\bar{a}\}x\} &&, \text{ by } (13.11) \text{ and } \{cyc\} = 0. \end{aligned}$$

Setting $a = \{uvc\}$ and using $\{\overline{uvc}\} = \{uvc\}$ the foregoing leads to

$$\{\{uvc\}c\{xyc\}\} = \{cy\{c\{vuc\}x\}\} + \{c\{yc\{vuc\}\}x\}.$$

Replacing this expression in A gives (AT 1).

(AT 2):

$$\begin{aligned} \langle xy\langle xyz\rangle\rangle &= \{\{xyc\}c\{\{xyc\}cz\}\} \\ &= \{P(\{xyc\})ccz\} &&, \text{ by } (15.10e) \end{aligned}$$

On the other hand we have

$$\langle\langle xyx\rangle yz\rangle = \{\{P(x)yyc\}cz\} \quad , \text{ by } (16.7)$$

We are done if we can show

$$P(\{xyc\})c = \{P(x)y\ y\ c\}.$$

Using (13.8) we get

$$(P(\{xyc\})c = P(x,c)P(y)P(x,c)c$$

(since $P(\mathfrak{A}_{1/2})c = 0$ and $P(c)y = 0$ (see (15.11)))

$= \{x\ P(y)x\ c\}$, since $\{ccx\} = x$

$= \{P(x)y\ y\ c\}$, by (13.2) .

Since we already proved (AT 1) we know that (AT 3) is equivalent to (16.2). We shall prove (16.2). We set $a := \{uvc\}$. Then

$$\langle x\langle vuy\rangle x\rangle = P(x)\{\bar{a}cy\} \quad \text{, by (16.7)}$$

$$= P(x)L(y,c)\bar{a} = P(\{cyx\},x)\bar{a} \quad \text{, by (13.10) and } P(x)\bar{a} = 0$$

$$= \{\{xyc\}\bar{a}x\} = \{\{xyc\}c\{acx\}\} \quad \text{, by (15.10g)}$$

$$= \{\{xyc\}c\{\{uvc\}cx\}\} = \langle xy\langle uvx\rangle\rangle.$$

This completes the proof.

16.3. Let $\mathcal{W}$ be an alternative triple system. We denote

$$\ell(x,y)z = \langle xyz\rangle = r(z,y)x = p(x,z)y.$$

Furthermore we denote

$$F(x,y) = (\ell(x,y),\ell(y,x)) \quad \mathrm{End}_{\Phi}\mathcal{W}\times \mathrm{End}_{\Phi}\mathcal{W} =: \mathcal{E}.$$

Let $\mathcal{M}_o$ be the submodule of $\mathcal{E}$ generated by all $F(x,y), x,y\in\mathcal{W}$, and $\mathcal{M} = \Phi E + \mathcal{M}_o$, where $E = (\mathrm{Id}_{\mathcal{W}}, \mathrm{Id}_{\mathcal{W}})$. Clearly the canonical involution $A = (A_1,A_2)\mapsto \bar{A} = (A_2,A_1)$ of the quadratic Jordan algebra $\mathcal{E}$ maps $\mathcal{M}$ onto $\mathcal{M}$, moreover $\overline{F(x,y)} = F(y,x)$. Finally we define an action of $\mathcal{M}$ on $\mathcal{W}$ by $A\cdot x = A_1x$ if $A = (A_1,A_2)$. With these notations we list a set of formulas which can be read off from (AT 1) - (16.6).

(16.8) $A\cdot\langle xyz\rangle = \langle (A\cdot x)yz\rangle + \langle x(\bar{A}\cdot y)z\rangle - \langle xy(A\cdot z)\rangle$

(16.9) $AF(x,y) + F(x,y)A = F(Ax,y) + F(x,\bar{A}y)$

(16.10) $F(x,y)F(x,y) = F(\langle xyx\rangle,y)$

(16.11) $A\cdot\langle xyx\rangle = \langle (A\cdot x)yx\rangle$

(16.12) $$\langle xy(A\cdot x)\rangle = \langle x(\bar{A}\cdot y)x\rangle$$

(16.13) $$F(\langle xux\rangle,y) = F(x,\langle yxu\rangle)$$

(For this equation we have to use (16.3) and (16.4).)

(16.14) $$F(\langle xyu\rangle,y) = F(x,\langle yuy\rangle)$$

(which is the "conjugate" of (16.13)).

(16.10) and (16.9) show that $\mathcal{M}$ is closed under the squaring $A \mapsto A^2$. We wish to show that it is also closed under $(A,B) \mapsto ABA$, and therefore is a subalgebra of the quadratic Jordan algebra $\mathcal{L}$.

A linearization of (16.11) is

$$A_1\langle xyz\rangle + A_1\langle zyx\rangle = \langle (A_1\ x)yz\rangle + \langle (A_1\ z)yx\rangle$$

In operator form $(A = (A_1, A_2))$

(16.15) $$\ell(A_1x,y) = A_1\ell(x,y) + [A_1,r(x,y)].$$

We shall use this formula to prove

(16.16) $$AF(x,y)A = F(A\cdot x,\bar{A}\cdot y)$$

(16.17) $$F(x,y)AF(x,y) = F(\langle xy(A\cdot x)\rangle,y) = F(x,\langle yx(\bar{A}\cdot y)\rangle)$$

(16.18) $$ABF(x,y) + F(x,y)BA = F(A\cdot B\cdot x,y) + F(x,\bar{A}\cdot\bar{B}\cdot y)$$

<u>Proof</u>: $A = (A_1, A_2)$.

$$\ell(A\cdot x,\bar{A}\cdot y) = A_1\ell(A\cdot x,y) + \ell(A\cdot x,y)A_1 - \ell(A^2\cdot x,y) \quad ,\text{by (16.8)}$$

$$= A_1^2\ell(x,y) + A_1[A_1,r(x,y)] + A_1\ell(x,y)A_1$$

$$+ [A_1,r(x,y)]A_1 - A_1^2\ell(x,y) - [A_1^2,r(x,y)], \quad \text{by (16.15)}$$

$$= A_1\ell(x,y)A_1.$$

Then by interchanging A, $\bar{A}$ and x and y, we also get

$$\ell(\bar{A}\cdot y,\ A\cdot x) = A_2\ell(y,x)A_2.$$

These two equations clearly give (16.16). We apply (16.15) twice to derive

$$\begin{aligned}\ell(\langle xy(A\cdot x)\rangle ,y) &= \ell(x,y)\ell(A\cdot x,y) + [\ell(x,y),r(Ax,y)]\\ &= \ell(x,y)A_1\ell(x,y) + \ell(x,y)[A_1r(x,y)]\\ &\quad + [\ell(x,y),r(Ax,y)].\end{aligned}$$

The right hand side equals $\ell(x,y)A_1\ell(x,y)$, iff

$$\langle xy(A\langle zyx\rangle)\rangle - \langle xy\langle (Az)yx\rangle\rangle + \langle xy\langle zy(Ax)\rangle\rangle - \langle\langle xyz\rangle y(Ax)\rangle = 0.$$

Using (16.8) for the first term, we see that this is equivalent to

$$\langle xy\langle z(\bar{A}y)x\rangle\rangle = \langle\langle xyz\rangle y(Ax)\rangle.$$

We have

$$\begin{aligned}\langle xy\langle z(\bar{A}y)x\rangle\rangle &= \langle x\langle (\bar{A}y)zy\rangle x\rangle && \text{, by (16.2)}\\ &= \langle x\bar{A}\langle yzy\rangle x\rangle && \text{, by (16.11)}\\ &= \langle x\langle yzy\rangle (Ax)\rangle && \text{, by (16.12)}\\ &= \langle\langle xyz\rangle y(Ax)\rangle && \text{, by (16.4)}\end{aligned}$$

which is the desired result. Therefore

(a) $\ell(x,y)A_1\ell(x,y) = \ell(\langle xy(Ax)\rangle ,y)$.

In the next step we shall prove

(b) $\ell(\langle xy(Ax)\rangle ,y) = \ell(x,\langle yx(Ay)\rangle)$.

Using (16.12) and (16.13) gives

$$\ell(\langle xy(Ax)\rangle ,y) = \ell(\langle x(\bar{A}y)x\rangle ,y) = \ell(x,\langle yx(\bar{A}y)\rangle).$$

Clearly (16.17) is a trivial consequence of (a) and (b).

Using (16.9) and a linearization of (16.16) we derive

$$F(ABu,v) + F(u,\bar{A}\,\bar{B}v)$$

$$= AF(Bu,v) + F(Bu,v)A - F(Bu,\bar{A}v)$$

$$+ AF(u,\bar{B}v) + F(u,\bar{B}v)A - F(Au,\bar{B}v)$$

$$= ABF(u,v) + AF(u,v)B + BF(u,v)A + F(u,v)BA - BF(u,v)A - AF(u,v)B$$

$$= ABF(u,v) + F(u,v)BA .$$

We shall need another formula, which says that $(A,\bar{A})$ is in the structure monoid of the Jts $\mathcal{Q}^+$. We claim

(16.19) $$<(Ax)y(Ax)> = A<x(\bar{A}y)x>.$$

<u>Proof</u>: $A<x(\bar{A}y)x> = A<xy(Ax)>$, by (16.12)

$$= <(Ax)y(Ax)> + <x(\bar{A}y)(Ax)> - <xy(A^2x)>$$

$$= <(Ax)y(Ax)>,$$

since $<x(\bar{A}y)(Ax)> = <x(\bar{A}^2y)x> = <x\overline{(A^2y)}x> = <xy(A^2x)>$.

<u>Note</u>: As an application of (16.19) we get that $\mathcal{R} := \mathrm{Rad}\,\mathcal{Q}^+$ <u>is an ideal of</u> $\mathcal{Q}$. From (16.19) and Lemma 13.1 we get $<\mathcal{Q}\mathcal{Q}\mathcal{R}> \subset \mathcal{R}$ and then $<\mathcal{R}\mathcal{Q}\mathcal{Q}> \subset \mathcal{R}$, since $\mathcal{R}$ is ideal in $\mathcal{Q}^+$. Furthermore, from (16.19) we then get for $x, y \in \mathcal{Q}$, $z \in \mathcal{R}$,

$$<<xzy>u<xzy>> = <xz<y<zxu>y>> \in \mathcal{R}$$

since $<zxu> \in \mathcal{R}$ implies $<y<zxu>y> = P(u)<zxu> \in \mathcal{R}$. Thus $P(<xzy>)\mathcal{Q}^+ \subset \mathcal{R}$ and then $<xzy> \in \mathcal{R}$, by (13.25). Thus $<\mathcal{Q}\mathcal{R}\mathcal{Q}> \subset \mathcal{R}$.

16.4. Now we are ready for the main result of this section. We define on the direct sum

$$\mathcal{A}(\mathcal{Q}) = \mathcal{M} \oplus \mathcal{Q}$$

a quadratic operator P by

(16.20) $$P(A \oplus x)(B \oplus y) = A\bar{B}A + F(x,\bar{A}\cdot y) \oplus <xyx> + A\cdot\bar{B}\cdot x$$

This is in particular,

$$
(16.21)\quad
\begin{aligned}
&\text{i)}\quad P(A)B = A\bar{B}A; P(x)y = \langle xyx\rangle \\
&\text{ii)}\quad P(A)x = P(x)A = 0 \\
&\text{iii)}\quad P(A,x)B = \{ABx\} = A\cdot\bar{B}\cdot x; P(A,x)y = \{xyA\} = F(x,\bar{A}\cdot y).
\end{aligned}
$$

<u>Theorem 3.</u> i) $\mathfrak{A} = (\mathfrak{A}(\mathfrak{Q}), P)$ <u>is a Jordan triple system</u>,

ii) $E = (\mathrm{Id}, \mathrm{Id})$ <u>is an idempotent of</u> $\mathfrak{A}$ <u>such that</u>

$$\mathfrak{A}_{1/2}(E) = \mathfrak{Q},\ \mathfrak{A}_1(E) = \mathfrak{M},\ \mathfrak{A}_o(E) = 0$$

iii) $\langle xyz\rangle = \{\{xyE\}Ez\}$

iv) $\{\mathfrak{Q}\mathfrak{Q}\mathfrak{M}\} = \mathfrak{M}_o$ <u>and</u> $\mathfrak{M}_o \oplus \mathfrak{Q}$ <u>is an ideal of</u> $\mathfrak{A}$.

<u>Proof.</u> We first observe that P restricted to $\mathfrak{M}$ defines on $\mathfrak{M}$ the structure of a Jts and also P restricted to $\mathfrak{Q}$ gives the Jts $\mathfrak{Q}^+$. Also (AT 1) - (AT 3) remain valid under all extensions, therefore we need not worry about scalar extensions. We have to verify the axioms for a Jts.

<u>I.</u> $$L(A+x,B+y)P(A+x) = P(A+x)L(B+y,A+x).$$

By comparing degrees in elements of $\mathfrak{M}$ and $\mathfrak{Q}$ respectively, we see that this equation is equivalent to

1) $L(A,B)P(A) = P(A)L(B,A)$

2) $L(x,v)P(x) = P(x)L(v,x)$

3) $P(A)L(B,x) + P(A,x)L(B,A) = L(x,B)P(A) + L(A,B)P(A,x)$

4) $P(A,x)L(B,x) + P(x)L(B,A) = L(x,B)P(A,x) + L(A,B)P(x)$

5) $P(x)L(B,x) = L(x,B)P(x)$

6) $L(x,y)P(A,x) + L(A,y)P(x) = P(A,x)L(y,x) + P(x)L(y,A)$

7) $L(A,y)P(A,x) + L(x,y)P(A) = P(A,x)L(y,A) + P(A)L(y,x)$

8) $L(A,y)P(A) = P(A)L(y,A)$.

Proof. Formula 1) applied to elements of $\mathfrak{M}$ is valid, since $(\mathfrak{M}, P)$ is a Jts, $P(A)B = A\bar{B}A$ (see 16.3.). Applied to $y \in \mathfrak{Q}$ we get zero on both sides, by definition. The same argument applies to prove formula 2), by Theorem 1.

3) applied to $C \in \mathfrak{M}$:

$$A\cdot(\overline{[B\bar{A}C + C\bar{A}B]}\cdot x) = A\bar{C}A\cdot\bar{B}\cdot x + A(\bar{B}(A\bar{C}x)),$$

this equation clearly holds.

3) applied to $y \in \mathfrak{Q}$:

$P(A)F(y,\bar{B}\cdot x) + P(A,x)B\cdot\bar{A}\cdot y = L(A,B)F(x,\bar{A}\cdot y)$, equivalently

$$AF(\bar{B}\cdot x,y)A + F(x,\bar{A}B\bar{A}y) = A\bar{B}F(x,\bar{A}y) + F(x,\bar{A}y)\bar{B}A.$$

We use (16.16) for the first term on the left hand side and (16.18) for the right hand side to see that this is a valid equation.

4) applied to elements in $\mathfrak{M}$ is zero on both sides (by definition) and applied to $y \in \mathfrak{Q}$ it is equivalent to

$$P(A,x)F(y,\bar{B}\cdot x) + P(x)B\cdot\bar{A}\cdot y = L(x,B)F(x,\bar{A}\cdot y) + A\cdot\bar{B}\ P(x)y, \text{ or}$$

$$A\langle(\bar{B}\cdot x)yx\rangle + \langle x(B\bar{A}\cdot y)x\rangle = \langle x(\bar{A}\cdot y)\bar{B}\cdot x)\rangle + A\bar{B}\cdot\langle xyx\rangle.$$

Using (16.11) this reduces to $\langle x(B\bar{A}\cdot y)x\rangle = \langle x(\bar{A}\cdot y)(\bar{B}\cdot x)\rangle$ which is valid by (16.12).

5) applied to $C \in \mathfrak{M}$ and $y \in \mathfrak{Q}$ gives zero on both sides.

6) applied to $B \in \mathfrak{M}$ is

$$L(x,y)A\cdot\bar{B}\cdot x = P(A,x)F(y,\bar{B}\cdot x) + P(x)B\bar{A}y, \text{ or}$$

$$\langle xy(A\cdot\bar{B}x)\rangle + \langle(A\bar{B}x)yx\rangle = \langle x(B\bar{A}y)x\rangle + A\langle(\bar{B}x)yx\rangle$$

$$= \langle x(\bar{A}y)(\bar{B}x)\rangle + A\cdot\langle(\bar{B}x)yx\rangle$$

, by (16.12)

$$= \langle x(\bar{A}y)(\bar{B}x)\rangle + \langle(A\bar{B}x)yx\rangle + \langle(\bar{B}x)(\bar{A}y)x\rangle - \langle(\bar{B}x)y(Ax)\rangle$$

, by (16.8).

What remains is just a linearization of (16.12).

6) applied to $z \in \mathcal{Q}$:

$$F(x,\langle(\bar{A}\cdot z)xy\rangle) + F(\langle xzx\rangle,\bar{A}y) = F(x,\bar{A}\langle yxz\rangle) + F(x,\bar{A}\langle zxy\rangle).$$

$$= F(x,\langle(\bar{A}y)xz\rangle) + F(x,\langle(\bar{A}z)xy\rangle),$$ by a linearization of (16.11),

and this identity clearly holds, by (16.13).

7) applied to $B \in \mathcal{M}$ is

$$F(A\bar{B}x,\bar{A}y) + F(x,\bar{A}B\bar{A}y) = F(x,\bar{A}B\bar{A}y) + AF(\bar{B}x,y)A,$$

which holds by (16.16).

7) applied to $z \in \mathcal{Q}$ gives on both sides zero.

Finally 8) holds, since by definition both sides are zero.

<u>II</u>. $L(P(A+x)(B+y),B+y) = L(A+x,P(B+y)(A+x))$

Using (16.20) and comparing the parts of equal degree in elements of $\mathcal{M}$ and $\mathcal{Q}$, we have to show

1) $L(P(A)\bar{B},B) = L(A,P(B)\bar{A})$

2) $L(\langle xyx\rangle,y) = L(x,\langle yxy\rangle)$

3) $L(P(A)\bar{B},y) = L(A,B\cdot\bar{A}\cdot y)$

4) $L(A\cdot\bar{B}x,B) = L(x,P(B)\bar{A})$

5) $L(F(x,\bar{A}y),B) + L(A\bar{B}x,y) = L(A,F(y,\bar{B}x)) + L(x,B\bar{A}y)$

6) $L(F(x,\bar{A}y),y) = L(A,\langle yxy\rangle)$

7) $L(\langle xyx\rangle,B) = L(x,F(y,\bar{B}x))$

<u>Proof</u>: 1) applied to $C \in \mathcal{M}$ holds since $(\mathcal{M},P)$ is a Jts., applied to $x \in \mathcal{Q}$ gives $P(A)\bar{B}(\bar{B}\cdot x) = A\cdot\overline{P(B)\bar{A}}\,x$, which holds.

2) applied to $z \in \mathcal{Q}$ holds by Theorem 1, and applied to $A \in \mathcal{M}$ gives

$$F(\langle xyx\rangle, \bar{A}\cdot y) = F(x, \bar{A}\cdot\langle yxy\rangle) ,$$

which holds by (16.11) and (16.13).

3) applied to elements in $\mathcal{M}$ is zero on both sides and applied to $x \in \mathcal{Q}$ is $F(x,\bar{A}B\bar{A}y) = F(x,\bar{A}\cdot B\cdot\bar{A}\cdot y)$, thus 3) holds. For similar trivial reasons we get that 4) holds.

5) applied to $C \in \mathcal{M}$ is

$$F(x,\bar{A}y)\bar{B}C + C\bar{B}F(x,\bar{A}y) + F(A\bar{B}x,\bar{C}y)$$

$$= AF(\bar{B}x,y)C + CF(\bar{B}x,y)A + F(x,\bar{C}\cdot B\cdot\bar{A}y)$$

Using (16.18) on the left hand side and a linearization of (16.16) on the right hand side we get a trivial identity.

5) applied to $z \in \mathcal{Q}$ is

$$\langle x(\bar{A}y)(\bar{B}z)\rangle + \langle(A\cdot\bar{B}x)yz\rangle + \langle zy(A\bar{B}x)\rangle$$

$$= A\langle(\bar{B}x)yz\rangle + \langle x(B\bar{A}y)z\rangle + \langle z(B\bar{A}y)x\rangle.$$

We use (16.8) for the first term on the right, then (16.12) to see that this equation holds.

6) and 7) are verified in the same manner, we only have to use (16.11) in both cases.

<u>III</u>. $\quad P(P(A+x)(B+y)) = P(A+x)P(B+y)P(A+x).$

First we notice that in a quadratic triple system the fundamental formula will follow from a) $L(x,y)P(x) = P(x)L(y,x)$, b) $L(P(x)y,y) = L(x,P(y)x)$ and c) $P(P(x)y)y = P(x)P(y)P(x)y$ (in all extensions).

Since a linearization of c) is

$$P(P(x)y)z + P(P(x)y,P(x)z)y = P(x)P(y)P(x)z + P(x)P(y,z)P(x)y,$$

and from a) and b) we get

$$P(P(x)y,P(x)z)y = \{P(x)y\ y\ P(x)z\} = (\text{by b})L(x,P(y)x)P(x)z$$
$$= (\text{by a})P(x)L(P(y)x,x)z = (\text{by b})P(x)L(y,P(x)y)z$$
$$= P(x)P(y,z)P(x)y.$$

We therefore have to prove

<u>III'.</u> $P(P(A+x)(B+y))(B+y) = P(A+x)P(B+y)P(A+x)(B+y).$

We expand both sides of this equation and compare terms of equal degree. Using again that ($\mathfrak{M}$,P) and $\mathfrak{Q}^+$ are Jts we have to show

1) $P(F(x,\bar{A}y))B + F(\langle xyx\rangle,\bar{A}B\bar{A}y) + F(A\bar{B}x,\langle(\bar{A}y)xy\rangle)$
$= AF(\bar{B}\langle xyx\rangle,y)A + F(x,\bar{A}\langle y(A\bar{B}x)y\rangle) + F(x,\bar{A}B\langle(\bar{A}y)xy\rangle)$

2) $F(A\bar{B}x,\bar{A}B\bar{A}y) + A\bar{B}A\bar{B}F(x,\bar{A}y) + F(x,\bar{A}y)\bar{B}A\bar{B}A$
$= AF(\bar{B}A\bar{B}x,y)A + F(x,\bar{A}B\bar{A}B\bar{A}y) + A\bar{B}F(x,\bar{A}y)\bar{B}A$

3) $F(\langle xyx\rangle,\langle(\bar{A}y)xy\rangle) = F(x,\bar{A}\langle y\langle xyx\rangle y\rangle)$

4) $\langle(A\bar{B}x)y(A\bar{B}x)\rangle + A\bar{B}A\bar{B}\langle xyx\rangle + \langle x(\bar{A}y)(\bar{B}A\bar{B}x)\rangle$
$= \langle x(B\bar{A}B\bar{A}y)x\rangle + A\bar{B}\langle x(\bar{A}y)(\bar{B}x)\rangle + A\langle(\bar{B}A\bar{B}x)yx\rangle$

5) $\langle x(\bar{A}y)(B\langle xyx\rangle)\rangle + \langle\langle xyx\rangle y(A\bar{B}x)\rangle + \langle(A\bar{B}x)y\langle xyx\rangle\rangle$
$= \langle x\langle y(A\bar{B}x)y\rangle x\rangle + A\langle(\bar{B}\langle xyx\rangle)yx\rangle + \langle x(B\langle(\bar{A}y)xy\rangle)x\rangle.$

<u>Proof:</u> Using (16.11) and (16.13) we get

$$F(x,\bar{A}B\langle(\bar{A}y)xy\rangle) = F(x,\bar{A}B\bar{A}\langle yxy\rangle) = F(x,\langle(\bar{A}B\bar{A}y)xy\rangle)$$
$$= F(\langle xyx\rangle,\bar{A}B\bar{A}y) ,$$

and (16.11), (16.14) and (16.16) imply

$$AF(\bar{B}\langle xyx\rangle,y)A = AF(\langle(\bar{B}x)yx\rangle,y) = AF(\bar{B}x,\langle yxy\rangle)A$$
$$= F(A\bar{B}x,\bar{A}\langle yxy\rangle) = F(A\bar{B}x,\langle(\bar{A}y)xy\rangle).$$

Thus 1) reduces to

$$F(x,\bar{A}y)\bar{B}F(x,\bar{A}y) = F(x,\bar{A}\langle y(A\bar{B}x)y\rangle).$$

But we have

$$F(x,\bar{A}y)\bar{B}F(x,Ay) = F(x,<(\bar{A}y)x(B\bar{A}y>) \qquad \text{by (16.17)}$$

$$= F(x,<(\bar{A}y)(\bar{B}x)(\bar{A}y)>) \qquad \text{by (16.12)}$$

$$= F(x,\bar{A}<y(A\bar{B}x)y>) \qquad \text{by (16.19)}$$

This completes the proof of formula 1.

2): From (16.16) we get

$$A\bar{B}F(x,\bar{A}y)\bar{B}A = F(A\bar{B}x,\bar{A}B\bar{A}y)$$

and (16.18) shows ($A\bar{B}A \in \mathcal{M}$)

$$A\bar{B}A\bar{B}F(x,\bar{A}y) + F(x,\bar{A}y)\bar{B}A\bar{B}A$$

* $$= F(A\bar{B}A\bar{B}x,\bar{A}y) + F(x,\bar{A}B\bar{A}B\bar{A}y)$$

$$= AF(\bar{B}A\bar{B}x,y)A + F(x,\bar{A}B\bar{A}B\bar{A}y) \quad \text{, by (16.16).}$$

This proves formula 2).

3): $F(x,\bar{A}<y<xyx>y>)$

$$= F(x,<(\bar{A}y)<xyx>y>) \qquad \text{, by (16.11)}$$

$$= F(x,<<(\bar{A}y)xy>xy>) \qquad \text{, by (16.5)}$$

$$= F(<xyx>,<(\bar{A}y)xy>) \qquad \text{, by (16.13).}$$

4): (16.12) shows $<x(\bar{A}y)(\bar{B}A\bar{B}x)> = <x(B\bar{A}B\bar{A}y)x>$ and (16.11) gives $A<(\bar{B}A\bar{B}x)yx> = A\bar{B}A\bar{B}<xyx>$. Thus 4) reduces to

$$<(A\bar{B}x)y(A\bar{B}x)> = A\bar{B}<x(\bar{A}y)(\bar{B}x)>$$

but this is an immediate consequence of (16.19) and (16.12).

5): Using (16.2),(16.11) and (16.12) we get

$$<x(\bar{A}y)(\bar{B}<xyx>) = <x(\bar{A}y)<(\bar{B}x)yx>> = <x<y(\bar{B}x)(\bar{A}y)>x>$$

$$= <x<y(A\bar{B}x)y>x>.$$

Therefore 5) reduces to an equation of the form

$$<xv(Au)> + <(Au)vx> = A\cdot<uvx> + <x(\bar{A}v)u>$$

(setting $v = \langle yxy \rangle$, $u = \bar{B}x$ and using (AT 2), (16.2), (16.11) and (16.12.) Substituting $A \cdot \langle uvx \rangle = \langle (Au)vx \rangle + \langle u(\bar{A}v)x \rangle - \langle uv(Ax) \rangle$ leads to a linearization of (16.12).

This completes the proof of the first part of Theorem 3. The verification of the other parts is straightforward and is left as an easy exercise.

16.5. Next we study the connections between the ideals of $\mathcal{D}$ (as alternative triple system) and the Jordan triple ideals of the standard embedding $\mathcal{A} = \mathcal{A}(\mathcal{D})$.

Since $\mathcal{M}$ and $\mathcal{D}$ are Peirce modules, we have for any ideal $\mathcal{L}$ of $\mathcal{A} = \mathcal{A}(\mathcal{D})$

$$\mathcal{L} = (\mathcal{L} \cap \mathcal{M}) \oplus \mathcal{L} \cap \mathcal{D}$$

And clearly, since $\langle xyz \rangle = \{\{xyE\}Ez\}$, $x,y,z \in \mathcal{D}$, $\mathcal{U} := \mathcal{L} \cap \mathcal{D}$ is an ideal of $\mathcal{D}$, and since $P(A)B = A\bar{B}A$, the submodule $\mathcal{L} \cap \mathcal{M}$ is an ideal of $\mathcal{M}$ such that $\overline{\mathcal{L} \cap \mathcal{M}} \subset \mathcal{L} \cap \mathcal{M}$.

<u>Lemma 1</u>: The <u>submodule</u> $\mathcal{L} = \mathcal{N} \oplus \mathcal{U}$, $\mathcal{N} \subset \mathcal{M}$, $\mathcal{U} \subset \mathcal{D}$, <u>is a Jordan triple ideal of</u> $\mathcal{A}(\mathcal{D})$, <u>iff</u>

1) $\mathcal{U}$ <u>is an ideal of</u> $\mathcal{D}$ (<u>as alternative triple system</u>)

2) $\mathcal{N}$ <u>is a Jts ideal of</u> $\mathcal{M}$ <u>such that</u> $\overline{\mathcal{N}} = \mathcal{N}$, <u>and</u>

3) a) $\mathcal{N} \cdot \mathcal{D} \subset \mathcal{U}$

b) $F(\mathcal{U}, \mathcal{D}) + F(\mathcal{D}, \mathcal{U}) \subset \mathcal{N}$

<u>Proof</u>: If $\mathcal{L}$ is an ideal, then clearly 1) and 2) hold by the foregoing remarks. But also $\{\mathcal{N}E\mathcal{D}\} = \mathcal{N} \cdot \mathcal{D} \subset \mathcal{L} \cap \mathcal{D}$ and $F(\mathcal{U}, \mathcal{D}) + F(\mathcal{D}, \mathcal{U}) = \{\mathcal{U}\mathcal{M}\mathcal{D}\} + \{\mathcal{M}\mathcal{U}\mathcal{D}\} \subset \mathcal{L} \cap \mathcal{M}$.

Conversely let $\mathcal{L} = \mathcal{N} \oplus \mathcal{U}$ have the properties 1) - 3). We have to show

$$P(\mathcal{A})\mathcal{L} \subset \mathcal{L}, \quad P(\mathcal{L})\mathcal{A} \subset \mathcal{L}, \quad \{\mathcal{A}\mathcal{A}\mathcal{L}\} \subset \mathcal{L}.$$

Comparing components in $\mathfrak{M}$ and $\mathfrak{T}$, these conditions are equivalent to

1) $P(\mathfrak{M})\mathfrak{N}\subset\mathfrak{N}$,	2) $P(\mathfrak{M},\mathfrak{T})\mathfrak{U}\subset\mathfrak{N}$,
3) $P(\mathfrak{N})\mathfrak{M}\subset\mathfrak{N}$,	4) $P(\mathfrak{N},\mathfrak{U})\mathfrak{T}\subset\mathfrak{N}$,
5) $\{\mathfrak{M}\mathfrak{M}\mathfrak{N}\}\subset\mathfrak{N}$,	6) $\{\mathfrak{M}\mathfrak{T}\mathfrak{U}\}\subset\mathfrak{N}$,
7) $\{\mathfrak{T}\mathfrak{T}\mathfrak{N}\}\subset\mathfrak{N}$,	8) $P(\mathfrak{T})\mathfrak{U}\subset\mathfrak{U}$,
9) $P(\mathfrak{M},\mathfrak{T})\mathfrak{N}\subset\mathfrak{U}$,	10) $P(\mathfrak{U})\mathfrak{T}\subset\mathfrak{U}$,
11) $P(\mathfrak{N},\mathfrak{U})\mathfrak{M}\subset\mathfrak{U}$,	12) $\{\mathfrak{M}\mathfrak{M}\mathfrak{U}\}\subset\mathfrak{U}$,
13) $\{\mathfrak{T}\mathfrak{M}\mathfrak{N}\}\subset\mathfrak{U}$,	14) $\{\mathfrak{T}\mathfrak{T}\mathfrak{U}\}\subset\mathfrak{U}$.

Since $\mathfrak{N}$ is an ideal in $\mathfrak{M}$ we know that 1), 3) and 5) hold. By definition

$\{\mathfrak{M}\mathfrak{U}\mathfrak{T}\} + \{\mathfrak{U}\mathfrak{T}\mathfrak{M}\} = F(\mathfrak{T},\mathfrak{M}\cdot\mathfrak{U}) + F(\mathfrak{U},\mathfrak{M}\cdot\mathfrak{T})$

$= F(\mathfrak{T},\mathfrak{U}) + F(\mathfrak{U},\mathfrak{T})$, since $\mathfrak{M}\cdot\mathfrak{U}\subset\mathfrak{U}$ ($\mathfrak{U}$ is ideal). Thus 3b) implies 2) and 6). 4) is a particular case of 6).

$\{\mathfrak{T}\mathfrak{T}\mathfrak{N}\} = F(\mathfrak{T},\mathfrak{N}\cdot\mathfrak{T})\subset F(\mathfrak{T},\mathfrak{U})$, by 3a), thus 7) holds.

$\mathfrak{U}$ being an ideal in $\mathfrak{T}$ implies 8) and 10). And $\{\mathfrak{M}\mathfrak{N}\mathfrak{T}\} = \mathfrak{M}\cdot(\mathfrak{N}\cdot\mathfrak{T})$ $\subset\mathfrak{M}\cdot\mathfrak{U}$ (by 3a) $\subset\mathfrak{U}$ (since $\mathfrak{U}$ is ideal). Finally

$\{\mathfrak{N}\mathfrak{M}\mathfrak{U}\}\subset(\mathfrak{N}\cdot\mathfrak{M})\cdot\mathfrak{U}\subset\mathfrak{U}$.

Corollary. <u>If $\mathfrak{U}$ is an ideal in the alternative triple system $\mathfrak{T}$, then $J(\mathfrak{U}) := F(\mathfrak{U},\mathfrak{T}) + F(\mathfrak{T},\mathfrak{U}) \oplus \mathfrak{U}$ is an ideal in</u> $\mathfrak{A}(\mathfrak{T})$.

<u>Proof</u>. We only have to show that $\mathfrak{N} := F(\mathfrak{U},\mathfrak{T}) + F(\mathfrak{T},\mathfrak{U})$ is an ideal in $\mathfrak{M}$, since by the choice of $\mathfrak{U}$ and $\mathfrak{N}$ the conditions 1) and 3) of the above Lemma are fulfilled and also $\overline{\mathfrak{N}} = \mathfrak{N}$.

Let $A \in \mathfrak{M}$, then $P(A)\mathfrak{N} = A[\overline{F(\mathfrak{U},\mathfrak{V}) + F(\mathfrak{V},\mathfrak{U})}]A$

$AF(\mathfrak{U},\mathfrak{V})A + AF(\mathfrak{V},\mathfrak{U})A \subset F(A\cdot\mathfrak{U},\bar{A}\cdot\mathfrak{V}) + F(A\cdot\mathfrak{V},\bar{A}\cdot\mathfrak{U})$
$\subset \mathfrak{N}$. (We used (16.16) and $\mathfrak{M}\cdot\mathfrak{U}\subset\mathfrak{U}$)
If $A,B\in\mathfrak{M}$, $u\in\mathfrak{U}$, then $\{ABF(u,v)\} = A\bar{B}F(u,v) + F(u,v)\bar{B}A$
$=$ (by 16.18) $F(A\cdot B\cdot u,v) + F(u,\bar{A}\cdot\bar{B}\cdot v) \in F(\mathfrak{U},\mathfrak{V})$. Similarly
$\{ABF(v,u)\} \in F(\mathfrak{V},\mathfrak{U})$. Thus $\{\mathfrak{M}\mathfrak{M}\mathfrak{N}\}\subset\mathfrak{N}$. Using (16.17) we obtain

$$P(F(u,v))\mathfrak{M} = F(u,v)\mathfrak{M}F(u,v) = F(\langle uv(\mathfrak{M}\cdot u)\rangle,v)$$
$\subset F(\mathfrak{U},\mathfrak{V})$ similarly $P(F(v,u))\mathfrak{M}\subset\mathfrak{N}$ and finally
$\{F(u,v)\mathfrak{M}F(u',v')\}\subset\mathfrak{N}$, by what we already proved ($\{\mathfrak{M}\mathfrak{M}\mathfrak{N}\}\subset\mathfrak{N}$).

<u>Theorem 4.</u> $\mathfrak{V}$ <u>is simple, iff</u> $\mathfrak{A}(\mathfrak{V})$ <u>has no other ideals than</u> 0, $\mathfrak{M}_0 \oplus \mathfrak{V}$ <u>and</u> $\mathfrak{A}$. <u>In this case</u> $\mathfrak{M}$ <u>acts faithfully on</u> $\mathfrak{V}$.

<u>Proof.</u> If $\mathfrak{A}$ has no other ideals, then $\mathfrak{V}$ is simple, by the corollary. Conversely, let $\mathfrak{V}$ be simple. We shall first show that $\mathfrak{M}$ acts faithfully on $\mathfrak{V}$. Let $A = (A_1,A_2)\in\mathfrak{M}$ and $A\cdot x = A_1x = 0$ for all $x\in\mathfrak{V}$ (i.e. $A_1 = 0$) then $\langle x(\bar{A}\cdot y)z\rangle = 0$ for all $x,y,z\in\mathfrak{V}$, this follows from (16.8). We consider

$$\mathfrak{U} := \{y\in\mathfrak{V},\ \langle xyz\rangle = 0 \text{ for all } x,z\in\mathfrak{V}\}.$$

Since for $y\in\mathfrak{U}$ we have $\langle xyz\rangle\in\mathfrak{U}$ by definition and $\langle vuy\rangle\in\mathfrak{U}$ (for all $u,v\in\mathfrak{V}$ by (AT 1) and also $\langle yuv\rangle\in\mathfrak{U}$ for all $u,v\in\mathfrak{V}$ (again by (AT 1), (interchange v and y)), we see that $\mathfrak{U}$ is an ideal and therefore $\mathfrak{U} = 0$ by simplicity of $\mathfrak{V}$. Since $\bar{A}\cdot y\in\mathfrak{U}$ we obtain $A_2y = 0$, i.e. $A_2 = 0$, thus $A = 0$.

Now let $\mathfrak{L} = \mathfrak{N}\oplus\mathfrak{U}$ be an ideal of $\mathfrak{A}$, then $\mathfrak{U} = 0$ or $\mathfrak{U} = \mathfrak{V}$. If $\mathfrak{U} = 0$, then $\mathfrak{N}\cdot\mathfrak{V}\subset\mathfrak{U} = 0$ (by Lemma 1) and thus $\mathfrak{N} = 0$, since $\mathfrak{M}$ acts faithfully. In the second case $\mathfrak{U} = \mathfrak{V}$ and $F(\mathfrak{V},\mathfrak{V}) = \mathfrak{M}\subset\mathfrak{N}$. This completes the proof.

16.6. Now let $\mathfrak{A}$ be a simple and semi-simple Jordan triple system and c a maximal idempotent of $\mathfrak{A}$.

$$\mathfrak{A} = \mathfrak{A}_1(c) \oplus \mathfrak{A}_{1/2}(c).$$

Assume $\mathfrak{A}_{1/2}(c) \neq 0$.

Lemma 2: $\mathfrak{A}_1 = \{\mathfrak{A}_{1/2}\,\mathfrak{A}_{1/2}c\}$.

Proof: Verify that $\{\mathfrak{A}_{1/2}\,\mathfrak{A}_{1/2}c\} \oplus \mathfrak{A}_{1/2}$ is ideal in $\mathfrak{A}$. (The verification is left as an exercise.)

Since $\mathfrak{A}$ is simple, the alternative triple system $\mathfrak{A}_{1/2}(c)$ is simple and $\mathfrak{M} = F(\mathfrak{A}_{1/2}, \mathfrak{A}_{1/2})$ operates faithfully on $\mathfrak{A}_{1/2}$, or, in other words, the projection $F(x,y) \mapsto \ell(x,y)$ is an isomorphism (of Jordan triple systems). We identify $\mathfrak{M}$ and $\ell(\mathfrak{A}_{1/2}, \mathfrak{A}_{1/2})$ via this isomorphism. Since $\langle xyz \rangle = \{\{xyc\}cz\}$, we have $\ell(x,y) = L(\{xyc\},c)$ (restricted to $\mathfrak{A}_{1/2}$). $\overline{\ell(x,y)} = \ell(y,x) = L(\{yxc\},c) = L(\overline{\{xyc\}},c)$ by (15.11). These considerations and the above Lemma show

$$\mathfrak{M} = L(\mathfrak{A}_1, c)\,\big|\,\mathfrak{A}_{1/2}, \quad \overline{L(a,c)} = L(\bar{a},c).$$

<u>Theorem 5</u>: <u>Let $\mathfrak{A}$ be a simple and semi-simple Jordan triple system and</u> c <u>an idempotent of $\mathfrak{A}$ such that $\mathfrak{A} = \mathfrak{A}_1(c) \oplus \mathfrak{A}_{1/2}(c)$</u>, $\mathfrak{A}_{1/2} \neq 0$, <u>then $\mathfrak{A}$ is isomorphic to the standard imbedding $\mathfrak{A}(\mathfrak{A}_{1/2})$ of the alternative triple system $\mathfrak{A}_{1/2}$</u>.

<u>Proof.</u> The above considerations show

$$\mathfrak{A}(\mathfrak{A}_{1/2}) \cong L(\mathfrak{A}_1, c) \oplus \mathfrak{A}_{1/2}$$

(where, for this proof only, the L's denote the restriction of the left multiplication in $\mathfrak{A}$ to $\mathfrak{A}_{1/2}$). Let $a, b \in \mathfrak{A}_1$, $x,y,z \in \mathfrak{A}_{1/2}$. The Jt-composition is given by

$$P(L(a,c))L(b,c) = L(a,c)L(\bar{b},c)L(a,c)\ ,$$
$$P(L(a,c),x)y = L(\{x\{\bar{a}cy\}c\},c)\ \ (\Leftrightarrow\ F(x,\overline{L(a,c)}\cdot y))\ ,$$
$$P(L(a,c),x)L(b,c) = L(a,c)L(\bar{b},c)x = \{ac\{\bar{b}cx\}\}\ ,$$
$$P(x)y = \langle xyx\rangle .$$

Using

$$\{abx\} = \{\{ac\{\bar{b}cx\}\}$$
$$\{xya\} = \{x\{\bar{a}cy\}c\}$$

and (13.13) we can immediately verify that φ: $a \oplus x \longmapsto L(a,c) \oplus x$ defines a homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}(\ \mathfrak{A}_{1/2})$, (the verification is left as an exercise). Since $\mathfrak{A}$ is simple and $\varphi \neq 0$ the kernel of φ (which is an ideal) has to be zero.

XVII. Peirce Decomposition in Alternative Triple Systems.

17.1. We recall some facts about the Peirce decomposition in alternative algebras. Let $\mathfrak{A}$ be an alternative algebra, i.e. a Φ-module together with multiplication $(x,y)\mapsto xy$ satisfying

(17.1) $$x^2y = x(xy) \; ; \quad yx^2 = (yx)x \quad ; \quad (xy)x = x(yx) \; .$$

$c \in \mathfrak{A}$ is an idempotent, if $c^2 = c$.

As in the case of associative algebras (see 2.5.) we have the Peirce decomposition of $\mathfrak{A}$ relative c

$$\mathfrak{A} = \mathfrak{A}_{11} \oplus \mathfrak{A}_{10} \oplus \mathfrak{A}_{01} \oplus \mathfrak{A}_{00}$$

where $\mathfrak{A}_{ij} = \{x_{ij} \in \mathfrak{A} \; , \; cx_{ij} = ix_{ij}, \; x_{ij}c = jx_{ij}\}$. The decomposition of any element x into its Peirce components is

$$x = cxc + (cx - cxc) + (xc - cxc) + (x - cx - cx + cxc)$$

Note: For the verification of this statement we have to use 17.1.

The following multiplication rules are valid

(17.2) $$\mathfrak{A}_{ij} \; \mathfrak{A}_{jk} \subset \mathfrak{A}_{ik} \; ; \; \mathfrak{A}_{ij} \; \mathfrak{A}_{k\ell} = 0 \; , \; j \neq k \text{ unless } (i,j)=(k,\ell)$$
$$\mathfrak{A}_{ij} \; \mathfrak{A}_{ij} \subset \mathfrak{A}_{ji} \; , \; x_{ij}^2 = 0 \qquad (i \neq j).$$

Exercise: Prove (17.2).

17.2. Let $\mathfrak{A}$ be an alternative triple system over Φ. c is an idempotent in $\mathfrak{A}$, if <ccc> = c. In this case c is an idempotent of the alternative algebra $\mathfrak{A}_c$, $(x,y) \mapsto xy = \langle xcy \rangle$, and the above results apply.

If

$$\mathcal{Q}_{ij} := \{x_{ij} \in \mathcal{Q}, \langle ccx_{ij} \rangle = ix_{ij}, \langle x_{ij}cc \rangle = jx_{ij}\}$$

then

$$\mathcal{Q} = \mathcal{Q}_{11} \oplus \mathcal{Q}_{10} \oplus \mathcal{Q}_{01} \oplus \mathcal{Q}_{00}$$

and we have the following composition rules

$$
\begin{aligned}
& \langle \mathcal{Q}_{ij}\, \mathcal{Q}_{kj}\, \mathcal{Q}_{k\ell} \rangle \subset \mathcal{Q}_{i\ell} & & \\
& \langle \mathcal{Q}_{ij}\, \mathcal{Q}_{kj}\, \mathcal{Q}_{ik} \rangle \subset \mathcal{Q}_{ki} & & i \neq k \\
(17.3) \quad & \langle \mathcal{Q}_{ij}\, \mathcal{Q}_{ji}\, \mathcal{Q}_{i\ell} \rangle \subset \mathcal{Q}_{j\ell} & & i \neq j \\
& \langle \mathcal{Q}_{ij}\, \mathcal{Q}_{ji}\, \mathcal{Q}_{ji} \rangle \subset \mathcal{Q}_{ij} & & i \neq j
\end{aligned}
$$

while all other compositions are zero.

<u>Exercise</u>: Prove (17.3).

<u>Note</u>: One also has a Peirce decomposition of $\mathcal{Q}$ relative to an orthogonal set $c_1,\ldots,c_r$ of idempotents in $\mathcal{Q}$. (Idempotents c_1, c_2 are <u>orthogonal</u>, if $c_1 \in \mathcal{Q}_{00}(c_2)$.)

Set $c := \sum c_r$, and

$$\mathcal{Q}_{ij} = \{x \in \mathcal{Q}, \langle c_k c x \rangle = \delta_{ik} x, \langle x c c_k \rangle = \delta_{jk} x\}$$

then $\mathcal{Q} = \oplus\, \mathcal{Q}_{ij}$ and the composition rules (17.3) hold, for all $i,j\,(1 \le i,j \le r)$.

<u>Exercise</u>: An idempotent $c \in \mathcal{Q}$ is an idempotent of the Jordan triple system $\mathcal{Q}^+$. Show:

$$\mathcal{Q}_1^{\,+}(c) = \mathcal{Q}_{11}, \quad \mathcal{Q}_0^{\,+}(c) = \mathcal{Q}_{00},$$

$$\mathcal{Q}_{1/2}^{\,+}(c) = \mathcal{Q}_{10} \oplus \mathcal{Q}_{01}.$$

17.3. We shall restrict ourselves to the study of the Peirce decomposition relative to a maximal idempotent. Clearly, the idempotent c is <u>maximal</u>, if $\mathcal{A}_{00} = 0$. Let c be maximal. Then

$$\mathcal{A} = \mathcal{A}_{11} \oplus \mathcal{A}_{10} \oplus \mathcal{A}_{01}$$

and we have the following composition rules

(17.4)

$$\begin{array}{llll}
(1) & \langle \mathcal{A}_{11}\ \mathcal{A}_{11}\ \mathcal{A}_{11} \rangle \subset \mathcal{A}_{11} & (8) & \langle \mathcal{A}_{01}\ \mathcal{A}_{11}\ \mathcal{A}_{01} \rangle \subset \mathcal{A}_{10} \\
(2) & \langle \mathcal{A}_{11}\ \mathcal{A}_{11}\ \mathcal{A}_{10} \rangle \subset \mathcal{A}_{10} & (9) & \langle \mathcal{A}_{11}\ \mathcal{A}_{01}\ \mathcal{A}_{10} \rangle \subset \mathcal{A}_{01} \\
(3) & \langle \mathcal{A}_{01}\ \mathcal{A}_{11}\ \mathcal{A}_{11} \rangle \subset \mathcal{A}_{01} & (10) & \langle \mathcal{A}_{10}\ \mathcal{A}_{01}\ \mathcal{A}_{11} \rangle \subset \mathcal{A}_{01} \\
(4) & \langle \mathcal{A}_{10}\ \mathcal{A}_{10}\ \mathcal{A}_{11} \rangle \subset \mathcal{A}_{11} & (11) & \langle \mathcal{A}_{01}\ \mathcal{A}_{10}\ \mathcal{A}_{01} \rangle \subset \mathcal{A}_{11} \\
(5) & \langle \mathcal{A}_{11}\ \mathcal{A}_{01}\ \mathcal{A}_{01} \rangle \subset \mathcal{A}_{11} & (12) & \langle \mathcal{A}_{01}\ \mathcal{A}_{10}\ \mathcal{A}_{10} \rangle \subset \mathcal{A}_{01} \\
(6) & \langle \mathcal{A}_{10}\ \mathcal{A}_{10}\ \mathcal{A}_{10} \rangle \subset \mathcal{A}_{10} & (13) & \langle \mathcal{A}_{10}\ \mathcal{A}_{01}\ \mathcal{A}_{01} \rangle \subset \mathcal{A}_{10} \\
(7) & \langle \mathcal{A}_{01}\ \mathcal{A}_{01}\ \mathcal{A}_{01} \rangle \subset \mathcal{A}_{01} & &
\end{array}$$

while all other compositions are zero.

Furthermore, $\mathcal{A}_{11}$ is an alternative algebra with unit element c and involution $a \mapsto \bar{a} = \langle cac \rangle$, and the following formulas hold for $x,y \in \mathcal{A}$, $a,b \in \mathcal{A}_{11}$, $f,g,h \in \mathcal{A}_{10}$ and $u,v,w \in \mathcal{A}_{01}$:

(17.5) $$\langle xay \rangle = (x \cdot \bar{a}) \cdot y$$

(17.6) $$a \cdot (b \cdot f) = (a \cdot b) \cdot f, \quad (u \cdot a) \cdot b = u \cdot (a \cdot b)$$

(17.7) $$\left\{\begin{array}{l}
\overline{\langle fgc \rangle} = \langle gfc \rangle \\
a \cdot \langle fgc \rangle = \langle (a \cdot f) gc \rangle \\
\langle fgb \rangle \cdot a = \langle fg(b \cdot a) \rangle \\
\langle fga \rangle = \langle f(\bar{a} \cdot g) c \rangle
\end{array}\right.$$

(17.8) $$\left\{\begin{array}{l}
\overline{\langle cuv \rangle} = \langle cvu \rangle \\
\langle buv \rangle \cdot a = \langle bu(v \cdot a) \rangle \\
a \cdot \langle buv \rangle = \langle (a \cdot b) uv \rangle \\
\langle auv \rangle = \langle c(u \cdot \bar{a}) v \rangle
\end{array}\right.$$

(17.9) $\langle fua\rangle = -\langle auf\rangle$

$\langle fua\rangle \cdot b = \langle fu(a\cdot b)\rangle = \langle (a\cdot f)ub\rangle = \langle f(u\cdot \bar{a})b\rangle$

(17.10) $u\cdot u = 0,\ a\cdot(u\cdot v) = (u\cdot a)\cdot v = u\cdot(v\cdot a) = \langle u\bar{a}v\rangle$

(17.11) $\langle fgh\rangle = \langle fgc\rangle \cdot h$

(17.12) $\langle uvw\rangle = u\cdot\langle cvw\rangle + \langle cv(u\cdot w)\rangle$

(17.13) $\langle ufv\rangle = -\langle (u\cdot v)fc\rangle$

(17.14) $\langle ufg\rangle = u\cdot\langle fgc\rangle$

(17.15) $\langle fuv\rangle = \langle fuc\rangle \cdot v$

Proof. (AT 1) gives

(a) $\langle cc\langle xyz\rangle\rangle + \langle xy\langle ccz\rangle\rangle = \langle\langle ccx\rangle yz\rangle + \langle x\langle ccy\rangle z\rangle$

$= \langle\langle xyc\rangle cz\rangle + \langle c\langle yxc\rangle z\rangle.$

A linearization of (AT 3) shows

(b) $\langle xy\langle zcc\rangle\rangle + \langle xy\langle ccz\rangle\rangle = \langle\langle xyz\rangle cc\rangle + \langle\langle xyc\rangle cz\rangle.$

And from a linearization of (16.4) we get

(c) $\langle\langle xcc\rangle yz\rangle + \langle\langle xyc\rangle cz\rangle = \langle x\langle ycc\rangle z\rangle + \langle x\langle ccy\rangle z\rangle.$

Taking $x = x_{ij}, y = y_{k\ell}, z = z_{rs}$, then (a) implies

(d) $\langle cc\langle x_{ij}y_{k\ell}z_{rs}\rangle\rangle = (i+k-r)\langle x_{ij}y_{k\ell}z_{rs}\rangle.$

From b) and c) we get

(e) $\langle\langle x_{ij}y_{k\ell}z_{rs}\rangle cc\rangle = (s+r+j-\ell-k)\langle x_{ij}y_{k\ell}z_{rs}\rangle.$

Now (d) and (e) imply the composition rules (17.3) and then (17.4), (1) - (13). In particular the $\mathcal{U}_{ij}$ are subsystems, and $\mathcal{Q}_{11}$ is a subalgebra of $\mathcal{Q}_c$ with unit element c. We consider $a \mapsto \bar{a} = \langle cac \rangle$ in $\mathcal{Q}_{11}$.

First

$$\bar{\bar{a}} = \langle c\langle cac\rangle c\rangle = \langle cc\langle acc\rangle\rangle \quad \text{, by (16.2)}$$
$$= a \quad \text{, since } a \in \mathcal{Q}_{11}.$$

Then for arbitrary $x, y \in \mathcal{Q}$, $a \in \mathcal{Q}_{11}$

$$\langle x\bar{a}y\rangle = \langle x\langle cac\rangle y\rangle = \langle\langle xca\rangle cy\rangle \quad \text{, by (16.4)}$$
$$= (x\cdot a)\cdot y.$$

This is (17.5). Using this formula we derive

$$\bar{a}\cdot\bar{b} = (c\cdot\bar{a})\cdot\bar{b} = \langle ca\bar{b}\rangle \quad \text{, by (17.5)}$$
$$= \langle ca\langle cbc\rangle\rangle = \langle c\langle bca\rangle c\rangle \quad \text{, by (16.2)}$$
$$= \overline{b\cdot a}.$$

Let $x \in \mathcal{Q}_{10}$ and $z \in \mathcal{Q}$ arbitrary, then $\langle cxz\rangle = \langle\langle ccc\rangle xz\rangle = \langle c\langle xcc\rangle c\rangle = 0$, by (16.3) and $x \in \mathcal{Q}_{10}$. Also $\langle xcz\rangle = \langle x\langle ccc\rangle z\rangle = \langle\langle xcc\rangle cz\rangle = 0$, by (16.4). Then for $a \in \mathcal{Q}_{11}$ we get $\langle axz\rangle = \langle\langle c\bar{a}c\rangle xz\rangle = \langle c\langle xc\bar{a}\rangle z\rangle = 0$ and $\langle xaz\rangle = \langle x\langle c\bar{a}c\rangle a\rangle = \langle\langle xc\bar{a}\rangle cz\rangle = 0$. Thus

$$\text{(f)} \qquad \langle \mathcal{Q}_{11}\mathcal{Q}_{10}\mathcal{Q}\rangle = \langle\mathcal{Q}_{10}\mathcal{Q}_{11}\mathcal{Q}\rangle = 0.$$

We still have to show

$$\text{(g)} \qquad \langle abx_{01}\rangle = \langle ax_{01}b\rangle = \langle x_{01}y_{10}a\rangle = \langle x_{01}y_{01}a\rangle = \langle x_{01}y_{01}z_{10}\rangle$$
$$= \langle x_{10}y_{10}z_{01}\rangle = 0,\ x_{ij}, y_{ij} \in \mathcal{Q}_{ij}, a, b \in \mathcal{Q}.$$

The other cases are covered by (f) and $\mathcal{Q}_{00} = 0$. Using (16.16) we obtain

$\langle x_{1j} y_{1k} z_{01}\rangle = \langle\langle ccx_{1j}\rangle\langle ccy_{1k}\rangle z_{01}\rangle = \langle cc\langle xy\langle ccz\rangle\rangle\rangle = 0$; in particular $\langle abx_{01}\rangle = 0$ and $\langle x_{10} y_{10} z_{01}\rangle = 0$.

Again by (16.2) we get

$\langle c\langle ax_{01}b\rangle c\rangle = \langle cb\langle x_{01}ac\rangle\rangle = 0$, since $\langle x_{01}ac\rangle \in \mathcal{Q}_{01}$ and $\langle cby_{01}\rangle = 0$ (we just proved it.) Thus $\langle ax_{01}b\rangle \in \mathcal{Q}_{10} \oplus \mathcal{Q}_{01}$. Then $\langle cc\langle ax_{01}b\rangle\rangle = 0$ shows that $\langle ax_{01}b\rangle \in \mathcal{Q}_{01}$. But by (e) we get $\langle\langle ax_{01}b\rangle cc\rangle = 2\langle ax_{01}b\rangle$; thus $\langle ax_{01}b\rangle = 0$. The same argument applies to $\langle x_{01}y_{10}a\rangle$. We have $\langle c\langle x_{01}y_{10}a\rangle c\rangle = \langle ca\langle y_{10}x_{01}c\rangle\rangle = 0$, since $\langle y_{10}x_{01}c\rangle \in \mathcal{Q}_{01}$, by (10). Again $\langle cc\langle x_{01}y_{10}a\rangle\rangle = 0$ shows that $\langle x_{01}y_{10}a\rangle \in \mathcal{Q}_{01}$. Again by e) $\langle\langle x_{01}y_{10}a\rangle cc\rangle = 2\langle x_{01}y_{10}a\rangle$ and consequently $\langle x_{01}y_{10}a\rangle = 0$.

$\langle x_{01}y_{01}a\rangle = \langle x_{01}y_{01}c\bar{a}c\rangle\rangle = \langle\langle x_{01}y_{01}c\rangle\bar{a}c\rangle$. Therefore consider $\langle x_{01}y_{01}c\rangle = \langle\langle x_{01}cc\rangle y_{01}c\rangle = \langle x_{01}c\langle cy_{01}c\rangle\rangle = 0$, since $\langle cy_{01}c\rangle = 0$. (kernel $P(c) = \mathcal{Q}_{01} \oplus \mathcal{Q}_{10}$). Then we get $\langle x_{01}y_{01}a\rangle = 0$.
Next $\langle c\langle x_{01}y_{01}z_{10}\rangle c\rangle = \langle cz_{10}\langle y_{01}x_{01}c\rangle\rangle = 0$, thus $\langle x_{01}y_{01}z_{10}\rangle \in \mathcal{Q}_{01} \oplus \mathcal{Q}_{10}$. From (e) we get $\langle\langle x_{01}y_{01}z_{10}\rangle cc\rangle = \langle x_{01}y_{01}z_{10}\rangle$, thus $\langle x_{01}y_{01}z_{10}\rangle \in \mathcal{Q}_{01}$. But then $0 = \langle cc\langle x_{01}y_{01}z_{10}\rangle\rangle = -\langle x_{01}y_{01}z_{10}\rangle$, by (a). Finally $\langle c\langle x_{10}y_{10}z_{01}\rangle c\rangle = \langle cz_{01}\langle y_{10}x_{10}c\rangle\rangle = 0$, since $\langle y_{10}x_{10}c\rangle \in \mathcal{Q}_{11}$ (by (4)) and $\langle cz_{01}\mathcal{Q}_{11}\rangle = 0$ (we proved it before.) Consequently $\langle x_{10}y_{10}z_{01}\rangle \in \mathcal{Q}_{10} \oplus \mathcal{Q}_{01}$; and $\langle\langle x_{10}y_{10}z_{01}\rangle cc\rangle = 0$ (by (e)) implies $\langle x_{10}y_{10}z_{01}\rangle \in \mathcal{Q}_{10}$. But $\langle cc\langle x_{10}y_{10}z_{01}\rangle\rangle = 2\langle x_{10}y_{10}z_{01}\rangle$, thus $\langle x_{10}y_{10}z_{01}\rangle = 0$.

We still have to prove (17.6) - (17.15). Since $\mathcal{A}_c$ is alternative we get (from a linearization of the flexible law) $a\cdot(b\cdot x) - (a\cdot b)\cdot x = x\cdot(a\cdot b) - (x\cdot a)\cdot b$. Choosing $x = f \in \mathcal{A}_{10}$ or $x = u \in \mathcal{A}_{01}$ gives (17.6) since $\langle fca\rangle = \langle acu\rangle = 0$ (for all $a \in \mathcal{A}_{11}$), by (f) and (g).

(17.7) and (17.8): $\langle c\langle fgc\rangle c\rangle = \langle cc\langle gfc\rangle\rangle$, by (16.2)

$= \langle gfc\rangle$, by (4)(=(17.4,(4))).

A linearization of (AT 3) shows $\langle ac\langle xyb\rangle\rangle + \langle ac\langle byx\rangle\rangle = \langle\langle acx\rangle yb\rangle + \langle\langle acb\rangle yx\rangle$, from which follows the second equation in (17.7) and the third equation in (17.8), since $\langle \mathcal{A}_{11}\mathcal{A}_{10}\mathcal{A}\rangle = \langle \mathcal{A}_{01}\mathcal{A}_{01}\mathcal{A}_{11}\rangle = 0$. Using a linearization of (16.3) we obtain $\langle\langle xyb\rangle ca\rangle + \langle\langle byx\rangle ca\rangle = \langle x\langle cby\rangle a\rangle + \langle b\langle cxy\rangle a\rangle$ and therefore (again using $\langle \mathcal{A}_{11}\mathcal{A}_{10}\mathcal{A}\rangle = 0$) $\langle fgb\rangle\cdot a = \langle f\langle cbg\rangle a\rangle = \langle fg(b\cdot a)\rangle$, by (16.2). Next, by a linearization of (AT 3) we get $\langle\langle bxy\rangle ca\rangle + \langle\langle bxa\rangle cy\rangle = \langle bx\langle yca\rangle\rangle + \langle bx\langle acy\rangle\rangle$, which implies $\langle buv\rangle\cdot a = \langle bu(v\cdot a)\rangle$. And $\langle c\langle cuv\rangle c\rangle = \langle cv\langle ucc\rangle\rangle = \langle cvu\rangle$.

Now we have $\overline{\langle fga\rangle} = \langle\overline{\langle fgc\rangle\cdot a}\rangle = \bar{a}\cdot\langle gfc\rangle$

$= \langle(\bar{a}\cdot g)fc\rangle$,

$\langle fga\rangle = \langle f(\bar{a}\cdot g)c\rangle$.

Similarly we prove the last equation of (17.8).

(17.9): Again a linearization of (16.3) shows

$$\langle\langle fua\rangle cc\rangle + \langle\langle auf\rangle cc\rangle = \langle f\langle cau\rangle c\rangle + \langle a\langle cfu\rangle c\rangle$$

$$= 0 \qquad \text{, by (f) and (g).}$$

Now the first part of (17.9) follows from (9) and (10). From (16.4) we get $\langle fua\rangle\cdot b + \langle(f\cdot a)ub\rangle = \langle f\langle cau\rangle b\rangle + \langle f\langle uac\rangle b\rangle$.

This implies (using (17.5), $\langle \mathcal{A}_{10}\, \mathcal{A}_{11}\, \mathcal{A} \rangle = \langle \mathcal{A}_{11}\, \mathcal{A}_{11}\, \mathcal{A}_{01} \rangle$ $= 0$)

$$\langle fua\rangle \cdot b = \langle f(u\cdot\bar{a})b\rangle .$$

In particular $\langle fuc\rangle \cdot b = \langle fub\rangle$.

(17.6) implies

$$\langle fua\rangle \cdot b = (\langle fuc\rangle \cdot a)\cdot b = \langle fuc\rangle \cdot (a\cdot b)$$
$$= \langle fu(a\cdot b)\rangle .$$

The remaining identity follows from (AT 1): $\langle ac\langle fub\rangle\rangle + \langle fu\langle acb\rangle\rangle$ $= \langle\langle acf\rangle ub\rangle + \langle f\langle cau\rangle b\rangle$.

(17.10): $\langle ucu\rangle = \langle u\langle ccc\rangle u\rangle = \langle uc\langle ccu\rangle\rangle = 0.$

$\langle u\bar{a}v\rangle = (u\cdot a)\cdot v$, by (17.5).

The other identity again follows from (AT 1): $\langle uc\langle acv\rangle\rangle +$ $\langle ac\langle ucv\rangle\rangle = \langle\langle uca\rangle cv\rangle + \langle a\langle cuc\rangle v\rangle$.

Then $a\cdot(u\cdot v) = -\, a\cdot(v\cdot u)$ (since $u\cdot v = -\, v\cdot u$)

$$= -\,(v\cdot a)\cdot u = u\cdot(v\cdot a).$$

(17.11) follows from (c) page 204 .

(17.12): $\langle uc\langle cvw\rangle\rangle + \langle cv\langle ucw\rangle\rangle$

$= \langle\langle ucc\rangle vw\rangle + \langle c\langle cuv\rangle w\rangle$, by (AT 1).

But $\langle c\langle cuv\rangle w\rangle = 0$ since $\langle cuv\rangle \in \mathcal{A}_{11}$.

(17.13): By (16.4)

$\langle ufv\rangle cc\rangle + \langle\langle ucv\rangle fc\rangle$

$= \langle u\langle fvc\rangle c\rangle + \langle u\langle cvf\rangle c\rangle = 0$, by (17.9).

This proves (17.13) since $\langle ufv\rangle \in \mathcal{A}_{11}$, by (17.4).

(17.14): $\langle\langle ucc\rangle fg\rangle + \langle\langle ucf\rangle gc\rangle = \langle uc\langle cfg\rangle\rangle + \langle uc\langle fgc\rangle\rangle$.
by (AT 3)

Since $\langle ucc\rangle = u, \langle cf\,\mathcal{A}\rangle = 0$ and $\langle ucf\rangle \in \mathcal{A}_{00} = 0$ we have the desired result. Finally,

$$\langle fuv\rangle = \langle cc\langle fuv\rangle \qquad \text{, by (13)}$$

$$= \langle\langle fuc\rangle cv\rangle \qquad \text{, by (AT 1) and } \langle ufc\rangle = 0.$$

<u>Lemma 1</u>.

a) $\mathcal{A}_{10}$ <u>is associative (of 2nd kind)</u>

b) If $\mathcal{A}_{01} \cdot \mathcal{A}_{01} = 0$ <u>then</u> $\mathcal{A}_{01}$ <u>is associative</u>.

<u>Proof</u>. a) For $x,y,u,v,w \in \mathcal{A}_{10}$ we have

$$\langle xy\langle uvw\rangle\rangle = \langle xyc\rangle\cdot(\langle uvc\rangle\cdot w) \text{ , by (17.11)}$$

$$= (\langle xyc\rangle\cdot\langle uvc\rangle)\cdot w \text{ , by (17.6)}$$

$$= \langle\langle\langle\langle xyc\rangle cu\rangle vc\rangle\cdot w \text{ , by (17.7)}$$

$$= \langle\langle xyu\rangle vw\rangle \qquad \text{, again (17.11).}$$

The equation $\langle xy\langle uvw\rangle\rangle = \langle x\langle vuy\rangle w\rangle$ now follows from (AT 1).

b) If $x,y,u,v,w \in \mathcal{A}_{01}$, then

$$\langle xy\langle uvw\rangle\rangle = x\cdot\langle cy(u\cdot\langle cvw\rangle)\rangle \text{ , by (17.12) and } \mathcal{A}_{01}\cdot\mathcal{A}_{01} = 0,$$

$$= x\cdot\langle\langle cyu\rangle c\langle cvw\rangle\rangle \text{ , by (17.8)}$$

$$= \langle\langle xc\langle cyu\rangle\rangle c\langle cvw\rangle\rangle\text{, by (17.6)}$$

$$= \langle\langle xyu\rangle vw\rangle.$$

<u>Theorem 1</u>. $\mathcal{A} = \mathcal{A}_{11} \oplus \mathcal{A}_{01} \oplus \mathcal{A}_{10}$ ($\mathcal{A}_{ij}$ <u>as above</u>) <u>is an associative triple system (of 2nd kind), iff</u> $\mathcal{A}_{11}$ <u>is associative</u> and the compositions (8) - (13) (<u>in</u> (17.4)) <u>are zero</u>.

<u>Proof</u>. If $\mathcal{A}$ is associative, then clearly $\mathcal{A}_{11}$ is associative.

Also, $\quad <cc<x_{ij}y_{k\ell}z_{rs}>> = <<ccx_{ij}>y_{k\ell}z_{rs}> = i<x_{ij}y_{k\ell}z_{rs}>$ and

$<x_{ij}y_{k\ell}z_{rs}>cc> = <x_{ij}y_{k\ell}<z_{rs}cc>> = s<x_{ij}y_{k\ell}z_{rs}>$, thus

$<x_{ij}y_{k\ell}z_{rs}> \in \mathcal{A}_{is}$.

Comparing this result with (8) - (13) completes one direction of the proof.

From (AT 1) we see that

$$<xy<uvz>> = <<xyu>vz> \quad , \text{ iff}$$
$$<xy<uvz>> = <x<vuy>z> \quad (\text{for all } x,y,u,v,z).$$

Therefore it is sufficient to show

(*) $<x_{ij}y_{k\ell}<u_{rs}v_{pq}z_{mn}>> = <<x_{ij}y_{k\ell}u_{rs}>v_{pq}z_{mn}>$, where $x_{ij} \in \mathcal{A}_{ij}$ etc. We observe that by assumption $<\mathcal{A}_{ij}\,\mathcal{A}_{kj}\,\mathcal{A}_{ks}> \subset \mathcal{A}_{is}$, while all other products vanish (the second indices of the first two factors are equal and the first indices of the last two factors).

From a linearizarion of (AT 3) we get

$$<x_{ij}y_{k\ell}<u_{rs}v_{pq}z_{mn}>> + <x_{ij}y_{k\ell}<z_{mn}v_{pq}u_{rs}>>$$
$$= <<x_{ij}y_{k\ell}u_{rs}>v_{pq}z_{mn}> + <<x_{ij}y_{k\ell}z_{mn}>v_{pq}u_{rs}>.$$

If $j \neq \ell$ or $s \neq q$ or $p \neq m$ or $k \neq r$ then by assumption the first terms on both sides are zero and we have associativity, and if moreover $p \neq r$ or $k \neq m$ or $n \neq q$ then the second terms on both sides vanish. Therefore it remains to show (*) for $j = \ell$, $s = q = n$, $p = m = k = r$.

$$\langle x_{ij} y_{kj} \langle u_{ks} v_{ks} z_{ks} \rangle\rangle = \langle\langle x_{ij} y_{kj} u_{ks} \rangle v_{ks} z_{ks} \rangle.$$

(AT 1) shows

$$\langle x_{ij} y_{kj} \langle u_{ks} v_{ks} z_{ks} \rangle\rangle + \langle u_{ks} v_{ks} \langle x_{ij} y_{kj} z_{ks} \rangle\rangle$$

$$= \langle\langle x_{ij} y_{kj} u_{ks} \rangle v_{ks} z_{ks} \rangle + \langle u_{ks} \langle y_{kj} x_{ij} v_{ks} \rangle z_{ks} \rangle.$$

If $i \neq k$, again the second terms vanish, thus we may assume $k = i$. We have to show $\langle x_{ij} y_{ij} \langle u_{is} v_{is} z_{is} \rangle\rangle = \langle\langle x_{ij} y_{ij} u_{is} \rangle v_{is} z_{is} \rangle$. From a complete linearization of (16.5) we find for $j \neq s$

$$\langle u_{is} v_{is} \langle x_{ij} y_{ij} z_{is} \rangle\rangle = \langle u_{is} \langle y_{ij} x_{ij} v_{is} \rangle z_{is} \rangle.$$

Then (in this case)(AT 1) gives the desired result. Now the Theorem follows from Lemma 1 and assumption.

17.4. Next we are interested in the ideal structure of $\mathcal{Q}$. Let $\mathfrak{U}$ be an ideal of $\mathcal{Q}$. From 17.1, it is clearly seen that the Peirce components of element in $\mathfrak{U}$ are in $\mathfrak{U}$ again i.e.

$$\mathfrak{U} = \mathfrak{U}_{11} \oplus \mathfrak{U}_{10} \oplus \mathfrak{U}_{01}, \quad \mathfrak{U}_{ij} \subset \mathcal{Q}_{ij},$$

(direct sum of submodules).

<u>Lemma 2.</u> <u>$\mathfrak{U} = \mathfrak{U}_{11} \oplus \mathfrak{U}_{10} \oplus \mathfrak{U}_{01}$ is an ideal in</u> $\mathcal{Q}$,

<u>iff</u> 1) $\mathfrak{U}_{11} \cdot \mathcal{Q}_{11} \subset \mathfrak{U}_{11}$ and $\overline{\mathfrak{U}}_{11} = \mathfrak{U}_{11}$

2) $\mathfrak{U}_{11} \cdot \mathcal{Q}_{10} \subset \mathfrak{U}_{10}$, $\mathcal{Q}_{11} \cdot \mathfrak{U}_{10} \subset \mathfrak{U}_{10}$

3) $\mathcal{Q}_{01} \cdot \mathfrak{U}_{11} \subset \mathfrak{U}_{01}$, $\mathfrak{U}_{01} \cdot \mathcal{Q}_{11} \subset \mathfrak{U}_{01}$

4) $\langle \mathcal{U}_{10}\mathcal{O}_{10}\, c\rangle \subset \mathcal{U}_{11}$, $\langle c\, \mathcal{O}_{01}\mathcal{U}_{01}\rangle \subset \mathcal{U}_{11}$

5) $\langle \mathcal{O}_{10}\, \mathcal{U}_{01}\, c\rangle \subset \mathcal{U}_{01}$, $\langle \mathcal{U}_{10}\, \mathcal{O}_{01}\, c\rangle \subset \mathcal{U}_{01}$

6) $\mathcal{U}_{01}\cdot \mathcal{O}_{01} + \mathcal{O}_{01}\cdot \mathcal{U}_{01} \subset \mathcal{U}_{10}$.

The proof is left as an exercise (one has to use the formulas (17.4 - 17.15)).

Corollary 1: If $\mathcal{R}$ is an invariant (i.e. $\overline{\mathcal{R}} = \mathcal{R}$) ideal of the alternative algebra $\mathcal{O}_{11}$, then

$$\mathcal{L} := \mathcal{R} \oplus \mathcal{R}\cdot \mathcal{O}_{10} \oplus \mathcal{O}_{01}\cdot \mathcal{R} \text{ is an ideal of } \mathcal{O}.$$

The proof of this application of Lemma 2 is also left as an exercise (use the formulas to show that 3) - 6) of the Lemma are fulfilled.)

Corollary 2. If $\mathcal{O}$ is simple, then $(\mathcal{O}_{11}, x \mapsto \bar{x})$ is simple (i.e. the alternative algebra $\mathcal{O}_{11}$ has no proper invariant ideal.).

17.5. From now on let $\mathcal{O}$ be simple.

Theorem 2. a) $\mathcal{O}$ is an associative triple system of second kind, iff $\mathcal{O}_{11}$ is associative and $\mathcal{O}_{10}$ or $\mathcal{O}_{01}$, or both, are zero.

b) If $\mathcal{O}_{01} \neq 0$ or $\mathcal{O}_{10} \neq 0$ then $\mathcal{O}_{11}$ is associative.

Proof. If $\mathcal{O}_{11}$ is associative and $\mathcal{O}_{10}$ or $\mathcal{O}_{01}$ is zero then all compositions (8) - (13) in (17.4) are zero and therefore $\mathcal{O}$ is associative by Theorem 1. Conversely, assume $\mathcal{O}$ associative; then it is easily checked that

$\langle \mathcal{O}_{10}\, \mathcal{O}_{10}\, \mathcal{O}_{11}\rangle \oplus \mathcal{O}_{10}$ and $\langle \mathcal{O}_{11}\, \mathcal{O}_{01}\mathcal{O}_{01}\rangle \oplus \mathcal{O}_{01}$ are ideals in $\mathcal{O}$. This proves a).

b) Using (17.7), (17.8) and Lemma 2 we derive easily that $\langle \mathcal{A}_{10}\mathcal{A}_{10}\mathcal{A}_{11}\rangle + \langle \mathcal{A}_{11}\mathcal{A}_{01}\mathcal{A}_{01}\rangle \oplus \mathcal{A}_{10} \oplus \mathcal{A}_{01}$ is an ideal of $\mathcal{A}$. Since $\mathcal{A}$ is simple and $\mathcal{A}_{10}$ or $\mathcal{A}_{01} \neq 0$ we get

$$\mathcal{A}_{11} = \langle \mathcal{A}_{10}\mathcal{A}_{10}\mathcal{A}_{11}\rangle + \langle \mathcal{A}_{11}\mathcal{A}_{01}\mathcal{A}_{01}\rangle .$$

But the last equations in (17.7) and (17.8) show $\langle \mathcal{A}_{10}\mathcal{A}_{10}\mathcal{A}_{11}\rangle = \langle \mathcal{A}_{10}\mathcal{A}_{10}c\rangle$, $\langle \mathcal{A}_{11}\mathcal{A}_{01}\mathcal{A}_{01}\rangle = \langle c\,\mathcal{A}_{01}\mathcal{A}_{01}\rangle$. Thus $\mathcal{A}_{11}$ is (linearly) generated by the elements of the form $\langle fgc\rangle, f,g \in \mathcal{A}_{10}$ and $\langle cuv\rangle, u,v \in \mathcal{A}_{01}$. Let $a,b \in \mathcal{A}_{11}$. Then

$$\begin{aligned} a\cdot(b\cdot\langle fgc\rangle) &= \langle (a\cdot(b\cdot f))gc\rangle && \text{, by (17.7)}\\ &= \langle ((a.b).f)gc\rangle && \text{, by (17.6)}\\ &= (a\cdot b)\cdot\langle fgc\rangle && \text{, again (17.7)}\end{aligned}$$

and

$$\begin{aligned} a\cdot(b\cdot\langle cuv\rangle) &= a\cdot\langle buv\rangle = \langle (a\cdot b)uv\rangle \\ &= (a\cdot b)\langle cuv\rangle && \text{, by (17.8).}\end{aligned}$$

This completes the proof.

Due to Theorem 2, the classification of simple alternative triple systems (with maximal idempotent) reduces to the classification of the following (distinct) types:

I. $\mathcal{A}$ is an associative triple system of second kind. The structure theory for ats (with dcc) has been presented in Chapter IV.

II. $\mathcal{A} = \mathcal{A}_{11}$ is a simple alternative (an not associative) algebra with involution.

The classification of these algebras (and involutions) is known and can be found (for example) in [18]

III. $\mathcal{A} = \mathcal{A}_{11} \oplus \mathcal{A}_{10} \oplus \mathcal{A}_{01}$, $\mathcal{A}_{01}$ and $\mathcal{A}_{10} \neq 0$..

17.6. Since the cases I and II are (essentially) known, we shall assume in this section $\mathcal{A}$ <u>is simple with maximal idempotent,</u> $\mathcal{A}_{01}$ <u>and</u> $\mathcal{A}_{10} \neq 0$. Then $\mathcal{A}_{11}$ is associative, but $\mathcal{A}$ is not associative. We also note that Rad $\mathcal{A}^+ = 0$ since it is an ideal of $\mathcal{A}$ (see remark at the end of 16.3) and in the case $\mathcal{A} =$ Rad $\mathcal{A}^+$ there is no idempotent $\neq 0$ in $\mathcal{A}$, by Theorem 13.9. Hence $\mathcal{A}_{11}{}^+$ and $(\mathcal{A}_{01} \oplus \mathcal{A}_{10})^+$ are semi-simple, in particular $\mathcal{A}_{01} \oplus \mathcal{A}_{10}$ has no trivial element. We shall use these remarks to prove

$$\mathcal{V} := \mathcal{A}_{01} \cdot \mathcal{A}_{01} \neq 0.$$

<u>Proof</u>: If $\mathcal{V} = 0$ then

a) $\langle \mathcal{A}_{01} \mathcal{A}_{11} \mathcal{A}_{01} \rangle = 0$, by (17.10),

b) $\langle \mathcal{A}_{10} \mathcal{A}_{01} \mathcal{A}_{01} \rangle = 0$, by (17.15),

c) $\langle \mathcal{A}_{01} \mathcal{A}_{10} \mathcal{A}_{01} \rangle = 0$, by (17.13).

If $a \in \mathcal{A}_{11}$, $u,v \in \mathcal{A}_{01}$, $f,g \in \mathcal{A}_{10}$ then $\langle\langle auf\rangle (g+v) \langle auf\rangle\rangle$ $= \langle au\langle f\langle ua(v+g)\rangle f\rangle\rangle$ (by (16.19)) = 0, since $\langle uav\rangle = 0$ (by a)) and $\langle uag\rangle = 0$ by (17.4). Consequently

d) $\langle \mathcal{A}_{11} \mathcal{A}_{01} \mathcal{A}_{10} \rangle = 0$, by the above remarks.

Also

e) $\langle \mathcal{A}_{10} \mathcal{A}_{01} \mathcal{A}_{11} \rangle = 0$, by (17.9).

And finally $\langle ufg\rangle = \langle uc\langle fgc\rangle\rangle$ (by (17.14)) $= \langle\langle ucf\rangle gc\rangle + \langle f\langle cug\rangle c\rangle$ $- \langle fgu\rangle$ (by (AT 1)) = 0, by (17.4) and d). Thus

f) $\langle \mathcal{A}_{01} \mathcal{A}_{10} \mathcal{A}_{10} \rangle = 0$

and all products (8) - (13) in (17.4) vanish. Then $\mathcal{A}$ is associative (by Theorem 1) which contradicts our assumption.

<u>Lemma 3.</u> $\mathcal{D}_{11}$ <u>is commutative</u>.

<u>Proof.</u> We define

$$\mathcal{R} := \{a \in \mathcal{D}_{11}\ ,\ a \cdot \mathcal{U} = \bar{a} \cdot \mathcal{U} = 0\}$$

where $\mathcal{U} = \langle \mathcal{D}_{01}\ c\ \mathcal{D}_{01} \rangle$. Since $\mathcal{D}_{11} \cdot \mathcal{U} = \mathcal{U} \neq 0$, by (17.14), clearly $\mathcal{R} \neq \mathcal{D}_{11}$. Using (17.6) and (17.10) we derive for $a,b \in \mathcal{D}_{11}$ and $u,v \in \mathcal{D}_{01}$:

$$\begin{aligned}(a \cdot b) \cdot (u \cdot v) &= a \cdot (b \cdot (u \cdot v)) = a \cdot ((u \cdot b) \cdot v) \\ &= ((u \cdot b) \cdot a) \cdot v = (u \cdot (b \cdot a)) \cdot v \\ &= (b \cdot a) \cdot (u \cdot v).\end{aligned}$$

Thus $\mathcal{R}$ is an invariant ideal of $\mathcal{D}_{11}$ containing all commutators $a \cdot b - b \cdot a$. Since $\mathcal{R} \neq \mathcal{D}_{11}$ it has to be zero by corollary 2 in 17.3. Consequently $a \cdot b = b \cdot a$ for all $a,b \in \mathcal{D}_{11}$.

<u>Corollary.</u> $\mathcal{D}_{11}$ <u>is either a field or direct sum of two isomorphic fields</u>.

<u>Proof.</u> $\mathcal{D}_{11}$ is either simple or a direct sum of two antiisomorphic (= isomorphic, since $\mathcal{D}_{11}$ is commutative) ideals. The statement follows from the fact that a simpel associative, commutative algebra is a field.

We set

$\mathcal{Z} := \mathcal{D}_{11}$. With $a \cdot x = a \cdot x_{11} + a \cdot x_{10} + x_{01} \cdot a$, $\mathcal{D}$ is a $\mathcal{Z}$-module. Define $\varphi : \mathcal{D} \times \mathcal{D} \to \mathcal{Z}$ by

$$\varphi(x,y) = \bar{a} \cdot b + \langle fgc \rangle + \langle uvc \rangle$$

if $x = a + f + u, y = b + g + v, a,b \in \mathcal{Z}$, $f,g \in \mathcal{D}_{10}$, $uv \in \mathcal{D}_{01}$.

<u>Lemma 4</u>. <u>φ is a non degenerate sesquilinear form on $\mathcal{V}$</u> .

<u>Proof</u>. $\overline{\varphi(x,y)} = \varphi(y,x)$ and for $\alpha \in \mathcal{K}$ we have $\alpha \cdot \varphi(x,y)$
$= \varphi(x,\alpha\cdot y)$, by (17.7), (17.8) (observe $\alpha\cdot u := u\cdot\alpha$).
Clearly is non degenerate iff

$(u,v) \mapsto \langle cuv\rangle$ and $(f,g) \mapsto \langle fgc\rangle$ are non degenerate.

Assume $\langle cuv\rangle = 0$ for all $u \in \mathcal{V}_{01}$. Then $\langle v(\mathcal{V}_{10})v\rangle = 0$ by (17.13) and $\langle v(\mathcal{V}_{01})v\rangle = 0$ by (17.12) and assumption. Consequently $v = 0$ since $\mathcal{V}_{1/2}^{+}$ has no trivial elements $\neq 0$. Similarly $\langle fgc\rangle = 0$ for all $g \in \mathcal{V}_{10}$ implies (by (17.11)) that f is a trivial element, thus $f = 0$.

<u>Lemma 5</u>. a) $\mathcal{V}_{10}$ <u>is 1-dimensional over</u> $\mathcal{K}$

b) $\mathcal{V}_{01}$ <u>is a free</u> $\mathcal{K}$-<u>module</u>.

<u>Proof</u>. We set $\mathcal{U} = \mathcal{V}_{10}$, $\mathcal{W} = \mathcal{V}_{01}$. For $f \in \mathcal{U}$ we define

$$J_f : \mathcal{W} \to \mathcal{W} \text{ by}$$

$$J_f(u) = \langle cuf\rangle .$$

If $f,g \in \mathcal{U}$ then (AT 1) gives $\langle uc\langle fgc\rangle\rangle + \langle fg\langle ucc\rangle\rangle = \langle\langle ucf\rangle gc\rangle + \langle f\langle cug\rangle c\rangle$ and using $\langle fgu\rangle = \langle ucf\rangle = 0$ (by (17.4)) and (17.9) we obtain

$$J_f J_g(u) = -u\cdot\langle fgc\rangle . \tag{17.16}$$

Case 1: $\mathcal{K}$ is a field. Then b) is trivial. Assume $\dim \mathcal{U} > 1$. If $\varphi|_{\mathcal{U}}$ $(f,g) \mapsto \langle fgc\rangle$ is non alternating, then there exist f_1, f_2 such that $\langle f_i f_i c\rangle \neq 0$ but $\langle f_1 f_2 c\rangle = 0$. Then by (17.16) $J_{f_1} J_{f_2} = 0$ and $J_{f_i}{}^2 = \alpha_i \mathrm{Id}$ $(0 \neq \alpha_i \in \mathcal{K})$. This is a contradiction. If $\varphi|_{\mathcal{U}}$ is alternating then there exist f_1, f_2 such that $\langle f_i f_i c\rangle = 0$ but $\langle f_1 f_2 c\rangle = 1$. Again by (17.16) $J_{f_i}^2 = 0, J_{f_1} J_{f_2} = -\mathrm{Id}$, and again we have a contradiction. Thus a) must hold.

Case 2: $\mathcal{L}$ is a direct sum of 2 isomorphic fields (involution is the exchange involution):

$$\mathcal{L} = Fe_1 \oplus \overline{Fe_1}, \; e_2 := \bar{e}_1.$$

Then $\mathcal{U} = \mathcal{U}_1 \oplus \mathcal{U}_2$ where $\mathcal{U}_i = \langle e_i c \, \mathcal{U}\rangle$. Since $\langle (a\cdot f) g c\rangle = \langle f(\bar{a}\cdot g) c\rangle = \varphi(f, \bar{a}\cdot g)$ we have $\varphi(\mathcal{U}_i, \mathcal{U}_i) = 0$ and $\varphi(f,g) = \varphi(f_1, g_2) e_1 + \varphi(f_2, g_1) e_2, f_i, g_i \in \mathcal{U}_i$. Since $\varphi|_{\mathcal{U}}$ is non degenerate we can choose f_1, f_2 such that $\varphi(f_1, f_2) e_1 = e_1$ (then $\varphi(f_2, f_1) e_2 = e_2$). We define $f := f_1 + f_2$ and get $\varphi(f,f) = c$. Since φ defines a non degenerate pairing $\mathcal{U}_1 \times \mathcal{U}_2 \to \mathcal{L}$ we can write

$$\mathcal{U}_1 = Ff_1 \oplus f_2^{\perp} \quad , \; f_2^{\perp} = \{g \in \mathcal{U}_1, \; \varphi(g, f_2) = 0\}$$
$$\mathcal{U}_2 = Ff_2 \oplus f_1^{\perp} \; .$$

If $f_i^{\perp} \neq 0$ then there are $g_1, g_2 \in f_2^{\perp}$, $f_1^{\perp}$ (resp.) such that $\varphi(g_1, g_2) e_1 = e_1$ and for $g := g_1 + g_2$ we have $\varphi(g,g) = c$, $\varphi(f,g) = 0$. Then $J_f \cdot J_g = 0$, $J_f^2 = J_g^2 = -\mathrm{Id}$, which is a contradiction. This completes the proof of a).

For b), let $= Fe_1 \oplus \overline{Fe_1}, e_2 := \overline{e_1}, \mathcal{W} = \mathcal{W}_{01}$. Choose $f \in \mathcal{W}_{10}$ such that $\langle ffc\rangle = c$ (as in part a)). Set $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$, $\mathcal{W}_i = \langle \mathcal{W} c e_i\rangle$. Then $\varphi(u,v) = \varphi(u_1, v_2) e_1 + \varphi(u_2, v_1) e_2, u, v \in \mathcal{W}$.

For J_f , defined as above, we get $J_f^2 = -\mathrm{Id}$, and

$$J_f(u) = \langle e_1 u f\rangle + \langle e_2 u f\rangle.$$

By (17.9) and (AT 3) we get

$$\langle e_i uf\rangle = -\langle fue_i\rangle = -\langle fu\langle e_i ce_i\rangle\rangle$$
$$= -\langle\langle fuc_i\rangle ce_i\rangle = \langle\langle e_i uf\rangle ce_i\rangle,$$

i.e. $\langle e_i uf\rangle \in \mathfrak{M}_i$.

Moreover, if $u = \langle uce_i\rangle \in \mathfrak{M}_i$, then

$$J_f(u) = \langle c\langle uce_i\rangle f\rangle = \langle\langle ce_i c\rangle uf\rangle \quad \text{, by (16.3)}$$
$$= \langle e_j uf\rangle,\ i \neq j.$$

Thus $J_f\colon \mathfrak{M}_i \to \mathfrak{M}_j$, $i \neq j$, $J_f{}^2 = -\,\mathrm{Id}$, and J_f is a 1-1 linear map of $\mathfrak{M}_i$ onto $\mathfrak{M}_j$. $\mathfrak{M}_i$ are vector spaces over F resp. $\bar{F}$. Let $(u_\alpha, \alpha \in I)$ be an F - basis of $\mathfrak{M}_i$; then the $J_f(u_\alpha)$ are a basis of $\mathfrak{M}_j$ and $v_\alpha := u_\alpha + J_f(u_\alpha)$ a $\mathfrak{C}$- basis of $\mathfrak{M}$. Since $v_\alpha \cdot e_1 = u_\alpha$, $v_\alpha \cdot e_2 = J_f(u_\alpha)$ we get

$$x = x_1 + x_2 = \sum \xi_\alpha u_\alpha + \sum \eta_\alpha J_f(u_\alpha)$$
$$= \sum \xi_\alpha v_\alpha \cdot e_1 + \sum \eta_\alpha v_\alpha \cdot e_2$$
$$= \sum (\xi_\alpha e_1 + \eta_\alpha e_2)\cdot v_\alpha.$$

This completes the proof.

So far we have $\mathfrak{V}$ is a free $\mathfrak{C}$- module, $\mathfrak{V} = \mathfrak{C}\cdot c \oplus \mathfrak{C}\cdot f \oplus \mathfrak{M}$, and on $\mathfrak{V}$ we have a non degenerate hermitian form $\varphi : \mathfrak{V} \times \mathfrak{V} \to \mathfrak{C}$ such that

$$\varphi(u,v) = \langle cuv\rangle,\ \varphi(c,f) = 0,$$
$$\varphi(c,c) = c\ ,\ \varphi(f,f) = \langle ffc\rangle = \gamma c,\ (\gamma = \bar{\gamma},\ \gamma \neq 0)$$
$$\varphi(\mathfrak{C}\cdot c + \mathfrak{C}\cdot f, \mathfrak{M}) \doteq 0.$$

Now we define a $\mathfrak{C}$ - antilinear map $J\colon \mathfrak{V} \to \mathfrak{V}$ by $J(u) = J_f(u)$, $u \in \mathfrak{M}$, $J(c) = f$, $J(f) = -\gamma c$. Then $J^2 = -\gamma\,\mathrm{Id}$ (since $J_f{}^2 = -\gamma\,\mathrm{Id}|_{\mathfrak{M}}$).

Moreover, if we define a skew symmetric $\mathfrak{C}$ - bilinear form α on $\mathfrak{V}$ by

$$\alpha(u,v)\cdot f = u\cdot v \ , \ u, \ v \in \mathcal{W}$$

$$\alpha(c,f) = 1, \ \alpha(\mathcal{K}c + \mathcal{K}f, \mathcal{W}) = 0$$

then it is easily verified that

$$\varphi(x,y) + \alpha(J(x),y) = 0.$$

(Note: This equation can be used as definition of α). Using (17.5) - (17.15) one verifies that the triple product in $\mathcal{A}$ is given by

$$\langle xyz\rangle = x\,\varphi(y,z) + J(y)\,\alpha(x,z).$$

<u>Exercise</u>: Prove this formula. (One has to verify it for 13 cases according to (17.4).)

<u>Lemma 6.</u> <u>If $\mathcal{A}$ is of type</u> III, <u>then $\mathcal{A}$ is a free $\mathcal{K}$- module with hermitian sesquilinear $\mathcal{K}$- form φ, skew symmetric $\mathcal{K}$- bilinear form α and $\mathcal{K}$- antilinear map</u> J <u>defined as above such that for all $x,y,z \in \mathcal{A}$</u>

$$\langle xyz\rangle = x\,\varphi(y,z) + J(y)\,\alpha(x,z).$$

($\mathcal{K}$ is a field or direct sum of two isomorphic fields). We still need some information about $\mathcal{K}$.

17.6. Let $\mathcal{A}$ be an alternative triple system, $\ell(x,y)z = \langle xyz\rangle = r(z,y)x = p(x,z)y$. Let $\mathcal{T}$ be the associative algebra generated by all $\ell(x,y), r(x,y), p(x,y)$ in $\mathrm{End}_{\Phi}\mathcal{A}$, and $\mathcal{R}$ the subalgebra generated by all $\ell(x,y), r(x,y)$. The <u>centroid</u> $Z(\mathcal{A})$ of $\mathcal{A}$ is the centralizer of $\mathcal{T}$ in $\mathrm{End}\,\mathcal{A}$. $Z(\mathcal{A}) = \{\alpha \in \mathrm{End}\,\mathcal{A}, \ \alpha\tau = \tau\alpha$ for all $\tau \in \mathcal{T}\}$. The <u>metacentroid</u> $C(\mathcal{A})$ is the centralizer of $\mathcal{R}$ in $\mathrm{End}\,\mathcal{A}$. $C(\mathcal{A}) = \{\beta \in \mathrm{End}\,\mathcal{A}, \ \beta\rho = \rho\beta$ for all $\rho \in \mathcal{R}\}$. Clearly $\mathcal{R} \subset \mathcal{T}$ and $Z(\mathcal{A}) \subset C(\mathcal{A})$.

Lemma 7. a) If $\langle \mathcal{A}\mathcal{A}\mathcal{A} \rangle = \mathcal{A}$ then $C(\mathcal{A})$ is commutative.

b) If $\mathcal{A}$ is simple then $Z(\mathcal{A})$ is a field.

Proof. a) If $\alpha, \beta \in C(\mathcal{A})$ then $\alpha\beta\langle xyz \rangle = \alpha\langle xy(\beta z)\rangle = \langle(\alpha x)y(\beta z)\rangle = \beta\alpha\langle xyz\rangle$.

b) $\mathcal{A}$ simple $\Leftrightarrow$ $\mathcal{A}$ is an irreducible $\mathcal{T}$ - module. Then $Z(\mathcal{A})$ is a division ring, by Schur's lemma; since it is commutative (by a)) it is a field.

Theorem 3. Let $\mathcal{A}$ be simple. Then $\mathcal{R}$ acts either irreducibly, or $\mathcal{A}$ is the direct sum of two inequivalent irreducible $\mathcal{R}$ - modules $\mathcal{A}_1$, $\mathcal{A}_2$. In this case $\mathcal{A}_1 = \langle \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_1 \rangle$, $\mathcal{A}_2 = \langle \mathcal{A}_2 \mathcal{A}_1 \mathcal{A}_2 \rangle$, $\langle \mathcal{A}_i \mathcal{A}_i \mathcal{A} \rangle = \langle \mathcal{A}\mathcal{A}_i \mathcal{A}_i \rangle = 0$.

Proof. Similar to Thm XI,4 . Let $\mathcal{A}_1$ be a proper $\mathcal{R}$- submodule. Set $\mathcal{A}_2 = \langle \mathcal{A}\mathcal{A}_1\mathcal{A} \rangle$. Then $\mathcal{A}_2$ is a $\mathcal{R}$- submodule (follows from AT 1) and $\langle \mathcal{A}\mathcal{A}_2\mathcal{A} \rangle \subset \mathcal{A}_1$. This show that $\mathcal{A}_1 \cap \mathcal{A}_2$ and $\mathcal{A}_1 + \mathcal{A}_2$ are ideals of $\mathcal{A}$ and then $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$, $\mathcal{A}_1 \cap \mathcal{A}_2 = 0$. The rest is straightforward.

Corollary 1. $C(\mathcal{A})$ is a field or a direct sum of two fields (each containing $Z(\mathcal{A})$).

Corollary 2. If $\mathcal{A}$ is simple and finite dimensional over $F = Z(\mathcal{A})$, then $C(\mathcal{A})$ is either F, a quadratic extension of F or $F \oplus F$.

Proof. Let $\bar{F}$ be the algebraic closure of F, $\overline{\mathcal{A}} = \mathcal{A} \otimes_F \bar{F}$; then $Z(\overline{\mathcal{A}}) = \bar{F}$ since $Z(\overline{\mathcal{A}})$ is a finite extension of $\bar{F}$; by Corollary 1, $C(\overline{\mathcal{A}}) = \bar{F}$ or $\bar{F} \oplus \bar{F}$. In particular $\dim_{\bar{F}} (C(\overline{\mathcal{A}}) \leq 2$. Since $\dim_F C(\mathcal{A}) \leq \dim_{\bar{F}} C(\overline{\mathcal{A}})$ we get the desired result.

Theorem 4. *If c is a maximal idempotent of* $\mathfrak{A}$ *and* C *the center of* $\mathfrak{A}_{11}(c)$ (as algebra), $C^{+} = \{\alpha \in C, \bar{\alpha} = \alpha\}$ *then the map*

$$\zeta \mapsto \zeta(c)$$

is an isomorphism of $C(\mathfrak{A})$ *onto* C *which maps* $Z(\mathfrak{A})$ *onto* C^{+}.

Proof. If $\zeta \in C(\mathfrak{A})$ then $\zeta(c) = \zeta\langle ccc\rangle = \langle\zeta(c)cc\rangle = \langle cc\zeta(c)\rangle$.
Thus $\zeta(c) \in \mathfrak{A}_{11}$. Also
$\zeta(c)\cdot a = \langle\zeta(c)ca\rangle = \zeta a = \zeta\langle acc\rangle = a\cdot\zeta(c)$,
$(\zeta(c)\cdot a)\cdot b = \zeta(a\cdot b) = \zeta(c)\cdot(a\cdot b)$, etc. Thus $\zeta(c) \in$ Center of $\mathfrak{A}_{11}$.
If $\zeta \in Z(\mathfrak{A})$ then $\overline{\zeta(c)} = \langle c\zeta(c)c\rangle = \zeta c$.

Conversely, if $z \in C$, then φ_z defined by $\varphi_z(a) = z\cdot a$, $\varphi_z(x_{10}) = z\cdot x_{10}$, $\varphi_z(y_{01}) = y_{01}\cdot z$ is in $C(\mathfrak{A})$ and $\varphi_z \in Z(\mathfrak{A})$ if $z \in C^{+}$. Moreover $\varphi_z(c) = z$ and $\varphi_{\zeta(c)} = \zeta$. Also for $\zeta, \rho \in C(\mathfrak{A})$ we have $(\zeta\rho)(c) = \zeta(c)\cdot\rho(c)$.

17.7. We collect our previous results on the classification of simple alternative triple systems.

Theorem 5. *Let* $\mathfrak{A}$ *be a simple alternative triple system with maximal idempotent; then* $\mathfrak{A}$ *is either*

1) *a simple associative triple system of 2nd kind,*
2) *an alternative algebra* $\mathfrak{A}_c$ *with involution* $x \mapsto \bar{x}$ *such that* $(\mathfrak{A}_c, x \mapsto \bar{x})$ *is a simple pair and the triple product is given by* $\langle xyz\rangle = (x\cdot\bar{y})z$
3) $\mathfrak{A}$ *is a free C - module, where* C *is the metacentroid of* $\mathfrak{A}$, *which is either a field (with involution) or a direct sum of two isomorphic fields (with exchange involution), and the triple product is given by*

$$\langle xyz\rangle = \varphi(y,z)x + \alpha(x,z)Jy ,$$

where φ is a non degenerate hermitian sesquilinear form of $\mathcal{U}$ in C, α a skew symmetric C - bilinear form of $\mathcal{U}$ in $C^+ = \{x \in C, \bar{x} = x\}$ and J a C - antilinear map of $\mathcal{U}$ such that $J^2 = -\gamma \mathrm{Id}$, $0 \neq \gamma \in C^+$, $\varphi(x,y) + \alpha(Jx,y) = 0$.

Exercise: Show that an alternative triple system of type 3) is simple.

XVIII. Classification of Jordan Triple Systems

18.1. There are still many open questions concerning the classification of simple Jts's with dcc on inner ideals. The crucial point is the existence or non existence of maximal idempotents. For example $V = \mathbb{R}$ (the reals) over $\mathbb{R}$ together with $P(x)y = -x^2y$ is simple but has no idempotents.

Under the assumption that $\mathfrak{A}$ is a finite dimensional Jts over an algebraically closed field F, char $F \neq 2$ one can show that for a non nilpotent element $x \in \mathfrak{A}$ the subtriple system generated by x contains an idempotent; and if moreover $\mathfrak{A}$ is semi simple then $\mathfrak{A}$ contains a maximal idempotent.

These results and the classification of finite dimensional simple Jts's over an algebraically closed field of char $\neq 2$ are beautifully presented in O. Loos's notes "Lectures on Jordan Triples" (The University of British Columbia, 1971) and we won't waste more paper by copying his exposition. But to show the reader what he can expect we outline the conclusive results in Loos's notes.

Let $\mathfrak{A}$ be a finite dimensional simple Jts over F of char $\neq 2$; then by Theorem 16.6 either $\mathfrak{A} = \mathfrak{A}_1(c)$, in which case $\mathfrak{A}$ is a simple Jordan algebra with involution $x \to \bar{x}$ and the triple product is given by $P(x)y = Q(x)\bar{y}$, (Q denotes the quadratic representation of the Jordan algebra $\mathfrak{A}$), or $\mathfrak{A}$ is isomorphic to the standard imbedding $\mathfrak{M} \oplus \mathfrak{A}_{1/2}$ of the alternative triple system $\mathfrak{A}_{1/2}(c)$, where c is a maximal idempotent.

Therefore the classification reduces to the classification of all simple pairs $(\mathfrak{A}, j)$ where $\mathfrak{A}$ is a Jordan algebra and j an involution in

$\mathfrak{A}$, or to the classification of all simple alternative triple systems and the determinations of their standard imbeddings. Under the given assumptions one derives from Theorem 17.5 together with the known results on associative triple systems of the second kind and the results on Jordan algebras the following result

Theorem. A simple finite dimensional Jordan triple system over an algebraically closed field F of char. $\neq 2$ is isomorphic to one of the following.

(1) Hermitian, skew hermitian or rectangular matrices over $\mathcal{L}$, $\mathcal{L} = F$, $F \oplus F$ or the quaternions over F, with $P(x)y = x\bar{y}^t x$. ($y \to \bar{y}$ is induced from the canonical involution on $\mathcal{L}$.)

(2) Symmetric or skewsymmetric matrices over $F \oplus F$ with $P(x)y = x\bar{y}^t x$.

(3) F^n with $P(x)y = 2(x,y)x - (x,sx)sy$ where $(x,y) = \Sigma x_i y_i$ and s is a reflection in a subspace F^p of F^n.

(4) $(F \oplus F)^n$ with $P(x)y = 2(x,\bar{y})x - (x,x)\bar{y}$, where $(x,y) = \Sigma x_i y_i$.

(5) 1×2 matrices over $(\mathcal{C}, j)$ where $\mathcal{C}$ is the Cayley algebra, j involution, or over ($\mathcal{C} \oplus \mathcal{C}^{op}$, exchange involution), with $P(x)y = x(\bar{y}^t x)$.

(6) $\mathfrak{H}_3(\mathcal{C})$, the exceptional Jordan algebra with quadratic representation Q, the triple product is given by $P(x)y = Q(x)j(y)$, j an involution.

(7) $\mathfrak{H}_3(\mathcal{C}) \oplus \mathfrak{H}_3(\mathcal{C})$ with $P(x)y = Q(x)j(y)$, where j is the exchange involution.

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