π-complemented algebras

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Abstract

For an ideal U of a nonassociative algebra A, the π-closure of U is defined by $\overline{U} = \text{Ann}(\text{Ann}(U))$, where Ann(.) denotes the annihilator relative to the algebra A. An algebra A is said to be π-complemented if for every π-closed ideal U of A there exists a π-closed ideal V of A such that $A = U \oplus V$. In this paper we shall develop a structure theory for π-complemented algebras.

Key Words: Semiprime algebras; Closure operations; Complemented algebras; Decomposable algebras.

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Introduction and preliminaries.

Throughout this paper, we will deal with algebras over a fixed field $\mathbb{K}$ which are not necessarily associative. Let us start by recalling some well known concepts, as well as the description of complemented algebras. An algebra A is said to be *complemented* if every ideal of A is a direct summand of A, that is: for every ideal U of A there exists an ideal V of A such that $A = U \oplus V$. Examples of complemented algebras are null algebras and decomposable algebras. Recall that a *null algebra* is an algebra with zero product; a *decomposable algebra* is an algebra that is isomorphic to a direct sum of simple algebras; and a *simple algebra* is a non-null algebra lacking nonzero proper ideals. By regarding any algebra as a left module over its multiplication algebra, the standard characterization of completely reducible modules can be rewritten in this case as follows:

Description theorem for complemented algebras. *For a non-null algebra A the following assertions are equivalent:*

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(i) A is complemented,

(ii) A is isomorphic to an algebra of the form $B_0 \oplus B$, where B_0 is a null algebra and B is a decomposable algebra.

Given an algebra A and a closure operation $\sim$ on the complete lattice $\mathcal{I}_A$ of all ideals of A, borrowing terminology from the theory of Banach spaces, we will say that an $\sim$-closed ideal U of A is *$\sim$-complemented* (resp. *$\sim$-quasicomplemented*) in A if there exists a $\sim$-closed ideal V of A such that

$$A = U \oplus V \qquad (\text{resp.} \quad A = (U \oplus V)^{\sim}).$$

In such a case, V is called a *$\sim$-complement* (resp. *$\sim$-quasicomplement*) of U. We say that A is a *$\sim$-complemented* (resp. *$\sim$-quasicomplemented*) algebra when every $\sim$-closed ideal of A is $\sim$-complemented (resp. $\sim$-quasicomplemented) in A. Clearly every $\sim$-complemented algebra is $\sim$-quasicomplemented. It is also clear that in the case in which $\sim$ is the discrete closure (that is, $\widetilde{U} = U$ for all $U \in \mathcal{I}_A$), the concepts of complementarity, $\sim$-complementarity, and $\sim$-quasicomplementarity agree.

Naturally the task of the description of the lattice of the $\sim$-closed ideals often turns out to be involved. However, the study of the lattice of all $\| \, . \, \|$-closed ideals and the complementarity in such a lattice has been widely discussed in specific Banach algebras: algebras of functions, and particularly algebras of sequences, (see, e.g., [3, Chapter 4] and [9]); and operator algebras (see, e.g., [7] and [8]). It is worth pointing out that $\| \, . \, \|$-quasicomplemented algebras are generalized annihilator normed algebras, which were introduced and studied by B. Yood in the associative context [10], and by A. Fernández and A. Rodríguez in a nonassociative setting [5], and they were revisited by the authors in [1]. In this latter paper, by considering every algebra as a left module over its multiplication algebra, the ε-closure was introduced, and it was proved that the ε-quasicomplemented algebras with zero annihilator are precisely the multiplicativatively semiprime algebras.

Our aim in this paper is to study those algebras which are complemented with respect to the classical π-closure. For S_1, S_2 subspaces of an algebra A, we denote by S_1S_2 the subspace of A generated by all the products xy, for $x \in S_1$ and $y \in S_2$. For the sake of brevity, we write S^2 instead of SS. As usual, for each ideal U of A, the largest ideal V of A satisfying the conditions $UV = VU = 0$ is called the *annihilator* of U in A and is denoted by $\mathrm{Ann}(U)$. The following properties are immediately verifiable and will be used without further mention throughout the rest of this paper.

(1) If U, V are ideals of A such that $U \subseteq V$, then $\mathrm{Ann}(V) \subseteq \mathrm{Ann}(U)$.

(2) $U \subseteq \mathrm{Ann}(\mathrm{Ann}(U))$, for every ideal U of A.

(3) $\mathrm{Ann}(U) = \mathrm{Ann}(\mathrm{Ann}(\mathrm{Ann}(U)))$, for every ideal U of A.

(4) $\mathrm{Ann}(\sum U_i) = \bigcap \mathrm{Ann}(U_i)$, for every family $\{U_i\}$ of ideals of A.

The π-*closure* of an ideal U of A is defined by

$$\overline{U} = \mathrm{Ann}(\mathrm{Ann}(U)).$$

Note that the above property (3) can be read as follows:

$$\overline{\mathrm{Ann}(U)} = \mathrm{Ann}(\overline{U}) = \mathrm{Ann}(U),$$

for every $U \in \mathcal{I}_A$. Of course, the π-closure is a closure operation on $\mathcal{I}_A$, that is, it satisfies the following properties:

(1) If U, V are ideals of A such that $U \subseteq V$, then $\overline{U} \subseteq \overline{V}$.

(2) $U \subseteq \overline{U}$, for every ideal U of A.

(3) $\overline{U} = \overline{\overline{U}}$, for every ideal U of A.

(4) $\bigcap U_i$ is a π-closed ideal of A, for every family $\{U_i\}$ of π-closed ideals of A.

The set $\mathcal{I}_A^\pi$ of all π-closed ideals of A is a complete lattice for the meet and join operations given by

$$\bigwedge U_i = \bigcap U_i \quad \text{and} \quad \bigvee U_i = \overline{\sum U_i}.$$

Moreover, $\mathrm{Ann}(A)$ and A respectively are the smallest and the largest elements in $\mathcal{I}_A^\pi$.

The set of all maximal elements of $\mathcal{I}_A^\pi$ is denoted by $\mathbf{M}_A^\pi$ and the set

$$\pi\text{-}\mathrm{Rad}(A) := \bigcap_{M \in \mathbf{M}_A^\pi} M$$

is called the π-*radical* of A. We say that A is a π-*radical algebra* whenever $\mathbf{M}_A^\pi = \emptyset$. The set of all minimal elements of $\mathcal{I}_A^\pi$ is denoted by $\mathbf{m}_A^\pi$, and the set

$$\pi\text{-}\mathrm{Soc}(A) := \sum_{B \in \mathbf{m}_A^\pi} B$$

is called the π-*socle* of A. We say that A is a π-*decomposable algebra* whenever

$$A = \overline{\pi\text{-}\mathrm{Soc}(A)}.$$

The first three sections of this paper are devoted to π-quasicomplemented algebras. In section one it is shown that the π-quasicomplemented algebras are precisely the semiprime algebras. We also study the π-closure in a direct sum. In section two we discuss the π-closure in the quotient algebra A/U, for a semiprime algebra A and a π-closed ideal U of A. As a consequence we will obtain properties of π-Rad(A/U) and π-Soc(A/U). In section three we provide a description of the lattice $\mathcal{I}_A^\pi$ whenever A is an essential subdirect product of a family of semiprime algebras. Moreover, we prove that every semiprime algebra is a subdirect product of two semiprime algebras, one of which is π-radical and the other is π-decomposable. Section four provides different characterizations of π-complemented algebras. We prove that the π-complemented algebras are precisely the semiprime algebras in which the π-closure is additive. We will also prove that every π-complemented algebra is the direct sum of a π-radical π-complemented algebra with a π-decomposable π-complemented algebra. Moreover, we show that the algebra of all Lebesgue measurable functions on the unit interval is π-radical π-complemented. In the final section we set out to describe π-decomposable π-complemented algebras. Our approach relies on the structure theory for π-decomposable algebras due to A. Fernández, E. García, and M. I. Tocón [4, 6]. We show that, up to isomorphism, π-decomposable π-complemented algebras are the subalgebras of the direct product of a family of prime algebras which contain the direct sum of the family and remain invariant under all the block-projections.

1 π-quasicomplemented algebras

Our first goal is to prove that the π-quasicomplemented algebras are precisely the semiprime algebras. Recall that an algebra A is said to be *semiprime* if 0 is the unique ideal U of A with $U^2 = 0$.

Lemma 1.1. *Let A be an algebra with zero annihilator and U, V be ideals of A. If $A = \overline{U \oplus V}$, then $A = \overline{U \oplus \mathrm{Ann}(U)}$ and $\overline{V} = \mathrm{Ann}(U)$.*

Proof. Assume that $A = \overline{U \oplus V}$. Since $U \cap V = 0$ it follows that $UV = VU = 0$, and hence $V \subseteq \mathrm{Ann}(U)$. Thus, $A = \overline{U + \mathrm{Ann}(U)}$, and we see that

$$\mathrm{Ann}(A) = \mathrm{Ann}(\overline{U + \mathrm{Ann}(U)}) = \mathrm{Ann}(U) \cap \overline{U}.$$

Therefore $U \cap \mathrm{Ann}(U) \subseteq \mathrm{Ann}(A)$, hence $U \cap \mathrm{Ann}(U) = 0$, and so

$$A = \overline{U \oplus \mathrm{Ann}(U)}.$$

On the other hand, from $A = \overline{U \oplus V}$ it follows immediately that

$$\mathrm{Ann}(U) \cap \mathrm{Ann}(V) = \mathrm{Ann}(\overline{U + V}) = \mathrm{Ann}(A) = 0,$$

and so $\mathrm{Ann}(U) \subseteq \overline{V}$. Finally, from the chain $V \subseteq \mathrm{Ann}(U) \subseteq \overline{V}$ it follows that $\overline{V} = \mathrm{Ann}(U)$. □

Lemma 1.2. *If A is a π-quasicomplemented algebra, then* $\mathrm{Ann}(A) = 0$.

Proof. Since $\mathrm{Ann}(A)$ is a π-closed ideal of A, there exists a π-closed ideal V of A such that
$$A = \overline{\mathrm{Ann}(A) \oplus V}.$$

Therefore

$$\mathrm{Ann}(A) = \mathrm{Ann}(\overline{\mathrm{Ann}(A) \oplus V}) = \mathrm{Ann}(\mathrm{Ann}(A)) \cap \mathrm{Ann}(V) =$$

$$A \cap \mathrm{Ann}(V) = \mathrm{Ann}(V),$$

hence
$$A = \mathrm{Ann}(\mathrm{Ann}(A)) = \mathrm{Ann}(\mathrm{Ann}(V)) = \overline{V} = V,$$

and so
$$\mathrm{Ann}(A) = \mathrm{Ann}(A) \cap A = \mathrm{Ann}(A) \cap V = 0.$$

□

Proposition 1.3. *Let A be an algebra. Then the following assertions are equivalent:*

(i) A is π-quasicomplemented.

(ii) $A = \overline{U \oplus \mathrm{Ann}(U)}$ for every ideal U of A.

(iii) A is semiprime.

In this case, for each π-closed ideal U of A, $\mathrm{Ann}(U)$ is the unique π-quasicomplement of U.

Proof. (i) $\Rightarrow$ (ii). By Lemma 1.2, $\mathrm{Ann}(A) = 0$. Now, by Lemma 1.1, we see that $A = \overline{\overline{U} \oplus \mathrm{Ann}(U)}$, and consequently $A = \overline{U \oplus \mathrm{Ann}(U)}$ for every ideal U of A.

(ii) $\Rightarrow$ (iii). If U is an ideal of A with $U^2 = 0$, then $U \subseteq \mathrm{Ann}(U)$, and hence $U = 0$.

(iii) $\Rightarrow$ (i). Let U be a π-closed ideal of A. By semiprimeness we see that $\mathrm{Ann}(U) \cap U = 0$, hence $\mathrm{Ann}(U \oplus \mathrm{Ann}(U)) = 0$, and so $A = \overline{U \oplus \mathrm{Ann}(U)}$.

Let us assume that A satisfies the equivalent conditions in the statement. Then it is clear that, for each π-closed ideal U of A, $\mathrm{Ann}(U)$ is a π-quasicomplement of U. By Lemma 1.1, $\mathrm{Ann}(U)$ is the unique π-quasicomplement of U. □

From [6, Theorem 2.4] we deduce the following consequence:

Corollary 1.4. *An algebra A with zero annihilator is semiprime if, and only if, $\mathcal{I}_A^\pi$ is a boolean algebra.*

For a given algebra A, we denote by $\mathbf{D}_A^\pi$ the set of all π-dense ideals of A. It is immediately verified that

$$\mathbf{D}_A^\pi = \{U \in \mathcal{I}_A \, : \, \mathrm{Ann}(U) = \mathrm{Ann}(A)\}.$$

Corollary 1.5. *If A is a semiprime algebra, then*

$$\mathbf{D}_A^\pi := \{U \oplus \mathrm{Ann}(U) \, : \, U \in \mathcal{I}_A\}.$$

Proof. Let A be a semiprime algebra. By Proposition 1.3, we see that $U \oplus \mathrm{Ann}(U) \in \mathbf{D}_A^\pi$ for every $U \in \mathcal{I}_A$. Conversely, if $D \in \mathbf{D}_A^\pi$, then $\mathrm{Ann}(D) = \mathrm{Ann}(A) = 0$, and hence $D = D \oplus \mathrm{Ann}(D)$. $\square$

Remark 1.6. For every algebra A, the map

$$U \mapsto \mathrm{Ann}(U)$$

is an order-reversing involutive bijection from $\mathcal{I}_A^\pi$ onto itself. Therefore,

$$\mathbf{m}_A^\pi = \{\mathrm{Ann}(M) \, : \, M \in \mathbf{M}_A^\pi\} \text{ and } \mathbf{M}_A^\pi = \{\mathrm{Ann}(B) \, : \, B \in \mathbf{m}_A^\pi\}.$$

As a consequence, note that

(1) $\mathrm{Ann}(\pi{-}\mathrm{Soc}(A)) = \pi{-}\mathrm{Rad}(A)$.

Indeed,

$$\mathrm{Ann}(\pi{-}\mathrm{Soc}(A)) = \mathrm{Ann}\left(\sum_{B \in \mathbf{m}_A^\pi} B\right) = \bigcap_{B \in \mathbf{m}_A^\pi} \mathrm{Ann}(B) =$$

$$\bigcap_{M \in \mathbf{M}_A^\pi} M = \pi{-}\mathrm{Rad}(A).$$

In particular, in the case in which $\mathrm{Ann}(A) = 0$ we have

(2) A is π-decomposable if, and only if, $\pi{-}\mathrm{Rad}(A) = 0$.

(3) A is π-radical if, and only if, $\pi{-}\mathrm{Soc}(A) = 0$. $\square$

As another consequence of Proposition 1.3 we deduce from assertion (1) in the above remark the following statement:

Corollary 1.7. *If A is a semiprime algebra, then*

$$A = \overline{\pi{-}\mathrm{Soc}(A) \oplus \pi{-}\mathrm{Rad}(A)}.$$

For an ideal U of an algebra A, we set

$$(U : A) := \{a \in A \,:\, aA + Aa \subseteq U\}.$$

It is clear that $(U : A)$ is an ideal of A containing U.

Lemma 1.8. *Let A be an algebra. If U is an ideal of A such that $U \cap \mathrm{Ann}(U) = 0$, then $\mathrm{Ann}(U) = \mathrm{Ann}(U : A)$. If in addition U is π-closed, then $(U : A) = U$.*

Proof. Let U be an ideal of A such that $U \cap \mathrm{Ann}(U) = 0$. Since

$$(U : A)\mathrm{Ann}(U) + \mathrm{Ann}(U)(U : A) \subseteq U \cap \mathrm{Ann}(U),$$

it follows that

$$(U : A)\mathrm{Ann}(U) = \mathrm{Ann}(U)(U : A) = 0.$$

Therefore $\mathrm{Ann}(U) \subseteq \mathrm{Ann}(U : A)$. The converse inclusion is clear. Hence $\mathrm{Ann}(U) = \mathrm{Ann}(U : A)$, and as a consequence $\overline{U} = \overline{(U : A)}$. If in addition U is π-closed, then we have

$$(U : A) \subseteq \overline{(U : A)} = \overline{U} = U \subseteq (U : A),$$

and hence $U = (U : A)$. □

In the next statement we recall the well-known description of the lattice of all ideals of a direct sum.

Lemma 1.9. *Let $\{A_i\}_{i \in I}$ be a nonempty family of nonzero algebras, and set $A = \bigoplus_{i \in I} A_i$. Then $\mathcal{I}_A$ is the set of all subspaces U of A for which there exists a family $\{U_i\}_{i \in I}$, where each U_i is an ideal of A_i, satisfying*

$$\bigoplus_{i \in I} U_i \subseteq U \subseteq \bigoplus_{i \in I} (U_i : A_i).$$

Proof. If U is a subspace of A such that

$$\bigoplus_{i \in I} U_i \subseteq U \subseteq \bigoplus_{i \in I} (U_i : A_i),$$

where each U_i is an ideal of A_i, then it is fairly evident that U is an ideal of A. In order to prove the converse let us fix an ideal U of A and set $U_i := U \cap A_i$ for each $i \in I$. Clearly each U_i is an ideal of A_i, and it is immediately verifiable that

$$\bigoplus_{i \in I} U_i \subseteq U \subseteq \bigoplus_{i \in I} (U_i : A_i).$$

□

If A and B are algebras, and p is an algebra homomorphism from A onto B, then it is evident that $p(\mathrm{Ann}(U)) \subseteq \mathrm{Ann}(p(U))$ for every ideal U of A. Recall that an ideal D of an algebra A is said to be *essential* if for every nonzero ideal U of A we have $U \cap D \neq 0$. Now, we give a description of the π-closed ideals in the direct sum of a family of semiprime algebras.

Theorem 1.10. *Let $\{A_i\}_{i\in I}$ be a nonempty family of nonzero algebras, and set $A = \bigoplus_{i\in I} A_i$. Then*

(1) If $\{U_i\}_{i\in I}$ is a family, where each U_i is an ideal of A_i, then

$$\mathrm{Ann}(\bigoplus_{i\in I} U_i) = \bigoplus_{i\in I} \mathrm{Ann}_{A_i}(U_i) \quad \textit{and} \quad \overline{\bigoplus_{i\in I} U_i} = \bigoplus_{i\in I} \overline{U_i}.$$

As a consequence, regarding each A_i inside of A,

$$\mathcal{I}^{\pi}_{A_i} = \{U \in \mathcal{I}^{\pi}_A : U \subseteq A_i\}.$$

(2) A has zero annihilator if, and only if, A_i has zero annihilator for all i in I. In this case,

$$\mathbf{m}^{\pi}_A = \bigcup_{i\in I} \mathbf{m}^{\pi}_{A_i},$$

and the essential ideals of A are only those containing one of the form $\bigoplus_{i\in I} D_i$, where D_i is an essential ideal of A_i for each $i \in I$.

(3) A is semiprime if, and only if, A_i is semiprime for all i in I. In this case,

$$\mathcal{I}^{\pi}_A = \{\bigoplus_{i\in I} U_i \ : \ U_i \in \mathcal{I}^{\pi}_{A_i} \ \ \textit{for every} \ \ i \in I\}.$$

Proof. (1) For a given family $\{U_i\}_{i\in I}$, it is clear that

$$\bigoplus_{i\in I} \mathrm{Ann}_{A_i}(U_i) \subseteq \mathrm{Ann}(\bigoplus_{i\in I} U_i).$$

For each $i_0 \in I$, denote by p_{i_0} the projection from A onto A_{i_0}, and note that

$$p_{i_0}(\mathrm{Ann}(\bigoplus_{i\in I} U_i)) \subseteq \mathrm{Ann}_{A_{i_0}}(p_{i_0}(\bigoplus_{i\in I} U_i)) = \mathrm{Ann}_{A_{i_0}}(U_{i_0}).$$

Therefore

$$\mathrm{Ann}(\bigoplus_{i\in I} U_i) \subseteq \bigoplus_{i\in I} \mathrm{Ann}_{A_i}(U_i),$$

and we conclude that

$$\mathrm{Ann}(\bigoplus_{i\in I} U_i) = \bigoplus_{i\in I} \mathrm{Ann}_{A_i}(U_i).$$

Taking annihilators in this equality, we deduce that

$$\overline{\bigoplus_{i\in I} U_i} = \bigoplus_{i\in I} \overline{U_i},$$

and hence the consequence in the statement is clear.

(2) By part (1), $\mathrm{Ann}(A) = \bigoplus_{i\in I} \mathrm{Ann}_{A_i}(A_i)$, and hence the first clause is clear. Now, assume that $\mathrm{Ann}(A) = 0$. For a fixed i, from (1) it is clear that $\mathbf{m}^\pi_{A_i} \subseteq \mathbf{m}^\pi_A$. Conversely, given $U \in \mathbf{m}^\pi_A$ and given a family of ideals $\{U_i\}$ such that $\bigoplus_{i\in I} U_i \subseteq U \subseteq \bigoplus_{i\in I}(U_i : A_i)$, on account of (1), we can assume that $U_i \in \mathcal{I}^\pi_{A_i}$ for all $i \in I$, that is $\bigoplus_{i\in I} U_i \in \mathcal{I}^\pi_A$. If $\bigoplus_{i\in I} U_i = 0$, then $U_i = 0$, hence $(U_i : A_i) = \mathrm{Ann}_{A_i}(A_i) = 0$ for all i, and we reach the contradiction $U = 0$. Therefore $\bigoplus_{i\in I} U_i \neq 0$, and, by minimality, there exists $i_0 \in I$ such that $U = U_{i_0} \in \mathbf{m}^\pi_{A_{i_0}}$ because of (1).

Suppose that D_i is an essential ideal of A_i for each i, and U is an ideal of A such that $U \cap (\bigoplus_{i\in I} D_i) = 0$. If $\{U_i\}$ is a family of ideals such that $\bigoplus_{i\in I} U_i \subseteq U \subseteq \bigoplus_{i\in I}(U_i : A_i)$, then we see that $(\bigoplus_{i\in I} U_i) \cap (\bigoplus_{i\in I} D_i) = 0$, hence $U_i \cap D_i = 0$, and so $U_i = 0$ for all i. Therefore $U = 0$. Thus $\bigoplus_{i\in I} D_i$ is an essential ideal of A. On the other hand, given an essential ideal D of A, taking into account that the ideals of each A_i are ideals of A, it is clear that each $D_i := D \cap A_i$ is an essential ideal of A_i. Moreover, the inclusion $\bigoplus_{i\in I} D_i \subseteq D$ is obvious.

(3) The ideals of each A_i are ideals of A, and hence the semiprimeness of A yields to the semiprimeness of all A_i's. Conversely assume that all A_i's are semiprime algebras and U is an ideal of A such that $U^2 = 0$. If $\{U_i\}$ is a family of ideals such that $\bigoplus_{i\in I} U_i \subseteq U \subseteq \bigoplus_{i\in I}(U_i : A_i)$, then, for each i, we see that $U_i^2 = 0$, hence $U_i = 0$, and so $U = 0$. Thus A is semiprime. Finally assume that A is semiprime and U is a π-closed ideal of A. Consider a family $\{U_i\}_{i\in I}$ such that $\bigoplus_{i\in I} U_i \subseteq U \subseteq \bigoplus_{i\in I}(U_i : A_i)$. From (1) we can assume that $U_i \in \mathcal{I}^\pi_{A_i}$ for all $i \in I$. Since, by Lemma 1.8, $(U_i : A_i) = U_i$, we see that $U = \bigoplus_{i\in I} U_i$. Thus we find the inclusion

$$\mathcal{I}^\pi_A \subseteq \{\bigoplus_{i\in I} U_i \;:\; U_i \in \mathcal{I}^\pi_{A_i} \;\text{ for every }\; i \in I\}.$$

The opposite inclusion is a direct consequence of part (1). □

Corollary 1.11. *Let $\{A_i\}_{i\in I}$ be a nonempty family of nonzero algebras with zero annihilator, and set $A = \bigoplus_{i\in I} A_i$. Then*

$$\pi{-}\mathrm{Soc}(A) = \bigoplus_{i\in I} \pi{-}\mathrm{Soc}(A_i) \quad \textit{and} \quad \pi{-}\mathrm{Rad}(A) = \bigoplus_{i\in I} \pi{-}\mathrm{Rad}(A_i).$$

Proof. By the above theorem,

$$\mathbf{m}^\pi_A = \bigcup_{i\in I} \mathbf{m}^\pi_{A_i},$$

and as a consequence we have $\pi-\text{Soc}(A) = \bigoplus_{i\in I} \pi-\text{Soc}(A_i)$. By taking annihilators we deduce that $\pi-\text{Rad}(A) = \bigoplus_{i\in I} \pi-\text{Rad}(A_i)$.

$\square$

A subalgebra B of an algebra A is called an *essential subalgebra* of A whenever B contains an essential ideal of A.

Proposition 1.12. *Let A be an algebra, and B be an essential subalgebra of A. If B is semiprime, then so is A.*

Proof. Assume that B is semiprime and D is an essential ideal of A such that $D \subseteq B$. Let U be an ideal of A such that $U^2 = 0$. Then, $U \cap D$ is an ideal of B such that $(U \cap D)^2 = 0$. Since B is semiprime it follows that $U \cap D = 0$, and since D is essential we can confirm that $U = 0$. Thus, A is semiprime. $\square$

Corollary 1.13. *Let A be an algebra. Assume that U is a π-closed ideal of A such that $U \cap \text{Ann}(U) = 0$, and U and $\text{Ann}(U)$ are semiprime algebras. Then A is semiprime.*

Proof. By Theorem 1.10.(3), $U \oplus \text{Ann}(U)$ is a semiprime algebra. On the other hand, it is clear that $\text{Ann}(U \oplus \text{Ann}(U)) = \text{Ann}(U) \cap U = 0$, and hence $U \oplus \text{Ann}(U)$ is an essential ideal of A. Finally, by Proposition 1.12, A is semiprime.

$\square$

We make one final observation in this section. The π-closed ideals (even minimal π-closed ideals) and the essential subalgebras (even essential ideals) of a semiprime algebra may not be semiprime algebras.

Example 1.14. Let B be a non-null algebra and let F be a nonzero linear map from B into B with $F^2 = 0$. Consider the algebra B_F consisting of the vector space $B \times \mathbb{K}$ and the product defined by

$$(x, \lambda)(y, \mu) = (xy + \lambda F(y) + \mu F(x), \lambda\mu).$$

It is clear that the map $x \mapsto (x, 0)$ allows us to regard B as a subalgebra of B_F. It is easy to check that

$$\mathcal{I}_{B_F} = \{I \,:\, I \in \mathcal{I}_B \text{ with } F(I) \subseteq I\} \bigcup \{I \times \mathbb{K} \,:\, I \in \mathcal{I}_B \text{ with } F(B) \subseteq I\}.$$

Suppose that F also satisfies the condition: $F(I) \subseteq I$, for $I \in \mathcal{I}_B$, implies $I = 0$ or B. Then we see that

$$\mathcal{I}_{B_F} = \{0, B, B_F\},$$

and consequently B_F is a semiprime algebra. Moreover, B is an essential ideal of B_F and

$$\mathcal{I}^{\pi}_{B_F} = \{0, B_F\}.$$

Now consider the product algebra $B \times B$ and the linear map $F \times F$ from $B \times B$ into $B \times B$ given by $(F \times F)(x, y) = (F(x), F(y))$. Note that $F \times F \neq 0$ and $(F \times F)^2 = 0$. Let us denote by A the algebra $(B \times B)_{F \times F}$. Thus, A is the algebra consisting of the vector space $B \times B \times \mathbb{K}$ and the product defined by

$$(x, y, \lambda)(z, t, \mu) = (xz + \lambda F(z) + \mu F(x), yt + \lambda F(t) + \mu F(y), \lambda\mu).$$

Note that $0 = 0 \times 0$, $B_1 = B \times 0$, $B_2 = 0 \times B$, and $B_1 + B_2 = B \times B$ are $(F \times F)$-invariant ideals of $B \times B$, and hence ideals of A. It is easy to show that A is a semiprime algebra and

$$\mathcal{I}^{\pi}_{A} = \{0, B_1, B_2, A\}.$$

(Alternatively one can take into account that the map $(x, y, \lambda) \mapsto ((x, \lambda), (y, \lambda))$ is an algebra monomorphism from A into $B_F \times B_F$ and use Theorem 3.2.) Hence

$$\mathbf{m}^{\pi}_{A} = \mathbf{M}^{\pi}_{A} = \{B_1, B_2\}.$$

Let us now examine a specific pair B, F satisfying all the conditions stated above. Let B be the three-dimensional commutative algebra with generator $\{u, v, w\}$ given by the relations

$$u^2 = u,\ uv = v,\ v^2 = w,\ uw = vw = w^2 = 0.$$

It is immediately verifiable that

$$\mathcal{I}_B = \{0, \mathbb{K}w, \mathbb{K}v + \mathbb{K}w, B\},\quad \mathcal{I}^{\pi}_{B} = \{0, \mathbb{K}w, B\},$$

and $\mathrm{Ann}_B(B) = \mathbb{K}w$. Moreover, consider the linear map F from B into B given by $F(u) = 0$ and $F(v) = F(w) = u$. It is clear that $F \neq 0$, $F^2 = 0$, and 0 and B are the only F-invariant ideals of B. Since $B_1 \cong B$, $B_1 + B_2 \cong B \times B$, and B is an algebra with nonzero annihilator, we have provided an example of a minimal π-closed ideal and of an essential ideal of a semiprime algebra which are algebras with nonzero annihilator.

2 Quotients by a π-closed ideal

Let A be an algebra and U be an ideal of A. For a given nonempty subset $\mathcal{C}$ of $\mathcal{I}_A$, we denote by $h^{\mathcal{C}}(U)$ the *hull* of U relative to $\mathcal{C}$, that is

$$h^{\mathcal{C}}(U) = \{V \in \mathcal{C} : U \subseteq V\}.$$

We also use the set

$$\ell^{\mathcal{C}}(U) = \{V \in \mathcal{C} : V \subseteq U\}.$$

The quotient map $q : A \to A/U$ induces an order isomorphism from $h^{\mathcal{I}_A}(U)$ onto $\mathcal{I}_{A/U}$. The aim of this section is to study the relationship between the π-closures in A and A/U under this isomorphism. Concretely, we state the following

Theorem 2.1. *Let A be a semiprime algebra and let U be a π-closed ideal of A. Let $q : A \to A/U$ denote the quotient map. Then*

(1) A/U is a semiprime algebra.

(2) The map $V \mapsto q(V)$ is a bijection of $h^{\mathbf{D}^\pi_A}(U)$ onto $\mathbf{D}^\pi_{A/U}$, and of $h^{\mathcal{I}^\pi_A}(U)$ onto $\mathcal{I}^\pi_{A/U}$.

(3) The map

$$V \mapsto \overline{q(V)}$$

is a bijection of $\ell^{\mathcal{I}^\pi_A}(\mathrm{Ann}(U))$ onto $\mathcal{I}^\pi_{A/U}$.

As a direct consequence we have the following

Corollary 2.2. *If U is a π-closed ideal of a semiprime algebra A, then*

(1) The map

$$M \mapsto q(M)$$

is a bijection of $h^{\mathbf{M}^\pi_A}(U)$ onto $\mathbf{M}^\pi_{A/U}$.

(2) The map

$$B \mapsto \overline{q(B)}$$

is a bijection of $\ell^{\mathbf{m}^\pi_A}(\mathrm{Ann}(U))$ onto $\mathbf{m}^\pi_{A/U}$.

Before proceeding to the proof of Theorem 2.1 we need some previous results.

Lemma 2.3. *Let A be an algebra and U be an ideal of A. Then*

(1) $q^{-1}(\mathrm{Ann}(q(V)))V + Vq^{-1}(\mathrm{Ann}(q(V))) \subseteq U$, for every ideal V of A.

(2) $\mathrm{Ann}(q^{-1}(Q)) \subseteq q^{-1}(\mathrm{Ann}(Q)) \cap \mathrm{Ann}(U)$, for every ideal Q of A/U.

Proof. (1) Let V be an ideal of A. Since

$$q(q^{-1}(\mathrm{Ann}(q(V)))V) \subseteq q(q^{-1}(\mathrm{Ann}(q(V))))q(V) = \mathrm{Ann}(q(V))q(V) = 0,$$

it follows that $q^{-1}(\mathrm{Ann}(q(V)))V \subseteq U$. Analogously one can prove that

$$Vq^{-1}(\mathrm{Ann}(q(V))) \subseteq U.$$

(2) Let Q be an ideal of A/U. Since

$$q(\mathrm{Ann}(q^{-1}(Q)))) \subseteq \mathrm{Ann}(qq^{-1}(Q)) = \mathrm{Ann}(Q)$$

it follows that $\mathrm{Ann}(q^{-1}(Q)) \subseteq q^{-1}(\mathrm{Ann}(Q))$. Moreover, from the inclusion $U \subseteq q^{-1}(Q)$, it follows that $\mathrm{Ann}(q^{-1}(Q)) \subseteq \mathrm{Ann}(U)$. $\square$

Proposition 2.4. *Let A be an algebra and U be an ideal of A such that $U \cap \mathrm{Ann}(U) = 0$. Then*

(1) $q(\mathrm{Ann}(V)) = \mathrm{Ann}(q(V))$, for every $V \in \ell^{\mathcal{I}_A}(\mathrm{Ann}(U))$.

(2) $q(\overline{V}) = \mathrm{Ann}(q(\mathrm{Ann}(V)))$, for every ideal $V \in h^{\mathcal{I}_A}(U)$.

Proof. (1) Let V be an ideal of A such that $V \subseteq \mathrm{Ann}(U)$. Keeping in mind Lemma 2.3.(1) we see that

$$q^{-1}(\mathrm{Ann}(q(V)))V + Vq^{-1}(\mathrm{Ann}(q(V)))) \subseteq U \cap \mathrm{Ann}(U),$$

therefore

$$q^{-1}(\mathrm{Ann}(q(V)))V + Vq^{-1}(\mathrm{Ann}(q(V)))) = 0,$$

hence

$$q^{-1}(\mathrm{Ann}(q(V))) \subseteq \mathrm{Ann}(V),$$

and so $\mathrm{Ann}(q(V)) \subseteq q(\mathrm{Ann}(V))$. The opposite inclusion is obvious.

(2) If V is an ideal of A such that $U \subseteq V$, then $\mathrm{Ann}(V) \subseteq \mathrm{Ann}(U)$, and applying part (1) to $\mathrm{Ann}(V)$ we conclude the proof. $\square$

Lemma 2.5. *Let A be a semiprime algebra and D be an essential ideal of A. Then $\mathrm{Ann}(U \cap D) = \mathrm{Ann}(U)$ for every ideal U of A.*

Proof. Clearly $\mathrm{Ann}(U) \subseteq \mathrm{Ann}(U \cap D)$. Since A is semiprime we see that $U \cap D \cap \mathrm{Ann}(U \cap D) = 0$, and since D is an essential ideal of A we deduce that $U \cap \mathrm{Ann}(U \cap D) = 0$. Therefore, $\mathrm{Ann}(U \cap D) \subseteq \mathrm{Ann}(U)$. $\square$

Lemma 2.6. *Let A be a semiprime algebra and U be a π-closed ideal of A. Then*

(1) A/U is a semiprime algebra.

(2) If $D \in \mathbf{D}^{\pi}_{A}$, then $q(D) \in \mathbf{D}^{\pi}_{A/U}$.

(3) If $Q \in \mathbf{D}^{\pi}_{A/U}$, then $q^{-1}(Q) \in \mathbf{D}^{\pi}_{A}$.

Proof. (1) If V is an ideal of A satisfying $V^2 \subseteq U$, then we have

$$(V \cap \mathrm{Ann}(U))^2 \subseteq U \cap \mathrm{Ann}(U) = 0,$$

therefore $V \cap \mathrm{Ann}(U) = 0$, and so $V \subseteq \overline{U} = U$. Thus, A/U is a semiprime algebra.

(2) Let $D \in \mathbf{D}^\pi_A$. Taking into account Proposition 2.4.(1) and Lemma 2.5, we see that

$$\mathrm{Ann}(q(D \cap \mathrm{Ann}(U))) = q(\mathrm{Ann}(D \cap \mathrm{Ann}(U))) = q(\overline{U}) = q(U) = 0.$$

Since $\mathrm{Ann}(q(D)) \subseteq \mathrm{Ann}(q(D \cap \mathrm{Ann}(U)))$ we deduce that $\mathrm{Ann}(q(D)) = 0$, and consequently $q(D) \in \mathbf{D}^\pi_{A/U}$.

(3) Let $Q \in \mathbf{D}^\pi_{A/U}$. Since, by part (1), A/U is semiprime, we have $\mathrm{Ann}(Q) = 0$, and consequently, by Lemma 2.3.(2), $\mathrm{Ann}(q^{-1}(Q)) = 0$, and as a result $q^{-1}(Q) \in \mathbf{D}^\pi_A$. □

Proposition 2.7. *Let A be a semiprime algebra and U be a π-closed ideal of A. Then $\overline{q(V)} = \mathrm{Ann}(q(\mathrm{Ann}(V)))$ for each ideal V of A.*

Proof. Given an ideal V of A, by Corollary 1.5, $V + \mathrm{Ann}(V) \in \mathbf{D}^\pi_A$, and consequently, by Lemma 2.6.(2), $q(V) + q(\mathrm{Ann}(V)) \in \mathbf{D}^\pi_{A/U}$. On the other hand, since $q(\mathrm{Ann}(V)) \subseteq \mathrm{Ann}(q(V))$ and, by Lemma 2.6.(1), A/U is semiprime, it follows that

$$q(V) \cap q(\mathrm{Ann}(V)) = 0.$$

Therefore,

$$A/U = \overline{q(V) \oplus q(\mathrm{Ann}(V))}.$$

Finally, taking into account Lemma 1.1, we conclude that

$$\overline{q(V)} = \mathrm{Ann}(q(\mathrm{Ann}(V))).$$

□

Corollary 2.8. *Let A be a semiprime algebra and U be a π-closed ideal of A. Then*

(1) $\mathrm{Ann}(q(V)) = \overline{q(\mathrm{Ann}(V))}$ *for each ideal V of A.*

(2) $\overline{q(\overline{V})} = \overline{q(V)}$ *for each ideal V of A.*

(3) $q^{-1}(\overline{Q}) = \overline{q^{-1}(Q)}$ *for each ideal Q of A/U.*

Proof. (1) follows by taking annihilators in the equality given in Proposition 2.7.

(2) For a given ideal V of A, the equality $\overline{q(\overline{V})} = \overline{q(V)}$ follows by comparing the equality in Proposition 2.7 for V with the equality obtained by replacing V by $\overline{V}$.

(3) Let Q be an ideal of A/U. By Proposition 2.4.(2), $q(\overline{q^{-1}(Q)})$ is a π-closed ideal of A/U, and so, by the above part (2), we have

$$q(\overline{q^{-1}(Q)}) = \overline{q(q^{-1}(Q))} = \overline{Q} = q(q^{-1}(\overline{Q})).$$

Since $\overline{q^{-1}(Q)}, q^{-1}(\overline{Q}) \in h^{\mathcal{I}_A}(U)$, from the above it follows that $\overline{q^{-1}(Q)} = q^{-1}(\overline{Q})$. $\square$

We are now ready to establish the main result.

Proof of Theorem 2.1. (1) By Lemma 2.6.(1), A/U is a semiprime algebra.

(2) If $V \in h^{\mathcal{I}_A^\pi}(U)$, then, by Proposition 2.4.(2), $q(V) \in \mathcal{I}_{A/U}^\pi$. Conversely, for a given $Q \in \mathcal{I}_{A/U}^\pi$, by Corollary 2.8.(3) $q^{-1}(Q) \in h^{\mathcal{I}_A^\pi}(U)$. Analogous assertions for π-dense ideals are provided by Lemma 2.6.(2)-(3). Thus, the quotient map q induces a bijection of $h^{\mathbf{D}_A^\pi}(U)$ onto $\mathbf{D}_{A/U}^\pi$, and of $h^{\mathcal{I}_A^\pi}(U)$ onto $\mathcal{I}_{A/U}^\pi$.

(3) Note that the map $V \mapsto \mathrm{Ann}(V)$ induces a bijection of $\ell^{\mathcal{I}_A^\pi}(\mathrm{Ann}(U))$ onto $h^{\mathcal{I}_A^\pi}(U)$. Therefore, by part (2), the map $V \mapsto \mathrm{Ann}(q(\mathrm{Ann}(V)))$ is a bijection of $\ell^{\mathcal{I}_A^\pi}(\mathrm{Ann}(U))$ onto $\mathcal{I}_{A/U}^\pi$. Now, Proposition 2.7 allow us to conclude the proof. $\square$

Remark 2.9. Let A be an algebra with zero annihilator and U be a π-closed ideal of A. Then

(1) $\mathbf{m}_A^\pi = \ell^{\mathbf{m}_A^\pi}(U) \bigcup \ell^{\mathbf{m}_A^\pi}(\mathrm{Ann}(U))$.

Indeed, for a given $B \in \mathbf{m}_A^\pi$, it is clear that $B \cap U$ is a π-closed ideal of A contained in B. In the case $B \cap U \neq 0$ we see that $B \cap U = B$, and consequently $B \subseteq U$, whereas in the case $B \cap U = 0$ we clearly obtain that $B \subseteq \mathrm{Ann}(U)$.

As a consequence of (1) we have

(2) $\pi-\mathrm{Soc}(A) \subseteq U + \mathrm{Ann}(U)$.

Analogously, we may prove that

(3) $\mathbf{M}_A^\pi = h^{\mathbf{M}_A^\pi}(U) \bigcup h^{\mathbf{M}_A^\pi}(\mathrm{Ann}(U))$,

and as a consequence

(4) $\pi-\mathrm{Rad}(A) \supseteq U \cap \mathrm{Ann}(U)$.

Evidently, (3) can be deduced from (1), by taking into account Remark 1.6 and the following fact:

(5) $h^{\mathbf{M}_A^\pi}(U) = \{\mathrm{Ann}(B) \,:\, B \in \ell^{\mathbf{m}_A^\pi}(\mathrm{Ann}(U))\}$
and
$\ell^{\mathbf{m}_A^\pi}(U) = \{\mathrm{Ann}(M) \,:\, M \in h^{\mathbf{M}_A^\pi}(\mathrm{Ann}(U))\}$.

Note that from (5) it follows immediately that

(6) $U = \bigcap_{M \in h^{\mathbf{M}_A^\pi}(U)} M$ if, and only if, $\mathrm{Ann}(U) = \overline{\sum_{B \in \ell^{\mathbf{m}_A^\pi}(\mathrm{Ann}(U))} B}$.

□

We can now deduce several properties for the π-socle and the π-radical.

Corollary 2.10. *Let A be a semiprime algebra and U be a π-closed ideal of A. Then*

(1) $q(\pi\mathrm{-Soc}(A)) \subseteq \pi\mathrm{-Soc}(A/U) \subseteq \overline{q(\pi\mathrm{-Soc}(A))}$.

(2) $\pi\mathrm{-Rad}(A/U) = \overline{q(\pi\mathrm{-Rad}(A))}$.

(3) $q^{-1}(\pi\mathrm{-Rad}(A/U)) = \bigcap_{M \in h^{\mathbf{M}_A^\pi}(U)} M$.

(4) If $U \subseteq \pi\mathrm{-Rad}(A)$, *then*

$$\pi\mathrm{-Rad}(A/U) = \pi\mathrm{-Rad}(A)/U.$$

Proof. (1) Using Corollary 2.2.(2) and Assertion (1) in the above remark we see that

$$\pi\mathrm{-Soc}(A/U) = \sum_{C \in \mathbf{m}_{A/U}^\pi} C = \sum_{B \in \ell^{\mathbf{m}_A^\pi}(\mathrm{Ann}(U))} \overline{q(B)} = \sum_{B \in \mathbf{m}_A^\pi} \overline{q(B)}.$$

Therefore

$$\sum_{B \in \mathbf{m}_A^\pi} q(B) \subseteq \pi\mathrm{-Soc}(A/U) \subseteq \overline{\sum_{B \in \mathbf{m}_A^\pi} q(B)}.$$

Since

$$\sum_{B \in \mathbf{m}_A^\pi} q(B) = q\Big(\sum_{B \in \mathbf{m}_A^\pi} B\Big) = q(\pi\mathrm{-Soc}(A)),$$

it follows that

$$q(\pi\mathrm{-Soc}(A)) \subseteq \pi\mathrm{-Soc}(A/U) \subseteq \overline{q(\pi\mathrm{-Soc}(A))}.$$

(2) By taking annihilators in (1) we see that

$$\mathrm{Ann}(\pi\mathrm{-Soc}(A/U)) = \mathrm{Ann}(q(\pi\mathrm{-Soc}(A))).$$

On the other hand, by Corollary 2.8.(1) we have

$$\mathrm{Ann}(q(\pi{-}\mathrm{Soc}(A))) = \overline{q(\mathrm{Ann}(\pi{-}\mathrm{Soc}(A)))}.$$

From these equalities, taking into account Remark 1.6.(1), we deduce that

$$\pi{-}\mathrm{Rad}(A/U) = \overline{q(\pi{-}\mathrm{Rad}(A))}.$$

(3) By Corollary 2.2.(1),

$$\pi{-}\mathrm{Rad}(A/U) = \bigcap_{Q \in \mathbf{M}^{\pi}_{A/U}} Q = \bigcap_{M \in h^{\mathbf{M}^{\pi}_{A}}(U)} q(M).$$

Therefore,

$$q^{-1}(\pi{-}\mathrm{Rad}(A/U)) = q^{-1}\left(\bigcap_{M \in h^{\mathbf{M}^{\pi}_{A}}(U)} q(M)\right) =$$

$$\bigcap_{M \in h^{\mathbf{M}^{\pi}_{A}}(U)} q^{-1}(q(M)) = \bigcap_{M \in h^{\mathbf{M}^{\pi}_{A}}(U)} M.$$

(4) If $U \subseteq \pi{-}\mathrm{Rad}(A)$, then $h^{\mathbf{M}^{\pi}_{A}}(U) = \mathbf{M}^{\pi}_{A}$. Therefore, by part (3), $q^{-1}(\pi{-}\mathrm{Rad}(A/U)) = \bigcap_{M \in \mathbf{M}^{\pi}_{A}} M = \pi{-}\mathrm{Rad}(A)$, and hence $\pi{-}\mathrm{Rad}(A/U) = q(\pi{-}\mathrm{Rad}(A))$, as required. □

Corollary 2.11. *Let A be a semiprime algebra and U be a π-closed ideal of A. Then*

(1) The following assertions are equivalent:

- *(i) A/U is a π-radical algebra.*
- *(ii) $\pi{-}\mathrm{Soc}(A) \subseteq U$.*
- *(iii) $\mathrm{Ann}(U) \subseteq \pi{-}\mathrm{Rad}(A)$.*

(2) The following assertions are equivalent:

- *(i) A/U is a π-decomposable algebra.*
- *(ii) $U = \bigcap_{M \in h^{\mathbf{M}^{\pi}_{A}}(U)} M$.*
- *(iii) $\mathrm{Ann}(U) = \overline{\sum_{B \in \ell^{\mathbf{m}^{\pi}_{A}}(\mathrm{Ann}(U))} B}$.*
- *(iv) $\pi{-}\mathrm{Rad}(A) \subseteq U$.*

Proof. (1) (i) $\Rightarrow$ (ii). If A/U is a π-radical algebra, then, by Remark 1.6.(3), $\pi{-}\mathrm{Soc}(A/U) = 0$. Therefore, by Corollary 2.10.(1), we have $q(\pi{-}\mathrm{Soc}(A)) = 0$, and hence $\pi{-}\mathrm{Soc}(A) \subseteq U$.

(ii) $\Rightarrow$ (iii). This implication follows by taking annihilators.

(iii) $\Rightarrow$ (i). If $\mathrm{Ann}(U) \subseteq \pi{-}\mathrm{Rad}(A)$, then $\mathbf{M}_A^\pi = h^{\mathbf{M}_A^\pi}(\mathrm{Ann}(U))$ and $h^{\mathbf{M}_A^\pi}(U) = \emptyset$ because of Remark 2.9.(3) and Corollary 1.5. Now, by Corollary 2.2.(1), $\mathbf{M}_{A/U}^\pi = \emptyset$, and consequently A/U is π-radical.

(2) (i) $\Rightarrow$ (ii). If A/U is a π-decomposable algebra, then, by Remark 1.6.(2), $\pi{-}\mathrm{Rad}(A/U) = 0$. Therefore, by Corollary 2.10.(3), we have

$$U = q^{-1}(0) = q^{-1}(\pi{-}\mathrm{Rad}(A/U)) = \bigcap_{M \in h^{\mathbf{M}_A^\pi}(U)} M.$$

(ii) $\Rightarrow$ (iii). This implication was noted in Remark 2.9.(6).

(iii) $\Rightarrow$ (iv). If $\mathrm{Ann}(U) = \overline{\sum_{B \in \ell^{\mathbf{m}_A^\pi}(\mathrm{Ann}(U))} B}$, then $\mathrm{Ann}(U) \subseteq \overline{\pi{-}\mathrm{Soc}(A)}$, and taking annihilator we deduce that $\pi{-}\mathrm{Rad}(A) \subseteq U$.

(iv) $\Rightarrow$ (i). If $\pi{-}\mathrm{Rad}(A) \subseteq U$, then $q(\pi{-}\mathrm{Rad}(A)) = 0$. Therefore, by Corollary 2.10.(2), we have $\pi{-}\mathrm{Rad}(A/U) = 0$, and so A/U is π-decomposable because of Remark 1.6.(2). $\square$

On account of Remark 1.6, as a direct consequence of the above corollary we have the following result.

Corollary 2.12. *Let A be a semiprime algebra and U be a π-closed ideal of A. Then*

(1) If A is π-radical, then A/U is π-radical.

(2) If A is π-decomposable, then A/U is π-decomposable.

Example 1.14 allows us to answer some questions on π-socle and π-radical. If A and B denote the specific algebras in Example 1.14, then

$$\pi{-}\mathrm{Soc}(A) = B_1 + B_2 \cong B \times B \quad \text{and} \quad \pi{-}\mathrm{Rad}(A) = 0.$$

Since $A/\pi{-}\mathrm{Soc}(A) \cong \mathbb{K}$ it follows that $A/\pi{-}\mathrm{Soc}(A)$ is not π-radical. Note also that $\pi{-}\mathrm{Rad}(B) = \mathbb{K}w$, therefore $\pi{-}\mathrm{Rad}(B_1) \neq 0$, and so $\pi{-}\mathrm{Rad}(B_1) \not\subseteq \pi{-}\mathrm{Rad}(A)$.

3 π-closed ideals in an essential subdirect product

We begin this section by providing a description of the lattice of the π-closed ideals in an essential subdirect product. Recall that an algebra A is a *subdirect product* of a family of algebras $\{A_i\}_{i \in I}$ if there exists a monomorphism f from A into the full direct product $\prod_{i \in I} A_i$ such that, for every $i \in I$, $f_i = p_i \circ f$ maps onto A_i, where p_i is the canonical projection from $\prod_{i \in I} A_i$ onto A_i. In the case in which $f(A)$ is an essential subalgebra of $\prod_{i \in I} A_i$, we say that A is an *essential subdirect product.*

Lemma 3.1. *Let $\{A_i\}_{i\in I}$ be a nonempty family of nonzero semiprime algebras and D_i be an essential ideal of A_i for each $i \in I$. Assume that A is a subalgebra of $\prod_{i\in I} A_i$ containing $\bigoplus_{i\in I} D_i$ and such that $p_i(A) = A_i$ for each $i \in I$. Then*

(1) $\mathrm{Ann}\left((\prod_{i\in I} U_i) \cap A\right) = \mathrm{Ann}\left((\bigoplus_{i\in I} U_i) \cap A\right) = (\prod_{i\in I} \mathrm{Ann}_{A_i}(U_i)) \cap A$ *for each family $\{U_i\}$, where U_i is an ideal of A_i.*

(2) If U is an ideal of A, then $U_i := U \cap A_i$ is an ideal of A_i for each $i \in I$, and $\mathrm{Ann}(U) = (\prod_{i\in I} \mathrm{Ann}_{A_i}(U_i)) \cap A$. *As a consequence,* $\overline{U} = (\prod_{i\in I} \overline{U_i}) \cap A$.

Proof. (1) Let $\{U_i\}$ be a family, where each U_i is an ideal of A_i. Since

$$\bigoplus_{i\in I}(U_i \cap D_i) \subseteq \bigoplus_{i\in I} U_i \subseteq \prod_{i\in I} U_i,$$

it is obvious that

$$\mathrm{Ann}\left((\prod_{i\in I} U_i) \cap A\right) \subseteq \mathrm{Ann}\left((\bigoplus_{i\in I} U_i) \cap A\right) \subseteq \mathrm{Ann}\left(\bigoplus_{i\in I}(U_i \cap D_i)\right).$$

On the other hand, it is also clear that

$$\left(\prod_{i\in I} \mathrm{Ann}_{A_i}(U_i)\right) \cap A \subseteq \mathrm{Ann}\left((\prod_{i\in I} U_i) \cap A\right),$$

and therefore, to prove the equalities in the statement it is sufficient to show that

$$\mathrm{Ann}\left(\bigoplus_{i\in I}(U_i \cap D_i)\right) \subseteq \left(\prod_{i\in I} \mathrm{Ann}_{A_i}(U_i)\right) \cap A.$$

In order to do this, fix $j \in I$ and consider the projection p_j from A onto A_j. Since

$$p_j\left(\mathrm{Ann}(\bigoplus_{i\in I}(U_i \cap D_i))\right) \subseteq \mathrm{Ann}_{A_j}\left(p_j(\bigoplus_{i\in I}(U_i \cap D_i))\right) = \mathrm{Ann}_{A_j}(U_j \cap D_j),$$

by Lemma 2.5, we conclude that

$$p_j\left(\mathrm{Ann}(\bigoplus_{i\in I}(U_i \cap D_i))\right) \subseteq \mathrm{Ann}_{A_j}(U_j).$$

From this we deduce that $\mathrm{Ann}\left(\bigoplus_{i\in I}(U_i \cap D_i)\right) \subseteq \left(\prod_{i\in I} \mathrm{Ann}_{A_i}(U_i)\right) \cap A$, and so (1) is proved.

(2) Let us fix an ideal U of A and set $U_i := U \cap A_i$ for each $i \in I$. Clearly $A \cap A_i$ is an ideal of A, and hence $U_i = U \cap (A \cap A_i)$ is an ideal of A contained in A_i. Therefore $U_i = p_i(U_i)$ is an ideal of A_i. Note that

$$p_i(U)D_i = p_i(U)p_i(D_i) = p_i(UD_i) \subseteq p_i(U \cap D_i) = U \cap D_i \subseteq U_i,$$

and as a consequence $(p_i(U) \cap \mathrm{Ann}_{A_i}(U_i))D_i \subseteq U_i \cap \mathrm{Ann}_{A_i}(U_i) = 0$, and therefore $(p_i(U) \cap \mathrm{Ann}_{A_i}(U_i))D_i = 0$. Analogously we can prove that

$$D_i(p_i(U) \cap \mathrm{Ann}_{A_i}(U_i)) = 0.$$

Thus $p_i(U) \cap \mathrm{Ann}_{A_i}(U_i) \subseteq \mathrm{Ann}_{A_i}(D_i) = 0$, hence $p_i(U) \subseteq \overline{U_i}$, and so

$$U \subseteq \left(\prod_{i \in I} \overline{U_i} \right) \cap A.$$

Now, keeping in mind part (1), as a consequence of the inclusions

$$(\bigoplus_{i \in I} U_i) \cap A \subseteq U \subseteq \left(\prod_{i \in I} \overline{U_i} \right) \cap A$$

we see that

$$\mathrm{Ann}(U) = (\prod_{i \in I} \mathrm{Ann}_{A_i}(U_i)) \cap A.$$

From this, and again using (1), we also find that

$$\overline{U} = (\prod_{i \in I} \overline{U_i}) \cap A.$$

□

Theorem 3.2. *Let $\{A_i\}_{i \in I}$ be a nonempty family of nonzero algebras and D_i be an essential ideal of A_i for each $i \in I$. Assume that A is a subalgebra of $\prod_{i \in I} A_i$ containing $\bigoplus_{i \in I} D_i$ and such that $p_i(A) = A_i$ for each $i \in I$. Then the following assertions are equivalent:*

(i) A is a semiprime algebra.

(ii) A_i is a semiprime algebra for all $i \in I$.

In this case,

(1) $\mathcal{I}_A^\pi = \{(\prod_{i \in I} U_i) \cap A : U_i \in \mathcal{I}_{A_i}^\pi$ for each $i \in I\}$, and $(\prod_{i \in I} U_i) \cap A \neq (\prod_{i \in I} V_i) \cap A$ for families $\{U_i\} \neq \{V_i\}$.

(2) $\bigoplus_{i \in I} S_i \subseteq \pi-\mathrm{Soc}(A) \subseteq \bigoplus_{i \in I} \pi-\mathrm{Soc}(A_i)$, where each S_i is an ideal of A_i contained in D_i such that $\mathrm{Ann}_{A_i}(S_i) = \mathrm{Ann}_{A_i}(\pi-\mathrm{Soc}(A_i))$.

(3) $\pi-\mathrm{Rad}(A) = (\prod_{i \in I} \pi-\mathrm{Rad}(A_i)) \cap A$.

(4) The essential ideals of A are only those containing one of the form $\bigoplus_{i \in I} C_i$, where, for each $i \in I$, C_i is an essential ideal of A_i contained in D_i.

Proof. (i) $\Rightarrow$ (ii). Assume that U_i is an ideal of A_i such that $U_i^2 = 0$. Then $U_i \cap D_i$ is an ideal of A such that $(U_i \cap D_i)^2 = 0$. Since A is semiprime and D_i is essential in A_i it follows that $U_i = 0$. Thus A_i is semiprime.

(ii) $\Rightarrow$ (i). By Lemma 3.1.(1),

$$\operatorname{Ann}(A) = \operatorname{Ann}\left((\prod_{i\in I} A_i) \cap A \right) = (\prod_{i\in I} \operatorname{Ann}_{A_i}(A_i)) \cap A = 0.$$

Suppose that U is an ideal of A such that $U^2 = 0$ and define $U_i = U \cap A_i$ for each $i \in I$. By Lemma 3.1.(2), each U_i is an ideal of A_i, and clearly we have $U_i^2 = 0$. Therefore, by semiprimeness, $U_i = 0$ for each $i \in I$. Moreover, by Lemma 3.1.(2), we see that $\operatorname{Ann}(U) = (\prod_{i\in I} A_i) \cap A = A$, hence $U \subseteq \operatorname{Ann}(A)$, and as a consequence $U = 0$. Thus A is semiprime.

Now, assume that A is an algebra satisfying the equivalent conditions in the statement, and allow us to prove clauses (1)-(4).

(1) Note that the description of $\mathcal{I}_A^\pi$ is a direct consequence of the above lemma. If for families $\{U_i\}, \{V_i\}$ of π-closed ideals we have $(\prod_{i\in I} U_i) \cap A = (\prod_{i\in I} V_i) \cap A$, then $(\prod_{i\in I} U_i) \cap (\bigoplus_{i\in I} D_i) = (\prod_{i\in I} V_i) \cap (\bigoplus_{i\in I} D_i)$. Therefore, for each i, $U_i \cap D_i = V_i \cap D_i$, and taking into account Lemma 2.5 we see that $\operatorname{Ann}(U_i) = \operatorname{Ann}(V_i)$, and consequently $U_i = V_i$.

(2) By part (1) it follows that

$$\mathbf{m}_A^\pi = \bigcup_{i\in I} \{B_{ij} \cap A \,:\, B_{ij} \in \mathbf{m}_{A_i}^\pi\},$$

and hence

$$\pi\text{−}\operatorname{Soc}(A) = \bigoplus_{i\in I} \sum_{B_{ij}\in \mathbf{m}_{A_i}^\pi} (B_{ij} \cap A).$$

For each $i \in I$, consider

$$S_i := \sum_{B_{ij}\in \mathbf{m}_{A_i}^\pi} (B_{ij} \cap D_i).$$

It is clear that

$$S_i \subseteq \sum_{B_{ij}\in \mathbf{m}_{A_i}^\pi} (B_{ij} \cap A) \subseteq \sum_{B_{ij}\in \mathbf{m}_{A_i}^\pi} B_{ij} = \pi\text{−}\operatorname{Soc}(A_i),$$

and hence

$$\bigoplus_{i\in I} S_i \subseteq \pi\text{−}\operatorname{Soc}(A) \subseteq \bigoplus_{i\in I} \pi\text{−}\operatorname{Soc}(A_i).$$

Moreover, each S_i is an ideal of A_i and, keeping in mind Lemma 2.5, we see that

$$\operatorname{Ann}_{A_i}(S_i) = \operatorname{Ann}_{A_i}\Big(\sum_{B_{ij}\in \mathbf{m}_{A_i}^\pi} (B_{ij} \cap D_i)\Big) = \bigcap_{B_{ij}\in \mathbf{m}_{A_i}^\pi} \operatorname{Ann}_{A_i}(B_{ij} \cap D_i) =$$

$$\bigcap_{B_{ij}\in \mathbf{m}^{\pi}_{A_i}} \mathrm{Ann}_{A_i}(B_{ij}) = \mathrm{Ann}_{A_i}(\sum_{B_{ij}\in \mathbf{m}^{\pi}_{A_i}} B_{ij}) = \mathrm{Ann}_{A_i}(\pi\text{−Soc}(A_i)).$$

(3) This clause follows from (2) by taking annihilators and by using Lemma 3.1.(2).

(4) If C_i is an essential ideal of A_i contained in D_i, then, by Lemma 3.1.(1),

$$\mathrm{Ann}\left(\bigoplus_{i\in I} C_i\right) = (\prod_{i\in I} \mathrm{Ann}_{A_i}(C_i)) \cap A = 0,$$

and hence $\bigoplus_{i\in I} C_i$ is an essential ideal of A.

Let U be an essential ideal of A. Set $U_i = U \cap A_i$ and $C_i = U_i \cap D_i$. On account of Lemmas 3.1.(2) and 2.5 we have

$$0 = \mathrm{Ann}(U) = \left(\prod_{i\in I} \mathrm{Ann}_{A_i}(U_i)\right) \bigcap A = \left(\prod_{i\in I} \mathrm{Ann}_{A_i}(C_i)\right) \bigcap A,$$

therefore $\mathrm{Ann}_{A_i}(C_i) \cap D_i = 0$ for each $i \in I$. Since D_i is an essential ideal of A_i, it follows that $\mathrm{Ann}_{A_i}(C_i) = 0$, and consequently C_i is an essential ideal of A_i. Thus U contains the essential ideal $\bigoplus_{i\in I} C_i$. □

Our final goal in this section is to prove that every semiprime algebra is an essential subdirect product of a π-radical semiprime algebra and a π-decomposable semiprime algebra.

Lemma 3.3. *Let A be an algebra and U be a π-closed ideal of A. If $U \cap \mathrm{Ann}(U) = 0$, then $\mathrm{Ann}(q(\mathrm{Ann}(U))) = 0$, and consequently $q(\mathrm{Ann}(U))$ is an essential ideal of A/U.*

Proof. Assume that $U \cap \mathrm{Ann}(U) = 0$. By Proposition 2.4.(2) we have

$$\mathrm{Ann}(q(\mathrm{Ann}(U)))) = q(\overline{U}) = q(U) = 0,$$

therefore $\mathrm{Ann}(q(\mathrm{Ann}(U))) = 0$, and hence $q(\mathrm{Ann}(U))$ is an essential ideal of A/U. □

Proposition 3.4. *Let A be a semiprime algebra and U be a π-closed ideal of A. Then A is an essential subdirect product of the algebras A/U and $A/\mathrm{Ann}(U)$ and*

$$\mathcal{I}^{\pi}_A = \{V \cap W \,:\, V \in h^{\mathcal{I}^{\pi}_A}(U),\ W \in h^{\mathcal{I}^{\pi}_A}(\mathrm{Ann}(U))\}.$$

Proof. Let us denote by q_0 and q_1 the quotient maps from A onto $A_0 := A/U$ and $A_1 := A/\mathrm{Ann}(U)$ respectively. Consider the homomorphism $f : A \to A_0 \times A_1$ defined by $f(a) = (q_0(a), q_1(a))$. Since

$$\ker(f) = \ker(q_0) \cap \ker(q_1) = U \cap \mathrm{Ann}(U) = 0,$$

we see that f is injective. On the other hand, it is fairly evident that q_0 and q_1 are surjective, and

$$f(A) \supseteq f(U \oplus \mathrm{Ann}(U)) = q_0(\mathrm{Ann}(U)) \times q_1(U).$$

Thus, in order to prove that A is an essential subdirect product of A_0 and A_1, it is sufficient to show that $q_0(\mathrm{Ann}(U)) \times q_1(U)$ is an essential ideal of $A_0 \times A_1$. Since $\mathrm{Ann}_{A_0}(A_0) = (U : A)/U$ and $\mathrm{Ann}_{A_1}(A_1) = (\mathrm{Ann}(U) : A)/\mathrm{Ann}(U)$, by Lemma 1.8, it follows that both algebras A_0 and A_1 have zero annihilator. On the other hand, by Lemma 3.3, it follows that $q_0(\mathrm{Ann}(U))$ and $q_1(U)$ are essential ideals of A_0 and A_1 respectively. Now, taking into account Theorem 1.10.(2), we conclude that $q_0(\mathrm{Ann}(U)) \times q_1(U)$ is an essential ideal of $A_0 \times A_1$.

Finally, given $V \in h^{\mathcal{I}_A^\pi}(U)$ and $W \in h^{\mathcal{I}_A^\pi}(\mathrm{Ann}(U))$, it is easy to verify that $f(V \cap W) = (q_0(V) \times q_1(W)) \cap f(A)$. Since, by Theorems 3.2.(1) and 2.1.(2), we have $\mathcal{I}_{f(A)}^\pi = \{(q_0(V) \times q_1(W)) \cap f(A) : V \in h^{\mathcal{I}_A^\pi}(U),\ W \in h^{\mathcal{I}_A^\pi}(\mathrm{Ann}(U))\}$, we see that $\mathcal{I}_A^\pi = \{V \cap W : V \in h^{\mathcal{I}_A^\pi}(U),\ W \in h^{\mathcal{I}_A^\pi}(\mathrm{Ann}(U))\}$. □

Theorem 3.5. *Let A be an algebra. Then the following assertions are equivalent:*

(i) A is semiprime.

(ii) A is an essential subdirect product of two algebras A_0 and A_1, where A_0 is a π-radical semiprime algebra and A_1 is a π-decomposable semiprime algebra.

In this case,

$$\mathcal{I}_A^\pi = \{U \cap V : U \in h^{\mathcal{I}_A^\pi}(\pi{-}\mathrm{Soc}(A)),\ V \in h^{\mathcal{I}_A^\pi}(\pi{-}\mathrm{Rad}(A))\}.$$

Proof. The implication (ii) ⇒ (i) is a consequence of Theorem 3.2. The implication (i) ⇒ (ii) and the description of $\mathcal{I}_A^\pi$ follow from Proposition 3.4 by taking $U = \overline{\pi{-}\mathrm{Soc}(A)}$ and by considering the algebras $A_0 = A/\overline{\pi{-}\mathrm{Soc}(A)}$ and $A_1 = A/\pi{-}\mathrm{Rad}(A)$. Note that, by Theorem 2.1.(1), both algebras are semiprime and, by Corollary 2.11, A_0 is π-radical and A_1 is π-decomposable. □

4 π-complemented algebras

In this section we shall first establish the minimal character of the π-closure for the complementarity. Our main goal is to provide different characterizations of π-complemented algebras.

We start with the following observation.

Lemma 4.1. *Let A be an algebra with zero annihilator and U, V be ideals of A. If $A = U \oplus V$, then $V = \mathrm{Ann}(U)$ and $U = \mathrm{Ann}(V)$. Consequently U and V are π-closed.*

Proof. Assume that $A = U \oplus V$. From Lemma 1.1 it follows that $V \subseteq \mathrm{Ann}(U)$ and $U \cap \mathrm{Ann}(U) = 0$. Therefore, the equality $A = U \oplus V$ yields to the equality $A = U \oplus \mathrm{Ann}(U)$. From this it follows immediately that $V = \mathrm{Ann}(U)$. Finally, by interchanging the roles we also obtain that $U = \mathrm{Ann}(V)$. □

Now we can establish the minimal character of the π-closure for the complementarity.

Proposition 4.2. *Let A be an algebra with zero annihilator and let $\sim$ be a closure operation on $\mathcal{I}_A$. Assume that $\widetilde{U} \subseteq \overline{U}$ for all $U \in \mathcal{I}_A$. If A is $\sim$-complemented, then $\widetilde{U} = \overline{U}$ for all $U \in \mathcal{I}_A$.*

Proof. Suppose that A is $\sim$-complemented. Then, given $U \in \mathcal{I}_A$, we know that $A = \widetilde{U} \oplus V$ for a suitable $\sim$-closed ideal V of A. From Lemma 4.1 it follows that $\widetilde{U}$ is π-closed, and hence we see that

$$\overline{U} \subseteq \overline{\widetilde{U}} = \widetilde{U} \subseteq \overline{U}.$$

Thus, $\widetilde{U} = \overline{U}$. □

We say that the π-closure in an algebra A is *additive* whenever $\overline{U+V} = \overline{U} + \overline{V}$ for all U, V ideals of A. Given an ideal U of an algebra A, for each ideal I of U, we denote by $\mathrm{Ann}_U(I)$ the annihilator of I relative to the algebra U, and we denote by $\overline{I}^U$ the π-closure of I relative to the algebra U.

Theorem 4.3. *Let A be an algebra. Then the following assertions are equivalent:*

(i) *A is π-complemented*

(ii) *$A = \overline{U} \oplus \mathrm{Ann}(U)$ for every ideal U of A.*

(iii) *A is semiprime, and $U + V$ is a π-closed ideal of A for all U, V π-closed ideals of A.*

(iv) *A is semiprime, and $\mathrm{Ann}(U \cap V) = \mathrm{Ann}(U) + \mathrm{Ann}(V)$ for all U, V π-closed ideals of A.*

(v) *A is semiprime, and the π-closure is additive.*

In this case, every π-closed ideal U of A is a π-complemented algebra, and $\mathcal{I}_A^\pi = \{V \oplus W : V \in \mathcal{I}_U^\pi, W \in \mathcal{I}_{\mathrm{Ann}(U)}^\pi\}$. *Moreover,* $\mathrm{Ann}_U(I) = \mathrm{Ann}(I) \cap U$ *and* $\overline{I}^U = \overline{I}$ *for every ideal I of U.*

Proof. (i) $\Rightarrow$ (ii). Given an ideal U of A, there exists a π-closed ideal V of A such that $A = \overline{U} \oplus V$. By Lemma 4.1 we see that $V = \mathrm{Ann}(\overline{U}) = \mathrm{Ann}(U)$, and hence

$$A = \overline{U} \oplus \mathrm{Ann}(U).$$

(ii) $\Rightarrow$ (iii). The semiprimeness is a consequence of Proposition 1.3. In order to prove the second clause, assume first that U, V are π-closed ideals of A such that $U \cap V = 0$. Then $V \subseteq \mathrm{Ann}(U)$, and in particular V is an ideal of $\mathrm{Ann}(U)$. Since, by assumption $A = U \oplus \mathrm{Ann}(U)$, Theorem 1.10.(1) yields that V is a π-closed ideal of $\mathrm{Ann}(U)$, and $U \oplus V$ is a π-closed ideal of A in this case. Now, assume that U, V are arbitrary π-closed ideals of A. Keeping in mind that $A = U \oplus \mathrm{Ann}(U)$, parts (3) and (1) of Theorem 1.10 allow us to write $V = V_0 \oplus V_1$, where V_0 and V_1 are π-closed ideals of A contained in U and $\mathrm{Ann}(U)$ respectively. Therefore $U + V = U + (V_0 \oplus V_1) = U \oplus V_1$, and the first part of the argument entails $U + V$ is a π-closed ideal of A, as desired.

(iii) $\Rightarrow$ (iv). For all U, V π-closed ideals of A, by applying (iii) to the π-closed ideals $\mathrm{Ann}(U), \mathrm{Ann}(V)$ we have

$$\mathrm{Ann}(U) + \mathrm{Ann}(V) = \overline{\mathrm{Ann}(U) + \mathrm{Ann}(V)},$$

and hence

$$\mathrm{Ann}(U) + \mathrm{Ann}(V) = \mathrm{Ann}(\overline{U} \cap \overline{V}) = \mathrm{Ann}(U \cap V).$$

(iv) $\Rightarrow$ (v). For U, V ideals of A, by applying (iv) to the π-closed ideals $\mathrm{Ann}(U), \mathrm{Ann}(V)$ we have

$$\mathrm{Ann}(\mathrm{Ann}(U) \cap \mathrm{Ann}(V)) = \overline{U} + \overline{V},$$

and hence $\mathrm{Ann}(\mathrm{Ann}(U + V)) = \overline{U} + \overline{V}$, that is $\overline{U + V} = \overline{U} + \overline{V}$.

(v) $\Rightarrow$ (i). For every π-closed ideal U of A, by Proposition 1.3, we see that

$$A = \overline{U \oplus \mathrm{Ann}(U)} = \overline{U} \oplus \overline{\mathrm{Ann}(U)} = U \oplus \mathrm{Ann}(U).$$

Thus A is a π-complemented algebra.

Now, assume that A satisfies the equivalent conditions in the statement and suppose that U is a π-closed ideal of A. Since $A = U \oplus \mathrm{Ann}(U)$, by Theorem 1.10, we have $\mathcal{I}_A^\pi = \{V \oplus W : V \in \mathcal{I}_U^\pi, W \in \mathcal{I}_{\mathrm{Ann}(U)}^\pi\}$, and $\mathrm{Ann}_U(I) = \mathrm{Ann}(I) \cap U$ and $\overline{I}^U = \overline{I}$ for every ideal I of U. From these last two equalities we see that

$$U = U \cap A = U \cap (\overline{I} \oplus \mathrm{Ann}(I)) = \overline{I} \oplus (U \cap \mathrm{Ann}(I)) = \overline{I}^U \oplus \mathrm{Ann}_U(I).$$

Thus, U is a π-complemented algebra, and the proof is complete.

□

Corollary 4.4. *Let A be a π-complemented algebra and U be a π-closed ideal of A. Then*

(1) $\pi-\mathrm{Soc}(A) = \pi-\mathrm{Soc}(U) \oplus \pi-\mathrm{Soc}(\mathrm{Ann}(U))$ *and* $\pi-\mathrm{Rad}(A) = \pi-\mathrm{Rad}(U) \oplus \pi-\mathrm{Rad}(\mathrm{Ann}(U))$.

(2) A/U *is a π-complemented algebra,* $\mathcal{I}^{\pi}_{A/U} = \{q(V) : V \in \ell^{\mathcal{I}^{\pi}_A}(\mathrm{Ann}(U))\}$, $\pi-\mathrm{Soc}(A/U) = \pi-\mathrm{Soc}(A)/U$, *and* $\pi-\mathrm{Rad}(A/U) = \pi-\mathrm{Rad}(A)/U$.

Proof. (1) This assertion follows from Corollary 1.11 because $A = U \oplus \mathrm{Ann}(U)$.

(2) Since the decomposition $A = U \oplus \mathrm{Ann}(U)$ gives an isomorphism $q' : \mathrm{Ann}(U) \cong A/U$, the above theorem yields that A/U is a π-complemented algebra, and

$$\mathcal{I}^{\pi}_{A/U} = \{q'(V) : V \in \mathcal{I}^{\pi}_{\mathrm{Ann}(U)}\} = \{q(V) : V \in \ell^{\mathcal{I}^{\pi}_A}(\mathrm{Ann}(U))\},$$

where $q : A \to A/U$ is the quotient map. Moreover, from part (1) we deduce that

$$q(\pi-\mathrm{Soc}(A)) = q(\pi-\mathrm{Soc}(\mathrm{Ann}(U)))$$

and

$$q(\pi-\mathrm{Rad}(A)) = q(\pi-\mathrm{Rad}(\mathrm{Ann}(U))).$$

Since

$$q(\pi-\mathrm{Soc}(\mathrm{Ann}(U))) = q'(\pi-\mathrm{Soc}(\mathrm{Ann}(U))) = \pi-\mathrm{Soc}(A/U),$$

and

$$q(\pi-\mathrm{Rad}(\mathrm{Ann}(U))) = q'(\pi-\mathrm{Rad}(\mathrm{Ann}(U))) = \pi-\mathrm{Rad}(A/U),$$

we conclude that

$$q(\pi-\mathrm{Soc}(A)) = \pi-\mathrm{Soc}(A/U) \quad \text{and} \quad q(\pi-\mathrm{Rad}(A)) = \pi-\mathrm{Rad}(A/U).$$

□

Corollary 4.5. *If A is a finite-dimensional π-complemented algebra, then $A = \pi - \mathrm{Soc}(A)$.*

Proof. Assume that $A \neq 0$. Since every descending chain of π-closed ideals is a chain of subspaces of decreasing dimension, it follows that A has minimal π-closed ideals. Choose a minimal π-closed ideal B_1 of A. If $B_1 = A$, then the proof is concluded. Otherwise, by Theorem 4.3, $A = B_1 \oplus \mathrm{Ann}(B_1)$, $\mathrm{Ann}(B_1)$ is a nonzero π-complemented algebra, and $\mathbf{m}^{\pi}_A = \{B_1\} \cup \mathbf{m}^{\pi}_{\mathrm{Ann}(B_1)}$. On iterating the procedure, the finite dimension ensures that in finitely many steps we arrive at the desired decomposition. □

Given a π-complemented algebra A, for each π-closed ideal U of A, the decomposition $A = U \oplus \mathrm{Ann}(U)$ determines a projection P_U on A: For each $a \in$

A, $P_U(a)$ is the unique element b in U satisfying $a-b \in \mathrm{Ann}(U)$. Given a family $\{A_i\}_{i\in I}$ of π-complemented algebras and a family $\{U_i\}_{i\in I}$ of π-closed ideals, we consider the projection $P_{\{U_i\}}$ on $\prod_{i\in I} A_i$ given by $P_{\{U_i\}}(a_i) = (P_{U_i}(a_i))$.

Proposition 4.6. *Let $\{A_i\}_{i\in I}$ be a nonempty family of nonzero algebras. Assume that A is a subalgebra of $\prod_{i\in I} A_i$ containing $\bigoplus_{i\in I} A_i$. Then the following assertions are equivalent:*

(i) A is π-complemented.

(ii) A_i is π-complemented for all $i \in I$, and $P_{\{U_i\}}(A) \subseteq A$ for every family of π-closed ideals $\{U_i\}_{i\in I}$.

(iii) A_i is π-complemented for all $i \in I$, and $A = P_{\{U_i\}}(A) \oplus P_{\{\mathrm{Ann}_{A_i}(U_i)\}}(A)$ for every family of π-closed ideals $\{U_i\}_{i\in I}$.

In this case, $\mathcal{I}^\pi_A = \{P_{\{U_i\}}(A) : \{U_i\}_{i\in I} \in \mathcal{F}\}$, where $\mathcal{F} = \{\{U_i\}_{i\in I} : U_i \in \mathcal{I}^\pi_{A_i} \text{ for each } i \in I\}$.

Proof. (i) $\Rightarrow$ (ii). Since A is semiprime, by Theorem 3.2, each A_i is semiprime and $\mathcal{I}^\pi_A = \{(\prod_{i\in I} U_i) \cap A : U_i \in \mathcal{I}^\pi_{A_i} \text{ for each } i \in I\}$. Moreover, since each A_i is a π-closed ideal of A, by Theorem 4.3, A_i is a π-complemented algebra. Given a family $\{U_i\}$ with $U_i \in \mathcal{I}^\pi_{A_i}$, keeping in mind Lemma 3.1, we have the decomposition

$$A = \left[(\prod_{i\in I} U_i) \cap A\right] \bigoplus \left[(\prod_{i\in I} \mathrm{Ann}_{A_i}(U_i)) \cap A\right],$$

and as a consequence $P_{\{U_i\}}(A) \subseteq A$.

(ii) $\Rightarrow$ (iii). Let $\{U_i\}_{i\in I}$ be a family with $U_i \in \mathcal{I}^\pi_A$. From the definition of $P_{\{U_i\}}$, it is clear that $P_{\{U_i\}}(A) \subseteq \prod_{i\in I} U_i$. Since $U_i \cap \mathrm{Ann}_{A_i}(U_i) = 0$ for all $i \in I$, we see that $(\prod_{i\in I} U_i) \cap (\prod_{i\in I} \mathrm{Ann}_{A_i}(U_i)) = 0$, and consequently $P_{\{U_i\}}(A) \cap P_{\{\mathrm{Ann}_{A_i}(U_i)\}}(A) = 0$. For a given $a = (a_i) \in A$, writing each a_i in the form $a_i = b_i + c_i$ with $b_i \in U_i$ and $c_i \in \mathrm{Ann}_{A_i}(U_i)$, we see that $(b_i) = P_{\{U_i\}}(a)$ and $(c_i) = P_{\{\mathrm{Ann}_{A_i}(U_i)\}}(a)$. Hence $a \in P_{\{U_i\}}(A) \oplus P_{\{\mathrm{Ann}_{A_i}(U_i)\}}(A)$. Therefore $A \subseteq P_{\{U_i\}}(A) \oplus P_{\{\mathrm{Ann}_{A_i}(U_i)\}}(A)$. The converse inclusion follows from the assumption.

(iii) $\Rightarrow$ (i). Since each A_i is semiprime, by Theorem 3.2.(1),

$$\mathcal{I}^\pi_A = \{(\prod_{i\in I} U_i) \cap A : U_i \in \mathcal{I}^\pi_{A_i} \text{ for each } i \in I\}.$$

Given a family $\{U_i\}$ with $U_i \in \mathcal{I}^\pi_{A_i}$, note that $P_{\{U_i\}}(A) \subseteq \prod_{i\in I} U_i$, and so, as a consequence of the equality $A = P_{\{U_i\}}(A) \oplus P_{\{\mathrm{Ann}_{A_i}(U_i)\}}(A)$, we deduce that

$$A = \left[(\prod_{i\in I} U_i) \cap A\right] \bigoplus \left[(\prod_{i\in I} \mathrm{Ann}_{A_i}(U_i)) \cap A\right].$$

Thus, A is π-complemented.

Now, assume that A satisfies the equivalent conditions in the statement. Given a family $\{U_i\}$ with $U_i \in \mathcal{I}^{\pi}_{A_i}$, from the inclusion $P_{\{U_i\}}(A) \subseteq A$ it is fairly evident that $P_{\{U_i\}}(A) = (\prod_{i\in I} U_i) \cap A$, and hence the description of $\mathcal{I}^{\pi}_A$ follows from the argument used above. $\square$

Corollary 4.7. *If $\{A_i\}_{i\in I}$ is a nonempty family of nonzero algebras, then the following assertions are equivalent:*

(i) $\prod_{i\in I} A_i$ *is π-complemented.*

(ii) $\bigoplus_{i\in I} A_i$ *is π-complemented.*

(iii) A_i *is π-complemented for all $i \in I$.*

Theorem 3.5 can be perfected for π-complemented algebras as follows.

Theorem 4.8. *For every algebra A the following assertions are equivalent:*

(i) A is π-complemented.

(ii) A is isomorphic to $A_0 \oplus A_1$, where A_0 is a π-radical π-complemented algebra and A_1 is a π-decomposable π-complemented algebra.

In this case,

$$A_0 \cong \pi{-}\mathrm{Rad}(A) \quad \textit{and} \quad A_1 \cong \overline{\pi{-}\mathrm{Soc}(A)}.$$

Proof. (i) $\Rightarrow$ (ii). Let A be a π-complemented algebra and set $A_0 = \pi{-}\mathrm{Rad}(A)$ and $A_1 = \overline{\pi{-}\mathrm{Soc}(A)}$. Since A_0 is a π-closed ideal of A and $\mathrm{Ann}(A_0) = A_1$, by Theorem 4.3, it follows that $A = A_0 \oplus A_1$, and both A_0 and A_1 are π-complemented algebras which satisfy

$$\mathcal{I}^{\pi}_{A_0} = \ell^{\mathcal{I}^{\pi}_A}(A_0) \quad \text{and} \quad \mathcal{I}^{\pi}_{A_1} = \ell^{\mathcal{I}^{\pi}_A}(A_1).$$

It is clear that $\mathbf{m}^{\pi}_{A_0} = \emptyset$ and $\mathbf{m}^{\pi}_{A_1} = \mathbf{m}^{\pi}_A$. Hence A_0 is π-radical and A_1 is π-decomposable.

(ii) $\Rightarrow$ (i). Let A_0 be a π-radical π-complemented algebra and let A_1 be a π-decomposable π-complemented algebra. By Corollary 4.7, it follows that $A_0 \oplus A_1$ is a π-complemented algebra. Moreover, by Corollary 1.11, we have

$$\pi{-}\mathrm{Rad}(A_0 \oplus A_1) = \pi{-}\mathrm{Rad}(A_0) \oplus \pi{-}\mathrm{Rad}(A_1) = A_0.$$

By taking annihilators we see that $\overline{\pi{-}\mathrm{Soc}(A_0 \oplus A_1)} = A_1$, and the proof is complete. $\square$

To conclude this section we give an example of a π-radical π-complemented algebra. Note that, as a consequence of Corollary 4.5, such an algebra must necessarily be infinite dimensional. Our example is an algebra of measurable functions. For the definition and properties of the Lebesgue measure we refer the reader to the classical book [2] and merely establish some notation. We denote by $\mathcal{M}$ the σ-algebra of all Lebesgue measurable subsets of the unit interval $[0,1]$, and by $\lambda : \mathcal{M} \to [0,1]$ the Lebesgue measure.

We start with the following elemental fact.

Lemma 4.9. *For each subset A of $[0,1]$, there exist $A_*, A^* \in \mathcal{M}$ such that $A_* \subseteq A \subseteq A^*$ and $\lambda(E\backslash A_*) = \lambda(A^*\backslash F) = 0$, for all $E, F \in \mathcal{M}$ with $E \subseteq A \subseteq F$.*

Proof. Set

$$\alpha = \inf\{\lambda(F\backslash E) \,:\, E, F \in \mathcal{M} \text{ such that } E \subseteq A \subseteq F\}$$

and, for each $n \in \mathbb{N}$, choose $E_n, F_n \in \mathcal{M}$ such that $E_n \subseteq A \subseteq F_n$ and $\lambda(F_n\backslash E_n) \leq \alpha + \frac{1}{n}$. Now, consider the sets $A_* = \bigcup_{n\in\mathbb{N}} E_n$ and $A^* = \bigcap_{n\in\mathbb{N}} F_n$, and note that $A_*, A^* \in \mathcal{M}$ and $A_* \subseteq A \subseteq A^*$. Moreover, since $A^*\backslash A_* \subseteq F_n\backslash E_n$ for every n, it follows that

$$\alpha \leq \lambda(A^*\backslash A_*) \leq \lambda(F_n\backslash E_n) \leq \alpha + \frac{1}{n},$$

and as a result $\lambda(A^*\backslash A_*) = \alpha$. Given $E, F \in \mathcal{M}$ with $E \subseteq A \subseteq F$, note that $A_* \cup E \subseteq A \subseteq A^* \cap F$ and $(A^* \cap F)\backslash(A_* \cup E) \subseteq A^*\backslash A_*$, and hence $\lambda((A^* \cap F)\backslash(A_* \cup E)) = \alpha$. Moreover, by considering the decomposition

$$A^*\backslash A_* = (E\backslash A_*) \cup [(A^* \cap F)\backslash(A_* \cup E)] \cup (A^*\backslash F),$$

we see that

$$\begin{aligned}\alpha = \lambda(A^*\backslash A_*) &= \lambda(E\backslash A_*) + \lambda((A^* \cap F)\backslash(A_* \cup E)) + \lambda(A^*\backslash F) \\ &= \lambda(E\backslash A_*) + \alpha + \lambda(A^*\backslash F),\end{aligned}$$

hence $\lambda(E\backslash A_*) = \lambda(A^*\backslash F) = 0$, and the proof is complete. □

Example 4.10. The algebra $\mathbb{M}$ of all equivalence classes (under equality almost everywhere) of Lebesgue measurable functions on $[0,1]$ with pointwise operations is a π-radical π-complemented algebra.

Clearly $\mathbb{M}$ is a commutative associative algebra with a unit $\mathbf{1}$. Given $E \in \mathcal{M}$, note that the characteristic function χ_E is an idempotent of $\mathbb{M}$, $\chi_E\chi_{E^c} = 0$, and $\mathbf{1} = \chi_E + \chi_{E^c}$, where E^c denotes the complement of E in $[0,1]$. Therefore

$$\mathbb{M} = \chi_E\mathbb{M} \oplus \chi_{E^c}\mathbb{M}. \tag{1}$$

By Lemma 4.1, $\chi_E\mathbb{M} \in \mathcal{I}^\pi_\mathbb{M}$ and $\mathrm{Ann}(\chi_E\mathbb{M}) = \chi_{E^c}\mathbb{M}$. We claim that

$$\mathcal{I}^\pi_\mathbb{M} = \{\chi_E\mathbb{M} : E \in \mathcal{M}\}.$$

Given $f \in \mathbb{M}$, consider the measurable set $S_f := \{x \in [0,1] : f(x) \neq 0\}$, and note that $f = \chi_{S_f} f$ and $\chi_{S_f} = fg$, where g is the measurable function defined by $g(x) = f(x)^{-1}$ if $x \in S_f$ and $g(x) = 0$ otherwise. Therefore $f\mathbb{M} = \chi_{S_f}\mathbb{M}$. Note that, for a given π-closed ideal U of $\mathbb{M}$, we have

$$U = \mathrm{Ann}(\mathrm{Ann}(U)) = \mathrm{Ann}(\sum_{f\in\mathrm{Ann}(U)} \chi_{S_f}\mathbb{M})$$

$$= \bigcap_{f\in\mathrm{Ann}(U)} \mathrm{Ann}(\chi_{S_f}\mathbb{M}) = \bigcap_{f\in\mathrm{Ann}(U)} \chi_{S_f^c}\mathbb{M}.$$

Therefore

$$\mathcal{I}^\pi_\mathbb{M} = \{\bigcap_{i\in I} \chi_{E_i}\mathbb{M} : \{E_i : i \in I\} \subseteq \mathcal{M}\}.$$

Given a family $\{E_i : i \in I\} \subseteq \mathcal{M}$, it is immediately verified that $\bigcap_{i\in I} \chi_{E_i}\mathbb{M} = \chi_E\mathbb{M}$, where $E = (\bigcap_{i\in I} E_i)_*$, and so we have proved our claim. Now, from (1) we can confirm that $\mathbb{M}$ is π-complemented. Finally, note that $\chi_E\mathbb{M} = 0$ if, and only if, $\lambda(E) = 0$. Therefore, if U is a nonzero π-closed ideal, and $U = \chi_E\mathbb{M}$ for suitable $E \in \mathcal{M}$, then by choosing $F \subseteq E$ with $0 < \lambda(F) < \lambda(E)$ we see that $\chi_F\mathbb{M}$ is a nonzero π-closed ideal of $\mathbb{M}$ strictly contained in U. Thus, $\mathbb{M}$ is π-radical.

5 π-decomposable π-complemented algebras

The aim of this section is to obtain a description theorem for π-decomposable π-complemented algebras. The main tool is the structure theory of π-decomposable algebras developed in [4] and [6].

Let us start by determining the relationship between π-closed prime ideals and π-closed maximal ideals in a semiprime algebra. Recall that an algebra A is said to be *prime* if, for ideals U and V of A, the condition $UV = 0$ implies either $U = 0$ or $V = 0$. An ideal P of an algebra A is said to be a *prime ideal* if the quotient algebra A/P is a prime algebra. Clearly P is a prime ideal of A if, and only if, for ideals U and V of A, the condition $UV \subseteq P$ implies either $U \subseteq P$ or $V \subseteq P$. We set

$$\mathbf{P}_A := \{U : U \text{ is a proper prime ideal of } A\}$$

and

$$\mathbf{P}^\pi_A := \{U : U \text{ is a proper } \pi\text{-closed prime ideal of } A\}.$$

Proposition 5.1. *Let A be a semiprime algebra. Then*

(1) $\mathbf{P}_A \subseteq \mathbf{M}_A^\pi \cup \mathbf{D}_A^\pi$.

(2) $\mathbf{P}_A^\pi = \mathbf{M}_A^\pi$.

Proof. (1) Let $P \in \mathbf{P}_A$ and let U be a π-closed ideal of A such that $P \subseteq U$. Since $U\mathrm{Ann}(U) = 0$, it follows that either $U \subseteq P$ or $\mathrm{Ann}(U) \subseteq P$. In the first case we have $U = P$. In the second one, we see that $\mathrm{Ann}(U) \subseteq U$, therefore $\mathrm{Ann}(U) = 0$, and so $U = A$. In the particular case in which we take $U = \overline{P}$, the above reasoning shows that P is either π-closed or π-dense. Moreover, in the case in which P is π-closed, the above reasoning again shows that P is a maximal π-closed ideal of A.

(2) The inclusion $\mathbf{P}_A^\pi \subseteq \mathbf{M}_A^\pi$ follows from (1). In order to prove the opposite inclusion let us fix a maximal π-closed ideal M of A, and assume that U, V are ideals of A such that $UV \subseteq M$. If $U \nsubseteq M$, then $U \cap \mathrm{Ann}(M) \neq 0$. From this, taking into account that $\mathrm{Ann}(M)$ is a minimal π-closed ideal of A, it follows that $\overline{U \cap \mathrm{Ann}(M)} = \mathrm{Ann}(M)$. On the other hand, we see that $(U \cap \mathrm{Ann}(M))V \subseteq \mathrm{Ann}(M) \cap M = 0$, and hence $U \cap \mathrm{Ann}(M) \subseteq \mathrm{Ann}(V)$. Therefore, $\mathrm{Ann}(M) = \overline{U \cap \mathrm{Ann}(M)} \subseteq \mathrm{Ann}(V)$, and consequently $V \subseteq M$. Thus M is a prime ideal of A. $\square$

Now we show basic examples of π-decomposable π-complemented algebras.

Corollary 5.2. *For a nonnull algebra A the following assertions are equivalent:*

(i) A is prime

(ii) $\mathcal{I}_A^\pi = \{0, A\}$.

In this case, A is a π-decomposable π-complemented algebra.

Proof. (i) $\Rightarrow$ (ii). Clearly A is semiprime and $0 \in \mathbf{P}_A^\pi$. By Proposition 5.1, we have $0 \in \mathbf{M}_A^\pi$, and consequently $\mathcal{I}_A^\pi = \{0, A\}$.

(ii) $\Rightarrow$ (i). Since A is nonnull and $\mathrm{Ann}(A) \in \mathcal{I}_A^\pi$, it follows that $\mathrm{Ann}(A) = 0$. Let U be an ideal of A such that $U^2 = 0$. Then $U \subseteq \mathrm{Ann}(U)$. Since $\mathrm{Ann}(U)$ is π-closed, it follows that either $\mathrm{Ann}(U) = 0$ or $\mathrm{Ann}(U) = A$. In the first case, we have $U = 0$. In the second one, we see that $U \subseteq \mathrm{Ann}(A)$, and hence we also have $U = 0$. Therefore, A is semiprime. Since, by assumption, $0 \in \mathbf{M}_A^\pi$, from Proposition 5.1 it follows that $0 \in \mathbf{P}_A^\pi$, and so A is a prime algebra.

Finally, from (ii) it is clear that A is π-decomposable and π-complemented. $\square$

Corollary 5.3. *Let A be a nonzero algebra with zero annihilator. If a minimal π-closed ideal B of A is a direct summand of A, then B is a prime algebra. In particular, the minimal π-closed ideals of a π-complemented algebra are prime algebras.*

Proof. Assume that B is a minimal π-closed ideal of A and $A = B \oplus C$ for a suitable ideal C of A. Keeping in mind Theorem 1.10.(1)-(2) we see that $\mathcal{I}_B^\pi = \{0, B\}$ and $\mathrm{Ann}_B(B) = 0$, and hence B is a nonnull algebra. Thus, B is a prime algebra by the corollary above. □

By combining Theorem 1.10 with the corollaries above we obtain the following result.

Corollary 5.4. *Let A be a nonzero algebra with zero annihilator. Then the following assertions are equivalent:*

(i) $A = \pi\mathrm{-Soc}(A)$.

(ii) *A is isomorphic to a direct sum of a nonempty family of nonzero prime algebras.*

(iii) $\mathcal{I}_A^\pi = \{\bigoplus_{i\in J} B_i \, : \, J \subseteq I\}$*, where* $\{B_i\}_{i\in I}$ *is the family of all minimal π-closed ideals of A.*

In this case, A is a π-decomposable π-complemented algebra.

Proof. (i) $\Rightarrow$ (ii). This implication follows from the above corollary.

(ii) $\Rightarrow$ (iii). Let $\{A_i\}_{i\in I}$ be a nonempty family of nonzero prime algebras. By Corollary 5.2, $\mathcal{I}_{A_i}^\pi = \{0, A_i\}$ for each $i \in I$. Now, by Theorem 1.10.(3), we see that $\mathcal{I}^\pi_{\oplus_{i\in I} A_i} = \{\bigoplus_{i\in J} A_i \, : \, J \subseteq I\}$, and in particular $\mathbf{m}^\pi_{\oplus_{i\in I} A_i} = \{A_i \, : \, i \in I\}$.

(iii) $\Rightarrow$ (i). This implication is clear.

Finally, assume that A is an algebra satisfying the equivalent conditions in the statement. From (iii) it is clear that A is π-decomposable and π-complemented. □

Corollary 5.5. *For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the algebra c_{00} of all quasi-null sequences is a π-decomposable π-complemented algebra such that $c_{00} = \pi\mathrm{-Soc}(c_{00})$.*

Let us turn our attention to the structure theory for π-decomposable algebras. We begin with the following elemental fact.

Lemma 5.6. *Let A be a semiprime algebra. If $\{B_j\}_{j\in J}$ is a nonempty family of minimal π-closed ideals of A, then $\sum_{j\in J} B_j = \oplus_{j\in J} B_j$, and $\oplus_{j\in J\setminus\{j_0\}} B_j \subseteq \mathrm{Ann}(B_{j_0})$ for every $j_0 \in J$.*

Proof. Fix $j_0 \in J$ and note that $B_{j_0} \cap B_j = 0$, and hence $B_{j_0} B_j = 0$, for all $j \in J$ with $j \neq j_0$. Therefore, we have $B_{j_0}(\sum_{j\in J\setminus\{j_0\}} B_j) = 0$. From the semiprimeness of A it follows that $\sum_{j\in J\setminus\{j_0\}} B_j \subseteq \mathrm{Ann}(B_{j_0})$ and $B_{j_0} \cap (\sum_{j\in J\setminus\{j_0\}} B_j) = 0$. By running $j_0 \in J$ we conclude that $\sum_{j\in J} B_j = \oplus_{j\in J} B_j$. □

The next statement collect the structure theory of π-decomposable algebras, including Yood's π-decomposition theorem proved in [6, Theorem 6.3] for the more general context of pseudocomplemented lattices, and the description theorem for π-decomposable algebras proved in [4, Theorem 4.1]. We will give a proof of these results for the sake of completeness. Recall that a nonzero algebra A with zero annihilator is called *π-atomic* if each nonzero π-closed ideal of A contains a minimal π-closed ideal.

Theorem 5.7. *Let A be a nonzero algebra with zero annihilator. Then the following assertions are equivalent:*

(i) A is π-decomposable.

(ii) A is semiprime and π-atomic.

(iii) A is an essential subdirect product of a nonempty family of nonzero prime algebras.

(iv) $\mathcal{I}^\pi_A = \{\overline{\oplus_{i\in J} B_i} : J \subseteq I\}$, where $\{B_i\}_{i\in I}$ is the family of all minimal π-closed ideals of A.

Proof. (i) $\Rightarrow$ (ii). Let U be a π-closed ideal of A. Write $\mathbf{m}^\pi_A = \{B_i : i \in I\}$. If $B_i \nsubseteq U$ for all i, then, by minimality of B_i, we have $B_i \cap U = 0$, and so $B_i \subseteq \mathrm{Ann}(U)$ for all i. As a result,

$$A = \overline{\sum_{i\in I} B_i} \subseteq \mathrm{Ann}(U).$$

Therefore $U \subseteq \mathrm{Ann}(A)$, and so $U = 0$. Thus, A is π-atomic.

In order to prove that A is semiprime we begin by noting that $B_i^2 \neq 0$ for all $i \in I$. Indeed, if there exists $i_0 \in I$ such that $B_{i_0}^2 = 0$, then

$$B_{i_0} \subseteq \mathrm{Ann}\left(\sum_{i\in I} B_i\right) = \mathrm{Ann}\left(\overline{\sum_{i\in I} B_i}\right) = \mathrm{Ann}(A) = 0,$$

and so $B_{i_0} = 0$, which is a contradiction. Now, assume the existence of a nonzero ideal U of A such that $U^2 = 0$. From the equality $U^2 = 0$, it follows that $U \subseteq \mathrm{Ann}(U)$, therefore $\overline{U}^2 \subseteq \overline{U}\mathrm{Ann}(U) = 0$, and hence $\overline{U}^2 = 0$. Since A is π-atomic, there exists $i_0 \in I$ such that $B_{i_0} \subseteq \overline{U}$, and in this way we find the contradiction $B_{i_0}^2 = 0$. Thus A is semiprime.

(ii) $\Rightarrow$ (iii). Let us write $\mathbf{m}_A^\pi = \{B_i : i \in I\}$. From Proposition 5.1.(2) it follows that each $\mathrm{Ann}(B_i)$ is a prime ideal of A, and so $A_i := A/\mathrm{Ann}(B_i)$ is a prime algebra. For each $i \in I$, let $q_i : A \to A_i$ be the quotient map, and consider the map $f : A \to \prod_{i\in I} A_i$ given by $f(a) = (q_i(a))$. It is clear that f is an algebra homomorphism. Moreover,

$$\ker(f) = \bigcap_{i\in I} \ker(q_i) = \bigcap_{i\in I} \mathrm{Ann}(B_i),$$

and hence $B_i \nsubseteq \ker(f)$ for all $i \in I$. Since A is π-atomic, it follows that $\ker(f) = 0$, and so f is an isomorphism from A onto $f(A)$. Clearly $q_i(A) = A_i$, and hence we have that A is a subdirect product of the family $\{A_i\}_{i\in I}$. Moreover, taking into account Lemma 5.6, we see that

$$\bigoplus_{i\in I} q_i(B_i) = f(\bigoplus_{i\in I} B_i) \subseteq f(A).$$

Since $q_i(B_i)$ is a nonzero ideal of the prime algebra A_i, it follows that $q_i(B_i)$ is an essential ideal of A_i, and, taking into account Theorem 3.2.(4), we conclude that A is an essential subdirect product of the family $\{A_i\}_{i\in I}$.

(iii) $\Rightarrow$ (iv). Assume the existence of a nonempty family of nonzero prime algebras $\{A_i\}_{i\in I}$ such that A can be seen as an essential subalgebra of $\prod_{i\in I} A_i$ satisfying $p_i(A) = A_i$. Taking into account Corollary 5.2, from Theorem 3.2.(1) we have

$$\mathcal{I}_A^\pi = \{(\prod_{i\in J} A_i) \cap A \,:\, J \subseteq I\}.$$

Therefore, $\{A_i \cap A : i \in I\}$ is the family of all minimal π-closed ideals of A. For a fixed subset J of I, it is clear that $(\prod_{i\in J} A_i) \cap A$ is the smallest π-closed ideal of A containing $\bigoplus_{i\in J}(A_i \cap A)$, and so $\overline{\bigoplus_{i\in J}(A_i \cap A)} = (\prod_{i\in J} A_i) \cap A$.

(iv) $\Rightarrow$ (i). This implication is obvious.

□

Since every finite dimensional algebra with zero annihilator is π-atomic, we have the following consequence.

Corollary 5.8. *For every finite dimensional algebra A with zero annihilator the following assertions are equivalent:*

(i) A is π-decomposable.

(ii) A is semiprime.

π-decomposable (even finite-dimensional) algebras may not be π-complemented. The specific algebra A in Example 1.14 is such an example. An infinite dimensional example is provided by the algebra c of all convergent sequences over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Note that $\mathbf{m}_c^\pi = \{B_i : i \in \mathbb{N}\}$ and $c = \overline{\bigoplus_{i\in\mathbb{N}} B_i}$, where

$$B_i = \{\{a_n\} : a_n = 0 \text{ for all } n \neq i\}.$$

Thus c is π-decomposable. However, c is not π-complemented because

$$U = \{\{a_n\} \in c : a_{2n} = 0 \text{ for all } n \in \mathbb{N}\}$$

is a π-closed ideal of c with

$$\mathrm{Ann}(U) = \{\{a_n\} \in c : a_{2n-1} = 0 \text{ for all } n \in \mathbb{N}\}$$

and $c \neq U \oplus \mathrm{Ann}(U)$.

Given a family of algebras $\{A_i\}_{i\in I}$, for each $J \subseteq I$, we consider the block-projection $p_J : \prod_{i\in I} A_i \to \prod_{i\in I} A_i$ given by $p_J(a_i) = (b_i)$, where $b_i = a_i$ if $i \in J$ and $b_i = 0$ otherwise.

Our main result in this section is the following description theorem.

Theorem 5.9. *Let A be a nonzero algebra. Then the following assertions are equivalent:*

(i) A is π-decomposable π-complemented.

(ii) There exists a nonempty family of nonzero prime algebras $\{A_i\}_{i\in I}$ such that A can be regarded as a subalgebra of $\prod_{i\in I} A_i$ containing $\bigoplus_{i\in I} A_i$, and $p_J(A) \subseteq A$ for all $J \subseteq I$.

In this case, $\mathcal{I}_A^\pi = \{p_J(A) : J \subseteq I\}$.

Proof. (i) $\Rightarrow$ (ii). By Theorem 5.7, there exists a nonempty family of nonzero prime algebras $\{A_i\}_{i\in I}$ and an essential ideal D of $\prod_{i\in I} A_i$ such that A can be regarded as a subalgebra of $\prod_{i\in I} A_i$ containing D, and $p_i(A) = A_i$ for all $i \in I$. It is clear that $D_i := D \cap A_i$ is an essential ideal of A_i for each $i \in I$. By Theorem 3.2.(1), $\mathcal{I}_A^\pi = \{(\prod_{i\in I} U_i) \cap A : U_i \in \mathcal{I}_{A_i}^\pi \text{ for each } i \in I\}$. Since A is π-complemented, keeping in mind Lemma 3.1.(1), for each $i_0 \in I$ we have the decomposition

$$A = [A_{i_0} \cap A] \bigoplus \left[(\prod_{i \in I\setminus\{i_0\}} A_i) \cap A \right].$$

Therefore $A_{i_0} = p_{i_0}(A) = A_{i_0} \cap A$, and hence $A_{i_0} \subseteq A$. Thus, $\oplus_{i \in I} A_i \subseteq A$. For a given subset J of I, note that $p_J = P_{\{U_i\}}$ for the family $\{U_i\}$ where $U_i = A_i$ for $i \in J$ and $U_i = 0$ otherwise. By Proposition 4.6, it follows that $p_J(A) \subseteq A$.

(ii) $\Rightarrow$ (i). By Theorem 3.2, A is semiprime. Now, by Theorem 5.7, A is π-decomposable. Moreover, keeping in mind Corollary 5.2, by Proposition 4.6 we conclude that A is π-complemented and $\mathcal{I}_A^\pi = \{p_J(A) : J \subseteq I\}$. □

The following immediate consequence can also be deduced from Theorem 4.3 and Corollary 2.12.

Corollary 5.10. *Every nonzero π-closed ideal of a π-decomposable π-complemented algebra is a π-decomposable π-complemented algebra.*

Another consequences of Theorem 5.9 are the following:

Corollary 5.11. *The direct product of a family of prime algebras is a π-decomposable π-complemented algebra.*

Corollary 5.12. *For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the following algebras are examples of π-decomposable π-complemented algebras:*

(1) The algebra c_0 of all null sequences.

(2) The algebra ℓ_p $(1 \leq p < \infty)$ of all absolutely p-summable sequences.

(3) The algebra ℓ_∞ of all bounded sequences.

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