LIE ALGEBRAS WITH AN ALGEBRAIC ADJOINT REPRESENTATION REVISITED

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Dedicated to Professor Kevin McCrimmon on the occasion of his 70th birthday

Abstract. A well-known theorem due to Zelmanov proves that a Lie PI-algebras with an algebraic adjoint representation over a field of characteristic zero is locally finite-dimensional. In particular, a Lie algebra (over a field of characteristic zero) whose adjoint representation is algebraic of bounded degree is locally finite-dimensional.

Using recent results on Jordan structures in Lie algebras, we prove in this paper a proposition from which Zelmanov's theorem for Lie PI-algebras with an algebraic adjoint representation over an algebraically closed field of characteristic zero, and its corollary for Lie algebras with an algebraic adjoint representation of bounded degree (over an arbitrary field of characteristic zero) are easily derived.

1. Introduction

In [17], Zelmanov proves [17, Theorem 1] that a Lie algebra L over a field of characteristic zero with an algebraic adjoint representation is locally finite-dimensional, provided it satisfies a a polynomial identity, thus yielding a solution of the Kurosh problem for Lie algebras. As a corollary he obtains [17, Theorem 2] that a Lie algebra over a field of characteristic zero with an algebraic adjoint representation of bounded degree is locally finite-dimensional. In this note we prove:

Proposition 5.2. *Let $\tilde{L}$ be a nondegenerate Lie algebra over an algebraically closed field $\overline{\Phi}$ of characteristic zero, and let L be a Φ-subalgebra $\tilde{L}$, where Φ is a subfield of $\overline{\Phi}$, such that L has an algebraic adjoint representation and $\tilde{L}$ is $\overline{\Phi}$-spanned by L. Suppose that one of the following two conditions holds:*

(i) $\Phi = \overline{\Phi}$ *and* $L = \tilde{L}$ *satisfies a polynomial identity.*
(ii) *The algebraic adjoint representation of L is of bounded degree.*

Then $\tilde{L}$ is a subdirect product of a family of finite-dimensional simple Lie algebras of bounded dimension. As a consequence, $\tilde{L}$ satisfies all the identities which hold in some finite-dimensional Lie algebra.

This result enables us to shorten and simplify the proof of [17, Theorem 1] *under the additional assumption that Φ is algebraically closed*, and it also provides an independent proof of [17, Theorem 2]. To be more precise, let us give a brief outline of Zelmanov's original proof of [17, Theorem 1] and comment on the changes and new tools we use in the proof

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of Proposition 5.2. First Zelmanov observes that it suffices to prove that if in addition L is *nondegenerate* ($\mathrm{ad}_x^2 L = 0 \Rightarrow x = 0$, $x \in L$) and nonzero, then L contains a nonzero locally finite-dimensional ideal. Then the proof splits into two cases. If L is Engel, that is, any element of L is ad-nilpotent, he proves [17, Proposition 1] that L is locally nilpotent, so he can assume that L contains a non-Engel element, say a. Then he takes an algebraically closed extension F of Φ of sufficiently large cardinality, makes the scalar extension $L \otimes_\Phi F$, and takes the quotient algebra of $L \otimes_\Phi F$ by its Kostrikin radical, thus obtaining a Lie algebra $\overline{L}$ (over an algebraically closed field which is large for $\overline{L}$) which is nondegenerate, satisfies a polynomial identity, and has a nontrivial finite grading (that induced by the non-Engel element a). Moreover, he proves that L can be embedded in $\overline{L}$. At this point the proof becomes quite involved. Let $\overline{L}_\alpha$ be an extreme subspace of the grading in $\overline{L}$ defined by ad_a. From the fact that $\overline{L}$ satisfies a polynomial identity, he derives that the Jordan pair $V = (\overline{L}_\alpha, \overline{L}_{-\alpha})$ is PI. Moreover, V is nondegenerate, and since F is large for V, it follows from [17, Theorem JP1 and Lemma JP1] that V is actually a semiprimitive Jordan pair. By using deep results on Lie algebras with finite gradings (the most difficult part of the paper [17, pages 543-548]) and after a skilful manipulation of the primitive ideals of V, Zelmanov then proves: (i) the ideal I of $\overline{L}$ generated by $\overline{L}_\alpha$ can be embedded in a subdirect product of a family of simple Lie algebras of finite bounded dimension, and (ii) L intersects I nontrivially. Hence he obtains a nonzero ideal, $L \cap I$, of L that satisfies all the identities which hold in some finite-dimensional Lie algebra. The last step of the proof is the following result of independent interest [17, Lemmas 5, 6 and 7] (or [11, Theorem 5.4.6]): A Lie algebra (over a field of characteristic zero) with an algebraic adjoint representation and satisfying all the identities that hold in some finite-dimensional Lie algebra is locally finite-dimensional.

By applying a recent result proved in [7, Theorem 3.10] (actually this result, at least in its germinal state, could be attributed to Zelmanov himself), we reduce the proof of Proposition 5.2 to the case that $\tilde{L}$ is prime. Then, as a consequence of the Kostrikin lemma (in the Engel case) and of the existence of a nontrivial finite grading (in the non-Engel case), we obtain that $\tilde{L}$ contains a nonzero *Jordan element*, i.e., there exists a nonzero element x in $\tilde{L}$ such that, ($\mathrm{ad}_x^3 \tilde{L} = 0$). Then we consider the Jordan algebra $\tilde{L}_x$ of $\tilde{L}$ at x [4]. This Jordan algebra inherits primeness and nondegeneracy from $\tilde{L}$. Moreover, we manage to prove that $\tilde{L}_x$ satisfies a polynomial identity which is not an s-identity, i.e., $\tilde{L}_x$ is a Jordan PI-algebra, and, what is the most striking fact, that $\tilde{L}_x$ is an algebraic Jordan algebra (Proposition 4.2(iii) and Lemma 5.1). Using the structure theory of Jordan PI-algebras [16], and the transference of inner ideals from $\tilde{L}_x$ to $\tilde{L}$ [4], we obtain that $\tilde{L}$ contains an extremal element, say y, ($\mathrm{ad}_y \tilde{L} = \Phi y$). Then the socle of $\tilde{L}$ [3] is a locally finite-dimensional simple Lie algebra, and since $\tilde{L}$ satisfies a polynomial identity, $\mathrm{Soc}\,\tilde{L}$ is actually finite-dimensional, its dimension being bounded by a a number depending only of the degree of the polynomial identity. Hence $\tilde{L} = \mathrm{Soc}\,\tilde{L}$, since any derivation of a simple finite-dimensional Lie algebra over a field of characteristic zero is inner.

2. Lie algebras and Jordan algebras

1. Throughout this note, and unless specified otherwise, we will be dealing with Lie algebras L [9] and [10], with $[x, y]$ denoting the Lie bracket and ad_x the adjoint map determined by x, and with *linear* Jordan algebras J [12], with Jordan product $x \cdot y$, multiplication operators $m_x : y \mapsto$

$x \cdot y$, quadratic operators $U_x = 2m_x^2 - m_{x^2}$, and triple product $\{x, y, z\} = U_{x+z}y - U_xy - U_zy$, over a field Φ of characteristic 0. We set

$$[x_1] := x_1 \quad \text{and} \quad [x_1, x_2, \ldots, x_n] := [x_1, [x_2, \ldots, x_n]]$$

for $n > 1$ and $x_1, x_2, \ldots, x_n \in L$. Similarly, we set

$$x_1 \cdot x_2 \cdots x_n := x_1 \cdot (x_2 \cdots x_n)$$

for $n > 1$ and $x_1, x_2, \ldots, x_n \in J$.

Any associative algebra A gives rise to a Lie algebra A^-, with Lie bracket $[x, y] := xy - yx$, and a linear Jordan algebra A^+, with Jordan product $x \cdot y := 1/2(xy + yx)$. A Jordan algebra J is said to be *special* if it is isomorphic to a subalgebra of A^+ for some associative algebra A.

2. An *inner ideal* of J is a vector subspace B of J such that $U_B J \subseteq B$. Similarly, an *inner ideal* of L is a vector subspace B of L such that $[B, [B, L]] \subseteq B$. An *abelian inner ideal* of L is an inner ideal B which is also an abelian subalgebra, i.e., $[B, B] = 0$.

3. An element $x \in L$ is called *Engel* if ad_x is a nilpotent operator. In this case, the nilpotence index of ad_x is called the *index* of x. Engel elements of index at most 3 are called *Jordan elements.* Clearly, any element of an abelian inner ideal is a Jordan element. Conversely, by [2, Lemma 1.8], any Jordan element x generates the abelian inner ideal $\mathrm{ad}_x^2 L$. A good reason for this terminology is the following analogue of the fundamental identity for Jordan algebras:

$$\mathrm{ad}^2_{\mathrm{ad}_x^2 y} = \mathrm{ad}_x^2 \mathrm{ad}_y^2 \mathrm{ad}_x^2$$

which holds for any Jordan element x and any $y \in L$ [2, Lemma 1.7(iii)]. Another reason will be given Section 4.

4. A well-known lemma due to Kostrikin [11, Lemma 2.1.1] provides a method to construct Jordan elements by means of Engel elements. Under our assumption that the ground field Φ is of characteristic 0, this result reads as follows: If $x \in L$ is an Engel element of index n then, for any $a \in L$, $\mathrm{ad}_x^{n-1} a$ is Engel of index $\leq n - 1$. Recently, García and Gómez have given the following refinement of this result [6, Theorem 2.3 and Corollary 2.4].

Lemma 2.1. *If $x \in L$ is an Engel element of index n, then $\mathrm{ad}_x^{n-1} L$ is an abelian inner ideal of L. Hence, $\mathrm{ad}_x^{n-1} a$ is a Jordan element for any $a \in L$.*

5. Let Λ be a torsion free abelian group and let L be a Lie algebra. A Λ-grading $L = \sum_{\lambda \in \Lambda} L_\lambda$ of L is said to be *finite* if the set $\Lambda^* = \{\lambda \in \Lambda : L_\lambda \neq 0\}$ is finite, and *nontrivial* if Λ^* contains a nonzero element. Notice that if a Λ-grading is finite and nontrivial, then the subgroup $G = G(\Lambda^*)$ of Λ generated by Λ^* is free of finite rank, and therefore it is isomorphic to $\mathbb{Z}^r$ for some positive integer r.

Proposition 2.2. *Let L be a Lie algebra with a nontrivial finite Λ-grading. Then L contains a nonzero Jordan element.*

Proof. Take a basis $\{\lambda_1, \ldots, \lambda_r\}$ of G such that for some $\alpha \in \Lambda^*$, $\alpha = n_{\alpha_1}\lambda_1 + \ldots + n_{\alpha_r}\lambda_r$ with $n_{\alpha_1} \neq 0$, and let $\pi : G \to \mathbb{Z}$ be the homomorphism defined by putting $\pi(\lambda_1) = 1$ and $\pi(\lambda_i) = 0$ for $1 < i \leq r$. We may also assume that $|\pi(\beta)| \leq |\pi(\alpha)|$ $(\beta \in \Lambda^*)$. Then any $x \in L_\alpha$ is a Jordan element: for any $\beta \in \Lambda^*$, $\mathrm{ad}_x^3 L_\beta \subseteq L_{3\alpha+\beta} = 0$ since $|\pi(3\alpha + \beta)| > |\pi(\alpha)|$. $\square$

Corollary 2.3. *Let L be a Lie algebra over an algebraically closed field Φ of characteristic zero. If L has a nonzero element whose adjoint is algebraic, then L contains a nonzero Jordan element.*

Proof. Let a be a nonzero element of L such that ad_a is algebraic. If a is Engel of index, say n, we have by Lemma 2.1 that $\mathrm{ad}_a^{n-1}b$ is a nonzero Jordan element for some $b \in L$. Otherwise, by [9, Lemma 2.4.2(B)], ad_a yields a nontrivial finite $(\Phi, +)$-grading on L given by

$$L_\lambda = \{x \in L : (\mathrm{ad}_a - \lambda 1_L)^m x = 0 \text{ for some } m \geq 1\},$$

with $L_\lambda = 0$ if $\lambda \in \Phi$ is not an eigenvalue of ad_a. By Proposition 2.2, L contains a nonzero Jordan element. □

6. An element $x \in J$ is called an *absolute zero divisor* if $U_x = 0$. We say J is *nondegenerate* if it has no nonzero absolute zero divisors, *semiprime* if $B^2 = 0$ implies $B = 0$, and *prime* if $B \cdot C = 0$ implies $B = 0$ or $C = 0$, for any ideals B, C of J. Similarly, given a Lie algebra L, $x \in L$ is an *absolute zero divisor* of L if $\mathrm{ad}_x^2 = 0$ (for Lie algebras over a field of characteristic 2, standard definition of absolute zero divisor or cover of a thin sandwich requires $\mathrm{ad}_x^2 = \mathrm{ad}_x \mathrm{ad}_y \mathrm{ad}_x = 0$, $y \in L$); L is *nondegenerate* if it has no nonzero absolute zero divisors, *semiprime* if $[B, B] = 0$ implies $B = 0$, and *prime* if $[B, C] = 0$ implies $B = 0$ or $C = 0$, for any ideals B, C of L. *Simplicity,* for both Jordan and Lie algebras, means nonzero product and the absence of nonzero proper ideals.

7. A Jordan or Lie algebra is called *strongly prime* if it is prime and nondegenerate. The following elmental characterizations of strong primeness for Lie algebras L and Jordan algebras J were proved in [5, Theorems 1.6 and 2.3]:

(i) *L is strongly prime if and only if for every $x, y \in L$ such that $[x, [y, L]] = 0$ we have that $x = 0$ or $y = 0$.*

(ii) *J is strongly prime if and only if for every $x, y \in J$ such that $\{x, J, y\} = 0$ we have that $x = 0$ or $y = 0$.*

8. Following [11, Definition 5.4.1], the smallest ideal of a Lie algebra L whose associated quotient algebra is nondegenerate is called the *Kostrikin radical* of L, denoted by $K(L)$. Put $K_0(L) = 0$ and let $K_1(L)$ be the ideal generated by all absolute zero divisors. Using transfinite induction, a nondecreasing chain of ideals $K_\alpha(L)$ is defined by putting $K_\alpha(L) = \bigcup_{\beta<\alpha} K_\beta(L)$ if α is a limit ordinal, and $K_\alpha(L)/K_{\alpha-1}(L) = K_1(L/K_{\alpha-1}(L))$ otherwise. It is obvious that $K(L) = \bigcup_\alpha K_\alpha(L)$. The Jordan counterpart of the Kostrikin radical is the *McCrimmon radical* (also called *degenerate radical*) $Mc(J)$ [12, page 92].

The following result, proved by Grishkov in [8], can be found translated to English in [11, Theorem 5.4.2].

Theorem 2.4. *Let L be a Lie algebra over a field of characteristic zero. Then $K_1(L)$ is locally nilpotent. Hence, simple Lie algebras over a field of characteristic zero are nondegenerate.*

The following characterization of the Kostrikin radical was proved in [7, Theorem 3.10].

Theorem 2.5. *The Kostrikin radical $K(L)$ of a Lie algebra L over a field of characteristic zero is the intersection of all strongly prime ideals of L. Therefore, L is nondegenerate if, and only if, it is a subdirect product of strongly prime Lie algebras.*

9. The *socle* of a Jordan algebra is the sum of all its minimal inner ideals [13]. The *socle* of a Lie algebra L, $\operatorname{Soc} L$, is defined as the sum of all minimal inner ideals of L [3]. By [13, Theorem 17] (for Jordan algebras) and [3, Theorem 2.5] (for Lie algebras), the socle of a nondegenerate Jordan algebra (Lie algebra) is the direct sum of its minimal ideals, each of which is a simple Jordan algebra (Lie algebra).

10. Let L be a Lie algebra over a field Φ. Recall that a nonzero element $x \in L$ is said to be *extremal* if $\mathrm{ad}_x^2 L = \Phi x$, that is, if it generates a one-dimensional inner ideal.

11. The adjoint representation of a Lie algebra L is said to be *algebraic* if ad_x is an algebraic operator for each x in L. It was proved in [14] that a Lie algebra whose adjoint representation is algebraic contains a maximal locally finite-dimensional ideal and the quotient algebra over this ideal has no nonzero locally finite-dimensional ideals. A similar result also holds for *Engel Lie algebras* (any element is Engel) with respect to the so-called locally nilpotent radical.

3. Polynomial identities

Let $L(X)$ denote the free Lie Φ-algebra over a countable set of indeterminates X. By using the Jacobi identity, we see that any monomial of $L(X)$ can be written as a linear combination of *standard monomials* $[x_{i_1}, \ldots, x_{i_m}]$ (see (1) for notation). For each positive integer n, let $\mathfrak{S}_n$ denote the set of all permutations of $1, \ldots, n$.

Lemma 3.1. *Let n be a positive integer. Then there exists a function $f_n : \mathfrak{S}_n \to \{0, 1, -1\}$ such that, for any $x_1, \ldots, x_n, x_{n+1}$ in X,*

$$\mathrm{ad}_{[x_1,\ldots,x_n]} x_{n+1} = \sum_{\sigma \in \mathfrak{S}_n} f_n(\sigma)[x_{\sigma(1)}, \ldots, x_{\sigma(n)}, x_{n+1}].$$

Proof. By induction on n. The case $n = 1$ is trivial. Now

$$\mathrm{ad}_{[x_1,x_2,\ldots,x_{n+1}]} x_{n+2} = \mathrm{ad}_{[x_1,[x_2,\ldots,x_{n+1}]]} x_{n+2} = \mathrm{ad}_{x_1} \mathrm{ad}_{[x_2,\ldots,x_{n+1}]} x_{n+2} - \mathrm{ad}_{[x_2,\ldots,x_{n+1}]} \mathrm{ad}_{x_1} x_{n+2}.$$

Hence, by the induction hypothesis,

$$\begin{aligned} \mathrm{ad}_{[x_1,\ldots,x_{n+1}]} x_{n+2} &= \sum_{\sigma \in \mathfrak{S}_n} f_n(\sigma)[x_1, x_{\sigma(1)+1}, \ldots, x_{\sigma(n)+1}, x_{n+2}] \\ &- \sum_{\sigma \in \mathfrak{S}_n} f_n(\sigma)[x_{\sigma(1)+1}, \ldots, x_{\sigma(n)+1}, x_1, x_{n+2}] \\ &= \sum_{\tau \in \mathfrak{S}_{n+1}} f_{n+1}(\tau)[x_{\tau(1)}, \ldots, x_{\tau(n+1)}, x_{n+2}]. \end{aligned}$$

where f_n is defined inductively by

$$f_{n+1}(\tau) = \begin{cases} 0 & \text{if } \tau(1) \neq 1 \text{ and } \tau(n+1) \neq 1 \\ f_n(\sigma) & \text{if } \tau(1) = 1 \text{ and } \sigma \in \mathfrak{S}_n \text{ is defined by } \sigma(i) = \tau(i+1) - 1 \\ -f_n(\sigma) & \text{if } \tau(n+1) = 1 \text{ and } \sigma \in \mathfrak{S}_n \text{ is defined by } \sigma(i) = \tau(i) - 1 \end{cases}$$

∎

Let $p = p(x_1, \ldots, x_n)$ be an element of a free Lie Φ-algebra $L(X)$. We say that a Lie algebra L satisfies the identity $p = 0$ if $p(a_1, \ldots, a_n) = 0$ for any $a_1, \ldots, a_n$ in L. A Lie algebra satisfying a nontrivial polynomial identity is called a *Lie PI-algebra.*

Proposition 3.2. *Any Lie PI-algebra L satisfies a multilinear identity $p = 0$, where*

$$p(x_1, \ldots, x_n, x_{n+1}) = \sum_{\sigma \in \mathfrak{S}_n} \alpha_\sigma [x_{\sigma(1)}, \ldots, x_{\sigma(n)}, x_{n+1}] \quad (\alpha_\sigma \in \Phi).$$

Proof. As pointed out above, we may assume that L satisfies a polynomial identity $p = 0$, where p is a linear combination of standard monomials $[x_{i_1}, \ldots, x_{i_m}]$. Moreover, by [18, Corollary of Theorem 1.5.7], we can assume that p is multilinear. Finally, by Lemma 3.1, we can replace p by a polynomial having the required form. □

Recall that a Jordan polynomial $p = p(x_1, \ldots, x_n)$ of the free Jordan Φ-algebra $J(X)$ is said to be an *s-identity* if it is satisfied by all special Jordan algebras, but not by all Jordan algebras. A Jordan algebra J satisfying a polynomial identity which is not an s-identity is called a Jordan PI-algebra.

Proposition 3.3. *A nonzero Jordan polynomial of the form*

$$p(x_1, \ldots, x_n, x_{n+1}) = \sum_{\sigma \in \mathfrak{S}_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)} \cdot x_{n+1} \quad (\alpha_\sigma \in \Phi)$$

is never an s-identity.

Proof. By relabeling the variables we may assume that $\alpha_\sigma = 1$ for $\sigma = \mathrm{Id}$. Let $Y = \{y_1, y_2, \ldots\}$ be a countable set. Denote by S the free semigroup generated by $Y \cup \{0\}$ satisfying the relations $y_i y_j = 0\,(j \neq i + 1)$ and $y_i 0 = 0 y_i = 0\,(i \geq 1)$. Let A be the associative algebra defined by taking $S - \{0\}$ as a basis. It is easy to verify that $2^n p(y_1, \ldots, y_n, y_{n+1}) = y_1 \ldots y_n y_{n+1} \neq 0$. Thus the special Jordan algebra A^+ does not satisfies the identity $p(x_1, \ldots, x_n, x_{n+1}) = 0$, and therefore $p(x_1, \ldots, x_n, x_{n+1})$ is not an s-identity. □

4. The Jordan algebras of a Lie algebra

In [4] a Jordan algebra was attached to any Jordan element of a Lie algebra. As will be proved in the propositions below, many properties of a Lie algebra can be transferred to its Jordan algebras, as well as the nature of the Jordan element in question is reflected on the structure of the attached Jordan algebra. These facts turn out to be crucial for applications of Jordan theory to Lie algebras.

Proposition 4.1. *Let a be a Jordan element of a Lie algebra L over a field Φ of characteristic $\neq 2, 3$. Then L with the new product defined by $x \cdot_a y := \frac{1}{2}[[x, a], y]$ is a nonassociative algebra denoted by $L^{(a)}$, such that*

(i) $\mathrm{Ker}_L a := \{x \in L : [a, [a, x]] = 0\}$ *is an ideal of* $L^{(a)}$.
(ii) $L_a := L^{(a)}/\mathrm{Ker}_L a$ *is a Jordan algebra, called* the Jordan algebra of L at a.

Proof. [4, Theorem 2.4]. □

Proposition 4.2. *Let a be a Jordan element of a Lie algebra L.*

(i) *If L is nondegenerate (strongly prime), then L_a is nondegenerate (strongly prime).*

(ii) *If every element of L is Engel, then L_a is nil.*
(iii) *If L has an algebraic adjoint representation, then L_a is algebraic.*
(iv) *If L is PI, then L_a is PI.*

Proof. Inheritance of nondegeneracy (strong primeness) was proved in [4, Proposition 2.15(i)] ([5, Theorem 2.2(i)]); (ii) and (iii) are consequence of the identity $\overline{x}^n = (1/2)^{n-1}\overline{\mathrm{ad}^{n-1}_{[x,a]}x}$, which holds for any $x \in L$ and any positive integer n, with $x \to \overline{x}$ denoting the linear mapping of L onto L_a. It only remains to prove (iv). By Proposition 3.2, L satisfies a multilinear $p = 0$, where

$$p(x_1, \ldots, x_n, x_{n+1}) = \sum_{\sigma\in\mathfrak{S}_n} \alpha_\sigma [x_{\sigma(1)}, \ldots, x_{\sigma(n)}, x_{n+1}].$$

By replacing each x_i by $[x_i, a]$ and multiplying by $(1/2)^n$ we obtain that the Jordan polynomial

$$q(x_1, \ldots, x_n, x_{n+1}) = \sum_{\sigma\in\mathfrak{S}_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)} \cdot x_{n+1}$$

vanishes on L_a. Since by Proposition 3.3 this polynomial is not an s-identity, L_a is PI. □

Proposition 4.3. *Let L be a nondegenerate Lie algebra with an algebraic adjoint representation over an algebraically closed field Φ of characteristic zero. Then any abelian minimal inner ideal B of L is one-dimensional, so any nonzero element of B is extremal*

Proof. Let x be a nonzero element of B. Then $\mathrm{ad}_x^2 L = B$ and x is a Jordan element of L. By [4, (2.14)] together with the minimality of B, the Jordan algebra L_x of L at x has no nonzero proper inner ideals, that is, it is a division Jordan algebra, and by [4, Proposition 2.15(ii)], any $y \in L$ such that $[[x, y], x] = 2x$ yields the identity element $\overline{y}$ of L_x. Since L_x is algebraic (Proposition 4.2) and Φ is algebraically closed, $\overline{L} = L_x = \Phi\overline{y}$. Hence $B = \mathrm{ad}_x^2 L = \Phi x$. □

5. Main results

Given a subset X of a Lie algebra L, we will write ΦX to denote the Φ-subspace of L spanned by X.

Lemma 5.1. *Let $\tilde{L}$ be a Lie algebra over a field $\overline{\Phi}$ of characteristic zero, and let L be a Φ-subalgebra of $\tilde{L}$, where Φ is a subfield of $\overline{\Phi}$, such that $\tilde{L} = \overline{\Phi}L$ and L has an algebraic adjoint representation of bounded degree. Then for any Jordan element a of $\tilde{L}$ the Jordan algebra $\tilde{L}_a$ is algebraic.*

Proof. Let a be a Jordan element of of $\tilde{L}$ and let y be an arbitrary element of $\tilde{L}$. In virtue of the formula $\overline{y}^m = 1/2^{m-1}\overline{\mathrm{ad}^{m-1}_{[y,a]}y}$, $m \geq 1$, proving that $\overline{y}$ is an algebraic element of $\tilde{L}_a$ is equivalent to seeing that the set $\{\mathrm{ad}^s_{[y,a]}y \mid s \geq 0\}$ is linearly dependent over $\overline{\Phi}$.

Write a as $a = \alpha_1 a_1 + \ldots + \alpha_k a_k$, where $\alpha_i \in \overline{\Phi}$ and $a_i \in L$, and let x be an arbitrary element of L. For every $s \geq 1$ and $\beta_1, \ldots, \beta_k \in \overline{\Phi}$, we have

$$\mathrm{ad}^s_{[x,\beta_1 a_1+\ldots+\beta_k a_k]}x = (\beta_1 \mathrm{ad}_{[x,a_1]} + \ldots + \beta_k \mathrm{ad}_{[x,a_k]})^s x = \sum_{\substack{\{i_1,\ldots,i_s\},\\ 1\leq i_1\leq i_2\leq\ldots\leq i_s\leq k}} \beta_{i_1}\cdots\beta_{i_s}\left(\sum_{\sigma\in\mathfrak{S}_s} \mathrm{ad}_{[x,a_{i_{\sigma(1)}}]}\cdots\mathrm{ad}_{[x,a_{i_{\sigma(s)}}]}\right)x\,.$$

Since Φ is infinite, for each $s \geq 1$ we can find a finite subset T_s of Φ such that the finite-dimensional Φ-subspace of L spanned by all the elements $\mathrm{ad}^s_{[x,\gamma_1 a_1+\ldots+\gamma_k a_k]}x$, $\gamma_1,\ldots,\gamma_k \in T_s$, contains all the elements

$$\left(\sum_{\sigma\in\mathfrak{S}_s} \mathrm{ad}_{[x,a_{i_{\sigma(1)}}]}\cdots \mathrm{ad}_{[x,a_{i_{\sigma(s)}}]}\right)x \quad (1\leq i_1\leq\ldots\leq i_s\leq k)\,.$$

Hence we get that the sets $\{\mathrm{ad}^s_{[x,\beta_1 a_1+\ldots+\beta_k a_k]}x \mid (\beta_1,\ldots,\beta_k)\in\overline{\Phi}^k\}$,

$$\{\mathrm{ad}^s_{[x,\theta_1 a_1+\ldots+\theta_k a_k]}x \mid (\theta_1,\ldots,\theta_k)\in T_s^k\} \text{ and } \{\mathrm{ad}^s_{[x,\gamma_1 a_1+\ldots+\gamma_k a_k]}x \mid (\gamma_1,\ldots,\gamma_k)\in \Phi^k\}$$

span the same (finite-dimensional) $\overline{\Phi}$-subspace $W(x,s)$ of $\tilde{L}$.

Note also that we can choose the finite subset T_s on the base of standard ideas connected with Vandermonde determinant. Hence we can assume that the number of elements of T_s is less than or equal to some integer $l(s,k)$, $l(s,k)\leq(s+1)^k$, and, therefore, $\dim_{\overline{\Phi}} W(x,s)\leq l(s,k)$.

Since L has an algebraic adjoint representation of bounded degree, say n, it follows that for all $s\geq n$,

$$W(x,s)\ \subseteq\ \sum_{t=0}^{n-1} W(x,t).$$

Hence the $\overline{\Phi}$-subspace $W(x)$ of $\tilde{L}$ spanned by the set

$$\{\mathrm{ad}^s_{[x,\beta_1 a_1+\ldots+\beta_k a_k]}x \mid (\beta_1,\ldots,\beta_k)\in\overline{\Phi}^k,\ s\geq 0\}$$

has finite-dimension, with $\dim_{\overline{\Phi}} W(x)\leq 1+l(1,k)+\ldots+l(n-1,k)$.

In particular, it follows that the $\overline{\Phi}$-subspace $U(x)$ of $W(x)$ spanned by all the elements

$$\mathrm{ad}^s_{[x,a]}x\ =\ \mathrm{ad}^s_{[x,\alpha_1 a_1+\ldots+\alpha_k a_k]}x \quad (s\geq 0)\,,$$

has finite dimension, with $\dim_{\overline{\Phi}} U(x)\leq d$, where d is a positive integer depending on the degree n of the algebraic adjoint representation of L and on the number k determined by the chosen representation of the Jordan element a, but which is independent of the element x. In fact, since $\mathrm{ad}^s_{[x,a]}x=\mathrm{ad}^t_{[x,a]}\mathrm{ad}^{s-t}_{[x,a]}x$ for any positive integers s,t such that $s\geq t$, we have that

$$U(x)\ =\ \sum_{t=0}^{d-1}\overline{\Phi}\,\mathrm{ad}^t_{[x,a]}x \tag{$\star$}$$

Now let y be an arbitrary element of $\tilde{L}$. We can write $y=\psi_1 y_1+\ldots+\psi_l y_l$, where $\psi_i\in\overline{\Phi}$ and $y_i\in L$. For every $t\geq 1$ and $(\phi_1,\ldots,\phi_l)\in\overline{\Phi}^l$, we have the equation

$$\mathrm{ad}^t_{[\phi_1 y_1+\ldots+\phi_l y_l,a]}(\phi_1 y_1+\ldots+\phi_l y_l)\ =\ (\phi_1\mathrm{ad}_{[y_1,a]}+\ldots+\phi_l\mathrm{ad}_{[y_l,a]})^t(\phi_1 y_1+\ldots+\phi_l y_l)\ =$$
$$\sum_{\substack{\{i_1,\ldots,i_{t+1}\},\\ 1\leq i_1\leq i_2\leq\ldots\leq i_{t+1}\leq l}} \phi_{i_1}\cdots\phi_{i_{t+1}}\left(\sum_{\sigma\in\mathfrak{S}_{t+1}}(\mathrm{ad}_{[y_{i_{\sigma(1)}},a]}\cdots\mathrm{ad}_{[y_{i_{\sigma(t)}},a]})y_{i_{\sigma(t+1)}}\right).$$

Set $V_0=\Phi y_1+\ldots+\Phi y_l$. Using the same arguments as before, we can find for each $t\geq 1$ a finite subset H_t of Φ such that the finite-dimensional Φ-subspace of $\tilde{L}$ spanned by the elements

$\operatorname{ad}^t_{[\tau_1 y_1+\ldots+\tau_l y_l,a]}(\tau_1 y_1+\ldots+\tau_l y_l)$, $(\tau_1,\ldots,\tau_l)\in H^l_t$, contains all the elements

$$\sum_{\sigma\in\mathfrak{S}_{t+1}}(\operatorname{ad}_{[y_{i_{\sigma(1)}},a]}\cdots\operatorname{ad}_{[y_{i_{\sigma(t)}},a]})y_{i_{\sigma(t+1)}}\quad(1\le i_1\le\ldots\le i_{t+1}\le l)\,.$$

Hence the subsets $\{\operatorname{ad}^t_{[\tau_1 y_1+\ldots+\tau_l y_l,a]}(\tau_1 y_1+\ldots+\tau_l y_l)\mid(\tau_1,\ldots,\tau_l)\in H^l_t\}$,

$$\{\operatorname{ad}^t_{[v,a]}v\mid v\in V_0\}\ \text{ and }\ \{\operatorname{ad}^t_{[w,a]}w\mid w\in\overline{\Phi}V_0\}$$

span the same (finite-dimensional) $\overline{\Phi}$-subspace V_t of $\tilde{L}$. In particular, for each $s\ge 0$ there exists $v_1,\ldots v_{r_s}\in V_0$ such that $\operatorname{ad}^s_{[y,a]}y\in\sum_{j=1}^{r_s}U(v_j)$, and hence, by $(\star)$,

$$\operatorname{ad}^s_{[y,a]}y\in\sum_{j=1}^{r_s}U(v_j)\subseteq\sum_{j=1}^{r_s}\Big(\sum_{t=0}^{d-1}\overline{\Phi}\operatorname{ad}^t_{[v_j,a]}v_j\Big)=\sum_{t=0}^{d-1}\Big(\sum_{j=1}^{r_s}\overline{\Phi}\operatorname{ad}^t_{[v_j,a]}v_j\Big)\subseteq\sum_{t=0}^{d-1}V_t\,.$$

Since each V_t is a finite-dimensional $\overline{\Phi}$-subspace of $\tilde{L}$, the set $\{\operatorname{ad}^s_{[y,a]}y\mid s\ge 0\}$ is linearly dependent over $\overline{\Phi}$, as required. □

Proposition 5.2. *Let $\tilde{L}$ be a nondegenerate Lie algebra over an algebraically closed field $\overline{\Phi}$ of characteristic zero, and let L be a Φ-subalgebra $\tilde{L}$, where Φ is a subfield of $\overline{\Phi}$, such that L has an algebraic adjoint representation and $\tilde{L}$ is $\overline{\Phi}$-spanned by L. Suppose that one of the following two conditions holds:*

(i) $\Phi=\overline{\Phi}$ *and $L=\tilde{L}$ satisfies a polynomial identity.*

(ii) *The algebraic adjoint representation of L is of bounded degree.*

Then $\tilde{L}$ is a subdirect product of a family of finite-dimensional simple Lie algebras of bounded dimension. As a consequence, $\tilde{L}$ satisfies all the identities which hold in some finite-dimensional Lie algebra.

Proof. By Theorem 2.5, we may reduce the proof to the case that $\tilde{L}$ is strongly prime. Since in both cases, (i) and (ii), L is PI and $\tilde{L}$ is spanned by L, we have that $\tilde{L}$ satisfies a multilinear identity, say of degree n. Moreover, it follows from Corollary 2.3 that $\tilde{L}$ contains a nonzero Jordan element, say a. By Proposition 4.2, the Jordan algebra $\tilde{L}_a$ inherits primeness, nondegeneracy and the PI-condition from $\tilde{L}$. Moreover, $\tilde{L}_a$ is algebraic by Lemma 5.1. Using just the fact that $\tilde{L}_a$ is strongly prime and PI, it follows from [16, Theorems 5 and 7] that the centre $Z(\tilde{L}_a)$ of $\tilde{L}_a$ is nonzero and its central localization $Z(\tilde{L}_a)^{-1}L\tilde{L}_a$ is a simple unital Jordan algebra containing minimal inner ideal. Moreover, since $\tilde{L}_a$ is algebraic over $\overline{\Phi}$ and $\overline{\Phi}$ is algebraically closed, $Z(\tilde{L}_a)$ coincides with $\overline{\Phi}$ and $\tilde{L}_a$ is itself a simple unital Jordan algebra with minimal inner ideals. But minimal inner ideals of $\tilde{L}_a$ give rise to abelian minimal inner ideals of $\tilde{L}$ [4, (2.14)], so $\tilde{L}$ contains abelian minimal inner ideals and hence extremal elements by Proposition 4.3. Let e be an extremal element of $\tilde{L}$, and let S denote the ideal of $\tilde{L}$ generated by e. By [3, Proposition 1.15], S is a minimal ideal of $\tilde{L}$ which is simple as an algebra, and by [17, Lemma 15], S is locally finite-dimensional. Since the ground field is algebraically closed and of characteristic zero, and $\tilde{L}$ is PI, we have by [1, Theorem 2])(see also its proof) that S is actually finite-dimensional. Since no matrix algebra $M_r(\overline{\Phi})$ satisfies an identity of degree less than $2r$ and the Lie algebra $M_r(\overline{\Phi})^-$ can be embedded in each one of the simple Lie algebras A_r,B_r,C_r and D_r, S is isomorphic to one of the algebras G_2,F_4,E_6,E_7,E_8,A_r,

B_r, C_r, or D_r, where $r \leq [n/2]$. Finally, by primeness, $\tilde{L}$ can be embedded in $\mathrm{Der}(S)$ via the adjoint representation. Hence $\tilde{L} = S$ because every derivation of a simple finite-dimensional Lie algebra over a field of characteristic zero is inner. □

Corollary 5.3. *Let L be a Lie algebra with an algebraic adjoint representation over a field Φ of characteristic zero. Suppose that some of the following two conditions holds:*

(i) *Φ is algebraically closed and L satisfies a polynomial identity.*
(ii) *The algebraic adjoint representation of L is of bounded degree.*

Then L is locally finite-dimensional.

Proof. As in the proof of [17, Theorem 1], after factorizing by the largest locally finite-dimensional ideal (11), it suffices to prove that L contains a nonzero locally finite-dimensional ideal. Moreover, since $K_1(L)$ is locally nilpotent by Theorem 2.4 (in particular, locally finite-dimensional), we may suppose that L is nondegenerate.

Let $\overline{\Phi}$ be the algebraic closure of Φ and $\overline{L} = \overline{\Phi} \otimes_\Phi L$ ($\overline{\Phi} = \Phi$ and $\overline{L} = L$ if (i)), and set $\tilde{L} = \overline{L}/K(\overline{L})$. Now $K(L) = 0$ implies by [17, Proposition 2 and Corollary 1] that $L \cap K(\overline{L}) = 0$, so L can be regarded as a Φ-subalgebra of $\tilde{L}$. Since $\tilde{L}$ is $\overline{\Phi}$-spanned by L, it follows from Proposition 5.2 that L satisfies all the identities which hold in some finite-dimensional algebra. Hence, by [17, Lemma 7] (or [11, Theorem 5.4.6]), L is locally-finite dimensional. □

As a further illustration of our methods we give an alternative proof of following result.

Proposition 5.4. (Zelmanov) *Any Engel Lie PI-algebra L over a field of characteristic zero is locally nilpotent.*

Proof. Suppose that L is not locally nilpotent. As in Zelmanov's original proof, after factorizing L by its locally nilpotent radical we may assume that L is nonzero and nondegenerate. By Lemma 2.1, L contains a nonzero Jordan element a, and by Proposition 4.2, L_a is a nondegenerate Lie PI-algebra which is also nil. Then, by [15, Theorem 4], $L_a = Mc(L_a) = 0$, and hence $\mathrm{ad}_a^2 L = 0$, which is a contradiction since L is nondegenerate. □

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