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Discriminant algebras of finite rank algebras and quadratic trace modules

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Abstract Based on the construction of the discriminant algebra of an even-ranked quadratic form and Rost's method of shifting quadratic algebras, we give an explicit rational construction of the discriminant algebra of finite-rank algebras and, more generally, of quadratic trace modules, over arbitrary commutative rings. The discriminant algebra is a tensor functor with values in quadratic algebras, and a symmetric tensor functor with values in quadratic algebras with parity. The automorphism group of a separable quadratic trace module is a smooth, but in general not reductive, group scheme admitting a Dickson type homomorphism into the constant group scheme $\mathbf{Z}_2$.

Introduction

Consider an étale algebra E over a commutative ring k which is projective of rank r as a k-module. The discriminant of E is the bilinear form δ_E on $\bigwedge^r E$ given by

$$\delta_E(x_1 \wedge \cdots \wedge x_r,\, y_1 \wedge \cdots \wedge y_r) = \det\big(T(x_i y_j)\big),$$

where $T(x)$ denotes the trace of left multiplication $L(x)$ by x. A finer invariant is the discriminant algebra of E, a quadratic algebra for which various definitions have been proposed in the literature. E.g., Revoy [14] uses Galois theory while Waterhouse [16] gives a cohomological definition. For the case $r = 3$, Rost [15] constructs the discriminant algebra of E as a shift of the discriminant algebra of a suitable quadratic form. In [5], Deligne sketches an approach which uses sophisticated algebraic-geometric methods and is quite different from the more elementary one presented here.

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The present paper combines Rost's idea and the theory developed in [12] to give a new construction of the discriminant algebra offering the following features:

- It is rational over the base ring k in the sense that no extensions of k are required.
- It is constructive: If $E/k\cdot 1$ is free as a k-module then the discriminant algebra is a free quadratic algebra $k[\mathbf{t}]/(\mathbf{t}^2 - b\,\mathbf{t} + c)$, and we give explicit formulae for the coefficients b, c as polynomials in the structure constants of E.
- It works in greater generality: The assumption that E be étale is superfluous; in fact, E need not even be an algebra. Our construction makes sense in the following more general situation:

It is a simple but crucial observation that the discriminant of E (and, as it turns out, the discriminant algebra as well) depends only on the unit element, the trace and the quadratic trace, i.e., the quadratic form $Q(x) = \operatorname{trace}\bigwedge^2 L(x)$. Abstracting from their properties, we define a *quadratic trace module of rank* $r \geqslant 1$ as a quadruple $\mathfrak{X} = (X, Q, T, 1)$ consisting of a projective k-module X of rank r, a linear and a quadratic form T and Q on X and a unimodular vector $1 \in X$ satisfying

$$T(1) = r, \quad Q(1) = \binom{r}{2}, \quad B(1,x) = (r-1)T(x)$$

for all $x \in X$, where B is the polar form of Q. The zero module is considered as a quadratic trace module as well. Not all quadratic trace modules arise from an algebra, as soon as $r \geqslant 3$.

We construct a discriminant algebra $\operatorname{Dis}(\mathfrak{X})$ for such $\mathfrak{X}$ as follows. Consider the bilinear form $\Delta_{\mathfrak{X}}(x,y) = T(x)T(y) - B(x,y)$ on X. Put $\delta_{\mathfrak{X}} = \bigwedge^r \Delta_{\mathfrak{X}}$ and note that $\delta_E = \delta_{\mathfrak{X}}$ in the algebra case. First assume $r = 2n$ even. Then $\operatorname{Dis}(\mathfrak{X})$ is defined as the shift of the discriminant algebra $\mathfrak{D}(Q)$ of Q by $(-1)^{n-1}\lfloor n/2 \rfloor\, \delta_{\mathfrak{X}}$ (this choice of shift comes from the requirement that the discriminant of $\operatorname{Dis}(\mathfrak{X})$ should be $\delta_{\mathfrak{X}}$). If $r = 2n+1$ is odd, the discriminant algebra $\mathfrak{D}(Q)$ is a graded quadratic algebra of odd type which can only be separable if 2 is a unit in k. On the other hand, quadratic trace modules admit natural direct sums, so we define $\operatorname{Dis}(\mathfrak{X}) = \operatorname{Dis}(\mathfrak{E}_1 \oplus \mathfrak{X})$ where $\mathfrak{E}_1 = (k, 0, \mathrm{Id}_k, 1_k)$ is the unique quadratic trace module of rank 1. We also give an alternative construction of $\operatorname{Dis}(\mathfrak{X})$ in the odd rank case as a shift of the discriminant algebra of a suitable quadratic form on $X/k\cdot 1$, which generalizes Rost's definition in the rank three case (Theorem 3.8).

Quadratic trace modules form a symmetric tensor category $\mathbf{qtm}_k$ with the direct sum as the product operation. Likewise, quadratic algebras admit a natural product $\square$ with which they are a symmetric tensor category $\mathbf{qa}_k$. We show in Theorem 6.5 that the discriminant algebra functor is multiplicative:

$$\operatorname{Dis}(\mathfrak{X}_1 \oplus \mathfrak{X}_2) \cong \operatorname{Dis}(\mathfrak{X}_1) \,\square\, \operatorname{Dis}(\mathfrak{X}_2),$$

and in Theorem 6.6 that it is in fact a tensor functor. However, Dis is not a symmetric tensor functor, i.e., it does not commute with the symmetries of $\mathbf{qtm}_k$ and $\mathbf{qa}_k$, as foreseen by Deligne [5]. To remedy this defect, one must keep track of the parity of the rank of $\mathfrak{X}$ when passing to the discriminant algebra. (For the discriminant algebra $\mathfrak{D}(q)$ of a quadratic module (M,q) this is automatic because $\mathfrak{D}(q)$ is a graded algebra of even or odd type depending on the parity of the rank of M). We are thus led to introduce the category $\widetilde{\mathbf{qa}}_k$ of *quadratic algebras with parity* whose objects are pairs (D,p) consisting of a quadratic k-algebra D and an idempotent $p \in k$. They, too, form a symmetric tensor category, and the extended functor

$$\widetilde{\mathrm{Dis}}(\mathfrak{X}) = (\mathrm{Dis}(\mathfrak{X}), \mathrm{rk}(\mathfrak{X}) \pmod 2))$$

is a symmetric tensor functor from $\mathbf{qtm}_k$ to $\widetilde{\mathbf{qa}}_k$ (Theorem 7.7).

A quadratic trace module is called *separable* if $\Delta_{\mathfrak{X}}$ is nonsingular. This is the case if and only if there exists a faithfully flat and étale k-algebra R such that $\mathfrak{X} \otimes R$ is isomorphic to the split quadratic trace module of rank r (Theorem 8.8). In the last two sections we study the automorphism group $\mathbf{G}$ of a separable quadratic trace module and show first that it is a smooth group scheme of fibre dimension $\binom{r-1}{2}$ (Theorem 9.3), which admits a Dickson type homomorphism into the constant group scheme $\mathbf{Z}_2$ (Theorem 9.7). As an application, we show in 9.10 that our construction, when applied to an étale algebra, yields a concrete realization of Waterhouse's abstract approach. The centre of $\mathbf{G}$ is determined in Theorem 10.5; it is an open subgroup scheme of $\mathbf{Z}_2$ resp. $\boldsymbol{\mu}_2$, depending on the parity of r. Finally, we study the restriction homomorphism from $\mathbf{G}$ to the orthogonal group of the quadratic form induced by Q on the submodule of trace zero elements (Theorems 10.8 and 10.9) and obtain necessary and sufficient conditions for $\mathbf{G}$ to be reductive.

1. Basics

1.1. Definition. We work over an arbitrary commutative ring k and denote the category of commutative associative unital k-algebras by k-**alg**. Unadorned tensor products are taken over k.

A *quadratic trace module of rank* $r \geqslant 1$ over k is a quadruple $\mathfrak{X} = (X, Q, T, 1)$ consisting of a finitely generated and projective k-module X of rank r, a quadratic form Q with polar form B, a linear form T, called the *trace*, and a unimodular vector $1_X = 1 \in X$, the *unit element* or *base point*, satisfying the conditions

$$T(1) = r, \quad Q(1) = \binom{r}{2}, \quad B(1,x) = (r-1)T(x) \tag{1}$$

for all $x \in X$. The zero module, with the only possible choices of Q, T and 1, is also considered as a quadratic trace module. Morphisms between quadratic trace modules of the same rank are k-linear maps preserving quadratic forms, trace forms and base points. We do not allow morphisms between quadratic trace modules of different rank.

It is also possible to consider quadratic trace modules of variable rank. Then $r = \mathrm{rk}(X)\colon \mathrm{Spec}(k) \to \mathbb{N}$ is a locally constant function, and (1) has to be interpreted in an obvious way. However, by decomposing the base ring according to the values of r, it is no great restriction to assume r constant. The category of quadratic trace modules over k is denoted $\mathbf{qtm}_k$.

We let

$$\dot{X} := X/k \cdot 1 \quad \text{and} \qquad x \mapsto \dot{x}$$

denote the quotient of X by $k\cdot 1$ and the canonical map $X\to\dot X$. For $r\geqslant 1$ there is a canonical isomorphism

$$\bigwedge^{r-1}\dot X \xrightarrow{\cong} \bigwedge^{r} X, \tag{2}$$

given by $\dot x_1\wedge\cdots\wedge\dot x_{r-1} \mapsto 1\wedge x_1\wedge\cdots\wedge x_{r-1}$.

The *discriminant form* of $\mathfrak{X}$ is the symmetric bilinear form $\Delta=\Delta_{\mathfrak{X}}$ on X given by

$$\Delta(x,y) := T(x)T(y) - B(x,y). \tag{3}$$

Note that

$$\Delta(x,1) = rT(x) - (r-1)T(x) = T(x). \tag{4}$$

1.2. Special cases. (a) The *split quadratic trace module of rank r over k* is $\mathfrak{E}_r := (k^r, Q_r, T_r, 1_r)$ where $k^r=\bigoplus_{i=1}^r k\cdot e_i$ in the standard basis, $1_r = e_1+\cdots+e_r$, and T_r and Q_r are the first and second elementary symmetric polynomials in r variables:

$$T\Big(\sum_{i=1}^r x_i e_i\Big) = \sum_{i=1}^r x_i, \qquad Q\Big(\sum_{i=1}^r x_i e_i\Big) = \sum_{1\leqslant i<j\leqslant r} x_i x_j.$$

Here $\Delta(e_i,e_j)=\delta_{ij}$ so Δ is the standard scalar product on k^r.

(b) The only quadratic trace modules of rank 0 resp. 1 are $\mathfrak{E}_0 = (\{0\},0,0,0)$ and $\mathfrak{E}_1 = (k,0,\mathrm{Id}_k,1)$.

(c) Let $\mathfrak{X}$ be a quadratic trace module of rank 2. Then 1.1.1 shows that $\mathfrak{X}$ is entirely determined by X, Q and 1. Hence the quadratic trace modules of rank 2 are precisely the unital quadratic forms of rank 2 as in [11].

1.3. Algebras. Let A be a k-algebra with multiplication $xy=L_x(y)$, which is finitely generated and projective of rank r as a k-module, and which has a left unit element 1_A. We make no assumptions on associativity or commutativity of A. Then A determines a quadratic trace module

$$\mathfrak{X} = \mathrm{qt}(A) = (A,Q,T,1_A) \quad \text{where} \quad T(x)=\mathrm{tr}(L_x), \quad Q(x)=\mathrm{qtr}(L_x). \tag{1}$$

Here $\mathrm{qtr}(f)=\mathrm{tr}(\bigwedge^2 f)$ is the trace of the second exterior power of an endomorphism f of A. This may also be expressed by saying that $T(x)$ and $Q(x)$ are the coefficients of $\mathbf{t}$ and $\mathbf{t}^2$ in the polynomial $\det(\mathrm{Id}+\mathbf{t}L_x)$.

If A is associative and 1_A is the (two-sided) unit element of A, then

$$\Delta(x,y) = T(xy) \tag{2}$$

which follows from associativity and the well-known relation $\mathrm{tr}(f)\mathrm{tr}(g) = \mathrm{tr}(f\circ g) + \mathrm{qtr}(f,g)$ for the trace and quadratic trace of endomorphisms. Here $\mathrm{qtr}(f,g)$ denotes the polar form of the quadratic form $\mathrm{qtr}(f)$.

Not every quadratic trace module comes from an associative algebra via (1) unless $r\leqslant 2$, see below. Indeed, (2) says that the discriminant form must factor via T. Using this fact, it is easy to give examples of quadratic trace modules of rank $\geqslant$3 which are not obtained from an associative algebra. Also, $\mathrm{qt}(A)$ does not depend functorially on A because a homomorphism of algebras (even of the same rank) in general does not respect the trace and quadratic trace forms.

1.4. Quadratic algebras. Suppose $\mathfrak{X} = (X, Q, T, 1)$ is a quadratic trace module of rank 2. By the proof of [11, Prop. 1.6], there is a unique algebra structure D on X such that $\mathrm{qt}(D) = \mathfrak{X}$. Then Q and T are just the usual norm and trace of D. This yields a functor F from quadratic trace modules of rank 2 to quadratic algebras, i.e., unital algebras which are finitely generated and projective of rank 2 as k-modules. Such algebras are automatically associative and commutative. However, F is not an isomorphism of categories (contrary to the erroneous statement of [11, Prop. 1.6]), because algebra homomorphisms between quadratic algebras need not preserve norms and traces. We therefore introduce the category $\mathbf{qa}_k$ whose objects are quadratic k-algebras and whose morphisms are those algebra homomorphisms $D \to D'$ which preserve norms and traces; equivalently, which commute with the standard involutions of D and D'. Then the assignment $D \mapsto \mathrm{qt}(D)$ is an isomorphism between $\mathbf{qa}_k$ and the category of quadratic trace modules of rank 2, with inverse F.

1.5. Direct sums. The *direct sum* of quadratic trace modules $\mathfrak{X}$ and $\mathfrak{X}'$ is $\mathfrak{X}'' = (X \oplus X', Q'', T'', 1 \oplus 1')$ where

$$T''(x \oplus x') = T(x) + T'(x'), \quad Q''(x \oplus x') = Q(x) + Q'(x') + T(x)T'(x'). \tag{1}$$

Thus the quadratic form Q'' is not simply the orthogonal sum of Q and Q' but nearly so, because the difference between Q'' and $Q \perp Q'$ is just the product of two linear forms. The properties 1.1.1 for Q'' are easily verified. It is also straightforward to check that with the direct sum operation, $\mathbf{qtm}_k$ becomes a symmetric tensor category, with neutral object $\mathfrak{E}_0$ and the interchange of factors $\omega\colon X \oplus X' \to X' \oplus X$ as symmetry.

Direct sums commute with the assignment $A \mapsto \mathrm{qt}(A)$ described in 1.3, and from 1.1.3 one sees that the discriminant form satisfies

$$\Delta_{\mathfrak{X} \oplus \mathfrak{X}'} = \Delta_{\mathfrak{X}} \perp \Delta_{\mathfrak{X}'}, \tag{2}$$

the usual orthogonal sum of bilinear forms. The split quadratic trace module $\mathfrak{E}_r$ is just the direct sum of r copies of $\mathfrak{E}_1$.

1.6. Tensor products. The *tensor product* of quadratic trace modules $\mathfrak{X}$ and $\mathfrak{X}'$ is $\mathfrak{X}'' = (X \otimes X', Q'', T'', 1 \otimes 1')$ where

$$T'' = T \otimes T', \quad Q'' = T^{(2)} \otimes Q' + Q \otimes T'^{(2)} - Q \otimes Q'. \tag{1}$$

Here $T \otimes T'$ is the linear form $x \otimes x' \mapsto T(x)T'(x')$ on $X \otimes X'$, and $T^{(2)}$ the bilinear form on X given by $T^{(2)}(x,y) = T(x)T(y)$. Tensor products between bilinear forms and quadratic forms are defined as usual, see, e.g., [13] or [11, 2.1]. Again, tensor products are compatible with the assignment $A \mapsto \mathrm{qt}(A)$ of 1.3.

1.7. Remarks. If $r \in k^\times$ then X decomposes $X = k \cdot 1 \oplus \operatorname{Ker} T$ and $Q = \langle \binom{r}{2} \rangle \perp (Q \big| \operatorname{Ker} T)$. Thus in this case the category of quadratic trace modules of rank r is equivalent to the category of quadratic modules of rank $r-1$. If $r-1 \in k^\times$ then $T(x) = (r-1)^{-1} B(1,x)$ is determined by Q, and the category of quadratic trace modules of rank r is equivalent to the category of quadratic modules of rank r with a unimodular base point 1 which satisfies $Q(1) = \binom{r}{2}$. — In general, however, it does not seem possible to base the theory of quadratic trace modules on the quadratic form Q alone.

2. Discriminants

2.1. Definition. Let $\mathfrak{X} = (X, Q, T, 1)$ be a quadratic trace module of rank r. The *discriminant* of $\mathfrak{X}$ is the bilinear form

$$\delta_{\mathfrak{X}} := \bigwedge^r \Delta_{\mathfrak{X}} \tag{1}$$

on $\bigwedge^r X$, where $\Delta_{\mathfrak{X}}$ is the discriminant form of 1.1.3. For $r \leqslant 1$, we have $\bigwedge^r X = k$ and δ_X is just multiplication in k. If $\mathfrak{X} = \mathrm{qt}(A)$ comes from an associative algebra A as in 1.3, then it is clear from 1.3.2 that $\delta_{\mathfrak{X}} = \delta_A$, the usual discriminant of A, defined by

$$\delta_A(x_1 \wedge \cdots \wedge x_r, y_1 \wedge \cdots \wedge y_r) = \det\big(T(x_i y_j)\big). \tag{2}$$

We also note that the discriminant is multiplicative with respect to direct sums:

$$\delta_{\mathfrak{X} \oplus \mathfrak{X}'} = \delta_{\mathfrak{X}} \otimes \delta_{\mathfrak{X}'} \tag{3}$$

(tensor product of bilinear forms) after identifying $(\bigwedge^r X) \otimes (\bigwedge^{r'} X')$ and $\bigwedge^{r+r'}(X \oplus X')$ by $\xi \otimes \eta \mapsto \xi \wedge \eta$. This follows easily from 1.5.2.

We next express the (signed) discriminant δ_Q of Q in terms of $\delta_{\mathfrak{X}}$. The transpose of a matrix A with entries in k is denoted $A^\top$.

2.2. Lemma. *Let $\mathfrak{X}$ be a quadratic trace module of rank $r = m+1$ and let $x_1, \ldots, x_m \in X$. We put $\xi = 1 \wedge x_1 \wedge \cdots \wedge x_m$, $v = \big(T(x_1), \ldots, T(x_m)\big) \in k^m$ (row vector) and $D = \big(B(x_i, x_j)\big) \in \mathrm{Mat}_m(k)$. Then*

$$\delta_{\mathfrak{X}}(\xi, \xi) = \det\begin{pmatrix} r & v \\ v^\top & v^\top v - D \end{pmatrix} = (-1)^m \cdot \det\begin{pmatrix} r & v \\ m v^\top & D \end{pmatrix}. \tag{1}$$

If $r = 2n$ is even the discriminant of Q is given by

$$\delta_Q = (-1)^{n-1}(r-1)\,\delta_{\mathfrak{X}} = \big\{1 + 4 \cdot (-1)^{n-1} \lfloor n/2 \rfloor\big\}\,\delta_{\mathfrak{X}} \tag{2}$$

while it is

$$\delta_Q = (-1)^n n\,\delta_{\mathfrak{X}} \tag{3}$$

if $r = 2n+1$ is odd.

Remark. With the convention that the discriminant of the zero quadratic form on the zero module is just ordinary multiplication on k, formula (2) holds also for $r = 0$.

Proof. The first equation of (1) is immediate from the definitions. For the second, multiply the first row formally by $v^\top$ and subtract from the second row. This yields

$$\det\begin{pmatrix} r & v \\ v^\top & v^\top v - D \end{pmatrix} = \det\begin{pmatrix} r & v \\ -m v^\top & -D \end{pmatrix} = (-1)^m \cdot \det\begin{pmatrix} r & v \\ m v^\top & D \end{pmatrix}.$$

If $r=2n$ is even, δ_Q is $(-1)^n$ times the $2n$-th exterior power of the polar form B of Q. By 1.1.1, $B(1,1)=2\binom{r}{2}=rm$ and $B(1,x_i)=mT(x_i)$. Hence,

$$\delta_Q(\xi,\xi)=(-1)^n\det\begin{pmatrix} rm & mv \\ mv^\top & D\end{pmatrix}=m\,(-1)^n\det\begin{pmatrix} r & v \\ mv^\top & D\end{pmatrix}.$$

Since $(-1)^m=(-1)^{2n-1}=-1$, we have the first formula of (2), and the second follows from the observation that

$$(-1)^{n-1}(2n-1)=1+4\cdot(-1)^{n-1}\lfloor n/2\rfloor. \tag{4}$$

Next let $r=2n+1$ be odd and let U be the upper triangular matrix with entries $u_{ii}=Q(x_i)$ and $u_{ij}=B(x_i,x_j)$. Then $U+U^\top=D$ so by 11.3.5 and (1),

$$\delta_Q(\xi,\xi)=(-1)^n\mathrm{hdet}\begin{pmatrix} rn & 2nv \\ 0 & U\end{pmatrix}=(-1)^n\det\begin{pmatrix} rn & nv \\ mv^\top & D\end{pmatrix}=(-1)^n\,n\,\delta_{\mathfrak{X}}(\xi,\xi),$$

because now $(-1)^m=(-1)^{2n}=1$.

2.3. Lemma. *Let $\mathfrak{X}$ be of odd rank $r=2n+1$. There is a well-defined quadratic form $\dot{Q}$ on $\dot{X}$ given by*

$$\dot{Q}(\dot{x})=nT(x)^2-rQ(x)=n\Delta_{\mathfrak{X}}(x,x)-Q(x), \tag{1}$$

for all $x\in X$. The polar form $\dot{B}$ of $\dot{Q}$ is

$$\dot{B}(\dot{x},\dot{y})=2nT(x)T(y)-rB(x,y)=2n\Delta_{\mathfrak{X}}(x,y)-B(x,y). \tag{2}$$

Define $\varpi(n)$ by the equation

$$(-1)^n\,(2n+1)^{2n-1}=1+4\cdot(-1)^n\,\varpi(n). \tag{3}$$

Then $\varpi(n)\in\mathbb{N}$, and the discriminant of $\dot{Q}$ is given by

$$\delta_{\dot{Q}}=(-1)^n\,r^{r-2}\,\delta_{\mathfrak{X}}=\big\{1+4\cdot(-1)^n\,\varpi(n)\big\}\delta_{\mathfrak{X}}, \tag{4}$$

where we identify $\bigwedge^r X$ and $\bigwedge^{r-1}\dot{X}$ as in 1.1.2.

Proof. It follows easily from 1.1.1 that $\dot{Q}$ is a well-defined quadratic form on $\dot{X}$, and (2) is immediate from 1.1.3. It is elementary to check that $\varpi(n)\in\mathbb{N}$.

For the proof of (4) let $\dot{x}_1,\ldots\dot{x}_m\in\dot{X}$ where $m=r-1=2n$ and put $\xi=1\wedge x_1\wedge\cdots\wedge x_m$ and $\eta=\dot{x}_1\wedge\cdots\wedge\dot{x}_m$. Then, with v and D as in Lemma 2.2,

$$\begin{aligned}\delta_{\dot{Q}}(\eta,\eta)&=(-1)^n\det\big(-(rD-2nv^\top v)\big) && \text{(by (2))}\\ &=(-1)^{n+2n}r^{m-1}\det\begin{pmatrix} r & v \\ mv^\top & D\end{pmatrix} && \text{(by 11.3.3)}\\ &=(-1)^n r^{r-2}(-1)^m\delta_{\mathfrak{X}}(\xi,\xi). && \text{(by 2.2.1)}\end{aligned}$$

This is the asserted formula (4) since $(-1)^m=1$.

2.4. Restriction to and extension from complements of 1**.** Let $\mathfrak{X} = (X, Q, T, 1)$ be a quadratic trace module of rank $r \geqslant 1$ and fix a decomposition $X = k \cdot 1 \oplus M$ (which always exists because 1 is a unimodular vector). Let

$$q := Q\big|M, \qquad t := T\big|M. \tag{1}$$

Then Q and T can be reconstructed from q and t by the formulas

$$Q(\lambda 1 \oplus x) = \lambda^2 \binom{r}{2} + \lambda(r-1)t(x) + q(x), \tag{2}$$

$$T(\lambda 1 \oplus x) = \lambda r + t(x). \tag{3}$$

Conversely, it easy to see that, given a quadratic form q and a linear form t on M, these formulas determine a quadratic trace module $(X, Q, T, 1)$. Thus it must be possible to express invariants of $\mathfrak{X}$ by means of (q, t). We do this later for the discriminant $\delta_{\mathfrak{X}}$ (5.2) and the discriminant algebra $\mathrm{Dis}(\mathfrak{X})$ (5.3, 5.4). Note, however, that (q, t) depend on the choice of complement M. Putting this on a more formal basis amounts to a systematic study of the splittings of the exact sequence $0 \longrightarrow k \longrightarrow X \xrightarrow{\text{can}} \dot{X} \longrightarrow 0$, equivalently, of linear forms α on X with $\alpha(1) = 1$ (unital linear forms), as was done in [11] for unital quadratic forms. It is possible to develop the theory of the discriminant algebra in this way, but the proof of independence of the choice of splitting becomes rather complicated. Nevertheless, this approach will lead to effective computations of $\mathrm{Dis}(\mathfrak{X})$ in section 5.

The following easily established lemma will be useful to reduce proofs to characteristic zero:

2.5. Lemma. *Let $\mathfrak{X}$ be a quadratic trace module with $\dot{X}$ free, say with basis $\dot{x}_1, \ldots, \dot{x}_m$ where $m = r - 1$. Then also X is free with basis $1_X, x_1, \ldots, x_m$. Consider the polynomial ring $R = \mathbb{Z}[\mathbf{t}_i, \mathbf{a}_{ij} : 1 \leqslant i \leqslant j \leqslant m]$ and the quadruple $\mathfrak{X}' := (X', Q', T', 1')$ where X' is the free R-module with basis $1', x'_1, \ldots, x'_m$ and Q' and T' are the quadratic and linear form given by $Q'(1') = \binom{r}{2}$, $T'(1') = r$, and*

$$Q'(x'_i) = \mathbf{a}_{ii}, \quad B'(x'_i, x'_j) = \mathbf{a}_{ij}\ (i < j), \quad B'(1', x'_i) = (r-1)\mathbf{t}_i, \quad T'(x'_i) = \mathbf{t}_i.$$

Then $\mathfrak{X}'$ is a quadratic trace module by 2.4*, and the ring homomorphism $R \to k$ mapping $\mathbf{t}_i \mapsto T(x_i)$, $\mathbf{a}_{ii} \mapsto Q(x_i)$, $\mathbf{a}_{ij} \mapsto B(x_i, x_j)$ $(i < j)$ induces an isomorphism*

$$\mathfrak{X}' \otimes_R k \xrightarrow{\cong} \mathfrak{X}$$

of quadratic trace modules.

3. The discriminant algebra

As noted in 1.4, quadratic algebras (with morphisms respecting the involutions) are the same as quadratic trace modules of rank 2. Let D be a quadratic k-algebra, with unit $1 = 1_D$, trace T_D, involution $\sigma_D(x) = -x + T_D(x) \cdot 1$ and norm (=quadratic trace) N_D. We denote the canonical map $p \colon D \to \dot{D} = D/k \cdot 1$ by $x \mapsto \dot{x}$. The construction in (a) of the following lemma is due to Rost [15].

3.1. Lemma. (a) *Let ε be a bilinear form on $\dot{D}$. Then the k-module D becomes a new quadratic algebra with the same unit element, but with multiplication*

$$x * y = xy - \varepsilon(\dot{x}, \dot{y}) \cdot 1, \tag{1}$$

called the shift ("Verschiebung") *of D with respect to ε and denoted by*

$$D + \varepsilon.$$

Obviously,

$$(D + \varepsilon_1) + \varepsilon_2 = D + (\varepsilon_1 + \varepsilon_2). \tag{2}$$

The involution and the trace and norm forms of $D + \varepsilon$ are

$$\sigma_{D+\varepsilon} = \sigma_D, \qquad T_{D+\varepsilon} = T_D, \qquad N_{D+\varepsilon}(x) = N_D(x) + \varepsilon(\dot{x}, \dot{x}). \tag{3}$$

The discriminant of $D + \varepsilon$ is

$$\delta_{D+\varepsilon} = \delta_D - 4\varepsilon. \tag{4}$$

(b) *Conversely, let D and D' be quadratic algebras with the same underlying k-module, unit element and trace. Then D' is a shift of D.*

(c) *Suppose $\psi\colon D \to D'$ is a morphism of quadratic algebras and ε and ε' are bilinear forms on $\dot{D}$ and $\dot{D}'$, respectively. If the induced map $\dot{\psi}\colon \dot{D} \to \dot{D}'$ satisfies $\varepsilon' \circ (\dot{\psi} \times \dot{\psi}) = \varepsilon$, then $\psi\colon D + \varepsilon \to D' + \varepsilon'$ is again a morphism of quadratic algebras.*

Proof. (a) It is clear that (1) defines the structure of a quadratic algebra D' on D with unit $1_{D'} = 1_D$. Since

$$x * x = x^2 - \varepsilon(\dot{x}, \dot{x}) \cdot 1 = T_D(x)x - \big(N_D(x) + \varepsilon(\dot{x}, \dot{x})\big) \cdot 1 = T_{D'}(x)x - N_{D'}(x) \cdot 1,$$

we have (3). In (4), we identify $\bigwedge^1 \dot{D} = \dot{D} \cong \bigwedge^2 D$ via $\dot{x} \;\mapsto\; 1 \wedge x$ and thus consider the discriminant as a bilinear form on $\dot{D}$. Then

$$\delta_D(\dot{x}, \dot{y}) = \begin{vmatrix} 2 & T(x) \\ T(y) & T(xy) \end{vmatrix},$$

so

$$\delta_{D+\varepsilon}(\dot{x}, \dot{y}) = \begin{vmatrix} 2 & T(x) \\ T(y) & T(xy) - 2\varepsilon(\dot{x}, \dot{y}) \end{vmatrix} = \delta_D(\dot{x}, \dot{y}) - 4\varepsilon(\dot{x}, \dot{y}).$$

(b) Denoting the multiplication in D and D' by xy and $x * y$, respectively, $xy - x * y$ depends only on $\dot{x}$ and $\dot{y}$, because D and D' have the same unit element. Thus $\beta(\dot{x}, \dot{y}) := p(xy - x * y)$ is a bilinear form on $\dot{D}$. Since D and D' have the same trace, it follows that $\beta(\dot{x}, \dot{x}) = p(x^2 - x * x) = p\big((N'(x) - N(x)) \cdot 1\big) = 0$. Hence β is an alternating form on the rank one module $\dot{D}$ and therefore vanishes. It follows that $xy - x * y = \varepsilon(\dot{x}, \dot{y}) \cdot 1$ is a multiple of 1.

(c) This is immediate from the definitions.

3.2. Free quadratic algebras. Let D be a quadratic algebra whose underlying k-module is free. Then there exists a basis of the form $\{1,z\}$ of D [8, p. 14, Exercise 3], so $z^2 = bz - c1$ where $b,c \in k$, or $D \cong k[\mathbf{t}]/(\mathbf{t}^2 - b\mathbf{t} + c)$. We write this as

$$D = (\!(b : c]\!].$$

Note that the algebra D does not determine b and c uniquely; rather, we have

$$(\!(b : c]\!] \cong (\!(b' : c']\!] \quad \Longleftrightarrow \quad b' = \mu b + 2\lambda, \quad c' = \mu^2 c + \lambda\mu b + \lambda^2,$$

for some $\lambda \in k$, $\mu \in k^\times$. This corresponds to changing the basis of D to 1 and $z' = \lambda 1 + \mu z$.

The split quadratic algebra is $I := (\!(1 : 0]\!]$, often identified with $k \times k$ by mapping z to the first standard basis vector e_1 of k^2. The algebra of dual numbers is $(\!(0 : 0]\!]$. The discriminant of $(\!(b : c]\!]$ is

$$\delta_{(\!(b:c]\!]} = b^2 - 4c. \tag{1}$$

If $D = (\!(b : c]\!]$ is a free quadratic algebra, we identify $\dot{D} = D/k \cdot 1$ canonically with k via $\lambda \in k \mapsto \lambda\dot{z} \in \dot{D}$. Then a bilinear form ε on $\dot{D}$ is just a scalar $e \in k$, and the shift of D by e is

$$(\!(b : c]\!] + e = (\!(b : c + e]\!]. \tag{2}$$

3.3. The discriminant algebra of a quadratic form. We recall from [12] the construction of the discriminant algebra $\mathfrak{D}(q)$ of a quadratic module (M,q) of even rank $2n$.

Let first M be free with basis $x_1,\ldots,x_{2n}$, and let A be a $2n \times 2n$-matrix such that $a_{ii} = q(x_i)$ and $a_{ij} + a_{ji} = b(x_i,x_j)$ where b is the polar form of q. Then $\mathfrak{D}(q)$ is (isomorphic to) the free quadratic algebra

$$\mathfrak{D}(q) \;\cong\; (\!(\mathrm{Pf}(A - A^\top) : (-1)^{n+1}\,\mathrm{qdet}(A)]\!]$$

where Pf denotes the Pfaffian and qdet the quarter-determinant, cf. 11.1. A more intrinsic construction which works for arbitrary M goes as follows.

Let a,a' be alternating bilinear forms on M. The n-th *Pfaffian power* of a is the linear form $\pi_n(a)$ on $L := \bigwedge^{2n} M$ defined by

$$\pi_n(a)(\xi) = \mathrm{Pf}\big(a(x_i,x_j)\big), \tag{1}$$

where $\xi = x_1 \wedge \cdots \wedge x_{2n} \in L$. Let $\mathbf{t}$ be an indeterminate and define $\Pi_n(\mathbf{t},a,a')$ by

$$\pi_n(a + \mathbf{t}a') = \pi_n(a) + \mathbf{t}\,\Pi_n(\mathbf{t}, a, a'). \tag{2}$$

A *representative* of q is a bilinear form f such that $f(x,x) = q(x)$ for all $x \in M$, which we also express as $q = [f]$, thus identifying quadratic forms with equivalence classes of bilinear forms modulo alternating forms. For representatives f,g of q define linear forms on L by

$$\tau_f := \pi_n(f - f^\top), \qquad \kappa_{fg} := \Pi_n(-2,\, f - f^\top,\, f - g), \tag{3}$$

where $f^{\top}(x,y) = f(y,x)$. Then

$$2\kappa_{fg} = \tau_f - \tau_g, \qquad \kappa_{fg} + \kappa_{gh} = \kappa_{fh}. \tag{4}$$

There is a unique bilinear form γ_f on L satisfying

$$\gamma_f(\xi,\xi) = (-1)^{n+1}\,\mathrm{qdet}\big(f(x_i,x_j)\big), \tag{5}$$

where qdet is the quarter-determinant, see 11.1. Now $D := \mathfrak{D}(q)$ is, as a k-module, generated by 1 and symbols $s_f(\xi)$, linear in $\xi \in L$, subject to the relations

$$s_f(\xi) - s_g(\xi) = \kappa_{fg}(\xi)\cdot 1, \tag{6}$$

where f and g run over all representatives of q. There is an exact sequence

$$0 \longrightarrow k \xrightarrow{\ i\ } D \xrightarrow{\ p\ } L \longrightarrow 0$$

where $p(s_f(\xi)) = \xi$. Trace and norm, and hence the algebra structure of D, are determined by

$$T_D(s_f(\xi)) = \tau_f(\xi), \qquad N_D(s_f(\xi)) = \gamma_f(\xi,\xi). \tag{7}$$

3.4. Definition. Let $\mathfrak{X}$ be a quadratic trace module of rank r. If $r = 2n$, the discriminant algebra of $\mathfrak{X}$ is the shift

$$\mathrm{Dis}(\mathfrak{X}) := \mathfrak{D}(Q) + (-1)^{n-1}\lfloor n/2\rfloor\cdot\delta_{\mathfrak{X}} \quad (r = 2n), \tag{1}$$

where $\lfloor n/2\rfloor$ is the integer part of $n/2$. If $r = 2n+1$ is odd, it would not do to define $\mathrm{Dis}(\mathfrak{X})$ as a shift of the discriminant algebra of Q, because this would yield a graded quadratic algebra of odd type which cannot be separable unless 2 is a unit of k. Therefore, we define

$$\mathrm{Dis}(\mathfrak{X}) := \mathrm{Dis}(\mathfrak{E}_1 \oplus \mathfrak{X}) \quad (r = 2n+1), \tag{2}$$

cf. 1.2(b) and 1.5. — Let A be an associative commutative k-algebra which is projective of rank r as a k-module. Then we define the discriminant algebra of A as the discriminant algebra of the associated quadratic trace module $\mathrm{qt}(A)$, thus

$$\mathrm{Dis}(A) := \mathrm{Dis}(\mathrm{qt}(A)). \tag{3}$$

Clearly, $\mathrm{Dis}(\mathfrak{X})$ is compatible with arbitrary base change because this is so for the discriminant algebra of a quadratic form. It depends functorially on $\mathfrak{X}$ with respect to morphisms of quadratic trace modules. Indeed, consider first the even rank case. A morphism $\varphi\colon \mathfrak{X}' \to \mathfrak{X}$ of quadratic trace modules is in particular a similitude between the quadratic forms Q' and Q. By [12, Th. 2.3(b)], we have an induced homomorphism $\mathfrak{D}(\varphi)\colon \mathfrak{D}(Q') \to \mathfrak{D}(Q)$, given by $1 \mapsto 1$ and

$$s_{\varphi^*(f)}(\xi) \;\mapsto\; s_f\Big(\Big(\bigwedge^{r}\varphi\Big)(\xi)\Big), \tag{4}$$

for all representatives f of Q and $\xi \in \bigwedge^r X'$. Here $\varphi^*(f) = f\circ(\varphi\times\varphi)$ is the pullback of f to X'. The discriminant forms Δ' and Δ of $\mathfrak{X}'$ and $\mathfrak{X}$ are related by $\varphi^*(\Delta) = \Delta'$, whence $\delta_{\mathfrak{X}}\circ(\bigwedge^r\varphi\times\bigwedge^r\varphi) = \delta_{\mathfrak{X}'}$. By 3.1(c), the module homomorphism $\mathfrak{D}(\varphi)$ is in fact a morphism $\mathrm{Dis}(\varphi) : \mathrm{Dis}(\mathfrak{X}') \to \mathrm{Dis}(\mathfrak{X})$ of quadratic algebras. The odd rank case is similar.

3.5. Special cases. For $r = 0$ we have a natural isomorphism

$$\Theta_0 : I = k \times k \xrightarrow{\cong} \mathrm{Dis}(\mathfrak{E}_0). \tag{1}$$

Indeed, by 3.4.1, $\mathrm{Dis}(\mathfrak{E}_0) = \mathfrak{D}(0)$ is the discriminant algebra of the zero quadratic form on the zero module $\{0\}$. Since the Pfaffian and the quarter-determinant of an empty matrix are 1 and 0, respectively, and $\bigwedge^0\{0\} = k$, we have $\mathfrak{D}(0) = k \cdot 1 \oplus k \cdot s_0(1_k)$ with the relation $s_0(1_k)^2 = s_0(1_k)$, and we obtain (1) by mapping $e_1 \mapsto s_0(1_k)$.

For $r = 2$, $\mathrm{Dis}(\mathfrak{X}) = \mathfrak{D}(Q)$ is clear from 3.4.1. On the other hand, $\mathfrak{X} = \mathrm{qt}(D)$ is, by 1.4, the quadratic trace module determined by a quadratic algebra D. There is a canonical isomorphism

$$\Phi = \Phi_D : D \xrightarrow{\cong} \mathrm{Dis}(D) \tag{2}$$

of quadratic algebras as follows. Specializing 3.3 to the present situation, $\mathfrak{D}(Q)$ is presented as a k-module by generators 1 and $s_f(x \wedge y)$ where f runs over all representatives of Q, with relations $s_f(x \wedge y) - s_g(x \wedge y) = \kappa_{fg}(x \wedge y) \cdot 1$, where g is another representative of Q. Since $r = 2$, we have $\kappa_{fg}(x \wedge y) = f(x,y) - g(x,y)$. Hence there is a k-module homomorphism Φ: $D \to \mathfrak{D}(Q)$ given by

$$\Phi(1) = 1 \quad \text{and} \quad \Phi(x) = f(x,1) \cdot 1 + s_f(1 \wedge x). \tag{3}$$

A straightforward computation shows that Φ is an isomorphism of algebras.

In particular, let $D = I = k \cdot e_1 \oplus k \cdot e_2$ be the split quadratic algebra so that $\mathrm{qt}(I) = \mathfrak{E}_2$. Let $N_I(\lambda e_1 \oplus \mu e_2) = \lambda\mu$ be its norm form and f_0 the bilinear form with matrix $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ which represents N_I. Then $1 \wedge e_1 = (e_1 + e_2) \wedge e_1 = -e_1 \wedge e_2$ and $f_0(e_1, 1) = 0$. Hence Φ_I is given by

$$\Phi_I : I \xrightarrow{\cong} \mathrm{Dis}(\mathfrak{E}_2), \qquad \Phi_I(e_1) = -s_{f_0}(e_1 \wedge e_2). \tag{4}$$

Finally, for $r = 1$ we have $\mathfrak{X} = \mathfrak{E}_1$ and $\mathfrak{E}_1 \oplus \mathfrak{X} = \mathfrak{E}_2$, so 3.4.2 and (4) yield

$$\mathrm{Dis}(\mathfrak{E}_1) \cong I, \tag{5}$$

the split quadratic algebra.

We now show that our definitions give the correct discriminants and the expected result in the split case. Consistency with Rost's definition in case $r = 3$ will be proved in 3.8, and with Waterhouse's approach in case of étale algebras in 9.10.

3.6. Lemma. *The discriminant of* $\mathrm{Dis}(\mathfrak{X})$ *is* $\delta_{\mathfrak{X}}$.

Proof. By [12, Th. 2.3(d)], the discriminant of $\mathfrak{D}(q)$, where q is any quadratic form on an even-ranked module, is the signed discriminant δ_q of q. If $\mathrm{rk}(\mathfrak{X}) = 2n$ is even,

$$\delta_{\mathrm{Dis}(\mathfrak{X})} = \delta_Q - 4(-1)^{n-1}\lfloor n/2 \rfloor \delta_{\mathfrak{X}} = (-1)^{n-1}\big\{2n - 1 - 4\lfloor n/2 \rfloor\big\}\,\delta_{\mathfrak{X}} = \delta_{\mathfrak{X}},$$

by 3.1.4, 2.2.2, and 2.2.4. If $\mathrm{rk}(\mathfrak{X}) = 2n+1$ is odd, we have similarly

$$\delta_{\mathrm{Dis}(\mathfrak{X})} = \delta_{\mathrm{Dis}(\mathfrak{E}_1 \oplus \mathfrak{X})} = \delta_{\mathfrak{E}_1 \oplus \mathfrak{X}} = \delta_{\mathfrak{E}_1} \otimes \delta_{\mathfrak{X}} = \delta_{\mathfrak{X}}$$

by 2.1.3, since $\delta_{\mathfrak{E}_1}$ is simply the bilinear form $(\lambda, \mu) \mapsto \lambda\mu$ on k.

3.7. The split case. Let A be the algebra $k^r = k\cdot e_1 \oplus \cdots \oplus k\cdot e_r$ with componentwise operations, and $\mathfrak{E}_r = \mathrm{qt}(A)$ the associated split quadratic trace module over k as in 1.2 and 1.3, so T and Q are given by

$$T(e_i) = 1, \quad Q(e_i) = 0, \quad B(e_i, e_j) = 1 \quad (i \neq j).$$

In view of the definition of the discriminant algebra in the odd rank case and since $\mathfrak{E}_1 \oplus \mathfrak{E}_{2n+1} = \mathfrak{E}_{2n+2}$, it suffices to compute $\mathrm{Dis}(\mathfrak{E}_{2n})$. Let $\xi := e_1 \wedge \cdots \wedge e_{2n}$ and let f be the bilinear form on k^{2n} whose matrix with respect to the standard basis is the strict upper triangular matrix U_{2n} with 1 above the diagonal. Then f represents Q, so $\mathfrak{D}(Q)$ is the free k-algebra with basis 1 and $z := s_f(\xi)$ and the relation $z^2 = \tau_f(\xi)z - \gamma_f(\xi,\xi)1$, see 3.3. From 11.2.4 and 11.2.5 it follows that $\tau_f(\xi) = \mathrm{Pf}(U_{2n} - U_{2n}^\top) = 1$ and $\gamma_f(\xi,\xi) = (-1)^{n-1}\mathrm{qdet}(U_{2n}) = (-1)^n\lfloor n/2 \rfloor$. Hence $\mathfrak{D}(Q)$ is the free quadratic algebra

$$\mathfrak{D}(Q) = (\!(1 : (-1)^n\lfloor n/2\rfloor]\!].$$

Since $\Delta_{\mathfrak{E}_r}(e_i,e_j) = \delta_{ij}$, we have $\delta_{\mathfrak{E}_{2n}}(\xi,\xi) = 1$, so by 3.2.2,

$$\mathrm{Dis}(\mathfrak{E}^{2n}) = (\!(1 : (-1)^n\lfloor n/2\rfloor + (-1)^{n-1}\lfloor n/2\rfloor]\!] = (\!(1:0]\!] = k \times k,$$

the split quadratic algebra.

3.8. Theorem. *Let $\mathfrak{X}$ be a quadratic trace module of odd rank $r = 2n+1$ and let $\dot{Q}$ and $\varpi(n)$ be as in 2.3. Then there is a natural isomorphism*

$$\rho : \mathfrak{D}(\dot{Q}) + (-1)^n\varpi(n)\delta_{\mathfrak{X}} \xrightarrow{\cong} \mathrm{Dis}(\mathfrak{X})$$

of quadratic algebras as follows: Identify $\bigwedge^{2n}\dot{X} \cong \bigwedge^{2n+1} X \cong \bigwedge^{2n+2}(k\cdot e_1 \oplus X)$ via

$$\xi := \dot{x}_1 \wedge \cdots \wedge \dot{x}_{2n} \;\mapsto\; \tilde{\xi} := 1_X \wedge x_1 \wedge \cdots \wedge x_{2n} \;\mapsto\; \hat{\xi} := e_1 \wedge 1_X \wedge x_1 \wedge \cdots \wedge x_{2n}.$$

For a bilinear form f on $\dot{X}$ representing $\dot{Q}$, let $\tilde{f}$ be the bilinear form on X given by

$$\tilde{f}(x,y) = -f(\dot{x},\dot{y}) + n\Delta(x,y),$$

and let $\hat{f}$ be the bilinear form on $\hat{X} := k\cdot e_1 \oplus X$ defined by

$$\hat{f}(\lambda e_1 \oplus x, \mu e_1 \oplus y) = \lambda T(y) + \tilde{f}(x,y).$$

Then ρ is given by $1 \mapsto 1$ and $s_f(\xi) \mapsto (-1)^n s_{\hat{f}}(\hat{\xi}) - n\tau_f(\xi)\cdot 1$.

Remark. For $r = 3$ we have in particular $\mathrm{Dis}(\mathfrak{X}) \cong \mathfrak{D}(\dot{Q}) + (-\delta_{\mathfrak{X}})$. This is Rost's definition [15] of the discriminant algebra of a cubic étale algebra.

Proof. Let g be a second representative of $\dot{Q}$ and define $\tilde{g}$ and $\hat{g}$ as above. We first show that

$$\tau_{\hat{f}}(\hat{\xi}) = (2n+1)(-1)^n\tau_f(\xi), \tag{1}$$

$$\kappa_{\hat{f}\hat{g}}(\hat{\xi}) = (2n+1)(-1)^n\kappa_{fg}(\xi). \tag{2}$$

Indeed, let $v = \big(T(x_1),\dots,T(x_{2n})\big) \in k^{2n}$ and let F and G be the square matrices of size $2n$ with entries $f(\dot x_i,\dot x_j)$ and $g(\dot x_i,\dot x_j)$, respectively. Then, with the notations introduced in 3.3, it follows from the definition of $\hat f$ and from 11.6.1 that

$$\begin{aligned}
\pi_{n+1}\big(\hat f - \hat f^{\top} + \mathbf{t}(\hat f - \hat g)\big)(\hat{\hat\xi}) &= \mathrm{Pf}\begin{pmatrix} 0 & r & v \\ -r & 0 & 0 \\ -v^{\top} & 0 & F^{\top} - F + \mathbf{t}(G-F)\end{pmatrix}\\
&= \mathrm{Pf}\begin{pmatrix} 0 & r \\ -r & 0\end{pmatrix}\cdot(-1)^n\cdot \mathrm{Pf}\big(F - F^{\top} + \mathbf{t}(F-G)\big)\\
&= r\cdot(-1)^n\cdot \pi_n\big(f - f^{\top} + \mathbf{t}(f-g)\big)(\xi). \qquad (3)
\end{aligned}$$

Now (1) and (2) follow from (3), 3.3.2 and 3.3.3 by comparing coefficients at powers of $\mathbf{t}$.

From the definition of $\dot Q$ and $\tilde f$ it is immediate that $\tilde f$ is a representative of Q, and hence $\hat f$ is a representative of $\hat Q$, the quadratic form of $\mathfrak{E}_1 \oplus \mathfrak{X}$. Let us put $D' := \mathfrak{D}(\dot Q)$ and $D := \mathfrak{D}(\hat Q)$. There is a module isomorphism $\rho\colon D' \to D$ sending 1 to 1 and $s_f(\xi)$ to $(-1)^n s_{\hat f}(\hat{\hat\xi}) - n\tau_f(\xi)\cdot 1$. Indeed, by the defining relations 3.3.6, the equation $\tau_f - \tau_g = 2\kappa_{fg}$ (cf. 3.3.4) and (2), ρ is well-defined. Since ρ induces the isomorphism $\xi \mapsto (-1)^n\hat{\hat\xi}$ on the quotients $\dot D' = D'/k\cdot 1$ and $\dot D = D/k\cdot 1$, it is a module isomorphism. Furthermore, ρ preserves traces:

$$\begin{aligned}
T_D\big(\rho(s_f(\xi))\big) &= T_D\big((-1)^n s_{\hat f}(\hat{\hat\xi}) - n\tau_f(\xi)\cdot 1\big) = (-1)^n\tau_{\hat f}(\hat{\hat\xi}) - 2n\tau_f(\xi)\\
&= (2n+1-2n)\tau_f(\xi) \text{ (by (1)) } = T_{D'}\big(s_f(\xi)\big).
\end{aligned}$$

By Lemma 3.1(b), this already proves that D is isomorphic to a shift of D'. To determine this shift, we must compute the behaviour of the norms of D' and D under ρ. We claim that

$$\gamma_{\hat f}(\hat{\hat\xi},\hat{\hat\xi}) = \gamma_f(\xi,\xi) + n(n+1)\tau_f(\xi)^2 + (-1)^n\Big\{\varpi(n) - \Big\lfloor \frac{n+1}{2}\Big\rfloor\Big\}\delta_{\mathfrak{X}}(\tilde{\xi},\tilde{\xi}). \quad (4)$$

After localization, it suffices to prove this in case $\dot X$ is free, and by Lemma 2.5, we may assume that k has no 2-torsion. We show that four times (4) holds. Indeed, since the discriminant of the discriminant algebra of a quadratic form q with representative f is $\delta_q = \tau_f^2 - 4\gamma_f$ [12, 1.7] we have, using (1) in the second formula,

$$\delta_{\dot Q}(\xi,\xi) = \tau_f(\xi)^2 - 4\gamma_f(\xi,\xi), \qquad (5)$$

$$\delta_{\hat Q}(\hat{\hat\xi},\hat{\hat\xi}) = (2n+1)^2\tau_f(\xi)^2 - 4\gamma_{\hat f}(\hat{\hat\xi},\hat{\hat\xi}). \qquad (6)$$

On the other hand, by 2.3.4 and 2.2.2,

$$\delta_{\dot Q}(\xi,\xi) = \big(1 + 4(-1)^n\varpi(n)\big)\delta_{\mathfrak{X}}(\tilde\xi,\tilde\xi), \qquad (7)$$

$$\delta_{\hat Q}(\hat{\hat\xi},\hat{\hat\xi}) = \Big(1 + 4(-1)^n\Big\lfloor\frac{n+1}{2}\Big\rfloor\Big)\delta_{\hat{\mathfrak{X}}}(\hat\xi,\hat\xi). \qquad (8)$$

By 2.1.3, we have $\delta_{\mathfrak{X}}(\tilde{\xi},\tilde{\xi}) = \delta_{\hat{\mathfrak{X}}}(\hat{\tilde{\xi}},\hat{\tilde{\xi}})$. Now (4) follows by equating the difference $\delta_{\dot{Q}}(\xi,\xi) - \delta_{\hat{Q}}(\hat{\xi},\hat{\xi})$ computed from (5) −(6) and (7) −(8) and cancelling the factor 4.

Let $D'' = D' + (-1)^n\varpi(n)\delta_{\mathfrak{X}}$, and put $w := s_f(\xi)$ and $\hat{w} := (-1)^n s_{\hat{f}}(\hat{\xi})$ for short. Then

$$N_{D''}(w) = \gamma_f(\xi,\xi) + (-1)^n\varpi(n)\delta_{\mathfrak{X}}(\tilde{\xi},\tilde{\xi}),$$

while, because of (1),

$$\begin{aligned} N_D\big(\rho(w)\big) &= N_D\big(\hat{w} - n\tau_f(\xi)1\big) = N_D\big(\hat{w}\big) - nT_D(\hat{w})\tau_f(\xi) + n^2\tau_f(\xi)^2 \\ &= \gamma_{\hat{f}}(\hat{\xi},\hat{\xi}) - n(-1)^n\tau_{\hat{f}}(\hat{\xi})\tau_f(\xi) + n^2\tau_f(\xi)^2 = \gamma_{\hat{f}}(\hat{\xi},\hat{\xi}) - n(n+1)\tau_f(\xi)^2. \end{aligned}$$

The image of $\rho(w) = \hat{w} - n\tau_f(\xi)\cdot 1$ in $\dot{D}$ is $(-1)^n\hat{\tilde{\xi}}$. Hence

$$\begin{aligned} N_{\mathrm{Dis}(\mathfrak{X})}(\rho(w)) &= N_D\big(\rho(w)\big) + (-1)^n\lfloor (n+1)/2\rfloor\delta_{\hat{\mathfrak{X}}}(\hat{\tilde{\xi}},\hat{\tilde{\xi}}) \\ &= \gamma_{\hat{f}}(\hat{\xi},\hat{\xi}) - n(n+1)\tau_f(\xi)^2 + (-1)^n\lfloor (n+1)/2\rfloor\delta_{\hat{\mathfrak{X}}}(\hat{\tilde{\xi}},\hat{\tilde{\xi}}) \\ &= \gamma_f(\xi,\xi) + (-1)^n\varpi(n)\delta_{\mathfrak{X}}(\tilde{\xi},\tilde{\xi}) \text{ (by (4)) } = N_{D''}(w). \end{aligned}$$

Since ρ preserves the traces of D' and D, hence also those of their shifts D'' and $\mathrm{Dis}(\mathfrak{X})$, it follows that $\rho\colon D'' \to \mathrm{Dis}(\mathfrak{X})$ preserves norms and traces, hence is an isomorphism of quadratic algebras. It remains to show naturality of ρ which is left to the reader.

4. Quadratic-linear modules

4.1. Definition. It will be useful to have the following non-unital version of quadratic trace modules. A *quadratic-linear module* is a triple $\mathfrak{M} = (M,q,t)$ consisting of a finitely generated and projective k-module M and a quadratic form q and a linear form t on M. Morphisms are defined in the obvious way. Just like quadratic trace modules, quadratic-linear modules form a symmetric tensor category with the following direct sum operation. Let $\mathfrak{M}_i = (M_i,q_i,t_i)$ be quadratic-linear modules, denote by $t_1 \oplus t_2$ and t_1t_2 the linear resp. quadratic form on $M_1 \oplus M_2$ given by

$$(t_1 \oplus t_2)(x_1 \oplus x_2) = t_1(x_1) + t_2(x_2), \quad (t_1t_2)(x_1 \oplus x_2) = t_1(x_1)t_2(x_2),$$

and by $q_1 \perp q_2$ the usual orthogonal sum of q_1 and q_2 on $M_1 \oplus M_2$. Then

$$\mathfrak{M}_1 \oplus \mathfrak{M}_2 := (M_1 \oplus M_2,\ (q_1 \perp q_2) + t_1t_2,\ t_1 \oplus t_2).$$

There is an obvious forgetful functor from quadratic trace modules to quadratic-linear modules sending $\mathfrak{X} = (X,Q,T,1)$ to (X,Q,T). It is compatible with the direct sum operation. In the opposite direction, there is a functor from quadratic-linear modules to quadratic trace modules given by the construction of 2.4.

Let (M,q,t) be a quadratic-linear module of rank r. We define $(M,q,t)^\sharp = (M^\sharp,q^\sharp,t^\sharp)$ as the quadratic-linear module of rank $r+1$ where

$$M^\sharp = k \oplus M, \qquad q^\sharp(\lambda \oplus x) = \lambda t(x) + q(x), \quad t^\sharp(\lambda \oplus x) = \lambda + t(x).$$

(The notation $q^\sharp$ is incomplete because $q^\sharp$ depends on q *and* on t.) Of course, this is just the direct sum of $(k,0,\mathrm{Id}_k)$ and (M,q,t). This assignment becomes a functor $\sharp$ from quadratic-linear modules of rank r to those of rank $r+1$ by defining, for a morphism $\varphi\colon \tilde{\mathfrak{M}} \to \mathfrak{M}$, the morphism $\varphi^\sharp\colon \tilde{\mathfrak{M}}^\sharp \to \mathfrak{M}^\sharp$ by $\lambda u \oplus x \mapsto \lambda u \oplus \varphi(x)$.

4.2. Bilinear-linear modules. Replacing the quadratic form q above by a bilinear form, we also consider triples (M,f,t) consisting of a finitely generated and projective k-module M, a bilinear form f and a linear form t on M, called bilinear-linear modules or bl-modules. For them as well, we define a direct sum operation by

$$(M_1, f_1, t_1) \oplus (M_2, f_2, t_2) := (M_1 \oplus M_2,\ f_{12},\ t_1 \oplus t_2),$$

where

$$f_{12} := (f_1 \perp f_2) + t_1 \otimes t_2. \tag{1}$$

Here $f_1 \perp f_2$ is the usual orthogonal sum of f_1 and f_2, and $t_1 \otimes t_2$ denotes the bilinear form on $M_1 \oplus M_2$ given by

$$(t_1 \otimes t_2)(x_1 \oplus x_2, y_1 \oplus y_2) = t_1(x_1)t_2(y_2).$$

With this operation, bl-modules form a tensor category. In particular, after identifying the k-modules $(M_1 \oplus M_2) \oplus M_3$ and $M_1 \oplus (M_2 \oplus M_3)$, we have the associativity law

$$(f_{12} \perp f_3) + (t_1 \otimes t_2) \otimes t_3 = (f_1 \perp f_{23}) + t_1 \otimes (t_2 \otimes t_3). \tag{2}$$

However, bl-modules do not form a symmetric nor even braided tensor category. The reason lies in the asymmetry of the definition of $t_1 \otimes t_2$ above. This definition is of course not canonical; for instance, it would have been equally possible to put $(t_1 \otimes t_2)(x_1 \oplus x_2, y_1 \oplus y_2) = t_1(y_1)t_2(x_2)$.

There is a tensor functor from bl-modules to quadratic-linear modules given by $(M,f,t) \mapsto (M,[f],t)$ (where $[f]$ denotes the quadratic form $x \mapsto f(x,x)$). In particular, this means that if f_i is a representative of q_i then f_{12} is a representative of $(q_1 \perp q_2) + t_1 t_2$.

Just as before, we define $(M,f,t)^\sharp = (k \oplus M,\ f^\sharp,\ t^\sharp)$ where

$$f^\sharp(\lambda \oplus x, \mu \oplus y) = \lambda t(y) + f(x,y), \qquad t^\sharp(\lambda \oplus x) = \lambda + t(x).$$

This is the same as the direct sum of the 1-dimensional bl-module $\mathfrak{e}_1 := (k,0,\mathrm{Id}_k)$ and (M,f,t). Note that the n-fold direct sum $\mathfrak{e}_1 \oplus \cdots \oplus \mathfrak{e}_1$ is $(k^n, U_n, (1,\dots,1))$ where we identify bilinear and linear forms on k^n with $n \times n$-matrices and row vectors, respectively, and U_n is the strict upper triangular matrix with 1 above the diagonal.

4.3. Notation. Let $\mathfrak{M}_i = (M_i, q_i, t_i)$ be quadratic-linear modules of rank r_i and put $L_i = \bigwedge^{r_i} M_i$. For $x_1^{(i)}, \dots, x_{r_i}^{(i)} \in M_i$, let $\xi_i = x_1^{(i)} \wedge \cdots \wedge x_{r_i}^{(i)} \in L_i$. Let $M = M_1 \oplus M_2$ and identify

$$L_1 \otimes L_2 \xrightarrow{\cong} L := \bigwedge^r M$$

via $\xi_1 \otimes \xi_2 \mapsto \xi = \xi_1 \wedge \xi_2$. In case $\mathfrak{M}_1 = (k,0,\mathrm{Id}_k)$ and $\mathfrak{M}_2 = \mathfrak{M}$, we identify $\bigwedge^r M \cong \bigwedge^{r+1} M^\sharp$ by $\xi = x_1 \wedge \cdots \wedge x_r \mapsto \xi^\sharp := 1 \wedge \xi$.

For representatives f_i of q_i we introduce the square matrices $F_i = \big(f_i(x_j^{(i)}, x_l^{(i)})\big)$ of size $r_i \times r_i$ and the row vectors $x^{(i)} = \big(T_i(x_1^{(i)}), \dots, T_i(x_{r_i}^{(i)})\big) \in k^{r_i}$, and put $x := x^{(1)}$ and $y := x^{(2)}$. Then the matrices of $f' := f_1 \perp f_2$ and $f := f' + t_1 \otimes t_2$ with respect to the $x_j^{(1)}, x_l^{(2)}$ are

$$F' = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} F_1 & x^\top y \\ 0 & F_2 \end{pmatrix}.$$

4.4. Lemma. *Let $\mathfrak{M}_i = (M_i, q_i, t_i)$ be quadratic-linear modules of even rank $r_i = 2n_i$ and put $\mathfrak{M} := \mathfrak{M}_1 \oplus \mathfrak{M}_2$. Let f_i, g_i be bilinear forms on M_i representing q_i, define f' and f as in* 4.3 *and put similarly $g' = g_1 \perp g_2$ and $g = g' + t_1 \otimes t_2$. Then, with τ_f, κ_{fg} and γ_f as in* 3.3, *we have*

$$\tau_f = \tau_{f'}\,, \tag{1}$$

$$\kappa_{fg} = \kappa_{f'g'}\,, \tag{2}$$

$$\gamma_f = \gamma_{f'} \,+\, \delta_{q_1^\sharp} \otimes \delta_{q_2^\sharp}. \tag{3}$$

Here $q_i^\sharp$ is defined as in 4.1 *and $\delta_{q_i^\sharp}$ is identified with a bilinear form on L_i via the isomorphism $L_i = \bigwedge^{r_i} M_i \cong \bigwedge^{r_i+1} M_i^\sharp$ of* 4.3 *and hence $\delta_{q_1^\sharp} \otimes \delta_{q_2^\sharp}$ with a bilinear form on L.*

Proof. Define the matrices G_i for g_i like the F_i for f_i in 4.3 and let $\mathbf{t}$ be an indeterminate. Since $f - g = f' - g'$, we have, using 11.6.1, and with $n = n_1 + n_2$,

$$\begin{aligned}
\pi_n\big(f - f^\top + \mathbf{t}(f-g)\big)(\xi) &= \operatorname{Pf}\begin{pmatrix} F_1 - F_1^\top + \mathbf{t}(F_1 - G_1) & x^\top y \\ -y^\top x & F_2 - F_2^\top + \mathbf{t}(F_2 - G_2) \end{pmatrix} \\
&= \operatorname{Pf}\begin{pmatrix} F_1 - F_1^\top + \mathbf{t}(F_1 - G_1) & 0 \\ 0 & F_2 - F_2^\top + \mathbf{t}(F_2 - G_2) \end{pmatrix} \\
&= \pi_n\big(f' - f'^\top + \mathbf{t}(f' - g')\big)(\xi)
\end{aligned}$$

Then (1) and (2) follow by comparing coefficients at powers of $\mathbf{t}$ in view of 3.3.2 and 3.3.3. By 3.3.5 and Lemma 11.5,

$$\begin{aligned}
(-1)^n\{\gamma_f(\xi,\xi) - \gamma_{f'}(\xi,\xi)\} &= -\operatorname{qdet}\begin{pmatrix} F_1 & x^\top y \\ 0 & F_2 \end{pmatrix} + \operatorname{qdet}\begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} \\
&= \operatorname{hdet}\begin{pmatrix} 0 & x \\ 0 & F_1 \end{pmatrix} \operatorname{hdet}\begin{pmatrix} 0 & y \\ 0 & F_2 \end{pmatrix}. \qquad (4)
\end{aligned}$$

From 4.2 it follows that $\begin{pmatrix} 0 & x \\ 0 & F_1 \end{pmatrix}$ is the matrix, with respect to $1, x_1^{(1)}, \dots, x_{r_1}^{(1)}$, of a bilinear form $f_1^\sharp$ on $M_1^\sharp$ representing the quadratic form $q_1^\sharp$. Since $M_1^\sharp$ has odd rank $2n_1 + 1$, the discriminant of $q_1^\sharp$ is given by

$$\delta_{q_1^\sharp}(\xi_1^\sharp, \xi_1^\sharp) = (-1)^{n_1} \operatorname{hdet}\begin{pmatrix} 0 & x \\ 0 & F_1 \end{pmatrix}.$$

An analogous formula holds for $\delta_{q_2^\sharp}$, so (3) follows.

The following result will be crucial for the proof in §6 that the discriminant algebra is a tensor functor. The quadratic form $q = (q_1 \perp q_2) + t_1 t_2$ of the direct sum of two quadratic-linear modules is not quite the orthogonal sum of q_1 and q_2. This is reflected in its discriminant algebra $\mathfrak{D}(q)$ which is a shift of $\mathfrak{D}(q_1 \perp q_2)$.

4.5. Proposition. *Let $\mathfrak{M}_i$ be quadratic-linear modules of even rank and $\mathfrak{M} = (M_1 \oplus M_2, q, t_1 \oplus t_2)$ their direct sum as in* 4.1. *Then there is a module isomorphism $\psi\colon \mathfrak{D}(q_1 \perp q_2) \to \mathfrak{D}(q)$ which sends* 1 *to* 1 *and*

$$s_{f_1 \perp f_2}(\xi) \;\mapsto\; s_{f_{12}}(\xi) \tag{1}$$

where f_i is a representative of q_i and f_{12} is as in 4.2.1. *Moreover,*

$$\psi = \psi_{\mathfrak{M}_1 \mathfrak{M}_2} : \mathfrak{D}(q_1 \perp q_2) + \big(\delta_{q_1^\sharp} \otimes \delta_{q_2^\sharp}\big) \xrightarrow{\cong} \mathfrak{D}(q) \tag{2}$$

is an isomorphism of quadratic algebras which is natural in $\mathfrak{M}_1$ and $\mathfrak{M}_2$.

Proof. Let us put $D' := \mathfrak{D}(q_1 \perp q_2)$ and $D := \mathfrak{D}(q)$ for short. As a k-module, D is generated by 1 and all $s_f(\xi)$, subject to the relations 3.3.6 where f, g run over all representatives of q, and trace and norm of D are determined by 3.3.7. Analogous statements hold for D', with f, g replaced by representatives f', g' of $q' := q_1 \perp q_2$. Now let f_i, g_i be representatives of q_i and let h be a representative of q'. Then also $f' := f_1 \perp f_2$ is a representative of q', and $f := f_{12}$ is a representative of q. We claim that the expression

$$s_f(\xi) + \kappa_{hf'}(\xi) \cdot 1 \tag{3}$$

does not depend on the choice of the f_i. Indeed, let also g_i be representatives of q_i, and define g' and g like f' and f. Then by 3.3.6 and 4.4.2 and the cocycle relation 3.3.4 for κ,

$$\begin{aligned} s_f(\xi) + \kappa_{hf'}(\xi) \cdot 1 - s_g(\xi) - \kappa_{hg'}(\xi) \cdot 1 &= \big(\kappa_{fg}(\xi) + \kappa_{hf'}(\xi) - \kappa_{hg'}(\xi)\big) \cdot 1 \\ &= \big(\kappa_{f'g'}(\xi) - \kappa_{f'h}(\xi) - \kappa_{hg'}(\xi)\big) \cdot 1 = 0. \end{aligned}$$

To prove that there exists a well-defined module homomorphism ψ sending $s_h(\xi)$ to (3), it remains to show that ψ respects the defining relations of D'. Thus let also j be a representative of q'. Then $s_h(\xi) - s_j(\xi) = \kappa_{hj}(\xi) \cdot 1$ while, again by the cocycle relation for κ,

$$s_f(\xi) + \kappa_{hf'}(\xi) \cdot 1 - s_f(\xi) - \kappa_{jf'}(\xi) \cdot 1 = \kappa_{hj}(\xi) \cdot 1,$$

as desired. Now we have a well-defined module homomorphism $\psi\colon D' \to D$ and it satisfies (1) because $\kappa_{f'f'} = 0$. Also, ψ induces the identity on $L = \dot{D} = \dot{D}'$ and hence is a module isomorphism.

To prove (2), it suffices by Lemma 3.1 to show that the traces and norms of D and D' are related by

$$T_D(\psi(w)) = T_{D'}(w), \tag{4}$$

$$N_D(\psi(w)) = N_{D'}(w) + \big(\delta_{q_1^\sharp} \otimes \delta_{q_2^\sharp}\big)(\dot{w}, \dot{w}), \tag{5}$$

where $w \in D'$ and $\dot{w} = p'(w) \in D'/k \cdot 1 = L$. Observe that D' is spanned by 1 and all $s_h(\xi_1 \wedge \xi_2)$ where $\xi_i \in L_i$ is arbitrary and h is a fixed representative of q'. Moreover, $T_{D'}(1) = 2$ and $N_{D'}(1) = 1$. This allows us to assume $h = f' = f_1 \oplus f_2$ as above, and then (4) and (5) follow from 4.4.1 and 4.4.3. Finally, naturality of ψ is easily checked.

5. Explicit computations

In this section, we derive explicit formulas for the discriminant algebra in the free case, based on the remark made in 2.4. The following result says, roughly speaking, that shifting a quadratic form by a *symmetric* bilinear form is reflected by a shift of its discriminant algebra.

5.1. Lemma. *Let (M,q) be a quadratic module of rank $2n$ and let h be a symmetric bilinear form on M. Put $q'(x) := q(x) + h(x,x)$, i.e., $q' = q + [h]$.*

(a) *There is a well-defined isomorphism of k-modules $\varphi = \varphi_h \colon \mathfrak{D}(q) \to \mathfrak{D}(q')$ given by*

$$\varphi(1) = 1, \quad \varphi(s_f(\xi)) = s'_{f+h}(\xi) \tag{1}$$

in terms of the generators $s_f(\xi)$ of $D := \mathfrak{D}(q)$ and $s'_{f+h}(\xi)$ of $D' := \mathfrak{D}(q')$, for all f representing q and all $\xi \in L := \bigwedge^{2n} M$. Moreover, $\varepsilon_h := \gamma_{f+h} - \gamma_f$ is independent of the choice of f and thus is a well-defined bilinear form on L, depending only on h (and of course on q), and

$$\varphi_h \colon \mathfrak{D}(q) + \varepsilon_h \xrightarrow{\cong} \mathfrak{D}(q')$$

is an isomorphism of quadratic algebras. The discriminants of q and q' are related by

$$\delta_{q'} = \delta_q - 4\,\varepsilon_h. \tag{2}$$

(b) *Let h' be another symmetric bilinear form on M and put $q'' := q' + [h'] = q + [h + h']$. Then*

$$\varepsilon_{h+h'} = \varepsilon_h + \varepsilon_{h'}, \tag{3}$$

and the diagram

$$\begin{array}{ccc} \big(\mathfrak{D}(q) + \varepsilon_h\big) + \varepsilon_{h'} & \xrightarrow{\varphi_h} & \mathfrak{D}(q') + \varepsilon_{h'} \\ \Big\| & & \Big\downarrow{\scriptstyle \varphi_{h'}} \\ \mathfrak{D}(q) + \varepsilon_{h+h'} & \xrightarrow[\varphi_{h+h'}]{} & \mathfrak{D}(q'') \end{array} \tag{4}$$

is commutative.

Proof. (a) Recall from 3.3 the linear forms τ_f and κ_{fg} and the bilinear form γ_f on L. Since h is symmetric,

$$\tau_{f+h} = \pi_n(f+h-(f+h)^\top) = \pi_n(f-f^\top) = \tau_f, \tag{5}$$

$$\begin{aligned}\kappa_{f+h,g+h} &= \Pi_n(-2, f+h-(f+h)^\top, f+h-(g+h)) \\ &= \Pi_n(-2, f-f^\top, f-g) = \kappa_{fg}.\end{aligned} \tag{6}$$

Now it follows immediately from (6) and 3.3.6 that (1) defines a homomorphism of k-modules. As φ induces the identity on L, it is an isomorphism of k-modules. Moreover, from (5) and the definition of the trace of $\mathfrak{D}(q)$ and $\mathfrak{D}(q')$ (cf. 3.3.7) we see that φ preserves traces. Hence by Lemma 3.1(b), φ is an isomorphism of a shift $D+\varepsilon$ onto D', and by 3.1.3 and 3.3.7, ε is given by

$$\varepsilon(\xi,\xi) = N_{D'}\big(\varphi(s_f(\xi))\big) - N_D\big(s_f(\xi)\big) = \gamma_{f+h}(\xi,\xi) - \gamma_f(\xi,\xi).$$

Finally, (2) follows from 3.1.4 and the fact that the discriminant of $\mathfrak{D}(q)$ is δ_q.

(b) By (a), we have $\varepsilon_{h+h'} = \gamma_{f+(h+h')} - \gamma_f = \gamma_{(f+h)+h'} - \gamma_{f+h} + \gamma_{f+h} - \gamma_f = \varepsilon_{h'} + \varepsilon_h$. Now the commutativity of (4) follows immediately from (1).

5.2. Proposition. *Let $\mathfrak{X}$ be a quadratic trace module of rank $r \geqslant 1$. Fix a decomposition $X = k\cdot 1_X \oplus M$ and let $q := Q\big|M$ and $t := T\big|M$ as in* 2.4.1, *thus defining a quadratic-linear module $\mathfrak{M} = (M,q,t)$. Consider $\mathfrak{M}^\sharp = (M^\sharp, q^\sharp, t^\sharp)$ as in* 4.1 *and identify $M^\sharp = k\oplus M$ with X by $1_k \mapsto 1_X$. Denote the polar forms of q and $q^\sharp$ by b and $b^\sharp$, respectively, and identify $\bigwedge^{r-1} M$ and $\bigwedge^r X$ via $\eta = x_1\wedge\cdots\wedge x_{r-1} \mapsto \xi = 1\wedge\eta$. Then*

$$\delta_{\mathfrak{X}} = (-1)^{r-1}\big\{r\cdot \bigwedge^{r-1} b + (r-1)\cdot \bigwedge^{r} b^\sharp\big\}. \tag{1}$$

Depending on the parity of r, this can be rewritten as follows:

$$\delta_{\mathfrak{X}} = \delta_{q^\sharp} - 4\cdot(-1)^n\big\{\lfloor n/2\rfloor\cdot\delta_{q^\sharp} - n\cdot\delta_q\big\} \quad \text{if } r = 2n \text{ is even,} \tag{2}$$

$$\delta_{\mathfrak{X}} = \delta_q - 4\cdot(-1)^{n+1}\big\{\lfloor (n+1)/2\rfloor\cdot\delta_q + n\cdot\delta_{q^\sharp}\big\} \text{ if } r = 2n+1 \text{ is odd.} \tag{3}$$

Proof. We put $m = r-1$ and use the notations introduced in 2.2. By 2.2.1 and 11.3.2,

$$\begin{aligned}\delta_{\mathfrak{X}}(\xi,\xi) &= (-1)^m\cdot\begin{pmatrix} r & v\\ mv^\top & D\end{pmatrix} = (-1)^m\Big\{r\cdot\det D + m\cdot\det\begin{pmatrix}0 & v\\ v^\top & D\end{pmatrix}\Big\}\\ &= (-1)^m\big\{r\cdot(\bigwedge^m b)(\eta,\eta) + m\cdot(\bigwedge^r b^\sharp)(\xi,\xi)\big\},\end{aligned}$$

proving (1). Now we distinguish the cases r even and r odd.

(a) $r = 2n$ is even: Since M has odd rank $m = 2n-1$, the discriminant of q is $\delta_q = (-1)^{n-1}\bigwedge^m q$, where the bilinear form $\bigwedge^m q$ on $\bigwedge^m M$ is given by the half-determinant and satisfies $2\bigwedge^m q = \bigwedge^m b$. Furthermore, $\delta_{q^\sharp} = (-1)^n\bigwedge^{2n} b^\sharp$. Substituting this into (1) yields

$$\delta_{\mathfrak{X}} = (-1)^{2n-1}\Big\{(2n)(-1)^{n-1}2\delta_q + (2n-1)(-1)^n\delta_{q^\sharp}\Big\},$$

which together with 2.2.4 gives (2).

(b) $r = 2n+1$ is odd: Then M has even rank $2n$, so by interchanging the roles of q and $q^\sharp$ we now have $\bigwedge^{2n} b = (-1)^n \delta_q$. Furthermore, $q^\sharp$ is a quadratic form on the odd-ranked module X, so $\bigwedge^{2n+1} b^\sharp = 2 \cdot (-1)^n \delta_{q^\sharp}$. Substituting this into (1) yields

$$\delta_{\mathfrak{X}} = (-1)^{2n} \Big\{ (2n+1)(-1)^n \delta_q + (-1)^n 2 \cdot 2n \delta_{q^\sharp} \Big\}.$$

From 2.2.4 (with n replaced by $n+1$) we see $(2n+1)(-1)^n = 1 - 4 \cdot (-1)^{n+1} \times \lfloor (n+1)/2 \rfloor$. By substituting this in the above formula we obtain (3).

5.3. Proposition. *Let $\mathfrak{X}$ be a quadratic trace module of even rank $r = 2n \geqslant 2$. We fix a decomposition $X = k \cdot 1_X \oplus M$ and use the notations of Prop.* 5.2. *Then*

$$\mathrm{Dis}(\mathfrak{X}) \;\cong\; \mathfrak{D}(q^\sharp) + (-1)^n \Big\{ \Big\lfloor \frac{n}{2} \Big\rfloor \cdot \delta_{q^\sharp} - n \cdot \delta_q \Big\}. \tag{1}$$

Proof. Let us abbreviate

$$\varepsilon^\sharp = (-1)^n \big\{ \lfloor n/2 \rfloor \cdot \delta_{q^\sharp} - n \cdot \delta_q \big\}, \qquad \zeta = (-1)^{n-1} \lfloor n/2 \rfloor \cdot \delta_{\mathfrak{X}}.$$

Also, let $\alpha \colon X \to k$ be the linear form determined by $\alpha(1) = 1$ and $\operatorname{Ker} \alpha = M$. Consider the following symmetric bilinear form on X:

$$h(x,y) = (n-1) \big[\alpha(x) T(y) + \alpha(y) T(x) \big] - \Big[\binom{r}{2} - r \Big] \alpha(x) \alpha(y).$$

An easy verification shows that $Q = q^\sharp + [h]$. Hence Lemma 5.1 yields an isomorphism $\mathfrak{D}(q^\sharp) + \varepsilon_h \xrightarrow{\cong} \mathfrak{D}(Q)$ which induces in an obvious way an isomorphism

$$\big(\mathfrak{D}(q^\sharp) + \varepsilon_h \big) + \zeta = \mathfrak{D}(q^\sharp) + (\varepsilon_h + \zeta) \xrightarrow{\cong} \mathfrak{D}(Q) + \zeta = \mathrm{Dis}(\mathfrak{X}),$$

cf. 3.4.1. Comparing this with (1), we see that it remains to prove

$$\varepsilon^\sharp = \varepsilon_h + \zeta. \tag{2}$$

By Lemma 2.5 we may assume that the base ring has no 2-torsion, so it suffices to prove that four times (2) holds. By 2.2.2, 5.2.2 and 5.1.2 we have

$$\delta_Q - \delta_{\mathfrak{X}} = 4 \cdot \zeta, \qquad \delta_{\mathfrak{X}} - \delta_{q^\sharp} = -4 \cdot \varepsilon^\sharp, \qquad \delta_{q^\sharp} - \delta_Q = 4 \cdot \varepsilon_h.$$

Adding these equations yields $4 \cdot (\zeta - \varepsilon^\sharp + \varepsilon_h) = 0$.

5.4. Proposition. *Let $\mathfrak{X}$ be a quadratic trace module of odd rank $r = 2n+1$. Choose a decomposition $X = k\cdot 1_X \oplus M$ and use the notations of Prop.* 5.2. *Then*

$$\operatorname{Dis}(\mathfrak{X}) \;\cong\; \mathfrak{D}(q) \;+\; (-1)^{n+1}\Big\{\Big\lfloor \frac{n+1}{2}\Big\rfloor\cdot\delta_q + n\cdot\delta_{q^\sharp}\Big\}. \tag{1}$$

Proof. Let $\dot Q$ be the quadratic form on $\dot X = X/k\cdot 1$ introduced in 2.3. The canonical map $X\to\dot X$ induces an isomorphism $M\cong\dot X$ by which we identify $\dot Q$ with a quadratic form, again denoted $\dot Q$, on M. Define a bilinear form h on M by $h(x,y) = n\Delta_{\mathfrak{X}}(x,y)$ for all $x,y\in M$, so that $\dot Q = -q+[h]$. Let us note that $\mathfrak{D}(q)\cong\mathfrak{D}(-q)$ by mapping $s_f(\eta)$ to $(-1)^n s_{-f}(\eta)$ for a representative f of q and all $\eta\in\bigwedge^{2n}M$. This follows from the easily proved relations $\tau_{-f} = (-1)^n\tau_f$, $\kappa_{-f,-g} = (-1)^n\kappa_{fg}$ and $\gamma_{-f} = \gamma_f$; cf. 3.3 and [12, Theorem 2.3], applied to $\tilde q = -q$, $\varphi = \mathrm{Id}$, $\mu = -1$. Hence by Lemma 5.1, $\mathfrak{D}(q)+\varepsilon_h\cong\mathfrak{D}(-q)+\varepsilon_h\cong\mathfrak{D}(\dot Q)$ and

$$\big(\mathfrak{D}(q)+\varepsilon_h\big)+\zeta = \mathfrak{D}(q)+(\varepsilon_h+\zeta)\;\xrightarrow{\;\cong\;}\;\mathfrak{D}(\dot Q)+\zeta\;\cong\;\operatorname{Dis}(\mathfrak{X}),$$

where the last isomorphism comes from Th. 3.8 and we put $\zeta = (-1)^n\varpi(n)\cdot\delta_{\mathfrak{X}}$. Thus it remains to show that

$$\varepsilon := (-1)^{n+1}\Big\{\Big\lfloor\frac{n+1}{2}\Big\rfloor\delta_q + n\delta_{q^\sharp}\Big\} = \varepsilon_h+\zeta.$$

By the same argument as in the proof of Prop. 5.3, it suffices to prove four times this equation. By 2.3.4, 5.2.3 and 5.1.2, we have

$$\delta_{\dot Q}-\delta_{\mathfrak{X}} = 4\cdot\zeta,\qquad \delta_{\mathfrak{X}}-\delta_q = -4\cdot\varepsilon,\qquad \delta_q-\delta_{\dot Q} = 4\cdot\varepsilon_h.$$

As before, the assertion follows by adding these equations.

5.5. Proposition. *Let $\mathfrak{X}$ be a quadratic trace module of rank $r = m+1\geqslant 2$, choose a decomposition $X = k\cdot 1_X\oplus M$ and suppose that M is free as a k-module, with basis $x_1,\dots,x_m$. Let $v = (v_1,\dots,v_m)\in k^m$ where $v_i = T(x_i)$, let f be a bilinear form representing Q and put $a_{ij} := f(x_i,x_j)$ so that $a_{ii} = Q(x_i)$ and $a_{ij}+a_{ji} = B(x_i,x_j)$ for $i\neq j$. Finally, define*

$$A = (a_{ij})\in\operatorname{Mat}_m(k)\quad\textit{and}\quad \hat A = \begin{pmatrix}0 & v\\ 0 & A\end{pmatrix}\in\operatorname{Mat}_r(k).$$

Then $\operatorname{Dis}(\mathfrak{X}) = (\!(b:c]\!]$ *is a free quadratic algebra where $b,c\in k$ are given as follows:*

(a) *If $r = 2n$ is even,*

$$b = \operatorname{Pf}\big(\hat A-\hat A^\top\big), \tag{1}$$
$$c = (-1)^{n+1}\operatorname{qdet}(\hat A)+\lfloor n/2\rfloor\det\big(\hat A+\hat A^\top\big)+n\operatorname{hdet}(A) \tag{2}$$
$$= (2n-1)\operatorname{qdet}(\hat A)+(-1)^n\lfloor n/2\rfloor b^2+n\operatorname{hdet}(A). \tag{3}$$

(b) *If* $r = 2n+1$ *is odd,*

$$\begin{aligned} b &= \mathrm{Pf}(A - A^\top), && (4)\\ c &= (-1)^{n+1}\,\mathrm{qdet}(A) - \lfloor (n+1)/2 \rfloor \det\big(A + A^\top\big) - n\mathrm{hdet}(\hat{A}) && (5)\\ &= -(2n+1)\,\mathrm{qdet}(A) + (-1)^{n+1}\lfloor (n+1)/2 \rfloor b^2 - n\mathrm{hdet}(\hat{A}). && (6) \end{aligned}$$

Proof. We use the notations of 5.3 and 5.4.

(a) By [12, 2.7], $\mathfrak{D}(q^\sharp) = ((b : c_0]]$ where

$$b = \mathrm{Pf}(\hat{A} - \hat{A}^\top), \quad c_0 = (-1)^{n+1}\,\mathrm{qdet}(\hat{A}).$$

By Prop. 5.3 and 3.2.2, $\mathrm{Dis}(\mathfrak{X}) = ((b : c_0 + e]]$ where

$$e = (-1)^n \lfloor n/2 \rfloor \delta_{q^\sharp}(\xi, \xi) + (-1)^{n-1} n\, \delta_q(\eta, \eta),$$

where $\eta = x_1 \wedge \cdots \wedge x_m$ and $\xi = 1 \wedge \eta$. On the other hand,

$$\delta_{q^\sharp}(\xi, \xi) = (-1)^n \det(\hat{A} + \hat{A}^\top), \qquad \delta_q(\eta, \eta) = (-1)^{n-1}\mathrm{hdet}(A),$$

which yields (2). The alternative form (3) follows easily from the relations 11.1.2 and 2.2.4 because $b^2 = \det(\hat{A} - \hat{A}^\top)$.

(b) $r = 2n+1$: Here $\mathfrak{D}(q)$ is the free quadratic algebra $((b : c_0]]$ where $b = \mathrm{Pf}(A - A^\top)$ and $c_0 = (-1)^{n+1}\,\mathrm{qdet}(A)$. By 5.4, $\mathrm{Dis}(\mathfrak{X}) = ((b : c_0]] + e = ((b : c_0 + e]]$ where

$$e = (-1)^{n+1}\big\{ \lfloor (n+1)/2 \rfloor\, \delta_q(\eta, \eta) + n\, \delta_{q^\sharp}(\xi, \xi) \big\}.$$

Since q and $q^\sharp$ are quadratic forms in $2n$ and $2n+1$ variables, we have

$$\delta_q(\eta, \eta) = (-1)^n \det(A + A^\top), \quad \delta_{q^\sharp}(\xi, \xi) = (-1)^n \mathrm{hdet}(\hat{A}).$$

It follows that

$$\begin{aligned} c &= c_0 + (-1)^{n+1}\lfloor (n+1)/2 \rfloor\, \delta_{q_M}(\eta, \eta) + (-1)^{n+1} n \delta_{\hat{q}_M}(\xi, \xi)\\ &= (-1)^{n+1}\,\mathrm{qdet}(A) - \lfloor (n+1)/2 \rfloor \det\big(A + A^\top\big) - n\mathrm{hdet}\hat{A}. \end{aligned}$$

Again, (6) is an easy consequence of 2.2.4 and 11.1.2.

5.6. The case $r = 3$**.** We have $\mathrm{qdet}(A) = \det(A)$ for a 2×2-matrix. Moreover, because of 11.4.1,

$$\mathrm{hdet}(\hat{A}) = \det \begin{pmatrix} 0 & v \\ v^\top & A \end{pmatrix} = -a_{11}v_2^2 - a_{22}v_1^2 + (a_{12} + a_{21})v_1 v_2.$$

By 5.5(b) this yields

$$\begin{aligned} b &= a_{12} - a_{21},\\ c &= -3\,\mathrm{qdet}(A) + b^2 - \mathrm{hdet}(\hat{A})\\ &= -3a_{11}a_{22} + a_{12}a_{21} + a_{12}^2 + a_{21}^2 + a_{11}v_2^2 + a_{22}v_1^2 - (a_{12} + a_{21})v_1 v_2. \end{aligned}$$

5.7. The case $r = 4$. Here we use 5.5(a) and obtain

$$\begin{aligned} b &= \operatorname{Pf}(\hat{A} - \hat{A}^\top) = (a_{12} - a_{21})v_3 + (a_{31} - a_{13})v_2 + (a_{23} - a_{32})v_1, \\ c &= 3\,\operatorname{qdet}(\hat{A}) + b^2 + 2\operatorname{hdet}(A) \\ &= 3\det\begin{pmatrix} 0 & v \\ v^\top & A \end{pmatrix} + \det(\hat{A} - \hat{A}^\top) + \det(A + A^\top), \end{aligned}$$

by 11.1.1 and 11.4.2.

6. Multiplicativity of the discriminant algebra

6.1. The product of quadratic algebras. Recall from [11, 2.4, 2.6] that there is a natural product $D_1 \square D_2$ of quadratic algebras with which the category $\mathbf{qa}_k$ is a symmetric tensor category. The product is constructed using the machinery of unital linear forms (although there is a simpler description if the algebras are étale, see [10, III, (2.3.4)]). We recall this quickly. A unital linear form on a quadratic algebra D is a linear form α with $\alpha(1_D) = 1$. Then $D_1 \square D_2$ is generated as a k-module by 1 and symbols $\dot{x}_1 \square_{(\alpha_1,\alpha_2)} \dot{x}_2$ where $x_i \in D_i$, $\dot{x}_i = \operatorname{can}(x_i) \in \dot{D}_i = D_i/k\cdot 1$, and α_i is a unital linear form on D_i. These symbols are bilinear in $\dot{x}_1$ and $\dot{x}_2$ and satisfy relations for which we refer to [11, 2.1, 2.4]. There is an exact sequence

$$0 \longrightarrow k \xrightarrow{\;i\;} D_1 \square D_2 \xrightarrow{\;p\;} \dot{D}_1 \otimes \dot{D}_2 \longrightarrow 0 \tag{1}$$

where $i(1) = 1$ and $p(\dot{x}_1 \square_{(\alpha_1,\alpha_2)} \dot{x}_2) = \dot{x}_1 \otimes \dot{x}_2$. The product of free quadratic algebras is given by the formula

$$(\!(b_1 : c_1]\!] \square (\!(b_2 : c_2]\!] = (\!(b_1 b_2 : c_1(b_2^2 - 2c_2) + c_2(b_1^2 - 2c_1)]\!], \tag{2}$$

see [8, p. 30, p. 42, Exercise 14] and [11, Th. 2.4]. The split algebra $I = k\cdot e_1 \oplus k\cdot e_2$ (cf. 3.2) acts as a neutral element for the product $\square$: There are natural isomorphisms

$$\mathfrak{r}_D : D \square I \xrightarrow{\;\cong\;} D, \qquad \mathfrak{l}_D : I \square D \xrightarrow{\;\cong\;} D, \tag{3}$$

given by

$$\dot{x} \square_{(\alpha,\beta)} \dot{e}_1 \;\mapsto\; x - \alpha(x)1, \qquad \dot{e}_1 \square_{(\beta,\alpha)} \dot{x} \;\mapsto\; x - \alpha(x)1, \tag{4}$$

where β is the unital linear form on I with $\beta(e_1) = 0$ and $\beta(e_2) = 1$, and α is any unital linear form on D, see [11, 2.6.11]. We will also need the associativity constraints

$$\mathfrak{a} = \mathfrak{a}_{D_1 D_2 D_3} : (D_1 \square D_2) \square D_3 \xrightarrow{\;\cong\;} D_1 \square (D_2 \square D_3)$$

which are as follows. Let α_i be unital linear forms on D_i. Then there are unique unital linear forms α_{ij} on $D_i \square D_j$ which vanish on all $\dot{x}_i \square_{(\alpha_i,\alpha_j)} \dot{x}_j$. Since $(D_i \square$

$D_j)/k\cdot 1\cong \dot D_i\otimes \dot D_j$ by (1), $(D_1\,\Box\, D_2)\,\Box\, D_3$ is generated by 1 and the elements $(\dot x_1\otimes\dot x_2)\,\Box_{(\alpha_{12},\alpha_3)}\,\dot x_3$, and then $\mathfrak{a}$ is given by

$$(\dot x_1\otimes\dot x_2)\,\Box_{(\alpha_{12},\alpha_3)}\,\dot x_3 \;\mapsto\; \dot x_1\,\Box_{(\alpha_1,\alpha_{23})}\,(\dot x_2\otimes\dot x_3). \tag{5}$$

(Note that formula (7) of [11, 2.6] is incorrect and should read

$$\eta\big(u_1\,\Box_{\alpha_{1(23)}}\,(u_2\otimes u_3)\big)=(u_1\otimes u_2)\,\Box_{\alpha_{(12)3}}\,u_3.$$

Line -3 of [11, p. 59] has to be modified similarly.)

The product $\Box$ is a bifunctor: If $\varphi_i\colon D_i'\to D_i$ are homomorphisms, then $\varphi_1\,\Box\,\varphi_2$ is given by $1\;\mapsto\;1$ and

$$(\varphi_1\,\Box\,\varphi_1)(\dot x_1'\,\Box_{(\alpha_1',\alpha_2')}\,\dot x_2')=\dot\varphi_1(\dot x_1')\,\Box_{(\alpha_1,\alpha_2)}\,\dot\varphi(\dot x_2'), \tag{6}$$

where $\alpha_i'=\alpha_i\circ\varphi_i$ and $\dot x_i'\in D_i'$. — We show next that the product of shifted quadratic algebras is a suitable shift of their product:

6.2. Lemma. *Let D_i $(i=1,2)$ be quadratic algebras, let $\delta_i=\delta_{D_i}$ be their discriminants, and let ε_i be bilinear forms on $\dot D_i$. Then*

$$\big(D_1+\varepsilon_1\big)\,\Box\,\big(D_2+\varepsilon_2\big)=\big(D_1\,\Box\,D_2\big)+(\varepsilon_1\otimes\delta_2+\delta_1\otimes\varepsilon_2-4\varepsilon_1\otimes\varepsilon_2). \tag{1}$$

Proof. It follows from [11, 2.1, 2.11(b)] that the underlying module, the unit element and the trace of $D_1\,\Box\,D_2$ depend only on the modules D_i, their unit elements and the traces T_{D_i}, but not on their norms. By 3.1, the shifted algebras $D_i'=D_i+\varepsilon_i$ have the same underlying modules, unit elements and traces as D_i, hence so do $D_1'\,\Box\,D_2'$ and $D_1\,\Box\,D_2$. Thus the equality sign in the statement of (1) makes sense. For the proof, we may by localization assume that the $D_i=\langle\!\langle b_i:c_i\,]\!]$ are free. Then, after identifying $\dot D_i\cong k$ as in 3.2, the ε_i and δ_i are identified with scalars, and we have $D_i'=\langle\!\langle b_i:c_i+\varepsilon_i\,]\!]$. By 3.2.1, the discriminants of D_i are $\delta_i=b_i^2-4c_i$. Now (1) follows from 6.1.2 by a straightforward computation.

6.3. We will need the product $D_1\,\Box\,D_2$ in particular when the D_i are the discriminant algebras of quadratic forms of even rank or shifts of such algebras. Let (M_i,q_i) $(i=1,2)$ be quadratic modules of even rank $r_i=2n_i$. Choose representatives f_i,g_i of q_i and let $f'=f_1\perp f_2$ and $g'=g_1\perp g_2$ be their orthogonal sums, which are then representatives of $q'=q_1\perp q_2$. By 3.3, D_i has generators 1 and $s_{f_i}(\xi_i)$ where $\xi_i\in L_i=\bigwedge^{r_i}M_i$. By [12, 2.2], the f_i determine unital linear forms ρ_{f_i} on D_i satisfying $\rho_{f_i}\circ s_{g_i}=-\kappa_{f_ig_i}$. Also, $\dot D_i\cong L_i$ via $s_{f_i}(\xi_i)\;\mapsto\;\xi_i$. We put

$$\xi_1\,\Box_{f'}\,\xi_2:=\xi_1\,\Box_{(\rho_{f_1},\rho_{f_2})}\,\xi_2. \tag{1}$$

Then $D'':=D_1\,\Box\,D_2$ is generated by 1 and the symbols $\xi_1\,\Box_{f'}\,\xi_2$, bilinear in ξ_1 and ξ_2, subject to the relations

$$\xi_1\,\Box_{f'}\,\xi_2-\xi_1\,\Box_{g'}\,\xi_2=\kappa_{f'g'}(\xi_1\otimes\xi_2)\cdot 1. \tag{2}$$

By [12, Th. 2.11], $\mathfrak{D}$ is a symmetric tensor functor from even-ranked quadratic modules (with $\perp$) to quadratic algebras (with $\Box$), i.e., there are natural isomorphisms

$$\vartheta : \mathfrak{D}(q_1) \Box \mathfrak{D}(q_2) \xrightarrow{\cong} \mathfrak{D}(q_1 \perp q_2), \tag{3}$$

$$\vartheta_0 : I \xrightarrow{\cong} \mathfrak{D}(0). \tag{4}$$

They are given by $1 \mapsto 1$ and

$$\vartheta(\xi_1 \Box_{f'} \xi_2) = s_{f'}(\xi_1 \wedge \xi_2), \qquad \vartheta_0(e_1) = s_0(1_k). \tag{5}$$

6.4. Lemma. *Let (M,q) be a quadratic module of even rank r. Let f_0 be the bilinear form with matrix $\binom{0\,0}{1\,0}$ on k^2, let f be a representative of q and let $\xi \in \bigwedge^r M$. Then the composite isomorphism*

$$\mathrm{Dis}(\mathfrak{E}_2) \Box \mathfrak{D}(q) \xrightarrow[\cong]{\Phi^{-1}\Box \mathrm{Id}} I \Box \mathfrak{D}(q) \xrightarrow[\cong]{\iota_{\mathfrak{D}(q)}} \mathfrak{D}(q) \tag{1}$$

(where $\Phi = \Phi_I$ is as in 3.5.4) is given explicitly by

$$-(e_1 \wedge e_2) \Box_{f_0 \perp f} \xi \;\mapsto\; s_f(\xi). \tag{2}$$

Proof. Let β be the linear form on I given by $\beta(e_1) = 0$ and $\beta(e_2) = 1$. By 3.5.4 we have $\Phi(e_1) = -s_{f_0}(e_1 \wedge e_2)$ and hence $\dot{\Phi}(\dot{e}_1) = -e_1 \wedge e_2$. We claim that $\rho_{f_0} \circ \Phi = \beta$. This follows from $\rho_{f_0}(\Phi(e_1)) = -\rho_{f_0}(s_{f_0}(e_1 \wedge e_2)) = 0 = \beta(e_1)$ and $\rho_{f_0}(\Phi(1)) = \rho_{f_0}(1) = 1 = \beta(e_2) = \beta(e_1 + e_2)$. Now 6.1.6 and 6.3.1 imply

$$(\Phi \Box \mathrm{Id})(\dot{e}_1 \Box_{(\beta,\rho_f)} \xi) = -(e_1 \wedge e_2) \Box_{(\rho_{f_0},\rho_f)} \xi = -(e_1 \wedge e_2) \Box_{f_0 \perp f} \xi.$$

On the other hand, putting $x = s_f(\xi)$, we have $\dot{x} = \xi$ and $\rho_f(x) = 0$ so by 6.1.4,

$$\iota_{\mathfrak{D}(q)}(\dot{e}_1 \Box_{(\beta,\rho_f)} \xi) = s_f(\xi)$$

This implies (2).

6.5. Theorem. *Let $\mathfrak{X}_i = (X_i, Q_i, T_i, 1_i)$ be quadratic trace modules of rank r_i and $\mathfrak{X} = \mathfrak{X}_1 \oplus \mathfrak{X}_2 = (X, Q, T, 1)$ their direct sum. Then there are isomorphisms*

$$\Theta = \Theta_{\mathfrak{X}_1 \mathfrak{X}_2} : \mathrm{Dis}(\mathfrak{X}_1) \Box \mathrm{Dis}(\mathfrak{X}_2) \xrightarrow{\cong} \mathrm{Dis}(\mathfrak{X}_1 \oplus \mathfrak{X}_2) \tag{1}$$

of quadratic algebras, natural in $\mathfrak{X}_1$ and $\mathfrak{X}_2$, defined as follows: Choose representatives f_i of Q_i and let

$$f_{12} = (f_1 \perp f_2) + T_1 \otimes T_2$$

be the representative of Q as in 4.2.1. Also, let $\xi_i \in L_i = \bigwedge^{r_i} X_i$, and recall the notations $f^\sharp$ and $\xi_i^\sharp$ of 4.2 and 4.3.

(a) *If* r_1 *and* r_2 *are even,* $\mathrm{Dis}(\mathfrak{X})$ *is by the definition in* 3.4.1 *a shift of* $\mathfrak{D}(Q)$ *and* $\mathrm{Dis}(\mathfrak{X}_1)\,\square\,\mathrm{Dis}(\mathfrak{X}_2)$ *is, by Lemma* 6.2*, a shift of* $\mathfrak{D}(Q_1)\,\square\,\mathfrak{D}(Q_2)$*. Then* Θ *is, as a module homomorphism, the composition*

$$\mathfrak{D}(Q_1)\,\square\,\mathfrak{D}(Q_2) \xrightarrow{\ \vartheta\ } \mathfrak{D}(Q_1 \perp Q_2) \xrightarrow{\ \psi\ } \mathfrak{D}(Q)$$

of 6.3.3 *and* 4.5*. Explicitly, it is given by*

$$\xi_1\,\square_{f_1\perp f_2}\,\xi_2 \ \mapsto\ s_{f_{12}}(\xi_1 \wedge \xi_2). \tag{2}$$

(b) *If* r_1 *is odd and* r_2 *is even, we have* $\mathrm{Dis}(\mathfrak{X}_1) = \mathrm{Dis}(\mathfrak{E}_1 \oplus \mathfrak{X}_1)$ *and* $\mathrm{Dis}(\mathfrak{X}) = \mathrm{Dis}(\mathfrak{E}_1 \oplus \mathfrak{X})$ *by definition in* 3.4.2*. Then* Θ *is the isomorphism*

$$\mathrm{Dis}(\mathfrak{E}_1 \oplus \mathfrak{X}_1)\,\square\,\mathrm{Dis}(\mathfrak{X}_2) \xrightarrow{\ \cong\ } \mathrm{Dis}(\mathfrak{E}_1 \oplus \mathfrak{X})$$

of (1)*, where we canonically identify* $(\mathfrak{E}_1 \oplus \mathfrak{X}_1) \oplus \mathfrak{X}_2 = \mathfrak{E}_1 \oplus (\mathfrak{X}_1 \oplus \mathfrak{X}_2)$*. Explicitly, it is given by*

$$\xi_1^\sharp\,\square_{f_1^\sharp\perp f_2}\,\xi_2 \ \mapsto\ s_{f_{12}^\sharp}\big((\xi_1 \wedge \xi_2)^\sharp\big). \tag{3}$$

(c) *If* r_1 *is even and* r_2 *is odd, let* $j\colon \mathfrak{X}_1 \oplus \mathfrak{E}_1 \oplus \mathfrak{X}_2 \to \mathfrak{E}_1 \oplus \mathfrak{X}$ *be the switch* $x_1 \oplus \lambda \oplus x_2 \ \mapsto\ \lambda \oplus x_1 \oplus x_2$*. Then* Θ *is the composition*

$$\mathrm{Dis}(\mathfrak{X}_1)\,\square\,\mathrm{Dis}(\mathfrak{E}_1 \oplus \mathfrak{X}_2) \xrightarrow{\ \cong\ } \mathrm{Dis}(\mathfrak{X}_1 \oplus \mathfrak{E}_1 \oplus \mathfrak{X}_2) \xrightarrow{\mathrm{Dis}(j)} \mathrm{Dis}(\mathfrak{E}_1 \oplus \mathfrak{X}) = \mathrm{Dis}(\mathfrak{X}), \tag{4}$$

where the first isomorphism is as in (1)*. Explicitly,*

$$\xi_1\,\square_{f_1\perp f_2^\sharp}\,\xi_2^\sharp \ \mapsto\ s_{f_{12}^\sharp}\big((\xi_1 \wedge \xi_2)^\sharp\big). \tag{5}$$

(d) *If* r_1 *and* r_2 *are odd, we have* $\mathrm{Dis}(\mathfrak{X}_i) = \mathrm{Dis}(\mathfrak{E}_1 \oplus \mathfrak{X}_i)$ *by definition. Let* $\mathfrak{E}_2 = \mathfrak{E}_1 \oplus \mathfrak{E}_1$ *be the split quadratic trace module of rank* 2 *and let* $j\colon \mathfrak{E}_1 \oplus \mathfrak{X}_1 \oplus \mathfrak{E}_1 \oplus \mathfrak{X}_2 \to \mathfrak{E}_2 \oplus \mathfrak{X}$ *be the switch* $\lambda_1 \oplus x_1 \oplus \lambda_2 \oplus x_2 \ \mapsto\ \lambda_1 \oplus \lambda_2 \oplus x_1 \oplus x_2$*. Then* Θ *is the composition*

$$\begin{aligned}\mathrm{Dis}(\mathfrak{E}_1 \oplus \mathfrak{X}_1)\,\square\,\mathrm{Dis}(\mathfrak{E}_1 \oplus \mathfrak{X}_2) &\xrightarrow{\ \cong\ } \mathrm{Dis}(\mathfrak{E}_1 \oplus \mathfrak{X}_1 \oplus \mathfrak{E}_1 \oplus \mathfrak{X}_2) \xrightarrow{\mathrm{Dis}(j)} \mathrm{Dis}(\mathfrak{E}_2 \oplus \mathfrak{X}) \\ &\xrightarrow{\ \cong\ } \mathrm{Dis}(\mathfrak{E}_2)\,\square\,\mathrm{Dis}(\mathfrak{X}) \xrightarrow{\ \cong\ } \mathrm{Dis}(\mathfrak{X})\end{aligned} \tag{6}$$

where the first and third maps are as in (1) *and the last map is the isomorphism* 6.4.1*. Explicitly,* Θ *is given by the formula*

$$\xi_1^\sharp\,\square_{f_1^\sharp\perp f_2^\sharp}\,\xi_2^\sharp \ \mapsto\ s_{f_{12}}(\xi_1 \wedge \xi_2). \tag{7}$$

Proof. (a) Let $r_i = 2n_i$ and $r = r_1 + r_2 = 2n$. Put $D_i := \mathfrak{D}(Q_i)$, $D'' := D_1\,\square\,D_2$, $D' := \mathfrak{D}(Q_1 \perp Q_2)$, and $D := \mathfrak{D}(Q)$ for short. Recall from 3.4.1 that $\mathrm{Dis}(\mathfrak{X}_i) = D_i + \varepsilon_i$ and $\mathrm{Dis}(\mathfrak{X}) = D + \varepsilon$ where

$$\varepsilon_i := (-1)^{n_i - 1}\lfloor n_i/2\rfloor\,\delta_{\mathfrak{X}_i}, \qquad \varepsilon := (-1)^{n-1}\lfloor n/2\rfloor\,\delta_{\mathfrak{X}}.$$

We apply Prop. 4.5 in case $q_i = Q_i$, and thus have to compute $\delta_{Q_i^\sharp}$. This follows immediately from 2.2.3 and 2.1.3:

$$\delta_{Q_i^\sharp}(\xi_i^\sharp, \xi_i^\sharp) = (-1)^{n_i} n_i \delta_{\mathfrak{X}_i^\sharp}(\xi_i^\sharp, \xi_i^\sharp) = (-1)^{n_i} n_i \delta_{\mathfrak{X}_i}(\xi_i, \xi_i).$$

By 6.1.1, $D''/k \cdot 1 = L_1 \otimes L_2$ which is identified with $L = \bigwedge^r X$. Thus it makes sense to shift both sides of the composition $\psi \circ \vartheta \colon D'' \to D' \to D$ by ε which yields an algebra isomorphism

$$D'' + \big((-1)^n n_1 n_2\, \delta_{\mathfrak{X}} + \varepsilon\big) \xrightarrow{\cong} D + \varepsilon = \mathrm{Dis}(\mathfrak{X}),$$

and from 6.3.5 and 4.5.1 it is clear that (2) holds. On the other hand, by 6.2.1,

$$\begin{aligned} \mathrm{Dis}(\mathfrak{X}_1) \,\square\, \mathrm{Dis}(\mathfrak{X}_2) &= \big(D_1 + \varepsilon_1\big) \,\square\, \big(D_2 + \varepsilon_2\big) \\ &= D'' + (\varepsilon_1 \otimes \delta_2 + \delta_1 \otimes \varepsilon_2 - 4\varepsilon_1 \otimes \varepsilon_2) \end{aligned}$$

where $\delta_i = \delta_{D_i} = \delta_{Q_i}$, so it remains to show that

$$(-1)^n n_1 n_2\, \delta_{\mathfrak{X}} + \varepsilon = \varepsilon_1 \otimes \delta_2 + \delta_1 \otimes \varepsilon_2 - 4\varepsilon_1 \otimes \varepsilon_2. \tag{8}$$

By 2.2.2 and 2.1.3,

$$\delta_i = (-1)^{n_i - 1}(r_i - 1)\delta_{\mathfrak{X}_i}, \qquad \delta_{\mathfrak{X}} = \delta_{\mathfrak{X}_1} \otimes \delta_{\mathfrak{X}_2}.$$

Then (8) comes down to the formula

$$n_1 n_2 - \Big\lfloor \frac{n_1 + n_2}{2} \Big\rfloor = (2n_1 - 1)\Big\lfloor \frac{n_2}{2} \Big\rfloor + (2n_2 - 1)\Big\lfloor \frac{n_1}{2} \Big\rfloor - 4\Big\lfloor \frac{n_1}{2} \Big\rfloor \Big\lfloor \frac{n_2}{2} \Big\rfloor$$

for all natural numbers n_1, n_2. The elementary proof is left to the reader.

(b) From what was proved in (a), it is clear that Θ is an algebra isomorphism. By specializing 4.2.2 to the case where the first factor is the bilinear-linear module $\mathfrak{e}_1$ of rank one, we have

$$f_{12}^\sharp = (f_1^\sharp \perp f_2) + T_1^\sharp \otimes T_2.$$

Hence (3) follows from (2) after replacing f_1 by $f_1^\sharp$ and ξ_1 by $\xi_1^\sharp$ because $\xi_1^\sharp \wedge \xi_2 = (1 \wedge \xi_1) \wedge \xi_2 = 1 \wedge (\xi_1 \wedge \xi_2) = (\xi_1 \wedge \xi_2)^\sharp$ in the exterior algebra.

(c) Again, it is clear from (a) that Θ is an algebra isomorphism, so it remains to show (5). Put $g := (f_1 \perp f_2^\sharp) + T_1 \otimes T_2^\sharp$ and $h := f_{12}^\sharp$. A calculation shows that

$$\begin{aligned} g(x_1 \oplus \lambda \oplus x_2,\, y_1 \oplus \mu \oplus y_2) = {} & f_1(x_1, y_1) + f_2(x_2, y_2) + T_1(x_1)T_2(y_2) \\ & + \mu T_1(x_1) + \lambda T_2(y_2), \end{aligned}$$

$$\begin{aligned} j^*(h)(x_1 \oplus \lambda \oplus x_2,\, y_1 \oplus \mu \oplus y_2) = {} & f_1(x_1, y_1) + f_2(x_2, y_2) + T_1(x_1)T_2(y_2) \\ & + \lambda T_1(y_1) + \lambda T_2(y_2), \end{aligned}$$

for $\lambda, \mu \in k$, $x_i, y_i \in X_i$. We claim that

$$s_{j^*(h)}(\xi_1 \wedge \xi_2^\sharp) = s_g(\xi_1 \wedge \xi_2^\sharp). \tag{9}$$

By the defining relations 3.3.6 of $\mathfrak{D}(Q_1 \oplus Q_2^\sharp)$, this is equivalent to $\kappa_{g,j^*(h)}(\xi_1 \wedge \xi_2^\sharp) = 0$. Let $\mathbf{t}$ be an indeterminate and put $a = g - g^\top$ and $a' = g - j^*(h)$. We use the notations introduced in 3.3 and 4.3 and put $2n = r_1 + r_2 + 1$. Then by 11.6.1,

$$\pi_n(a + \mathbf{t}a')(\xi_1 \wedge \xi_2^\sharp) = \mathrm{Pf}\begin{pmatrix} F_1 - F_1^\top & (1+\mathbf{t})x^\top & x^\top y \\ -(1+\mathbf{t})x & 0 & y \\ -y^\top x & -y^\top & F_2 - F_2^\top \end{pmatrix}$$
$$= \mathrm{Pf}(F_1 - F_1^\top) \cdot \mathrm{Pf}\begin{pmatrix} 0 & y \\ -y^\top & F_2 - F_2^\top \end{pmatrix}$$

is independent of $\mathbf{t}$, whence $\kappa_{g,j^*(h)} = 0$, as asserted. Now we apply the definition of Θ in (4) and the formula for $\mathrm{Dis}(j)$ in 3.4.4 which yields

$$\begin{aligned} \xi_1 \square_{f_1 \perp f_2^\sharp}(\xi_2^\sharp) &\mapsto s_g(\xi_1 \wedge \xi_2^\sharp) = s_{j^*(h)}(\xi_1 \wedge \xi_2^\sharp) \\ &\mapsto s_h\big(\big(\bigwedge^{r_1+r_2+1} j\big)(\xi_1 \wedge \xi_2^\sharp)\big) = s_{f_{12}^\sharp}\big((\xi_1 \wedge \xi_2)^\sharp\big), \end{aligned}$$

because r_1 is even.

(d) It is clear that Θ is an algebra isomorphism so let us prove (7). Let $\mathfrak{E}_2 = (k^2, Q_0, T_0, 1)$ be the split quadratic trace module of rank 2, so $Q_0(\lambda e_1 + \mu e_2) = \lambda\mu$, $T_0(\lambda e_1 + \mu e_2) = \lambda + \mu$ and $1 = e_1 + e_2$, cf. 1.2(a). Let f_0 be the bilinear form on k^2 with matrix $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, a representative of Q_0, and put

$$g = (f_1^\sharp \perp f_2^\sharp) + T_1^\sharp \otimes T_2^\sharp, \qquad h = (f_0 \perp f_{12}) + T_0 \otimes (T_1 \oplus T_2).$$

These are bilinear forms on $X_1^\sharp \oplus X_2^\sharp$ and $k \oplus k \oplus X_1 \oplus X_2$, respectively. A computation shows that

$$\begin{aligned} g(\lambda_1 \oplus x_1 \oplus \lambda_2 \oplus x_2,\ \mu_1 \oplus y_1 \oplus \mu_2 \oplus y_2) &= f_1(x_1, y_1) + f_2(x_2, y_2) \\ &\quad + T_1(x_1)T_2(y_2) + \lambda_1\mu_2 + \lambda_1(T_1(y_1) + T_2(y_2)) + \mu_2 T_1(x_1) + \lambda_2 T_2(y_2), \end{aligned}$$

$$\begin{aligned} j^*(h)(\lambda_1 \oplus x_1 \oplus \lambda_2 \oplus x_2,\ \mu_1 \oplus y_1 \oplus \mu_2 \oplus y_2) &= f_1(x_1, y_1) + f_2(x_2, y_2) \\ &\quad + T_1(x_1)T_2(y_2) + \lambda_2\mu_1 + (\lambda_1 + \lambda_2)(T_1(y_1) + T_2(y_2)). \end{aligned}$$

We claim that

$$s_{j^*(h)}(\xi_1^\sharp \wedge \xi_2^\sharp) = s_g(\xi_1^\sharp \wedge \xi_2^\sharp). \tag{10}$$

Similarly as in the proof of (c), let $a = g - g^\top$ and $a' = g - j^*(h)$, and put $2n = r_1 + r_2 + 2$. Using the notations of 4.3, let

$$\tilde{R} = \begin{pmatrix} 0 & x \\ -x^\top & F_1 - F_1^\top \end{pmatrix}, \qquad \tilde{S} = \begin{pmatrix} 0 & y \\ -y^\top & F_2 - F_2^\top \end{pmatrix},$$

$\tilde{x} = (1, x) \in k^{n_1+1}$ and $\tilde{y} = (1 + \mathbf{t}, y) \in k[\mathbf{t}]^{n_2+1}$. Then

$$\pi_n(a + \mathbf{t}a')(\xi_1^\sharp \wedge \xi_2^\sharp) = \mathrm{Pf}\begin{pmatrix} \tilde{R} & \tilde{x}^\top \tilde{y} \\ -\tilde{y}^\top \tilde{x} & \tilde{S} \end{pmatrix} = \mathrm{Pf}(\tilde{R}) \cdot \mathrm{Pf}(\tilde{S})$$

(by 11.6.1) is independent of $\mathbf{t}$. Hence $\kappa_{g,j^*(h)} = 0$ which proves (10).

Now we can establish (7). Let us identify e_1 with $1_k \oplus 0 \oplus 0 \oplus 0$ and e_2 with $0 \oplus 1_k \oplus 0 \oplus 0$ in $\mathfrak{E}_2 \oplus \mathfrak{X}$. Then since r_1 is odd and $\xi_1 \in \bigwedge^{r_1} X_1$,

$$(\bigwedge^{2n} j)(\xi_1^\sharp \wedge \xi_2^\sharp) = (\bigwedge^{2n} j)(e_1 \wedge \xi_1 \wedge e_2 \wedge \xi_2) = -e_1 \wedge e_2 \wedge \xi_1 \wedge \xi_2$$

and hence Θ maps

$$\begin{aligned} \xi_1^\sharp \,\Box_{f_1^\sharp \perp f_2^\sharp}\, \xi_2^\sharp &\mapsto s_g(\xi_1^\sharp \wedge \xi_2^\sharp) = s_{j^*(h)}(\xi_1^\sharp \wedge \xi_2^\sharp) \\ &\mapsto -s_h(e_1 \wedge e_2 \wedge \xi_1 \wedge \xi_2) \in \mathrm{Dis}(\mathfrak{E}_2 \oplus \mathfrak{X}) \\ &\mapsto -(e_1 \wedge e_2) \,\Box_{f_0 \perp f_{12}}\, (\xi_1 \wedge \xi_2) \in \mathrm{Dis}(\mathfrak{E}_2) \,\Box\, \mathrm{Dis}(\mathfrak{X}), \\ &\mapsto s_{f_{12}}(\xi_1 \wedge \xi_2), \end{aligned}$$

where we used (2) in reverse in the last but one and 6.4.2 in the last step.

We finally show that the isomorphisms Θ are natural in $\mathfrak{X}_1$ and $\mathfrak{X}_2$. In case (a), this follows from naturality of ϑ and ψ (Prop. 4.5). The cases (b) and (c) follow easily from this, and in case (d) one uses the naturality of the isomorphism $\mathfrak{l}_{\mathfrak{D}(q)}$ which implies that also the isomorphism $\mathrm{Dis}(\mathfrak{E}_2) \,\Box\, \mathrm{Dis}(\mathfrak{X}) \cong \mathrm{Dis}(\mathfrak{X})$ of 6.4.1 is natural in $\mathfrak{X}$.

6.6. Theorem. *The functor* Dis *is a tensor functor from the category* $\mathbf{qtm}_k$ *of quadratic trace modules (with* $\oplus$*) to the category* $\mathbf{qa}_k$ *of quadratic algebras (with* $\Box$*).*

Proof. This means [9] that, in addition to the natural isomorphisms Θ of Th. 6.5, we have a natural isomorphism $\Theta_0 \colon I \cong \mathrm{Dis}(0)$ such that the following diagrams commute for all $\mathfrak{X}, \mathfrak{X}_i$, where we put $D_i := \mathrm{Dis}(\mathfrak{X}_i)$ for short:

$$\begin{array}{ccc} \mathrm{Dis}(\mathfrak{X}) \,\Box\, I & \xrightarrow{\mathfrak{r}_D} & \mathrm{Dis}(\mathfrak{X}) \\ \downarrow{\scriptstyle \mathrm{Id}\Box\Theta_0} & & \uparrow \\ \mathrm{Dis}(\mathfrak{X}) \,\Box\, \mathrm{Dis}(0) & \xrightarrow[\Theta]{} & \mathrm{Dis}(\mathfrak{X} \oplus 0) \end{array} \qquad \begin{array}{ccc} I \,\Box\, \mathrm{Dis}(\mathfrak{X}) & \xrightarrow{\mathfrak{l}_D} & \mathrm{Dis}(\mathfrak{X}) \\ {\scriptstyle \Theta_0\Box\mathrm{Id}}\downarrow & & \uparrow \\ \mathrm{Dis}(0) \,\Box\, \mathrm{Dis}(\mathfrak{X}) & \xrightarrow[\Theta]{} & \mathrm{Dis}(0 \oplus \mathfrak{X}) \end{array} \qquad (1)$$

$$\begin{array}{ccccc} (D_1 \,\Box\, D_2) \,\Box\, D_3 & \xrightarrow{\mathfrak{a}} & D_1 \,\Box\, (D_2 \,\Box\, D_3) & \xrightarrow{\mathrm{Id}\Box\Theta} & D_1 \,\Box\, \mathrm{Dis}(\mathfrak{X}_2 \oplus \mathfrak{X}_3) \\ {\scriptstyle \Theta\Box\mathrm{Id}}\downarrow & & & & {\scriptstyle \Theta}\downarrow \\ \mathrm{Dis}(\mathfrak{X}_1 \oplus \mathfrak{X}_2) \,\Box\, D_3 & \xrightarrow[\Theta]{} & \mathrm{Dis}\big((\mathfrak{X}_1 \oplus \mathfrak{X}_2) \oplus \mathfrak{X}_3\big) & \longrightarrow & \mathrm{Dis}\big(\mathfrak{X}_1 \oplus (\mathfrak{X}_2 \oplus \mathfrak{X}_3)\big) \end{array} \qquad (2)$$

The commutativity of (1) is easy (use 3.5.1 and 6.1.3) so we only do the commutativity of (2). The unnamed arrow on the lower right is treated as the identity. The map $\mathfrak{a}$ on the upper left is as in 6.1.5. According to the parities of the ranks of the $\mathfrak{X}_i$, there are eight cases in which commutativity of (2) needs to be checked, because Θ is defined differently in each case. We do the case where all $\mathfrak{X}_i$ have even rank. The others follow the same pattern and are left to the reader.

Since D_i is a shift of $\mathfrak{D}(Q_i)$ we have $\dot{D}_i = L_i = \bigwedge^{r_i} X_i$. Let f_i be representatives of Q_i. Specializing formula 6.1.5 to the present situation and using 6.3.1, one sees that $\mathfrak{a}$ is given by

$$(\xi_1 \otimes \xi_2) \,\square_{(f_1 \perp f_2) \perp f_3}\, \xi_3 \;\mapsto\; \xi_1 \,\square_{f_1 \perp (f_2 \perp f_3)}\, (\xi_2 \otimes \xi_3).$$

Let $(f_{ij}, T_{ij}) := (f_i, T_i) \oplus (f_j, T_j)$ as in 4.2. Thus $T_{ij} = T_i \oplus T_j$, $f_{ij} = (f_i \perp f_j) + T_i \otimes T_j$ and f_{ij} is a representative of the quadratic form $Q_{ij} = Q_i \perp Q_j + T_i T_j$ of $\mathfrak{X}_i \oplus \mathfrak{X}_j$. Formula 6.5.2 yields

$$\Theta(\xi_i \,\square_{f_i \perp f_j}\, \xi_j) = s_{f_{ij}}(\xi_i \wedge \xi_j),$$

and therefore the map $\dot{\Theta}\colon D_i \,\square\, D_j \to \mathrm{Dis}(\mathfrak{X}_i \oplus \mathfrak{X}_j)$ is given by $\dot{\Theta}(\xi_i \otimes \xi_j) = \xi_i \wedge \xi_j$. Now we can compute the effect of going across and down in (2):

$$\begin{aligned} (\xi_1 \otimes \xi_2) \,\square_{(f_1 \perp f_2) \perp f_3}\, \xi_3 \;&\xmapsto{\;\mathfrak{a}\;}\; \xi_1 \,\square_{f_1 \perp (f_2 \perp f_3)}\, (\xi_2 \otimes \xi_3) \\ &\xmapsto{\;\mathrm{Id}\square\Theta\;}\; \xi_1 \,\square_{f_1 \perp f_{23}}\, (\xi_2 \wedge \xi_3) \\ &\xmapsto{\;\Theta\;}\; s_{(f_1 \perp f_{23}) + T_1 \otimes T_{23}}\big(\xi_1 \wedge (\xi_2 \wedge \xi_3)\big). \end{aligned} \tag{3}$$

Going down and across is easier and results in

$$\begin{aligned} (\xi_1 \otimes \xi_2) \,\square_{(f_1 \perp f_2) \perp f_3}\, \xi_3 \;&\xmapsto{\;\Theta\square\mathrm{Id}\;}\; (\xi_1 \wedge \xi_2) \,\square_{f_{12} \perp f_3}\, \xi_3 \\ &\xmapsto{\;\Theta\;}\; s_{(f_{12} \perp f_3) + T_{12} \otimes T_3}\big((\xi_1 \wedge \xi_2) \wedge \xi_3\big). \end{aligned} \tag{4}$$

By 4.2.2 we have the associative law

$$(f_{12} \perp f_3) \,+\, T_{12} \otimes T_3 = (f_1 \perp f_{23}) \,+\, T_1 \otimes T_{23} \tag{5}$$

so the commutativity of (2) follows.

7. The discriminant algebra as a symmetric tensor functor

7.1. Notations. Let $\mathbf{F}_2$ be the functor from k-**alg** to the category of commutative rings which assigns to R the set $\mathbf{F}_2(R)$ of all continuous maps from $\mathrm{Spec}(R)$ to $\mathbb{F}_2$, the ring with two elements, with the obvious ring structure. We usually identify an element $f \in \mathbf{F}_2(R)$ with the idempotent $p \in R$ such that $f^{-1}(1) = \mathrm{Spec}(Rp)$. Then the addition in $\mathbf{F}_2(R)$ is given by

$$p \dotplus p' = p(1-p') + p'(1-p) = p + p' - 2pp',$$

while multiplication is the usual product of idempotents in R. We denote by $\mathbf{Z}_2$ the k-group functor assigning to $R \in k$-**alg** the additive group of $\mathbf{F}_2(R)$. There is a homomorphism

$$\chi\colon \mathbf{Z}_2 \to \boldsymbol{\mu}_2, \qquad p \;\mapsto\; 1 - 2p = (-1)^p,$$

where $\boldsymbol{\mu}_2$ is the k-group functor of second roots of unity.

7.2. Involutions of quadratic algebras. A quadratic algebra D has a natural involution $\sigma = \sigma_D$ given by $x + \sigma(x) = T_D(x) \cdot 1$. By [11, 5.3] there is a homomorphism

$$\mathfrak{h}_D \colon \mathbf{Z}_2 \to \mathbf{Aut}(D), \qquad \mathfrak{h}_D(p) := \sigma^p := (1-p) \cdot \mathrm{Id} + p \cdot \sigma. \tag{1}$$

Here $\mathbf{Aut}(D)$ is the k-group functor $R \mapsto \mathrm{Aut}(D \otimes R)$. Explicitly, this means

$$\mathfrak{h}_D(p) \cdot x = \sigma^p(x) = p\, T_D(x) \cdot 1 + (1-2p)x, \tag{2}$$

for all $p \in \mathbf{Z}_2(R)$, $x \in D \otimes R$, $R \in k$-**alg**. Hence the map induced by σ^p on $\dot{D}$ is given by

$$\chi(p) : \dot{x} \mapsto (1-2p)\dot{x} = (-1)^p \dot{x}. \tag{3}$$

Since $\dot{D} \cong \bigwedge^2 D$ under the map $\dot{x} \mapsto 1 \wedge x$, it follows that

$$\det \mathfrak{h}_D(p) = \chi(p) = (-1)^p.$$

Suppose in particular that $D = (\!(b : c]\!] = k \cdot 1 \oplus k \cdot z$ is free, and identify $\mathbf{GL}(D)$ with $\mathbf{GL}_2$ by means of the basis $1, z$. Then it is easily seen that $\mathbf{Aut}(D) \subset \mathbf{GL}_2$ is the subgroup of all matrices

$$h = \begin{pmatrix} 1 & \lambda \\ 0 & \mu \end{pmatrix} \tag{4}$$

where μ is a unit and

$$2\lambda = b(1-\mu), \qquad \lambda(b-\lambda) = c(1-\mu^2). \tag{5}$$

Also, (2) applied to $x = z$ shows that

$$\mathfrak{h}_D(p) = \begin{pmatrix} 1 & pb \\ 0 & 1-2p \end{pmatrix}. \tag{6}$$

7.3. Proposition. *Let D_1 and D_2 be quadratic algebras with product $D = D_1 \square D_2$. For automorphisms h_i of D_i let $h_1 \square h_2$ be the automorphism of D as in* 6.1.6.

(a) *The map $(h_1, h_2) \mapsto h_1 \square h_2$ induces a homomorphism of group functors*

$$\square : \mathbf{Aut}(D_1) \times \mathbf{Aut}(D_2) \to \mathbf{Aut}(D). \tag{1}$$

(b) *Let $D_i = (\!(b_i : c_i]\!] = k \cdot 1 \oplus k \cdot z_i$ be free and $D = (\!(b : c]\!]$ as in* 6.1.2. *Writing $h_i = \begin{pmatrix} 1 & \lambda_i \\ 0 & \mu_i \end{pmatrix}$ as in* 7.2.4, *we have*

$$\begin{pmatrix} 1 & \lambda_1 \\ 0 & \mu_1 \end{pmatrix} \square \begin{pmatrix} 1 & \lambda_2 \\ 0 & \mu_2 \end{pmatrix} = \begin{pmatrix} 1 & \lambda_1 b_2 + \lambda_2 b_1 - 2\lambda_1\lambda_2 \\ 0 & \mu_1\mu_2 \end{pmatrix}. \tag{2}$$

(c) *The following diagram is commutative:*

$$\begin{array}{ccc} \mathbf{Z}_2 \times \mathbf{Z}_2 & \xrightarrow{\ \mathfrak{h}_{D_1} \times \mathfrak{h}_{D_2}\ } & \mathbf{Aut}(D_1) \times \mathbf{Aut}(D_2) \\ {\scriptstyle \dot{+}}\big\downarrow & & \big\downarrow{\scriptstyle \square} \\ \mathbf{Z}_2 & \xrightarrow[\ \mathfrak{h}_{D_1 \square D_2}\ ]{} & \mathbf{Aut}(D_1 \square D_2) \end{array} \tag{3}$$

In particular, for $p \in \mathbf{Z}_2(k)$,

$$\sigma^p_{D_1} \square \mathrm{Id}_{D_2} = \mathrm{Id}_{D_1} \square \sigma^p_{D_2} = \sigma^p_D. \tag{4}$$

Proof. (a) This is clear because the product $\square$ of quadratic algebras is a bifunctor commuting with base change.

(b) Let $\alpha_i \colon D_i \to k$ be the linear form given by $\alpha_i(1) = 1$ and $\alpha_i(z_i) = 0$. Then $D = k \cdot 1 \oplus k \cdot z$ where $z = \dot z_1 \square_{(\alpha_1,\alpha_2)} \dot z_2$. Now put $\beta_i = \alpha_i \circ h_i$. Since $\dot h_i(\dot z_i) = \mu_i \dot z_i$, 6.1.6 shows

$$(h_1 \square h_2)(\dot z_1 \square_{(\beta_1,\beta_2)} \dot z_2) = \mu_1\mu_2(\dot z_1 \square_{(\alpha_1,\alpha_2)} \dot z_2).$$

Define $t_{\alpha_i} \in \dot D_i^*$ by $t_{\alpha_i}(\dot x) = T_{D_i}(x - \alpha_i(x)\cdot 1)$. Then

$$\beta_i(z_i) = \alpha_i(\lambda_i \cdot 1 + \mu_i z_i) = \lambda_i, \qquad t_{\alpha_i}(\dot z_i) = T_{D_i}(z_i - \alpha_i(z_i)) = b_i.$$

By the defining relations of $D_1 \square D_2$ (cf. [11, 2.1]),

$$\dot z_1 \square_{(\alpha_1,\alpha_2)} \dot z_2 = \dot z_1 \square_{(\beta_1,\beta_2)} \dot z_2 + c_{\alpha\beta}(\dot z_1 \otimes \dot z_2)\cdot 1,$$

where $c_{\alpha\beta} = c_{\alpha_1\beta_1} \otimes t_{\alpha_2} + t_{\alpha_1} \otimes c_{\alpha_2\beta_2} - 2c_{\alpha_1\beta_1} \otimes c_{\alpha_2\beta_2}$, see [11, 2.1.4]. Substituting the above data, we obtain $c_{\alpha\beta}(\dot z_1 \otimes \dot z_2) = \lambda_1 b_2 + \lambda_2 b_1 - 2\lambda_1\lambda_2$, and therefore, by an easy computation,

$$(h_1 \square h_2)(z) = (\lambda_1 b_2 + \lambda_2 b_1 - 2\lambda_1\lambda_2)\cdot 1 + \mu_1\mu_2 \cdot z.$$

In matrix notation, this is (2).

(c) Since everything is compatible with base change, it suffices to prove the commutativity of (3) when the group functors are evaluated at $R = k$. By localization, we may assume D_i and hence D free. Then the assertion follows by direct computation from 7.2.6 and (2).

7.4. Quadratic algebras with parity. The categories $\mathbf{qtm}_k$ of quadratic trace modules and $\mathbf{qa}_k$ of quadratic algebras are symmetric tensor categories. It is thus natural to ask whether the tensor functor $\mathrm{Dis}\colon \mathbf{qtm}_k \to \mathbf{qa}_k$ respects the symmetries. This is not the case, but becomes true after replacing $\mathbf{qa}_k$ with a bigger category which we now define. Let $\widetilde{\mathbf{qa}}_k$ be the direct product of $\mathbf{qa}_k$ and the discrete category $\mathbf{F}_2(k)$. Thus the objects of $\widetilde{\mathbf{qa}}_k$ are pairs $\tilde D = (D, p)$ consisting of a quadratic algebra D and an element $p \in \mathbf{F}_2(k)$, called the *parity* of $\tilde D$, and the morphisms are

$$\mathrm{Mor}\big((D,p),(D',p')\big) = \begin{Bmatrix} \mathrm{Mor}(D,D') & \text{if } p = p' \\ \emptyset & \text{if } p \neq p' \end{Bmatrix}.$$

An object $\tilde D \in \widetilde{\mathbf{qa}}_k$ will be called even or odd if its parity is 0 or 1, respectively. For $R \in k$-**alg**, the base change of $\tilde D$ is $\tilde D \otimes R = (D \otimes R, p \otimes 1_R)$. We leave it to the reader to show that $\widetilde{\mathbf{qa}}_k$ becomes a tensor category with product

$$(D,p) \square (D',p') = (D \square D',\, p \dotplus p'),$$

unit $\tilde{I} = (I,0)$ and the associativity, left and right unit constraints $\tilde{\mathfrak{a}}$, $\tilde{\mathfrak{l}}$, $\tilde{\mathfrak{r}}$ derived in the obvious way from the corresponding ones in 6.1. The symmetry $\tilde{\mathfrak{c}}$ of $\widetilde{\mathbf{qa}}_k$ is defined as follows. First, $\mathbf{qa}_k$ is a symmetric tensor category with the symmetry

$$\mathfrak{c} = \mathfrak{c}_{D_1D_2} : D_1 \square D_2 \xrightarrow{\cong} D_2 \square D_1, \qquad \dot{x}_1 \square_{(\alpha_1,\alpha_2)} \dot{x}_2 \ \mapsto\ \dot{x}_2 \square_{(\alpha_2,\alpha_1)} \dot{x}_1, \quad (1)$$

see [11, Th. 2.6]. Now define $\tilde{\mathfrak{c}}$: $\tilde{D}_1 \square \tilde{D}_2 \to \tilde{D}_2 \square \tilde{D}_1$ for $\tilde{D}_i = (D_i, p_i) \in \widetilde{\mathbf{qa}}_k$ by

$$\tilde{\mathfrak{c}}_{\tilde{D}_1\tilde{D}_2} := \mathfrak{c}_{D_1D_2} \circ \mathfrak{h}_{D_1\square D_2}(p_1p_2) = \mathfrak{h}_{D_2\square D_1}(p_1p_2) \circ \mathfrak{c}_{D_1D_2}. \quad (2)$$

7.5. Lemma. *With the symmetries $\tilde{\mathfrak{c}}$ defined as above, $\widetilde{\mathbf{qa}}_k$ is a symmetric tensor category.*

Proof. Since the automorphisms σ^p have period two, it is clear that $(\tilde{\mathfrak{c}}_{\tilde{D}_1,\tilde{D}_2})^{-1} = \tilde{\mathfrak{c}}_{\tilde{D}_2,\tilde{D}_1}$. It remains to show the commutativity of the diagram

$$\begin{array}{ccccc} (\tilde{D}_1 \square \tilde{D}_2) \square \tilde{D}_3 & \xrightarrow{\tilde{\mathfrak{c}}\square \mathrm{Id}} & (\tilde{D}_2 \square \tilde{D}_1) \square \tilde{D}_3 & \xrightarrow{\tilde{\mathfrak{a}}} & \tilde{D}_2 \square (\tilde{D}_1 \square \tilde{D}_3) \\ \tilde{\mathfrak{a}} \downarrow & & & & \downarrow \mathrm{Id}\square\tilde{\mathfrak{c}} \\ \tilde{D}_1 \square (\tilde{D}_2 \square \tilde{D}_3) & \xrightarrow[\tilde{\mathfrak{c}}]{} & (\tilde{D}_2 \square \tilde{D}_3) \square \tilde{D}_1 & \xrightarrow[\tilde{\mathfrak{a}}]{} & \tilde{D}_2 \square (\tilde{D}_3 \square \tilde{D}_1) \end{array} \quad (1)$$

By 7.3.4 and because the automorphisms $\mathfrak{h}(p) = \sigma^p$ commute with morphisms of $\mathbf{qa}_k$, we can collect the powers of σ in going around the diagram. This yields for the upper leg

$$(\mathrm{Id} \square \tilde{\mathfrak{c}}) \circ \tilde{\mathfrak{a}} \circ (\tilde{\mathfrak{c}} \square \mathrm{Id}) = \sigma^{p_1p_2} \circ \sigma^{p_1p_3} \circ (\mathrm{Id} \square \mathfrak{c}) \circ \mathfrak{a} \circ (\mathfrak{c} \square \mathrm{Id}),$$

while the lower leg results in

$$\tilde{\mathfrak{a}} \circ \tilde{\mathfrak{c}} \circ \tilde{\mathfrak{a}} = \sigma^{p_1(p_2 \dot{+} p_3)} \circ \mathfrak{a} \circ \mathfrak{c} \circ \mathfrak{a}.$$

Now $\sigma^{p_1p_2} \circ \sigma^{p_1p_3} = \sigma^{p_1p_2 \dot{+} p_1p_3} = \sigma^{p_1(p_2 \dot{+} p_3)}$ because $p \mapsto \sigma^p$ is a group homomorphism, and $(\mathrm{Id} \square \mathfrak{c}) \circ \mathfrak{a} \circ (\mathfrak{c} \square \mathrm{Id}) = \mathfrak{a} \circ \mathfrak{c} \circ \mathfrak{a}$ because $\mathbf{qa}_k$ is a symmetric tensor category.

7.6. Definition. Let $\mathfrak{X}$ be a quadratic trace module. The *discriminant algebra with parity of* $\mathfrak{X}$ is defined as

$$\widetilde{\mathrm{Dis}}(\mathfrak{X}) = \big(\mathrm{Dis}(\mathfrak{X}), \mathrm{rk}(\mathfrak{X}) \ (\mathrm{mod}\ 2)\big).$$

For example, the discriminant algebras with parity of the split quadratic trace modules $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are now different, namely

$$\widetilde{\mathrm{Dis}}(\mathfrak{E}_1) = (\mathrm{Dis}(\mathfrak{E}_2), 1) \cong (I, 1), \qquad \widetilde{\mathrm{Dis}}(\mathfrak{E}_2) = (\mathrm{Dis}(\mathfrak{E}_2), 0) \cong (I, 0) = \tilde{I},$$

while their ordinary discriminant algebras are the same. If $\mathfrak{X}$ has nonconstant rank, $\mathrm{rk}(\mathfrak{X})$: $\mathrm{Spec}(k) \to \mathbb{Z}$ is locally constant, so $\mathrm{rk}(\mathfrak{X}) \ (\mathrm{mod}\ 2) \in \mathbf{F}_2(k)$. Since by definition there are no morphisms between quadratic trace modules of different rank, it is clear that $\widetilde{\mathrm{Dis}}$ is a functor from $\mathbf{qtm}_k$ to $\widetilde{\mathbf{qa}}_k$.

7.7. Theorem. $\widetilde{\mathrm{Dis}}$: $\mathbf{qtm}_k \to \widetilde{\mathbf{qa}}_k$ *is a symmetric tensor functor.*

Proof. Let $\mathfrak{X}_i = (X_i, Q_i, T_i, 1_i)$ be quadratic trace modules with parities $p_i = \mathrm{rk}(\mathfrak{X}_i) \pmod 2$. Then $\mathrm{rk}(\mathfrak{X}_1 \oplus \mathfrak{X}_2) \pmod 2 = p_1 \dotplus p_2$, and hence the isomorphism $\Theta = \Theta_{\mathfrak{X}_1\mathfrak{X}_2}$ of Theorem 6.5 induces an isomorphism

$$\tilde{\Theta}\colon \widetilde{\mathrm{Dis}}(\mathfrak{X}_1) \,\square\, \widetilde{\mathrm{Dis}}(\mathfrak{X}_2) \to \widetilde{\mathrm{Dis}}(\mathfrak{X}_1 \oplus \mathfrak{X}_2).$$

It is easy to check, using Th. 6.6, that with these isomorphisms $\widetilde{\mathrm{Dis}}$ is a tensor functor from $\mathbf{qtm}_k$ to $\widetilde{\mathbf{qa}}_k$. The symmetry in $\mathbf{qtm}_k$ is the switch $\omega\colon \mathfrak{X}_1 \oplus \mathfrak{X}_2 \to \mathfrak{X}_2 \oplus \mathfrak{X}_1$, $x_1 \oplus x_2 \mapsto x_2 \oplus x_1$. Thus for $\widetilde{\mathrm{Dis}}$ to be a symmetric tensor functor, it remains to check that the diagrams

$$\begin{array}{ccc} \widetilde{\mathrm{Dis}}(\mathfrak{X}_1) \,\square\, \widetilde{\mathrm{Dis}}(\mathfrak{X}_2) & \xrightarrow{\tilde{\Theta}} & \widetilde{\mathrm{Dis}}(\mathfrak{X}_1 \oplus \mathfrak{X}_2) \\ \Big\downarrow{\scriptstyle \tilde{c}} & & \Big\downarrow{\scriptstyle \widetilde{\mathrm{Dis}}(\omega)} \\ \widetilde{\mathrm{Dis}}(\mathfrak{X}_2) \,\square\, \widetilde{\mathrm{Dis}}(\mathfrak{X}_1) & \xrightarrow[\tilde{\Theta}]{} & \widetilde{\mathrm{Dis}}(\mathfrak{X}_2 \oplus \mathfrak{X}_1) \end{array} \tag{1}$$

commute. After decomposing the base ring, we may assume that $r_i = \mathrm{rk}(\mathfrak{X}_i)$ is constant on $\mathrm{Spec}(k)$. Then there are four cases, depending on the parity p_i of r_i. We do the case where r_1 and r_2 are odd and leave the other cases, which follow a similar pattern but are easier, to the reader.

Let f_i be bilinear forms on X_i representing Q_i and let Q_{ij} be the quadratic form of $\mathfrak{X}_i \oplus \mathfrak{X}_j$. Then $f_{ij} := (f_i \perp f_j) + T_i \otimes T_j$ is a bilinear form on $X_i \oplus X_j$ representing Q_{ij}, cf. 4.2. Put $D_i = \mathrm{Dis}(\mathfrak{X}_i)$ and $D_{ij} = \mathrm{Dis}(\mathfrak{X}_i \oplus \mathfrak{X}_j)$ for short, and let $z = \xi_1^\sharp \,\square_{f_1^\sharp \perp f_2^\sharp}\, \xi_2^\sharp \in D_1 \,\square\, D_2$, where we use the notations of Th. 6.5(d). Since $\tilde{\Theta}$ is just Θ as a map on $D_1 \,\square\, D_2$, we have by 6.5.7 that $\tilde{\Theta}(z) = s_{f_{12}}(\xi_1 \wedge \xi_2)$. We claim that the effect of going across and down in (1) is

$$\widetilde{\mathrm{Dis}}(\omega)\big(\tilde{\Theta}(z)\big) = \sigma_{D_{21}}\big(s_{f_{21}}(\xi_2 \wedge \xi_1)\big). \tag{2}$$

Indeed, let us note first that $(\bigwedge^{r_1+r_2} \omega)(\xi_1 \wedge \xi_2) = (-1)^{r_1 r_2} \xi_2 \wedge \xi_1 = -\xi_2 \wedge \xi_1$, since $\xi_i \in \bigwedge^{r_i} X_i$ and both r_1 and r_2 are odd. Hence, by 3.4.4,

$$\mathrm{Dis}(\omega)\big(s_g(\xi_1 \wedge \xi_2)\big) = -s_{f_{21}}(\xi_2 \wedge \xi_1), \tag{3}$$

where the pullback $g := \omega^*(f_{21})$ to $X_1 \oplus X_2$ is

$$g(x_1 \oplus x_2, y_1 \oplus y_2) = f_1(x_1, y_1) + f_2(x_2, y_2) + T_1(y_1) T_2(x_2).$$

Let $f := f_{12}$ for short. By the defining relations 3.3.6 of D_{12}, we have $s_f - s_g = \kappa_{fg} \cdot 1$, so we compute next κ_{fg}. Let $2n = r_1 + r_2$, and use the notations introduced in 4.3. Then by 11.6.2 and 3.3.3,

$$\begin{aligned}
\pi_n\big(f-f^\top+\mathbf{t}(f-g)\big)(\xi_1\wedge\xi_2) &= \operatorname{Pf}\begin{pmatrix} F_1-F_1^\top & (1+\mathbf{t})x^\top y \\ -(1+\mathbf{t})y^\top x & F_2-F_2^\top \end{pmatrix} \\
&= \operatorname{Pf}\begin{pmatrix} 0 & (1+\mathbf{t})x \\ -(1+\mathbf{t})x^\top & F_1-F_1^\top \end{pmatrix}\cdot \operatorname{Pf}\begin{pmatrix} 0 & y \\ -y^\top & F_2-F_2^\top \end{pmatrix} \\
&= (1+\mathbf{t})\operatorname{Pf}\begin{pmatrix} 0 & x \\ -x^\top & F_1-F_1^\top \end{pmatrix}\cdot \operatorname{Pf}\begin{pmatrix} 0 & y \\ -y^\top & F_2-F_2^\top \end{pmatrix} \\
&= (1+\mathbf{t})\operatorname{Pf}\begin{pmatrix} F_1-F_1^\top & x^\top y \\ -y^\top x & F_2-F_2^\top \end{pmatrix} \\
&= (1+\mathbf{t})\pi_n(f-f^\top)(\xi_1\wedge\xi_2) = (1+\mathbf{t})\tau_f(\xi_1\wedge\xi_2).
\end{aligned}$$

On the other hand, by 3.3.2,

$$\pi_n\big(f-f^\top+\mathbf{t}(f-g)\big) = \pi_n(f-f^\top)+\mathbf{t}\,\Pi_n(\mathbf{t}, f-f^\top, f-g).$$

By comparing coefficients at $\mathbf{t}$ we see that $\Pi_n(\mathbf{t}, f-f^\top, f-g) = \tau_f$ is independent of $\mathbf{t}$. Therefore $\kappa_{fg} = \Pi_n(-2, f-f^\top, f-g) = \tau_f$ and $s_f - s_g = \tau_f\cdot 1$. Since the trace of $s_f(\xi_1\wedge\xi_2)$ is $\tau_f(\xi_1\wedge\xi_2)$, it follows that

$$\sigma_{D_{12}}\big(s_f(\xi_1\wedge\xi_2)\big) = \tau_f(\xi_1\wedge\xi_2)\cdot 1 - s_f(\xi_1\wedge\xi_2) = -s_g(\xi_1\wedge\xi_2),$$

so (3) implies

$$\operatorname{Dis}(\omega)\big(\sigma_{D_{12}}(s_f(\xi_1\wedge\xi_2))\big) = s_{f_{21}}(\xi_2\wedge\xi_1).$$

But $\operatorname{Dis}(\omega)$ commutes with the standard involutions of D_{12} and D_{21}, whence

$$\operatorname{Dis}(\omega)\big(s_f(\xi_1\wedge\xi_2)\big) = \sigma_{D_{21}}\big(s_{f_{21}}(\xi_2\wedge\xi_1)\big).$$

This proves (2). On the other hand, going down and across in (1) yields, since Θ commutes with the involutions,

$$\begin{aligned}
z &\stackrel{\tilde{\mathfrak{c}}}{\longmapsto} \sigma_{D_2\square D_1}(\mathfrak{c}(z)) = \sigma_{D_2\square D_1}(\xi_2^\sharp \square_{\xi_2^\sharp\perp\xi_1^\sharp} \xi_1^\sharp) \\
&\stackrel{\tilde{\Theta}}{\longmapsto} \sigma_{D_{21}}\big(\Theta(\xi_2^\sharp \square_{\xi_2^\sharp\perp\xi_1^\sharp} \xi_1^\sharp)\big) = \sigma_{D_{21}}\big(s_{f_{21}}(\xi_2\wedge\xi_1)\big).
\end{aligned}$$

This completes the proof.

8. Separable quadratic trace modules

8.1. Definition. A quadratic trace module $\mathfrak{X}$ is called *separable* if its discriminant form $\Delta = \Delta_{\mathfrak{X}}$ is nonsingular; equivalently, if the discriminant $\delta_{\mathfrak{X}}$ is nonsingular or if the discriminant algebra $\mathrm{Dis}(\mathfrak{X})$ is a separable quadratic algebra (Lemma 3.6). Separability is preserved under arbitrary base changes and descends from faithfully flat base changes.

Since the discriminant form of the split quadratic trace module $\mathfrak{E}_r$ is the standard scalar product on k^r, it is clear that $\mathfrak{E}_r$ is separable. The goal of this section is to show that, conversely, a separable quadratic trace module is, locally in the étale topology, isomorphic to $\mathfrak{E}_r$.

In the sequel, a k-functor means a set-valued covariant functor on the category k-**alg**. Following [6], schemes are considered as special k-functors. The affine k-scheme defined by a k-algebra A is $\mathbf{Spec}(A)(R) = \mathrm{Hom}_{k\text{-}\mathbf{alg}}(A,R)$. For a k-module X let $X_{\mathbf{a}}$ denote the k-functor $R \mapsto X \otimes_k R$. If X is finitely generated and projective then $X_{\mathbf{a}}$ is an affine finitely presented k-scheme whose affine algebra is the symmetric algebra over the dual module X^*.

For an arbitrary quadratic trace module $\mathfrak{X}$, define k-functors $\mathbf{Y}$ and $\mathbf{U}$ by

$$\mathbf{Y}(R) = \{x \in X \otimes R : T(x) = 1,\ Q(x) = 0\}, \tag{1}$$

$$\mathbf{U}(R) = \{u \in \mathbf{Y}(R) :\ 1-u \text{ unimodular}\}, \tag{2}$$

for all $R \in k$-**alg**. Since $X_{\mathbf{a}}$ is finitely presented and affine so is $\mathbf{Y}$. Let $\alpha_1,\dots,\alpha_n$ be a set of generators of the dual module X^* and let $\varphi_i(x) = \alpha_i(1-x)$. Then $\mathbf{U} \subset \mathbf{Y}$ is the union of the open affine subschemes $\mathbf{Y}_i$ of $\mathbf{Y}$ where φ_i does not vanish, i.e., $u \in \mathbf{U}(R) \iff u \in \mathbf{Y}(R)$ and $\sum_i R\varphi_i(u) = R$, see [6, I, §1, 3.6]. Hence $\mathbf{U}$ is a quasi-affine finitely presented k-scheme. We will see that it plays the role of the unit sphere in Euclidean geometry. Note that $\mathbf{Y} = \mathbf{Spec}(\{0\})$ is empty ($\mathbf{Y}(R) = \emptyset$ for all $R \neq \{0\}$) if $r = \mathrm{rk}(\mathfrak{X}) = 0$, and $\mathbf{Y} = \mathbf{Spec}(k)$ is the one-point functor for $r = 1$, while $\mathbf{U}$ is empty for $r = 0,1$.

8.2. Proposition. *Let $\mathfrak{X} = (X,Q,T,1)$ be a quadratic trace module of rank r and let $u \in \mathbf{U}(k)$ (hence $r \geqslant 2$). Let $X' := u^{\perp}$ with respect to $\Delta = \Delta_{\mathfrak{X}}$, put $1' := 1-u$ and denote by Q' and T' the restrictions of Q and T to X'. Then $\mathfrak{X}' := (X',Q',T',1')$ is a quadratic trace module of rank $r-1$ and*

$$\mathfrak{X} = k\cdot u \oplus \mathfrak{X}' \cong \mathfrak{E}_1 \oplus \mathfrak{X}', \tag{1}$$

the direct sum of quadratic trace modules as in 1.5*. Moreover, $\mathfrak{X}$ is separable if and only if $\mathfrak{X}'$ is separable.*

Proof. We have $\Delta(u,u) = T(u)^2 - 2Q(u) = 1$, so the direct sum of modules in (1) is clear. Next, $\Delta(u,1-u) = \Delta(u,1) - 1 = T(u) - 1$ (by 1.1.4) $= 0$, which proves $1' \in X'$.

Clearly X' is finitely generated and projective of rank $r-1$, and $1'$ is unimodular by definition of $\mathbf{U}$. Furthermore, by 1.1.1, $T'(1') = T(1-u) = T(1) - 1 = r-1$ and

$$Q'(1') = Q(1-u) = Q(1) - B(1,u) + Q(u) = \binom{r}{2} - (r-1) + 0 = \binom{r-1}{2}.$$

Finally, let $x' \in X'$. Then $B'(1',x') = B(1-u,x') = T(1-u)T(x') - \Delta(1-u,x') = (r-1)T(x') - \Delta(1,x') + 0 = (r-2)T(x')$, by 1.1.4. Hence $\mathfrak{X}'$ is a quadratic trace module of rank $r-1$.

We show that $\mathfrak{X} \cong \mathfrak{E}_1 \oplus \mathfrak{X}'$ as quadratic trace modules. Clearly $1 = u \oplus 1'$ and $T(\lambda u \oplus x') = \lambda T(u) + T(x') = \lambda + T'(x')$. Moreover, $Q(\lambda u \oplus x') = \lambda^2 Q(u) + \lambda B(u,x') + Q(x') = 0 + \lambda\big(T(u)T(x') - \Delta(u,x')\big) + Q(x') = \lambda T'(x') + Q'(x')$ because $u \perp x'$ with respect to Δ. The statement concerning separability follows from 1.5.2.

8.3. Lemma. *Let $\mathfrak{X} = (X,Q,T,1)$ be separable of rank r. If $r \geqslant 1$ then $T\colon X \to k$ is surjective. If $r \geqslant 2$, the restriction of Q to $X^0 := \operatorname{Ker} T$ is primitive.*

Proof. By separability, Δ induces an isomorphism between X and its dual X^*, and 1.1.4 shows that $T \in X^*$ is the image of 1_X. Since 1_X is unimodular so is T. Hence there exists $u \in X^{**} \cong X$ with $T(u) = 1$.

For the second statement, we may assume that k is an algebraically closed field and then have to show that $Q\big|X^0 \neq 0$. If $r = 2$, $\mathfrak{X} = \operatorname{qt}(D)$ is the quadratic trace module determined by a separable quadratic algebra, so $D \cong k^2 = k\cdot e_1 \oplus k \cdot e_2$, and $D^0 = k\cdot(e_1 - e_2)$, with $Q(e_1 - e_2) = Q(e_1) - B(e_1,e_2) + Q(e_2) = -1$. If $r \geqslant 3$, pick an element $u \in X$ with $T(u) = 1$ so that $X = k\cdot u \oplus X^0$. Assume that Q vanishes on X^0. After choosing a basis in X^0, the matrix of Δ has the form $\begin{pmatrix} \lambda & v \\ v^\top & 0 \end{pmatrix}$ where $\lambda = \Delta(u,u) \in k$ and v is a row vector of length $r-1 \geqslant 2$. Such a matrix must be singular, contradiction.

8.4. Lemma. *If $\mathfrak{X}$ is separable of rank $r \geqslant 2$ over an algebraically closed field K then $\mathbf{U}(K) \neq \emptyset$.*

Proof. By Lemma 8.3 there exist $x,y \in X$ with $T(y) = 1$, $T(x) = 0$ and $Q(x) \neq 0$. Put $u = y + \lambda x$ and determine the scalar $\lambda \in K$ by the requirement $Q(u) = 0$. This yields the quadratic equation

$$\lambda^2 Q(x) + \lambda B(x,y) + Q(y) = 0$$

which has a solution since K is algebraically closed. If $u \neq 1$ we are done. Otherwise, $\binom{r}{2} = Q(1) = Q(u) = 0$ in K which implies $r \geqslant 3$. Also, $T(1) = T(u) = 1$ so $X = K\cdot 1 \oplus X^0$. Since Q does not vanish on X^0 and $\dim X^0 \geqslant 2$, there exists a non-zero isotropic vector $z \in X^0$. Put $\tilde{u} = 1 + z$. Then $\tilde{u} \neq 1$, $T(\tilde{u}) = T(1) = 1$ and $Q(\tilde{u}) = Q(1) + B(1,z) + Q(z) = 0 + (r-1)T(z) + 0 = 0$, as required.

8.5. Corollary. *A separable quadratic trace module $\mathfrak{X}$ over an algebraically closed field is split.*

Proof. If $\mathfrak{X}$ has rank $\leqslant 1$ this is evident, so we assume $r = \operatorname{rk}(\mathfrak{X}) \geqslant 2$. Then the assertion follows by induction from Prop. 8.2 and Lemma 8.4.

8.6. Theorem. *Let $\mathfrak{X}$ be a separable quadratic trace module of rank $r \geqslant 2$. Then* **U** *is a smooth quasi-affine finitely presented k-scheme. The geometric fibres of* **U** *have dimension $r-2$. They consist of two points if $r = 2$ and are connected and non-empty for $r \geqslant 3$.*

Proof. As noted in 8.1, **U** is quasi-affine and finitely presented. By [6, I, §4, Cor. 4.6], **U** is smooth if and only if, for every $R \in k$-**alg** and every ideal $\mathfrak{n}$ of square zero in R, the canonical map $\mathbf{U}(R) \to \mathbf{U}(R/\mathfrak{n})$ is surjective. After a base change from k to R we may assume $R = k$ to simplify notation. Denote the canonical maps $k \to \overline{k} := k/\mathfrak{n}$ and $X \to \overline{X} := X/\mathfrak{n}X = X \otimes \overline{k}/\mathfrak{n}$ by a bar, and let $v \in \mathbf{U}(k/\mathfrak{n})$. Decompose $\overline{\mathfrak{X}} = \overline{k} \cdot v \oplus \mathfrak{X}'$ with $\mathfrak{X}'$ separable of rank $r-1$ as in Prop. 8.2. Choose $x \in X$ with $\overline{x} = v$. Then $T(x) = 1 + \delta$ and $Q(x) = \varepsilon$ where $\delta, \varepsilon \in \mathfrak{n}$. Since $\mathfrak{n}$ has square zero, $1 + \delta \in k^\times$ with $(1+\delta)^{-1} = 1 - \delta$. After replacing x by $(1-\delta)x$, we may assume $T(x) = 1$. Then $\Delta(x,x) = T(x)^2 - B(x,x) = 1 - 2\varepsilon \in k^\times$, so X decomposes $X = k \cdot x \oplus M$ where $M = x^\perp$ with respect to Δ, and $\overline{M} = \overline{X}'$. By Lemma 8.3, there exists $w \in \overline{X}'$ with $T(w) = 1$. Choose $y \in M$ with $\overline{y} = w$. Then $T(y) = 1 + \gamma \in 1 + \mathfrak{n}$ is invertible, so after replacing y by $(1+\gamma)^{-1}y$, we have found an element $y \in M$ with $T(y) = 1$. Since also $T(x) = 1$, it follows that

$$0 = \Delta(x,y) = T(x)T(y) - B(x,y) = 1 - B(x,y).$$

Now put $u := x + \varepsilon(x-y)$. Then $\overline{u} = \overline{x} = v$, $T(u) = 1 + \varepsilon(1-1) = 1$, and

$$Q(u) = Q(x) + \varepsilon B(x, x-y) = Q(x) + \varepsilon\big(2Q(x) - B(x,y)\big) = \varepsilon + \varepsilon(2\varepsilon - 1) = 0.$$

It remains to show that $1-u$ is unimodular. Since $\overline{1} - v$ is unimodular in $\overline{X}$, there exists a linear form $\beta \in \overline{X}^*$ with $\beta(\overline{1} - v) = 1$. Now $\overline{X}^* \cong X^* \otimes \overline{k}$ since X is finitely generated and projective, so there exists $\alpha \in X^*$ with $\overline{\alpha} = \beta$. This implies $\overline{\alpha(1-u)} = 1$ and therefore $\alpha(1-u) \in 1 + \mathfrak{n} \subset k^\times$, as required.

To determine the geometric fibres of **U** we may, after a base change, assume that $k = K$ is an algebraically closed field. By Lemma 8.4 and Cor. 8.5, $\mathbf{U}(K) \neq \emptyset$ and $\mathfrak{X} \cong \mathfrak{E}_r$ is split. Put $m = r-1$ and identify X with K^r by means of the standard basis $e_0, \ldots, e_m$. Let Greek indices run from 0 to m and Latin indices from 1 to m. Then $\mathbf{Y}(K) \subset K^{m+1}$ is described by the equations

$$\sum_\lambda x_\lambda = 1, \qquad \sum_{\lambda < \mu} x_\lambda x_\mu = 0, \tag{1}$$

and $x \in \mathbf{U}(K)$ if in addition $x \neq 1_X = (1, \ldots, 1)$. For $r = 2$, $\mathbf{Y}(K) = \mathbf{U}(K) = \{(1,0), (0,1)\}$ consists of two points. Assume $r \geqslant 3$ and use the first equation of (1), $x_0 = 1 - \sum x_i$, to eliminate x_0 from the second equation. Then $\mathbf{Y}(K)$ becomes identified with the affine quadric in K^m with equation

$$f(x_1, \ldots, x_m) = \sum_i x_i - \sum_{i \leqslant j} x_i x_j = 0. \tag{2}$$

Since $m \geqslant 2$, it is easily seen that the polynomial $f(x_1, \ldots, x_m) \in K[x_1, \ldots, x_m]$ is irreducible, so $\mathbf{Y}(K)$ is an irreducible algebraic variety. Hence $\mathbf{U}(K)$ is connected [2, II, §4.1, Prop. 1].

8.7. Remark. In the situation of Th. 8.6, define

$$\mathbf{S}(R) = \mathbf{Y}(R) \cap \{1_{X_R}\}$$

for all $R \in k$-**alg**. Then **S** is a closed subscheme of **Y**, isomorphic to $\mathbf{Spec}(k/kd_1)$ where

$$d_1 = d_1(r) = \begin{Bmatrix} 2n-1 & \text{if } r = 2n \text{ is even} \\ n & \text{if } r = 2n+1 \text{ is odd} \end{Bmatrix}. \tag{1}$$

Moreover, **Y** is geometrically the union of **U** and **S**, and **U** is precisely the set of points of **Y** where the canonical projection $\mathbf{Y} \to \mathbf{Spec}(k)$ is smooth. The proof is left to the reader. As a consequence, we note:

$$\mathbf{U} = \mathbf{Y} \quad \iff \quad d_1(r) \cdot 1_k \in k^\times.$$

Clearly $d_1(r) = 1$ if and only if $r = 2$ or $r = 3$. Hence $\mathbf{U} = \mathbf{Y}$ is affine for $r = 2, 3$.

8.8. Theorem. *Let $\mathfrak{X}$ be a quadratic trace module of rank r. Then $\mathfrak{X}$ is separable if and only if there exists an étale cover R of k (i.e., an étale and faithfully flat $R \in k$-**alg**) such that $\mathfrak{X} \otimes R \cong \mathfrak{E}_r \otimes R$.*

Proof. The condition is necessary because separability descends from faithfully flat base extensions. The proof of the converse is by induction on r. The cases $r = 0, 1$ being trivial, we assume $r \geqslant 2$. By Th. 8.6 and [7, Cor. 17.16.3(ii)], there exists an étale cover $k' \in k$-**alg** such that $\mathbf{U}(k') \neq \emptyset$. Choose $u \in \mathbf{U}(k')$. Then Prop. 8.2 shows that $\mathfrak{X} \otimes k' \cong (\mathfrak{E}_1 \otimes k') \oplus \mathfrak{X}'$ where $\mathfrak{X}'$ is separable of rank $r-1$ over k'. By induction, $\mathfrak{X}' \otimes_{k'} R \cong \mathfrak{E}_{r-1} \otimes_k R$ where R is an étale cover of k' and hence of k. It follows that $\mathfrak{X} \otimes_k R \cong (\mathfrak{E}_1 \otimes R) \oplus (\mathfrak{X}' \otimes_{k'} R) \cong \mathfrak{E}_r \otimes R$.

9. The automorphism group I

9.1. Definition. Let $\mathfrak{X} = (X, Q, T, 1)$ be a quadratic trace module of rank r. An automorphism of $\mathfrak{X}$ is an element g of $\mathrm{GL}(X)$ such that

$$g(1) = 1 \quad \text{and} \quad T(g(x)) = T(x), \quad Q(g(x)) = Q(x), \quad \text{for all } x \in X. \tag{1}$$

We denote by $\mathrm{Aut}(\mathfrak{X})$ the set of all automorphisms of $\mathfrak{X}$, and let $\mathbf{G} = \mathbf{Aut}(\mathfrak{X})$ be the k-group functor $\mathbf{G}(R) = \mathbf{Aut}(\mathfrak{X} \otimes R)$, for all $R \in k$-**alg**. Note that the automorphism group is trivial if $r \leqslant 1$.

From the fact that X is finitely generated and projective as a k-module it follows easily that **G** is an affine finitely presented k-group scheme.

9.2. Lemma. *Let $\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$ be the Lie algebra of $\mathbf{G} = \mathbf{Aut}(\mathfrak{X})$. The following conditions are equivalent for an element A of $\mathrm{End}(X)$:*

(i) $A \in \mathfrak{g}$,
(ii) $A(1) = 0$ *and* $T(A(x)) = B(x, A(x)) = 0$, *for all* $x \in X$,
(iii) $A(1) = 0$ *and* $\Delta(x, A(x)) = 0$, *for all* $x \in X$.

Proof. A belongs to $\mathfrak{g}$ if and only if $\mathrm{Id}+\varepsilon A \in \mathbf{G}(k(\varepsilon))$ where $k(\varepsilon)$ is the algebra of dual numbers. Now the equivalence of (i) and (ii) follows easily from 9.1.1. Next, we have $\Delta(x,A(x)) = T(x)T(A(x)) - B(x,A(x))$ which shows (ii) implies (iii). On the other hand, $\Delta(1,x) = T(x)$ by 1.1.4, and from $\Delta(x,A(x)) = 0$ for all x we get by linearization that $\Delta(x,A(y)) = -\Delta(A(x),y)$. Hence $A(1) = 0$ implies that $T(A(x)) = \Delta(1,A(x)) = -\Delta(A(1),x) = 0$, so that (iii) implies (ii).

9.3. Theorem. *Let $\mathfrak{X}$ be a separable quadratic trace module of rank $r \geqslant 2$.*

(a) $\mathbf{G} = \mathbf{Aut}(\mathfrak{X})$ *is smooth of fibre-dimension* $\binom{r-1}{2}$.

(b) *If* $\mathfrak{X} = \mathfrak{E}_r$ *is split,* $\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$ *is the set of alternating $r \times r$ matrices with all row sums equal to zero.*

(c) *The "unit sphere"* $\mathbf{U}$ *of* 8.1.2 *is a homogeneous space under* $\mathbf{G}$ *in the following sense: If* $\mathbf{U}(k) \neq \emptyset$*, choose* $u \in \mathbf{U}(k)$ *and decompose* $\mathfrak{X} = k \cdot u \oplus \mathfrak{X}'$ *as in Proposition* 8.2*. Then* $\mathbf{H} := \mathbf{Cent}_{\mathbf{G}}(u) \cong \mathbf{Aut}(\mathfrak{X}')$*, and the orbit map* $\beta\colon \mathbf{G} \to \mathbf{U}$*,* $g \mapsto g(u)$*, is smooth and surjective, so that* $\mathbf{G}/\mathbf{H} \cong \mathbf{U}$ *as étale sheaves. In general, the map* $\vartheta\colon \mathbf{G} \times \mathbf{U} \to \mathbf{U} \times \mathbf{U}$*,* $(g,u) \mapsto (u,g(u))$*, is smooth and surjective.*

Proof. (a), (b) By localization we may assume that $\dot{X} = X/k\cdot 1$ is free, so X has a basis of the form $x_0 = 1, x_1, \ldots, x_m$ where $m = r-1$. We first show that $g \in \mathrm{GL}(X)$ belongs to $\mathbf{G}(k)$ if and only if

$$g(1) = 1, \tag{1}$$
$$T(g(x_i)) = T(x_i), \quad \text{for } i = 1,\ldots,m, \tag{2}$$
$$Q(g(x_i)) = Q(x_i), \quad \text{for } i = 1,\ldots,m, \tag{3}$$
$$B(g(x_i),g(x_j)) = B(x_i,x_j), \quad \text{for } 1 \leqslant i < j \leqslant m. \tag{4}$$

These conditions are obviously necessary. Now suppose that they hold. Then (1) implies that (2) holds for $i = 0$ as well so we have $T \circ g = T$. Similarly, (3) holds for $i = 0$ and we also have $B(g(1),g(x_j)) = (r-1)T(g(x_j)) = (r-1)T(x_j) = B(1,x_j)$, for $j = 1,\ldots,m$. This implies $Q \circ g = Q$, so g is an automorphism.

Clearly, (1) – (3) are polynomial equations in the entries of g (where we identify g with its matrix with respect to the basis $x_0,\ldots,x_m$). Note that (1) amounts to r scalar equations for the components of $g(1)$. Since all this remains valid in any scalar extension, we see that $\mathbf{G}$ is defined by

$$r + 2(r-1) + \binom{r-1}{2} = r^2 - \binom{r-1}{2}$$

polynomial equations.

To prove smoothness of $\mathbf{G}$, it suffices by [6, II, §5, Prop. 2.7] that for every prime ideal $\mathfrak{p}$ of k, the Lie algebra of $\mathbf{G} \otimes \boldsymbol{\kappa}(\mathfrak{p})$ has dimension $\binom{r-1}{2}$. Since $\mathbf{G} \otimes \boldsymbol{\kappa}(\mathfrak{p}) \cong \mathbf{Aut}(\mathfrak{X} \otimes \boldsymbol{\kappa}(\mathfrak{p}))$ and $\mathfrak{X} \otimes \boldsymbol{\kappa}(\mathfrak{p})$ splits over the algebraic closure of $\boldsymbol{\kappa}(\mathfrak{p})$ by Cor. 8.5, this will follow once we have established the description of $\mathfrak{g}$ in (b). Let, then, $\mathfrak{X} = \mathfrak{E}_r$ be split and let e_i be the standard basis of k^r. By 1.2, Δ is the standard scalar product on k^r. Hence $A \in \mathrm{Mat}_r(k)$ is alternating if and only if $\Delta(x,A(x)) = 0$ for all $x \in k^r$, and since $1 = 1_X$ is the vector with all components equal to 1, $A(1) = 0$ means that all row sums of A zero. In view of Lemma 9.2(iii) this completes the proof of (a) and (b).

(c) Let $u \in \mathbf{U}(k)$. Since $\mathbf{G}$ leaves the discriminant form Δ invariant and $\mathfrak{X}'$ is the orthogonal complement of u with respect to Δ, we see that $\mathbf{H}$ is the isomorphic image of $\mathbf{Aut}(\mathfrak{X}')$ under the map $g' \mapsto \mathrm{Id}_{k\cdot u} \oplus g'$, so we identify $\mathbf{Aut}(\mathfrak{X}')$ and $\mathbf{H}$. We show that $\mathbf{G}/\mathbf{H} \cong \mathbf{U}$ as sheaves in the étale topology, i.e., that for every $R \in k\text{-}\mathbf{alg}$ and every $v \in \mathbf{U}(R)$ there exists an étale cover S of R and $g \in \mathbf{G}(S)$ such that $g(u) = v$. After a base change, we may assume $R = k$ for easier notation. By Prop. 8.2, $\mathfrak{X} = k \cdot v \oplus \mathfrak{X}''$ where $\mathfrak{X}''$ is also separable of rank $r-1$. Theorem 8.8 shows that there exists an étale cover E of k and an isomorphism $h \colon \mathfrak{X}' \otimes E \to \mathfrak{X}'' \otimes E$. Define $g(u) = v$ and $g\big|_{\mathfrak{X}'\otimes E} = h$. Then $g \in \mathbf{G}(E)$ and $g(u) = v$.

Since $\mathbf{H} \cong \mathbf{Aut}(\mathfrak{X}')$ is smooth by (a), in particular, flat, it follows from [6, III, §3, Proposition 2.5(a), Corollary 2.6] that β is faithfully flat and smooth. Hence so are $\beta \times \mathrm{Id}_{\mathbf{U}} \colon \mathbf{G} \times \mathbf{U} \to \mathbf{U} \times \mathbf{U}$ and $\mathrm{Id}_{\mathbf{G}} \times \beta \colon \mathbf{G} \times \mathbf{G} \to \mathbf{G} \times \mathbf{U}$. One checks that the diagram

$$\begin{array}{ccc} \mathbf{G} \times \mathbf{G} & \longrightarrow & \mathbf{G} \times \mathbf{U} \\ {\scriptstyle \mathrm{Id}_{\mathbf{G}} \times \beta}\big\downarrow & & \big\downarrow{\scriptstyle \vartheta} \\ \mathbf{G} \times \mathbf{U} & \xrightarrow[\beta \times \mathrm{Id}_{\mathbf{U}}]{} & \mathbf{U} \times \mathbf{U} \end{array}$$

is Cartesian, where the top arrow is given by $(g,h) \mapsto (hg^{-1}, g(u))$. By faithfully flat descent, ϑ is faithfully flat and smooth.

In general, there exists a faithfully flat $R \in k\text{-}\mathbf{alg}$ such that $\mathbf{U}(R) \neq \emptyset$ [7, Corollary 17.16.2]. Here we use the fact that the canonical projection $\mathbf{U} \to \mathbf{Spec}(k)$ is surjective, hence (by smoothness of $\mathbf{U}$) faithfully flat, cf. Theorem 8.6. By what we proved already, $\vartheta \otimes R$ is faithfully flat and smooth, and therefore so is ϑ by faithfully flat descent.

9.4. The Dickson homomorphism. Since the discriminant algebra $\mathrm{Dis}(\mathfrak{X})$ of a quadratic trace module $\mathfrak{X}$ depends functorially on $\mathfrak{X}$ (cf. 3.4) and is compatible with base change, there is a homomorphism

$$\mathrm{Dick} = \mathrm{Dick}_{\mathfrak{X}} \colon \mathbf{Aut}(\mathfrak{X}) \to \mathbf{Aut}\big(\mathrm{Dis}(\mathfrak{X})\big), \qquad g \mapsto \mathrm{Dis}(g), \tag{1}$$

called the *Dickson homomorphism*, because it has properties similar to the classical Dickson homomorphism for orthogonal groups of even rank.

Suppose that $\mathfrak{X}$ is separable, whence also $D = \mathrm{Dis}(\mathfrak{X})$ is a separable quadratic algebra by 8.1. Then the canonical homomorphism $\mathfrak{h}_D \colon \mathbf{Z}_2 \to \mathbf{Aut}(D)$ of 7.2 is an isomorphism [11, 5.3]. In fact, it is the *unique* isomorphism between $\mathbf{Z}_2$ and $\mathbf{Aut}(D)$ because the automorphism group of the group scheme $\mathbf{Z}_2$ is trivial. We then define

$$\mathrm{dick} = \mathrm{dick}_{\mathfrak{X}} := \mathfrak{h}_D^{-1} \circ \mathrm{Dick} \colon \mathbf{Aut}(\mathfrak{X}) \to \mathbf{Z}_2. \tag{2}$$

Let in particular $A \in k\text{-}\mathbf{alg}$ be finitely generated and projective as a k-module, and let $\mathfrak{X} = \mathrm{qt}(A)$ be the associated quadratic trace module as in 1.3. Clearly, $\mathbf{Aut}(A) \subset \mathbf{Aut}(\mathfrak{X})$, and we define the *Dickson homomorphism of A* as the restriction of $\mathrm{Dick}_{\mathfrak{X}}$:

$$\mathrm{Dick}_A = \mathrm{Dick}_{\mathfrak{X}}\big|_{\mathbf{Aut}(A)} \colon \mathbf{Aut}(A) \to \mathbf{Aut}(\mathrm{Dis}(A)). \tag{3}$$

If A is étale, we put in analogy to (2),

$$\mathrm{dick}_A = \mathfrak{h}_D^{-1} \circ \mathrm{Dick}_A : \mathbf{Aut}(A) \to \mathbf{Z}_2. \tag{4}$$

An explicit formula for $\mathrm{Dick}_{\mathfrak{X}}$ is as follows. First let $\mathfrak{X}$ have even rank $r = 2n$ and let $g \in \mathrm{Aut}(\mathfrak{X})$. Since $\mathrm{Dis}(\mathfrak{X})$ is a shift of $\mathfrak{D}(Q)$, the automorphism $\mathrm{Dick}(g)$ of D is the same as the automorphism $\mathfrak{D}(g) \in \mathrm{Aut}(\mathfrak{D}(Q))$ (cf. 3.1(c)). Therefore, [12, 2.4.1] shows that it is given by

$$\mathrm{Dick}(g) \cdot s_f(\xi) = \kappa_{f,g^*(f)}(\xi) \cdot 1 + \det(g) \cdot s_f(\xi) \tag{5}$$

where f is a representative of Q and $\xi \in \bigwedge^{2n} X$. If $\mathfrak{X}$ has odd rank $2n+1$ then by definition, $\mathrm{Dis}(\mathfrak{X}) = \mathrm{Dis}(\mathfrak{E}_1 \oplus \mathfrak{X})$, and $\mathrm{Dick}(g) = \mathrm{Dis}(\mathrm{Id}_{\mathfrak{E}_1} \oplus g)$. Hence the above formula applies with the appropriate modifications. Since $\bigwedge^r X = \mathrm{Dis}(\mathfrak{X})/k \cdot 1 \cong \bigwedge^2 \mathrm{Dis}(\mathfrak{X})$, (5) implies that

$$\det(g) = \det(\mathrm{Dick}(g)). \tag{6}$$

If $\mathfrak{X}$ is separable, the analogous formula for dick is

$$\det(g) = \chi(\mathrm{dick}(g)) = 1 - 2\,\mathrm{dick}(g), \tag{7}$$

cf. 7.1 and 7.2.3.

We use this to obtain an explicit formula for $p = \mathrm{dick}(g)$ in case $\mathfrak{X} = \mathfrak{E}_{2n}$ is split. Identify bilinear forms with matrices by means of the standard basis e_i and put $\xi = e_1 \wedge \cdots \wedge e_{2n}$. By 3.7, $f = U_{2n}$ is a representative of Q and $D = \mathrm{Dis}(\mathfrak{X}) = k \cdot 1 \oplus k \cdot z$ where $z = s_f(\xi)$ satisfies $z^2 = z$; in particular, $T_D(z) = 1$. Hence (5) and (7) show $\mathrm{Dick}(g) \cdot z = \kappa_{f,g^*(f)}(\xi) \cdot 1 + (1-2p)z$. On the other hand, by 7.2.2, $\mathrm{Dick}(g) \cdot z = \mathfrak{h}_D(p) \cdot z = pT_D(z) \cdot 1 + (1-2p)z$. By comparison, we obtain the formula

$$\mathrm{dick}(g) = \kappa_{f,g^*(f)}(e_1 \wedge \cdots \wedge e_{2n}). \tag{8}$$

9.5. Lemma. *Let $\mathfrak{X}_1, \mathfrak{X}_2$ be quadratic trace modules and $\mathfrak{X} = \mathfrak{X}_1 \oplus \mathfrak{X}_2$ their direct sum. Let $\mathbf{G}_i = \mathbf{Aut}(\mathfrak{X}_i)$, $\mathbf{G} = \mathbf{Aut}(\mathfrak{X})$ and put $D_i := \mathrm{Dis}(\mathfrak{X}_i)$ and $D := \mathrm{Dis}(\mathfrak{X})$. Denote by $\theta\colon \mathbf{Aut}(D_1 \,\square\, D_2) \to \mathbf{Aut}(D)$ the isomorphism induced by the isomorphism $\Theta\colon D_1 \,\square\, D_2 \to D$ of* 6.5, *and let $\mathrm{Dick}_i\colon \mathbf{G}_i \to \mathbf{Aut}(D_i)$ be the Dickson homomorphisms. Then the following diagram is commutative:*

$$\begin{array}{ccccc} \mathbf{G}_1 \times \mathbf{G}_2 & & \xrightarrow{\mathrm{Dick}_1 \times \mathrm{Dick}_2} & & \mathbf{Aut}(D_1) \times \mathbf{Aut}(D_2) \\ \oplus \downarrow & & & & \downarrow \square \\ \mathbf{G} & \xrightarrow[\mathrm{Dick}]{} & \mathbf{Aut}(D) & \xleftarrow[\theta]{} & \mathbf{Aut}(D_1 \,\square\, D_2) \end{array} \tag{1}$$

where the left vertical arrow is the embedding $(g_1, g_2) \mapsto g_1 \oplus g_2 = \left(\begin{smallmatrix} g_1 & 0 \\ 0 & g_2 \end{smallmatrix}\right)$ and the right vertical arrow is the homomorphism introduced in 7.3.1. *If the $\mathfrak{X}_i$ (and* *therefore $\mathfrak{X}$) are separable, the analogous diagram for* dick *is*

$$\begin{CD} \mathbf{G}_1 \times \mathbf{G}_2 @>{\mathrm{dick}_1 \times \mathrm{dick}_2}>> \mathbf{Z}_2 \times \mathbf{Z}_2 \\ @V{\oplus}VV @VV{\dot{+}}V \\ \mathbf{G} @>>{\mathrm{dick}}> \mathbf{Z}_2 \end{CD} \tag{2}$$

Proof. Let $g_i \in \mathbf{G}_i(k)$ and $h_i = \mathrm{Dick}_i(g_i) \in \mathrm{Aut}(D_i)$. Since Θ is functorial, the following diagram is commutative:

$$\begin{CD} D_1 \,\square\, D_2 @>{h_1 \square h_2}>> D_1 \,\square\, D_2 \\ @V{\Theta}V{\cong}V @V{\cong}V{\Theta}V \\ D @>>{\mathrm{Dick}(g_1 \oplus g_2)}> D \end{CD}$$

By definition of θ we have $\theta(h_1 \,\square\, h_2) = \Theta \circ (h_1 \,\square\, h_2) \circ \Theta^{-1}$. Hence $\theta(h_1 \,\square\, h_2) = \mathrm{Dick}(g_1 \oplus g_2)$, which shows (1) is commutative when the group functors are evaluated at $R = k$. Since everything is compatible with base change, we have (1).

In the separable case, (2) follows from 7.3.3, the definition of dick in 9.4.2 and the fact that $\theta \circ \mathfrak{h}_{D_1 \square D_2} \colon \mathbf{Z}_2 \to \mathbf{Aut}(D)$ is an isomorphism and therefore agrees with $\mathfrak{h}_D$.

9.6. Lemma. *Let $\mathfrak{X}$ be a quadratic trace module of rank two, thus $\mathfrak{X} = \mathrm{qt}(D)$ where D is a quadratic algebra, cf.* 1.4. *Then $\mathbf{Aut}(\mathfrak{X}) = \mathbf{Aut}(D)$, and* Dick: $\mathbf{Aut}(D) \to \mathbf{Aut}(\mathrm{Dis}(D))$ *is the isomorphism induced by the isomorphism $\Phi_D \colon D \to \mathrm{Dis}(D)$ of* 3.5.2. *Moreover, the following diagram is commutative:*

$$\begin{array}{ccc} \mathbf{Aut}(D) & \xrightarrow[\cong]{\ \mathrm{Dick}\ } & \mathbf{Aut}(\mathrm{Dis}(D)) \\ & {\scriptstyle \mathfrak{h}_D} \nwarrow \quad \nearrow {\scriptstyle \mathfrak{h}_{\mathrm{Dis}(D)}} & \\ & \mathbf{Z}_2 & \end{array} \tag{1}$$

If $\mathfrak{X}$ is separable, equivalently, if D is étale, then

$$\mathrm{dick} = \mathfrak{h}_D^{-1} : \mathbf{Aut}(D) \to \mathbf{Z}_2; \tag{2}$$

in particular,

$$\mathrm{dick}(\sigma_D) = 1 \in \mathbf{Z}_2(k), \tag{3}$$

where σ_D is the involution of D, cf. 7.2.

Proof. After a base change, it suffices to show that the diagram

$$\begin{CD} D @>{g}>> D \\ @V{\Phi}VV @VV{\Phi}V \\ \mathrm{Dis}(\mathfrak{X}) @>>{\mathrm{Dick}(g)}> \mathrm{Dis}(\mathfrak{X}) \end{CD}$$

is commutative for all $g \in \mathrm{Aut}(D) = \mathrm{Aut}(\mathfrak{X})$. Let f be a bilinear form representing Q and let $x \in D$. Then by 3.5.3 and since $g(1) = 1$,

$$\Phi(g(x)) = f(g(x),1)\cdot 1 + s_f(1 \wedge g(x)) = f(g(x),1)\cdot 1 + \det(g)\cdot s_f(1 \wedge x),$$

while by 9.4.5,

$$\begin{aligned}\mathrm{Dick}(g)\big(\Phi(x)\big) &= \kappa_{f,g^*(f)}(1\wedge x)\cdot 1 + f(x,1)\cdot 1 + \det(g)\cdot s_f(1\wedge x)\\ &= \big(f(1,x) - f(1,g(x)) + f(x,1)\big)\cdot 1 + \det(g)\cdot s_f(1\wedge x).\end{aligned}$$

This proves our assertion because the trace form $T(x) = f(1,x) + f(x,1)$ is invariant under g. Being an isomorphism of quadratic algebras, Φ respects the involutions. Hence $\mathrm{Dick}(\sigma_D) = \sigma_{\mathrm{Dis}(D)}$, from which (1) follows immediately. Finally, (2) and (3) follow easily from (1) and the definition of dick in 9.4.2.

9.7. Theorem. *Let $\mathfrak{X}$ be a separable quadratic trace module of rank $r \geqslant 2$, with automorphism group $\mathbf{G} = \mathbf{Aut}(\mathfrak{X})$ and Dickson homomorphism* dick: $\mathbf{G} \to \mathbf{Z}_2$, *and let $\mathbf{G}^+ = \mathbf{Ker}(\mathrm{dick})$.*

(a) *If $\mathfrak{X}$ splits off a direct summand of rank two; in particular, if $\mathfrak{X}$ is split, then* dick *has sections in the category of group schemes. In general,* dick *has sections locally in the étale topology, so the sequence*

$$1 \longrightarrow \mathbf{G}^+ \xrightarrow{\ \mathrm{inc}\ } \mathbf{G} \xrightarrow{\ \mathrm{dick}\ } \mathbf{Z}_2 \longrightarrow 0$$

is exact in the étale topology. Moreover, $\mathbf{G}^+$ is smooth and dick *is smooth and surjective.*

(b) *$\mathbf{G}^+$ has connected fibres. If $r \geqslant 3$ then $\mathbf{U}$ is a homogeneous space under $\mathbf{G}^+$; i.e., Theorem* 9.3(c) *holds for $\mathbf{G}^+$ and $\mathbf{H}^+ = \mathbf{G}^+ \cap \mathbf{H}$ instead of $\mathbf{G}$ and $\mathbf{H}$.*

Proof. (a) Assume that $\mathfrak{X} = \mathfrak{X}_1 \oplus \mathfrak{X}_2$ where $\mathfrak{X}_1$ has rank two, and let $\mathbf{G}_i = \mathbf{Aut}(\mathfrak{X}_i)$. By Lemma 9.6, $\mathrm{dick}_1\colon \mathbf{G}_1 \to \mathbf{Z}_2$ is an isomorphism. Now it follows from 9.5.2 that a section $s\colon \mathbf{Z}_2 \to \mathbf{G}$ of dick is given by $s(p) = \mathrm{dick}_1^{-1}(p) \oplus \mathrm{Id}_{\mathfrak{X}_2}$, for all $p \in \mathbf{Z}_2(R)$, $R \in k$-**alg**. Hence $\mathbf{G}$ is the semidirect product of $\mathbf{G}^+$ and $\mathbf{Z}_2$; in particular, $\mathbf{G}$ is isomorphic to $\mathbf{G}^+ \times \mathbf{Z}_2$ as a k-scheme. Since $\mathbf{G}$ is smooth by Th. 9.3 it follows that $\mathbf{G}^+$ is smooth as well. Hence dick is smooth and obviously surjective.

In general, $\mathfrak{X}$ splits over an étale cover of k by Theorem 8.8, so the assertions follow by descent from the split case.

(b) We proceed by induction on r. For $r = 2$, dick: $\mathbf{G} \to \mathbf{Z}_2$ is an isomorphism by Lemma 9.6, whence $\mathbf{G}^+$ is trivial. Now let $r \geqslant 3$. We show that $\mathbf{U}$ is a homogeneous space under $\mathbf{G}^+$. First, assume there exists a section $u \in \mathbf{U}(k)$, decompose $\mathfrak{X} = k\cdot u \oplus \mathfrak{X}'$ as in 8.2 and let $\mathbf{H}$ be the isotropy group of u in $\mathbf{G}$. By Th. 9.3(c), $\mathbf{G}' := \mathbf{Aut}(\mathfrak{X}')$ is isomorphic to $\mathbf{H}$ under the map $g' \mapsto \mathrm{Id}_{k\cdot u} \oplus g'$. From 9.5.2, specialized to the present situation (where now $\mathbf{G}_1 = \mathbf{Aut}(k\cdot u)$ is trivial and $\mathbf{G}_2 = \mathbf{G}'$) it follows that the restriction of dick to $\mathbf{H}$ corresponds to the Dickson homomorphism dick$'$ of $\mathbf{G}'$. This implies

$$\mathbf{H}^+ = \mathbf{Cent}_{\mathbf{G}^+}(u) \cong (\mathbf{G}')^+. \tag{1}$$

We show that $\mathbf{U} \cong \mathbf{G}^+/\mathbf{H}^+$ as étale sheaves. Let $v \in \mathbf{U}(R)$, $R \in k$-**alg**. By Th. 9.3(c), there exists an étale cover E of R and an element $g \in \mathbf{G}(E)$ with $g(u) = v$. We now modify g to an element g^+ having the same property.

Since $r-1 \geqslant 2$, $\mathrm{dick}'\colon \mathbf{G}' \to \mathbf{Z}_2$ is an étale epimorphism by (a), so there exists an étale cover E' of E and $g' \in \mathbf{G}'(E')$ such that $\mathrm{dick}'(g') = \mathrm{dick}(g)^{-1}$. Put $g^+ := g \circ (\mathrm{Id} \oplus g') \in \mathbf{G}(E')$. Then still $g^+(u) = v$ and furthermore,

$$\mathrm{dick}(g^+) = \mathrm{dick}(g) \dotplus \mathrm{dick}(\mathrm{Id} \oplus g') = \mathrm{dick}(g) \dotplus \mathrm{dick}'(g') = 0,$$

so $g^+ \in \mathbf{G}^+(E')$. This proves that $\mathbf{U} \cong \mathbf{G}^+/\mathbf{H}^+$ as étale sheaves. Now the same arguments as in the proof of Th. 9.3(c) show that the orbit map $\beta\colon \mathbf{G}^+ \to \mathbf{U}$ is smooth and surjective, and so is the map $\vartheta\colon \mathbf{G}^+ \times \mathbf{U} \to \mathbf{U} \times \mathbf{U}$ even when $\mathbf{U}(k)$ is empty.

By induction and (1), $\mathbf{H}^+ \cong (\mathbf{G}')^+$ has connected fibres, and by Th. 8.6 so does $\mathbf{U}$. Since $\beta\colon \mathbf{G}^+ \to \mathbf{U}$ is faithfully flat and therefore open it follows easily that $\mathbf{G}^+$ has connected fibres.

9.8. The sign homomorphism. The constant k-group scheme defined by the symmetric group $\mathfrak{S}_r$ is denoted $\boldsymbol{\mathfrak{S}}_r$. An element of $\boldsymbol{\mathfrak{S}}_r(R)$ can be considered as a locally constant map from the spectrum of R to $\mathfrak{S}_r$ or as a complete family of orthogonal idempotents $(\varepsilon_\pi)_{\pi \in \mathfrak{S}_r}$ of R, with multiplication $(\varepsilon_\pi) \cdot (\varepsilon'_\pi) = (\varepsilon''_\pi)$, where $\varepsilon''_\pi = \sum_{\sigma\tau=\pi} \varepsilon_\sigma \varepsilon'_\tau$. The sign homomorphism

$$\mathrm{sgn}\colon \mathfrak{S}_r \to \mathbb{Z}/2\mathbb{Z}, \qquad \mathrm{sgn}(\pi) = \begin{Bmatrix} 0 & \text{if } \pi \text{ is even} \\ 1 & \text{if } \pi \text{ is odd} \end{Bmatrix}$$

induces a homomorphism

$$\mathrm{sgn}\colon \boldsymbol{\mathfrak{S}}_r \to \mathbf{Z}_2, \qquad \mathrm{sgn}\big((\varepsilon_\pi)_{\pi \in \mathfrak{S}_r}\big) = \sum_{\pi \in \mathfrak{S}_r \setminus \mathfrak{A}_r} \varepsilon_\pi, \tag{1}$$

where $\mathfrak{A}_r$ denotes the alternating group.

Let $E_r = k^r$ be the split étale algebra of rank r, with standard basis $e_1, \ldots, e_r$, and let $P_\pi \in \mathrm{GL}_r(k)$ be defined by $P_\pi(e_i) = e_{\pi(i)}$. It is well known that the map

$$\eta_r\colon \boldsymbol{\mathfrak{S}}_r \to \mathbf{Aut}(E_r), \qquad (\varepsilon_\pi)_{\pi \in \mathfrak{S}_r} \;\mapsto\; \sum_{\pi \in \mathfrak{S}_r} \varepsilon_\pi P_\pi \tag{2}$$

is an isomorphism.

9.9. Lemma. *The Dickson homomorphism of E_r (cf.* 9.4.4*) is induced by the sign homomorphism; i.e., the diagram*

$$\begin{array}{ccc} \boldsymbol{\mathfrak{S}}_r & \xrightarrow{\ \mathrm{sgn}\ } & \mathbf{Z}_2 \\ {\scriptstyle \eta_r}\big\downarrow{\scriptstyle \cong} & \overset{\mathrm{dick}_{E_r}}{\nearrow} & \big\uparrow{\scriptstyle \mathrm{dick}_{\mathfrak{E}_r}} \\ \mathbf{Aut}(E_r) & \xrightarrow[\ \mathrm{inc}\ ]{} & \mathbf{Aut}(\mathfrak{E}_r) \end{array}$$

is commutative.

Proof. We may assume $r \geqslant 2$, otherwise both sgn and dick are trivial. Let $\tau_{12} \in \mathfrak{S}_r$ be the transposition of 1 and 2. Decompose $\mathfrak{E}_r = \mathrm{qt}(E_r) = \mathfrak{X}_1 \oplus \mathfrak{X}_2$ where $\mathfrak{X}_1 = \mathfrak{E}_2$ and $\mathfrak{X}_2 = \mathfrak{E}_{r-2}$. Then $\eta_2(\tau_{12})$ is the switch of factors in $E_2 = k \times k$, that is, $\eta_2(\tau_{12}) = \sigma_{E_2}$ is the involution of E_2. Hence $\mathrm{dick}_{E_2}(\eta_2(\tau_{12})) = 1 \in \mathbf{Z}_2(k)$ by 9.6.3, and 9.5.2 implies $\mathrm{dick}(\eta_r(\tau_{12})) = 1 = \mathrm{sgn}(\tau_{12})$.

An arbitrary transposition τ is conjugate to τ_{12} in the symmetric group $\mathfrak{S}_r$. As $\mathbf{Z}_2$ is abelian, $\mathrm{dick}(\eta_r(\tau)) = \mathrm{dick}(\eta_r(\tau_{12})) = 1 = \mathrm{sgn}(\tau)$. Since the transpositions generate $\mathfrak{S}_r$, it follows that $\mathrm{dick}(\eta_r(\pi)) = \mathrm{sgn}(\pi)$ for all $\pi \in \mathfrak{S}_r$. Finally, $\boldsymbol{\mathfrak{S}}_r$ is the sheaf in the Zariski topology associated to the constant functor $\mathbf{F}(R) = \mathfrak{S}_r$, for all $R \in k$-**alg**. We have shown that the morphisms sgn and $\mathrm{dick}_{E_r} \circ \eta_r$ from $\boldsymbol{\mathfrak{S}}_r$ to $\mathbf{Z}_2$ agree on $\mathbf{F}$. Hence they are equal because $\mathbf{Z}_2$ is a Zariski sheaf [6, III, §1, Prop. 1.7].

9.10. Torsors and cohomology. In 9.4, the Dickson homomorphism was deduced from the discriminant algebra functor. This can — to a certain extent — be reversed. As an application, it will be seen that our definition of the discriminant algebra of an étale algebra is compatible with Waterhouse's [16].

Let $\mathscr{T}$ be a Grothendieck topology on k-**alg**. Fix a quadratic trace module $\mathfrak{X}_0$ with discriminant algebra $D_0 = \mathrm{Dis}(\mathfrak{X}_0)$. Let $\mathbf{qtm}(\mathfrak{X}_0) \subset \mathbf{qtm}_k$ be the subcategory whose objects are quadratic trace modules $\mathscr{T}$-locally isomorphic to $\mathfrak{X}_0$ and whose morphisms are isomorphisms, and define $\mathbf{qa}(D_0) \subset \mathbf{qa}_k$ analogously. As the functor Dis commutes with base change, it restricts to a functor Dis: $\mathbf{qtm}(\mathfrak{X}_0) \to \mathbf{qa}(D_0)$. Let $\mathbf{G}_0 = \mathbf{Aut}(\mathfrak{X}_0)$ and $\mathbf{H}_0 = \mathbf{Aut}(D_0)$, and denote the categories of $\mathbf{G}_0$-torsors and $\mathbf{H}_0$-torsors (with respect to $\mathscr{T}$) over k by $\mathbf{tor}(\mathbf{G}_0)$ and $\mathbf{tor}(\mathbf{H}_0)$, respectively. Then the Dickson homomorphism Dick: $\mathbf{G}_0 \to \mathbf{H}_0$ induces a functor, likewise denoted Dick: $\mathbf{tor}(\mathbf{G}_0) \to \mathbf{tor}(\mathbf{H}_0)$, which assigns to a $\mathbf{G}_0$-torsor $\mathbf{X}$ the $\mathbf{H}_0$-torsor $\mathbf{X} \vee^{\mathbf{G}_0} \mathbf{H}_0$ [6, III, §4, 3.2]. There are equivalences of categories $\mathbf{qtm}(\mathfrak{X}_0) \to \mathbf{tor}(\mathbf{G}_0)$ and $\mathbf{qa}(D_0) \to \mathbf{tor}(\mathbf{H}_0)$ given by $\mathfrak{X} \mapsto \mathbf{Isom}(\mathfrak{X}_0, \mathfrak{X})$ and $D \mapsto \mathbf{Isom}(D_0, D)$, with quasi-inverses given by twisting $\mathfrak{X}_0$ resp. D_0 with a torsor [6, III, §5, Prop. 1.2]. Then the following diagram is commutative up to a natural isomorphism of functors:

$$\begin{array}{ccc} \mathbf{qtm}(\mathfrak{X}_0) & \xrightarrow{\mathrm{Dis}} & \mathbf{qa}(D_0) \\ {\scriptstyle \mathbf{Isom}(\mathfrak{X}_0,-)}\downarrow & & \downarrow{\scriptstyle \mathbf{Isom}(D_0,-)} \\ \mathbf{tor}(\mathbf{G}_0) & \xrightarrow[\mathrm{Dick}]{} & \mathbf{tor}(\mathbf{H}_0) \end{array} \qquad (1)$$

Indeed, let $\mathfrak{X} \in \mathbf{qtm}(\mathfrak{X}_0)$ and put $\mathbf{X} := \mathbf{Isom}(\mathfrak{X}_0, \mathfrak{X}) \in \mathbf{tor}(\mathbf{G}_0)$ as well as $\mathbf{Y} := \mathbf{Isom}(D_0, \mathrm{Dis}(\mathfrak{X})) \in \mathbf{tor}(\mathbf{H}_0)$. We must construct an isomorphism $\varphi = \varphi_{\mathfrak{X}} \colon \mathbf{X} \vee^{\mathbf{G}_0} \mathbf{H}_0 \to \mathbf{Y}$ of $\mathbf{H}_0$-torsors, natural in $\mathfrak{X}$. First, there is a morphism $\psi \colon \mathbf{X} \times \mathbf{H}_0 \to \mathbf{Y}$ as follows: Let $R \in k$-**alg**, $f \in \mathbf{X}(R)$ and $h \in \mathbf{H}_0(R)$; thus $f \colon \mathfrak{X}_0 \otimes R \to \mathfrak{X} \otimes R$ is an isomorphism and $h \in \mathrm{Aut}(D_0 \otimes R)$. Since Dis commutes with base change, we have an isomorphism

$$\psi(f,h) := \mathrm{Dis}(f) \circ h : D_0 \otimes R \xrightarrow{h} D_0 \otimes R \xrightarrow{\mathrm{Dis}(f)} \mathrm{Dis}(\mathfrak{X} \otimes R) \cong \mathrm{Dis}(\mathfrak{X}) \otimes R,$$

i.e., $\psi(f,h) \in \mathrm{Isom}(D_0 \otimes R, \mathrm{Dis}(\mathfrak{X}) \otimes R) = \mathbf{Isom}(D_0, \mathrm{Dis}(\mathfrak{X}))(R) = \mathbf{Y}(R)$. Next, for all $g \in \mathbf{G}_0(R) = \mathrm{Aut}(\mathfrak{X}_0 \otimes R)$,

$$\psi(f \circ g, h) = \mathrm{Dis}(f \circ g) \circ h = \mathrm{Dis}(f) \circ \mathrm{Dick}(g) \circ h = \psi(f, \mathrm{Dick}(g)h),$$

by functoriality of Dis and definition of Dick. It is immediate that $\psi(f, h \circ h') = \psi(f,h) \circ h'$, for all $h' \in \mathbf{H}_0(R)$. Now $\mathbf{X} \vee^{\mathbf{G}_0} \mathbf{H}_0$ is the quotient sheaf of $\mathbf{X} \times \mathbf{H}_0$ by the equivalence relation $(f \circ g, h) \sim (f, \mathrm{Dick}(g)h)$. Hence ψ induces a morphism $\varphi\colon \mathbf{X} \vee^{\mathbf{G}_0} \mathbf{H}_0 \to \mathbf{Y}$ of $\mathbf{H}_0$-torsors which is automatically an isomorphism [6, III, §4, Prop. 1.4]. Naturality of $\varphi_{\mathfrak{X}}$ is easily checked.

The preceding argument only required that Dis be a functor commuting with base change. Hence, analogous statements hold when $\mathbf{qtm}(\mathfrak{X}_0)$ is replaced by the category of finitely generated and projective k-algebras locally isomorphic to a fixed algebra A_0 and $\mathbf{G}_0$ by $\mathbf{Aut}(A_0)$, with Dis and Dick for algebras defined as in 3.4.3 and 9.4.3.

Let us specialize to the case where $\mathscr{T}$ is the étale topology and $\mathfrak{X}_0 = \mathfrak{E}_r$ is the split quadratic trace module of rank r. By Theorem 8.8, $\mathbf{qtm}(\mathfrak{X}_0)$ is the category of separable quadratic trace modules of rank r. Furthermore, $D_0 = \mathrm{Dis}(\mathfrak{E}_r) = k \times k$ by 3.7 and $\mathfrak{h}_{D_0}\colon \mathbf{Z}_2 \to \mathbf{H}_0 = \mathbf{Aut}(D_0)$ is an isomorphism. Denote as usual the pointed set of isomorphism classes of $\mathbf{G}$-torsors by $\mathrm{H}^1(k, \mathbf{G})$. Then (1) says that the assignment $\mathfrak{X} \mapsto \mathrm{Dis}(\mathfrak{X})$ gives, in the separable case, a concrete realization of the map $\mathrm{H}^1(\mathrm{dick})\colon \mathrm{H}^1(k, \mathbf{G}_0) \to \mathrm{H}^1(k, \mathbf{Z}_2)$ between the cohomology sets. Similarly, by Lemma 9.9, the assignment $E \mapsto \mathrm{Dis}(E)$ (where E is an étale algebra of rank r) realizes the map

$$\mathrm{H}^1(\mathrm{sgn})\colon \mathrm{H}^1(k, \mathfrak{S}_r) \to \mathrm{H}^1(k, \mathbf{Z}_2).$$

This proves that our definition of the discriminant algebra of an étale algebra is compatible with Waterhouse's cohomological definition [16].

10. The automorphism group II: Centre and restriction map

10.1. Lemma. *Let $a, b, c \in k$ be relatively prime. Consider the polynomials $P(\mathbf{t}) = a\mathbf{t}^2 - b\mathbf{t} + c$ and $\tilde{P}(\mathbf{t}) = \mathbf{t}^2 - b\mathbf{t} + ac$ and the k-algebras $C := k[\mathbf{t}]/(P)$ and $D := k[\mathbf{t}]/(\tilde{P}) = k \cdot 1 \oplus k \cdot z$. Let $\mathbf{C} := \mathbf{Spec}(C)$ and $\mathbf{D} := \mathbf{Spec}(D)$ be the affine schemes determined by C and D.*

(a) *Define $\iota\colon \mathbf{C} \to \mathbf{D}$ by $\iota(x) = ax$, for all $x \in \mathbf{C}(R)$, $R \in k$-**alg**. Then ι is an open immersion whose image is the open subscheme of $\mathbf{D}$ defined by the ideal $I = Da + D(b - z)$ of D.*

(b) *$\mathbf{C}$ is a flat, finitely presented and quasi-finite k-scheme. The image of $\mathbf{C}$ in $\mathbf{S} = \mathbf{Spec}(k)$ is the open subscheme defined by the ideal $ka + kb$. Hence $\mathbf{C}$ is faithfully flat over k if and only if a and b are relatively prime.*

(c) *If $b^2 - 4ac \in k^\times$ then $\mathbf{C}$ is étale over k. The converse holds if $ka + kb = k$.*

Proof. (a) We have $x \in \mathbf{C}(R)$ if and only if $x \in R$ and $P(x) = 0$. Hence $\tilde{P}(ax) = aP(x) = 0$, so $ax \in \mathbf{D}(R)$ and thus ι maps $\mathbf{C}$ to $\mathbf{D}$. Next, ι is a monomorphism: Let $x_1, x_2 \in \mathbf{C}(R)$ and $ax_1 = ax_2$. Then $0 = P(x_1) - P(x_2) = a(x_1 - x_2)(x_1 + x_2) - b(x_1 - x_2) = -b(x_1 - x_2)$, and $0 = P(x_1)(x_1 - x_2) = c(x_1 - x_2)$ imply $x_1 - x_2 = 0$

because a,b,c are relatively prime. The open subscheme $\mathbf{V}$ of $\mathbf{D}$ defined by I is the functor

$$\mathbf{V}(R) = \{y \in \mathbf{D}(R) : Ra + R(b-y) = R\},$$

cf. [6, I, §1, 3.5]. The values of ι lie in $\mathbf{V}$: Let $x \in \mathbf{C}(R)$ and let $J = Ra + R(b-ax)$. Then $a,b \in J$, hence also $c = x(b-ax) \in J$, so $J = R$ because a,b,c are relatively prime. Conversely, let $y \in \mathbf{V}(R)$, thus $y(b-y) = ac$ and $R = Ra + R(b-y)$. We must show that $y = ax$ for some $x \in \mathbf{C}(R)$. Choose $u,v \in R$ such that

$$1 = ua + v(b-y). \tag{1}$$

Multiplying this equation with y yields

$$y = yua + yv(b-y) = auy + (ac)v = a(uy + cv).$$

Put $x = uy + cv$. Then $ax = y$, and it remains to show that $x \in \mathbf{C}(R)$, i.e., that $x(b-ax) = c$. Now

$$\begin{aligned} x(b-ax) &= x(b-y) = (uy+cv)(b-y) = uy(b-y) + cv(b-y) \\ &= u(ac) + v(b-y)c = (ua + v(b-y))c = c, \end{aligned}$$

because of (1).

(b) Clearly, $\mathbf{D}$ is a flat and finite (of rank 2) k-scheme. Since ι is an open immersion, $\mathbf{C}$ is flat and quasi-finite over k, and it is obviously finitely presented. The fibre of $\mathbf{C}$ over a prime ideal $\mathfrak{p}$ of k is $\mathbf{Spec}(C \otimes \boldsymbol{\kappa}(\mathfrak{p}))$ which is empty if and only if $a,b \in \mathfrak{p}$. This proves the remaining statements.

(c) D is étale if and only if the discriminant $b^2 - 4ac$ of $\tilde{P}$ is a unit of k. Now the assertions follow readily from (a) and (b).

10.2. Definition. We define a family of groups $\mathbf{C}_r$ ($r \in \mathbb{N}$) which are open subgroups of $\mathbf{Z}_2$ resp. $\boldsymbol{\mu}_2$, depending on the parity of r. Let

$$d(r) := \frac{r}{\gcd(2,r)} = \left\{ \begin{matrix} n & \text{if } r = 2n \text{ is even} \\ r & \text{if } r = 2n+1 \text{ is odd} \end{matrix} \right\} \tag{1}$$

and consider the polynomials

$$P_r(\mathbf{t}) = d(r)\mathbf{t}^2 - \gcd(2, r-1)\mathbf{t} = \left\{ \begin{matrix} n\mathbf{t}^2 - \mathbf{t} & \text{if } r = 2n \\ r\mathbf{t}^2 - 2\mathbf{t} & \text{if } r = 2n+1 \end{matrix} \right\}. \tag{2}$$

The coefficients of P_r are relatively prime so Lemma 10.1 is applicable. Let $C_r := k[\mathbf{t}]/(P_r)$ and let $\mathbf{C}_r = \mathbf{Spec}(C_r)$ be the affine scheme defined by C_r, i.e., the set-valued functor on k-**alg** given by

$$\mathbf{C}_r(R) = \{\lambda \in R : P_r(\lambda) = 0\} \quad (R \in k\text{-}\mathbf{alg}). \tag{3}$$

Note that $\mathbf{C}_0 = \mathbf{Spec}(k)$, that $\mathbf{C}_2 = \mathbf{Z}_2$ and that $\mathbf{C}_1 \cong \boldsymbol{\mu}_2$, the group scheme of second roots of unity, under the map $\lambda \mapsto 1 - \lambda$.

10.3. Lemma. (a) $\mathbf{C}_r$ *is an affine faithfully flat finitely presented quasi-finite abelian k-group scheme with the group law*

$$\lambda \dotplus \lambda' = \lambda + \lambda' - r\lambda\lambda',$$

for all $\lambda, \lambda' \in \mathbf{C}_r(R)$, $R \in k$-**alg**. *Moreover,* $\mathbf{C}_{2n}$ *is étale while* $\mathbf{C}_{2n+1}$ *is étale if and only if* $2 \in k^\times$.

(b) *The maps*

$$\omega_n \colon \mathbf{C}_{2n} \to \mathbf{Z}_2, \qquad \omega_n(\lambda) = n\lambda,$$
$$\psi_n \colon \mathbf{C}_{2n+1} \to \boldsymbol{\mu}_2, \qquad \psi_n(\lambda) = 1 - (2n+1)\lambda$$

are open immersions and homomorphisms of group schemes. They are isomorphisms if and only if $d(r) \in k^\times$, *and are constant if and only if* $d(r) = 0$ *in k.*

(c) *The homomorphism*

$$\chi_r \colon \mathbf{C}_r \to \boldsymbol{\mu}_2, \qquad \chi_r = \left\{ \begin{matrix} \chi \circ \omega_n & \text{if } r = 2n \\ \psi_n & \text{if } r = 2n+1 \end{matrix} \right\}$$

(where $\chi \colon \mathbf{Z}_2 \to \boldsymbol{\mu}_2$ *is as in* 7.1*) is a monomorphism if* $r \equiv 1 \pmod 2$ *or* $r \equiv 0 \pmod 4$ *while the kernel of* χ_{4l+2} *is* $\mathbf{K}(R) = \{\lambda \in R : \lambda^2 = \lambda,\ 2\lambda = 0\}$. *It is an isomorphism if and only if* $r \in k^\times$.

Proof. (a) The scheme-theoretic properties of $\mathbf{C}_r$ follow from Lemma 10.1 and the rest is a straightforward computation.

(b) If $r = 2n$ then ω_n is the open immersion ι of Lemma 10.1. If $r = 2n+1$ then ψ_n is the composition of ι and the isomorphism $\mathbf{C}_1 \to \boldsymbol{\mu}_2$ given by $\mu \mapsto 1 - \mu$. Hence ω_n and ψ_n are open immersions. The homomorphism property is easily checked. The last statement is obvious.

(c) If r is odd then χ_r is a monomorphism by (b). Now let $r = 2n$ and assume that $\chi(\omega_n(\lambda)) = 1 - 2n\lambda = 1$. Then $2n\lambda = 0$ and therefore $2n\lambda^2 = 2\lambda = 0$. If $n = 2l$ then $\lambda = n\lambda^2 = 2l\lambda^2 = 0$. If $n = 2l+1$ then $\lambda = (2l+1)\lambda^2 = \lambda^2$, so $\lambda \in \mathbf{K}(R)$. Conversely, $\mathbf{K} \subset \mathbf{Ker}(\chi_{4l+2})$ is clear from the definitions. Finally, if $r \in k^\times$ then $\mu \mapsto r^{-1}(1-\mu)$ is the inverse of χ_r. Conversely, assume that χ_r is an isomorphism but that $r \notin k^\times$. After dividing by a suitable maximal ideal, we may assume $r = 0$ in k. Then χ_r is constant, but $\boldsymbol{\mu}_2$ is not the trivial group, contradiction.

10.4. The quadratic form Q^0 and the restriction map. Let $\mathfrak{X}$ be a separable quadratic trace module of rank $r \geqslant 2$, let $X^0 = \mathrm{Ker}(T)$ and define

$$Q^0 := -Q\big|_{X^0}. \tag{1}$$

By 8.3, X^0 is a direct summand of X of rank $r-1$ and Q^0 is primitive. The minus sign is introduced so that the polar form B^0 of Q^0 becomes the restriction of Δ to X^0:

$$B^0(x,y) = \Delta(x,y) \qquad (x, y \in X^0), \tag{2}$$

as is immediate from 1.1.3. Suppose $\mathfrak{X} = \mathfrak{E}_r^{\mathbb{Z}}$ split over $\mathbb{Z}$, with standard basis $e_1,\dots,e_r$. Then $v_i = e_i - e_{i-1}$ ($i = 2,\dots,r$) is a basis of X^0, and

$$Q^0(v_i) = 1, \qquad B^0(v_i, v_j) = \left\{ \begin{array}{rl} 2 & \text{if } i = j \\ -1 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise} \end{array} \right\}, \tag{3}$$

so Q^0 is the quadratic form of the root system A_{r-1}.

An automorphism g of $\mathfrak{X}$ leaves X^0 invariant and induces an orthogonal transformation of the quadratic form Q^0. We thus have a restriction homomorphism

$$\text{res: } \mathbf{G} = \mathbf{Aut}(\mathfrak{X}) \to \mathbf{O}(Q^0), \qquad g \mapsto g\big|X^0. \tag{4}$$

Note that

$$\det(\mathrm{res}(g)) = \chi(\mathrm{dick}(g)) = \det(g) \qquad (g \in \mathbf{G}(R),\ R \in k\text{-}\mathbf{alg}). \tag{5}$$

Indeed, pick an element $u \in X$ with $T(u) = 1$, write $X = k \cdot u \oplus X^0$ and thus identify g with a formal 2×2-matrix. Then $g(u) \equiv u \pmod{X^0}$, so $g = \left(\begin{smallmatrix} 1 & 0 \\ * & h \end{smallmatrix}\right)$ where $h = \mathrm{res}(g)$, which implies $\det(\mathrm{res}(g)) = \det(g) = \chi(\mathrm{dick}(g))$ by 9.4.7. Hence res induces a homomorphism

$$\text{res}^+\text{: } \mathbf{G}^+ = \mathbf{Ker}(\mathrm{dick}) \to \mathbf{SO}(Q^0). \tag{6}$$

10.5. Theorem. *Let $\mathfrak{X}$ be a separable quadratic trace module of rank $r \geqslant 2$ over k with automorphism group $\mathbf{G} = \mathbf{Aut}(\mathfrak{X})$, and let $\mathbf{Cent}(\mathbf{G})$ be the centre of $\mathbf{G}$ in the sense of group schemes [6, II, §1, 3.9].*

(a) *There is an isomorphism* cen: $\mathbf{C}_r \xrightarrow{\cong} \mathbf{Cent}(\mathbf{G})$ *given by*

$$\mathrm{cen}(\lambda)(x) = \chi_r(\lambda)x + \lambda T(x)1_X, \tag{1}$$

for all $\lambda \in \mathbf{C}_r(R)$, $x \in X \otimes R$, $R \in k$-$\mathbf{alg}$.

(b) *Let* res *be the restriction homomorphism of* 10.4.4 *and let* cen^0: $\boldsymbol{\mu}_2 \to \mathbf{O}(Q^0)$ *be the monomorphism* $\lambda \mapsto \lambda \cdot \mathrm{Id}_{X^0}$. *Then the diagram*

$$\begin{array}{ccc} \mathbf{C}_r & \xrightarrow{\ \mathrm{cen}\ } & \mathbf{G} \\ {\scriptstyle \chi_r}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{res}} \\ \boldsymbol{\mu}_2 & \xrightarrow[\ \mathrm{cen}^0\ ]{} & \mathbf{O}(Q^0) \end{array} \tag{2}$$

is commutative and in fact Cartesian, i.e., $\mathrm{res}^{-1}(\boldsymbol{\mu}_2 \cdot \mathrm{Id}_{X^0}) = \mathbf{Cent}(\mathbf{G})$. *Hence the kernel of* res *is central in* $\mathbf{G}$ *and the restriction* cen: $\mathbf{Ker}(\chi_r) \to \mathbf{Ker}(\mathrm{res})$ *is an isomorphism.*

(c) *The restriction of the determinant and the Dickson homomorphism to the centre of $\mathbf{G}$ are as follows:*

$$\det(\mathrm{cen}(\lambda)) = \left\{ \begin{array}{ll} \chi_r(\lambda) & \text{if } r = 2n \\ 1 & \text{if } r = 2n+1 \end{array} \right\}, \tag{3}$$

$$\mathrm{dick}(\mathrm{cen}(\lambda)) = \left\{ \begin{array}{ll} \omega_n(\lambda) & \text{if } r = 2n \\ 0 & \text{if } r = 2n+1 \end{array} \right\}, \tag{4}$$

for all $\lambda \in \mathbf{C}_r(R)$, $R \in k$-**alg**.

Proof. (a) A straightforward computation shows that cen: $\mathbf{C}_r \to \mathbf{GL}(X)$ is a group homomorphism. It is a monomorphism: If $\mathrm{cen}(\lambda) = \mathrm{Id}$ then $\mathrm{cen}(\lambda)\big|X^0 = \chi_r(\lambda)\cdot \mathrm{Id}_{X^0}$ which implies $\chi_r(\lambda) = 1$ because $\mathrm{rk}\, X^0 = r-1 \geqslant 1$. For an element u with $T(u) = 1$ (which exists by Lemma 8.3) it then follows that $u = \mathrm{cen}(\lambda)\cdot u = u + \lambda \cdot 1_X$ and hence $\lambda = 0$, because 1_X is unimodular.

We show next that the centralizer of $\mathbf{G}$ in $\mathrm{End}(X)$ consists of all endomorphism of the form

$$h_{\lambda,\mu}(x) = \mu x + \lambda T(x) 1_X \tag{5}$$

where $\lambda, \mu \in k$. Since 1_X and T are invariant under automorphisms, it is evident that $h_{\lambda,\mu}$ commutes with all automorphisms of $\mathfrak{X}$ in all base ring extensions. In particular, $\mathrm{cen}(\lambda) = h_{\lambda,1-r\lambda}$ centralizes $\mathbf{G}$. Conversely, let $h \in \mathrm{End}(X)$ have this property. After passing to a faithfully flat base extension we may assume that $\mathfrak{X}$ is split. Then h commutes with all permutations of the standard basis vectors e_i (cf. 9.8), so the matrix (a_{ij}) of h satisfies $a_{ij} = a_{\pi(i),\pi(j)}$ for all $\pi \in \mathfrak{S}_r$. This means $a_{ii} = a_{11}$ and $a_{ij} = a_{12}$ for all i and all $j \neq i$. Hence $h = h_{\lambda,\mu}$ for $\lambda = a_{12}$ and $\mu = a_{11} - a_{12}$.

To complete the proof of (a), it remains to show, after a base change, that

$$h = h_{\lambda,\mu} \in \mathrm{Aut}(\mathfrak{X}) \quad\Longleftrightarrow\quad \mu = 1 - r\lambda \text{ and } \lambda \in \mathbf{C}_r(k). \tag{6}$$

Since $T(1_X) = r$, we have $h(1_X) = (\mu + r\lambda)1_X$ and $T(h(x)) = (\mu + r\lambda)T(x)$, for all $x \in X$. Hence

$$h(1_X) = 1_X \text{ and } T \circ h = T \quad\Longleftrightarrow\quad \mu = 1 - r\lambda.$$

Assume that this is the case. Then a simple computation using 1.1.1 shows

$$\begin{aligned} Q(h(x)) - Q(x) &= (\mu^2 - 1)Q(x) + \Big[(r-1)\lambda\mu + \binom{r}{2}\lambda^2\Big]T(x)^2 \\ &= (r^2\lambda^2 - 2r\lambda)Q(x) + \Big[(r-1)\lambda - \binom{r}{2}\lambda^2\Big]T(x)^2 \\ &= F(\lambda)Q(x) - G(\lambda)T(x)^2 \end{aligned} \tag{7}$$

for all $x \in X$, where

$$F(\mathbf{t}) = (1 - r\mathbf{t})^2 - 1 = r^2\mathbf{t}^2 - 2r\mathbf{t} \quad\text{and}\quad G(\mathbf{t}) = \binom{r}{2}\mathbf{t}^2 - (r-1)\mathbf{t}.$$

This proves the implication from right to left of

$$Q \circ h_{\lambda,\mu} = Q \quad\Longleftrightarrow\quad F(\lambda) = G(\lambda) = 0.$$

For the implication from left to right, let first $x \in X^0$. Then (7) says $F(\lambda)\cdot Q^0 = 0$ and therefore $F(\lambda) = 0$, because Q^0 is primitive (8.3). Now choosing an x with

$T(x) = 1$ in (7) yields $G(\lambda) = 0$. — It is an elementary exercise to show that the ideal of $k[\mathbf{t}]$ generated by F and G is precisely the one generated by P_r. Hence

$$F(\lambda) = G(\lambda) = 0 \iff P(\lambda) = 0 \iff \lambda \in \mathbf{C}_r(k).$$

This completes the proof of (6) and hence of (a).

(b) Commutativity of (2) is evident from (1). Now let $g \in \mathbf{G}(k)$ with $\mathrm{res}(g) = \mu \cdot \mathrm{Id}_{X^0}$ where $\mu \in \boldsymbol{\mu}_2(k)$. Choose $u \in X$ with $T(u) = 1$ and put $w := g(u) - \mu u$. Then g is given by

$$g(x) = \mu x + T(x)w \tag{8}$$

for all $x \in X$. Indeed, for $x \in X^0$ this is clear, while $\mu u + T(u)w = \mu u + g(u) - \mu u = g(u)$, so the assertion follows from $X = k \cdot u \oplus X^0$. We claim that $\Delta(w, X^0) = 0$. Indeed, for all $x \in X^0$,

$$\Delta(w,x) = \Delta(g(u) - \mu u, x) = \Delta(u, g^{-1}(x)) - \Delta(u, \mu x) = 0,$$

since Δ is invariant under g and $g(x) = \mu x = g^{-1}(x)$. It follows that $w \in (X^0)^\perp = k \cdot 1_X$, because Δ is nondegenerate and $X^0 = 1_X^\perp$ by 1.1.4. Thus $w = \lambda 1_X$, and then (8) says that $g = h_{\lambda,\mu}$. By what we proved in (a), $g = \mathrm{cen}(\lambda)$ is central.

(c) Let $u \in X$ with $T(u) = 1$. Then $\mathrm{cen}(\lambda) \cdot u \equiv u \pmod{X^0}$ because $\mathrm{cen}(\lambda)$ preserves T, and $\mathrm{cen}(\lambda)$ induces $\chi_r(\lambda) \cdot \mathrm{Id}$ on X^0 by (2). As $X = k \cdot u \oplus X^0$ we have $\det(\mathrm{cen}(\lambda)) = \chi_r(\lambda)^{r-1}$. This proves (3) because $\chi_r(\lambda)$ is a second root of unity.

Let us compute the Dickson invariant. By faithfully flat descent and Theorem 8.8 we may assume $\mathfrak{X} = \mathfrak{E}_r$ split. Consider the split quadratic trace module $\mathfrak{E}_r^{\mathbb{Z}}$ over the integers, let $A = \mathbb{Z}[\mathbf{t}]/(P_r)$ be the coordinate ring of $\mathbf{C}_r^{\mathbb{Z}}$, let $t = \mathrm{can}(\mathbf{t}) \in A$ and $g := \mathrm{cen}(t) \in \mathrm{Aut}(\mathfrak{E}_r^{\mathbb{Z}} \otimes A)$. Then by (3), Lemma 10.3(c) and 9.4.7,

$$\det g = \left.\begin{cases} 1 - 2\,\omega_n(t) & \text{if } r = 2n \\ 1 & \text{if } r = 2n+1 \end{cases}\right\} = 1 - 2\,\mathrm{dick}(g). \tag{9}$$

By Lemma 10.3, A is flat over $\mathbb{Z}$, in particular, it it a torsion-free abelian group. Hence (9) shows that (4) holds in the special case $\lambda = t \in A$. Returning to $\mathfrak{X} = \mathfrak{E}_r$ over the ring k, let $\lambda \in \mathbf{C}_r(R)$. Then there is a ring homomorphism $A \to R$ sending t to λ, and since $(\mathfrak{E}_r^{\mathbb{Z}} \otimes_{\mathbb{Z}} A) \otimes_\varphi R = \mathfrak{E}_r^{\mathbb{Z}} \otimes_{\mathbb{Z}} R = \mathfrak{X} \otimes R$, we have (4) in general.

10.6. Corollary. *We keep the assumptions and notations of* 10.5.

(a) *If $r = 2n$ is even then the multiplication map* mult: $\mathbf{G}^+ \times \mathbf{Cent}(\mathbf{G}) \to \mathbf{G}$ *is an open immersion. Its image is the inverse image under* dick *of the image of ω_n in $\mathbf{Z}_2$, cf.* 10.3(b). *In particular,* $\mathbf{G}^+ \cap \mathbf{Cent}(\mathbf{G}) = \{1\}$.

(b) *If $r = 2n+1$ is odd then* $\mathbf{Cent}(\mathbf{G}) \subset \mathbf{G}^+$.

Proof. One checks, using the first formula of 10.5.4, that the diagram

$$\begin{array}{ccc} \mathbf{G}^+ \times \mathbf{C}_{2n} & \xrightarrow{\mathrm{pr}_2} & \mathbf{C}_{2n} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\omega_n} \\ \mathbf{G} & \xrightarrow[\mathrm{dick}]{} & \mathbf{Z}_2 \end{array}$$

is Cartesian, where $\varphi(g^+,\lambda) = g^+ \cdot \mathrm{cen}(\lambda)$. Hence φ is an open immersion, being the base change via dick of the open immersion ω_n (Lemma 10.3(b)). Since cen is an isomorphism, φ is isomorphic to mult: $\mathbf{G}^+ \times \mathbf{Cent}(\mathbf{G}) \to \mathbf{G}$. This proves (a) while (b) is immediate from the second formula of 10.5.4.

10.7. Lemma. *Let $\mathfrak{X}$ be a separable quadratic trace module of rank $r \geqslant 2$ and let $u \in X$ with $T(u) = 1$. Then the map $\eta \mapsto u \wedge \eta$ is an isomorphism $\bigwedge^{r-1} X^0 \to \bigwedge^r X$, independent of the choice of u. Treating this as an identification, the (signed) discriminant of Q^0 is*

$$\delta_{Q^0} = (-1)^{\binom{r-1}{2}} \cdot d(r) \cdot \delta_{\mathfrak{X}} \tag{1}$$

where $d(r)$ is defined in 10.2.1. *Hence Q^0 is separable if and only if $d(r) \in k^\times$.*

Here we call a quadratic form q on a finitely generated and projective module M *separable* if its discriminant is nonsingular. This means that its polar form b is nonsingular if M has even rank, and that q is semiregular [10, IV, §3] if M has odd rank.

Proof. Independence of u is easily seen. By Theorem 8.8, it suffices to consider the split quadratic trace module $\mathfrak{E}_r^{\mathbb{Z}}$ over $\mathbb{Z}$. Choose $u = e_1$. With respect to the basis $v_i = e_i - e_{i-1}$ ($i = 2, \ldots, r$) of X^0, the matrix $A = \big(B^0(v_i, v_j)\big)$ is the Cartan matrix of type A_{r-1} (cf. 10.4), so $\det A = r$ [4]. Put $\eta = v_2 \wedge \cdots \wedge v_r$ and $\xi = e_1 \wedge \eta = e_1 \wedge \cdots \wedge e_r$. Then for $r = 2n$ even,

$$2\delta_{Q^0}(\eta,\eta) = (-1)^{n-1} \det A = (-1)^{n-1} 2n \delta_{\mathfrak{X}}(\xi,\xi),$$

while for $r = 2n+1$ odd,

$$\delta_{Q^0}(\eta,\eta) = (-1)^n \det(A) = (-1)^n r \delta_{\mathfrak{X}}(\xi,\xi).$$

10.8. Theorem. *Let $\mathfrak{X}$ be a separable quadratic trace module of rank $r \geqslant 2$ with automorphism group $\mathbf{G} = \mathbf{Aut}(\mathfrak{X})$. Suppose that $d(r) \in k^\times$; equivalently, by* 10.7, *that Q^0 is separable.*

(a) *Let $r = 2n$ be even. Then Q^0 is a semiregular quadratic form of rank $2n-1$ and the maps*

$$\text{mult: } \mathbf{G}^+ \times \mathbf{Cent}(\mathbf{G}) \xrightarrow{\cong} \mathbf{G}, \tag{1}$$

$$\text{dick: } \mathbf{Cent}(\mathbf{G}) \xrightarrow{\cong} \mathbf{Z}_2, \tag{2}$$

$$\mathrm{res}^+ \colon \mathbf{G}^+ \xrightarrow{\cong} \mathbf{SO}(Q^0) \tag{3}$$

are isomorphisms.

(b) *Let $r = 2n+1$ be odd. Then Q^0 is a nonsingular quadratic form of rank $2n$. The centre of $\mathbf{G}$ is isomorphic to $\boldsymbol{\mu}_2$ and is contained in $\mathbf{G}^+$. The restriction map* res: $\mathbf{G} \to \mathbf{O}(Q^0)$ *is an isomorphism compatible with the Dickson homomorphisms, i.e., the diagram*

$$\begin{array}{ccc} \mathbf{G} & \xrightarrow[\cong]{\text{res}} & \mathbf{O}(Q^0) \\ & {\scriptstyle\text{dick}}\searrow \quad \swarrow{\scriptstyle\text{dick}} & \\ & \mathbf{Z}_2 & \end{array} \tag{4}$$

is commutative.

Here dick: $\mathbf{O}(Q^0) \to \mathbf{Z}_2$ is the usual Dickson homomorphism of the even orthogonal group.

Proof. (a) The first statement is clear from Lemma 10.7. As $d(r) = n \in k^\times$, ω_n is an isomorphism by Lemma 10.3(b). Now (1) follows from Cor. 10.6(a). Also, Theorem 10.5 shows that dick: $\mathbf{Cent}(\mathbf{G}) \to \mathbf{Z}_2$ is an isomorphism with inverse $\text{cen} \circ \omega_n^{-1}$, which proves (2).

It remains to prove (3). Let us put $\mathbf{H} := \mathbf{SO}(Q^0)$ and $\pi := \text{res}^+$ for simpler notation. First, π is a monomorphism because $\mathbf{Ker}(\text{res}) \subset \mathbf{Cent}(\mathbf{G})$ by Theorem 10.5(b) and $\mathbf{G}^+ \cap \mathbf{Cent}(\mathbf{G}) = \{1\}$ by Cor. 10.6(a). Next, suppose $k = K$ is a field. It is known that $\mathbf{H}$, which is a form of $\mathbf{SO}_{2n-1}$, is a connected smooth algebraic K-group of dimension $\binom{2n-1}{2}$. By Theorems 9.3 and 9.7, $\mathbf{G}^+$ is also smooth of the same dimension, and $\text{Lie}(\pi)\colon \text{Lie}(\mathbf{G}^+) = \text{Lie}(\mathbf{G}) \to \text{Lie}(\mathbf{H})$ is injective because π is a monomorphism. Since the dimension of a smooth K-group equals the dimension of its Lie algebra, $\text{Lie}(\pi)$ is an isomorphism. It follows that π is an open immersion [6, II, §5, Cor. 5.5(b)], and therefore even an isomorphism because $\mathbf{H}$ is connected.

Finally, return to the case of an arbitrary base ring k and let $\mathbf{G}^+$ act on $\mathbf{H}$ via $g^+ \cdot h = \pi(g^+)h$. Then π is the orbit map $g \mapsto g \cdot e$ where $e \in \mathbf{H}(k)$ is the unit element, and the centralizer of e in $\mathbf{G}^+$ is trivial because π is a monomorphism. Now [6, III, §3, Prop. 2.1] shows that π is an isomorphism.

(b) By Lemma 10.7, Q^0 is nonsingular if r is odd. The structure of $\mathbf{Cent}(\mathbf{G})$ is clear form Cor. 10.6(b) and the fact that $\chi_r\colon \mathbf{C}_r \to \boldsymbol{\mu}_2$ is an isomorphism, by Lemma 10.3(c), because $r = d(r) \in k^\times$. Since also $T(1_X) = r \in k^\times$, we have $X = k \cdot 1_X \oplus X^0$ as k-modules. Moreover, $Q = \langle \binom{r}{2} \rangle \perp (-Q^0)$ (orthogonal sum of quadratic forms) by 1.7. It follows easily that res is an isomorphism, with inverse $h \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & h \end{smallmatrix}\right)$ with respect to the above decomposition.

It remains to prove (4). Let $g \in \text{Aut}(\mathfrak{X})$ and $h = \text{res}(g) \in \text{O}(Q^0)$. Consider the quadratic form $\dot{Q}$ on $\dot{X}$ as in 2.3.1. Then the k-module isomorphism $\varphi\colon X^0 \to \dot{X}$, $\varphi(x) = \dot{x}$ (the canonical image of x in $\dot{X} = X/k \cdot 1_X$) satisfies $\dot{Q}(\varphi(x)) = r \cdot Q^0(x)$, and hence is an invertible similitude. By [12, Theorem 2.3(b)], it induces an isomorphism $\mathfrak{D}(\varphi)\colon \mathfrak{D}(Q^0) \to \mathfrak{D}(\dot{Q})$ of discriminant algebras. Observe that the isomorphism $\dot{h}$ of $\dot{X}$ induced by h is the same as the transformation $\dot{g}$ induced by

g. Hence by functoriality of $\mathfrak{D}$, the diagram of isomorphisms

$$\begin{array}{ccc} \mathfrak{D}(Q^0) & \xrightarrow{\mathfrak{D}(\varphi)} & \mathfrak{D}(\dot{Q}) \\ {\scriptstyle \mathfrak{D}(h)}\downarrow & & \downarrow{\scriptstyle \mathfrak{D}(\dot{g})} \\ \mathfrak{D}(Q^0) & \xrightarrow[\mathfrak{D}(\varphi)]{} & \mathfrak{D}(\dot{Q}) \end{array} \tag{5}$$

is commutative. By Theorem 3.8, there is a natural isomorphism $\rho\colon \mathfrak{D}(\dot{Q})+\varepsilon \to \mathrm{Dis}(\mathfrak{X})$ (where $\varepsilon = (-1)^n \varpi(n)\delta_{\mathfrak{X}}$) and hence a commutative diagram

$$\begin{array}{ccc} \mathfrak{D}(\dot{Q})+\varepsilon & \xrightarrow{\rho} & \mathrm{Dis}(\mathfrak{X}) \\ {\scriptstyle \mathfrak{D}(\dot{g})}\downarrow & & \downarrow{\scriptstyle \mathrm{Dick}(g)} \\ \mathfrak{D}(\dot{Q})+\varepsilon & \xrightarrow[\rho]{} & \mathrm{Dis}(\mathfrak{X}) \end{array} \tag{6}$$

Now observe that for a separable quadratic algebra D, and for a shift $D+\varepsilon$ of D which is also separable, we have $\mathbf{Aut}(D) = \mathbf{Aut}(D+\varepsilon)$. This follows from the isomorphisms $\mathfrak{h}_D$ and $\mathfrak{h}_{D+\varepsilon}$ from $\mathbf{Z}_2$ to $\mathbf{Aut}(D)$ (cf. 7.2 and 9.4) and the fact that D and $D+\varepsilon$ have the same standard involution, cf. 3.1.3. Applying this to $D = \mathfrak{D}(Q^0)$, we obtain from (5) and (6) the commutative diagram of isomorphisms

$$\begin{array}{ccc} \mathfrak{D}(Q^0)+\varepsilon' & \xrightarrow{\psi} & \mathrm{Dis}(\mathfrak{X}) \\ {\scriptstyle \mathfrak{D}(h)}\downarrow & & \downarrow{\scriptstyle \mathrm{Dick}(g)} \\ \mathfrak{D}(Q^0)+\varepsilon' & \xrightarrow[\psi]{} & \mathrm{Dis}(\mathfrak{X}) \end{array} \tag{7}$$

where ε' corresponds to ε under $\mathfrak{D}(\varphi)$ and $\psi = \rho \circ \mathfrak{D}(\varphi)$.

By definition, $\mathrm{Dick}(g) = \mathfrak{h}_{\mathrm{Dis}(\mathfrak{X})}(\mathrm{dick}(g))$ and $\mathfrak{D}(h) = \mathfrak{h}_{\mathfrak{D}(Q^0)}(\mathrm{dick}(h))$. Now $\mathrm{dick}(g) = \mathrm{dick}(h)$ follows from (7) and the fact that the isomorphism $\mathfrak{h}_D\colon \mathbf{Z}_2 \to \mathbf{Aut}(D)$ is unique, for a separable quadratic algebra D, as remarked in 9.4.

10.9. Theorem. *Let $\mathfrak{X}$ be separable of rank $r \geqslant 2$ with automorphism group $\mathbf{G} = \mathbf{Aut}(\mathfrak{X})$. Suppose that $d(r) = 0$ in k, equivalently, by* 10.7*, that Q^0 has zero discriminant.*

(a) *Then $r = 0$ in k and therefore $1_X \in X^0$. The quadratic form Q^0 induces a nonsingular quadratic form $\overline{Q}$ of rank $r-2$ on $\overline{X} := X^0/k\cdot 1_X$.*

(b) $\mathbf{G}$ *has trivial centre. Denote by $\mathbf{O}_1(Q^0)$ the isotropy group of 1_X in $\mathbf{O}(Q^0)$. Then the restriction map*

$$\mathrm{res}\colon \mathbf{G} \xrightarrow{\cong} \mathbf{O}_1(Q^0) \tag{1}$$

is an isomorphism, and there is a split exact sequence

$$0 \longrightarrow \overline{X}^*_{\mathbf{a}} \xrightarrow{i} \mathbf{O}_1(Q^0) \xrightarrow{p} \mathbf{O}(\overline{Q}) \longrightarrow 1 \tag{2}$$

described as follows: Denote the canonical map $X^0 \to \overline{X}$ *by* $x \mapsto \overline{x}$*. Then for a linear form* f *on* $\overline{X}$*,* $i(f) \in \mathrm{O}_1(Q^0)$ *is given by*

$$i(f)(x) = x + f(\overline{x}) \cdot 1_X \quad (x \in X^0),$$

while $p(h) = \overline{h}$ *is the map on* $\overline{X}$ *induced by* $h \in \mathrm{O}_1(Q^0)$*.*

Proof. (a) By 10.2.1, $d(r) = 0$ in k implies $r = 0$ in k. Hence $T(1_X) = r = 0$, so $1_X \in X^0$. By 1.1.4, $X^0 = 1_X^{\perp}$ with respect to Δ, and since Δ is nonsingular, $k \cdot 1_X = (X^0)^{\perp}$. As B^0 is the restriction of Δ to X^0 by 10.4.2, the kernel of B^0 is $k \cdot 1_X$. Now 1_X is unimodular, so $\overline{X} = X^0/k \cdot 1_X$ is finitely generated and projective of rank $r-2$. Moreover, $\binom{r}{2} = d(r)d(r-1)$ shows that $Q^0(1_X) = -\binom{r}{2} = 0$ in k, hence Q^0 induces a quadratic form $\overline{Q}$ on $\overline{X}$ whose polar form $\overline{B}$ is the nonsingular symmetric bilinear form induced by B^0 on $\overline{X}$.

(b) From 10.2.2, 10.2.3 and $d(r) = 0$ it is clear that $\mathbf{C}_r$ is the trivial group and hence so is $\mathbf{Cent}(\mathbf{G})$ by Theorem 10.5(a). Since automorphisms fix 1_X, res maps $\mathbf{G}$ to $\mathbf{O}_1(Q^0)$. The kernel of res, being central (Theorem 10.5(b)), is trivial, so res is a monomorphism. To determine its image, choose $u \in X$ with $T(u) = 1$, decompose $X = k \cdot u \oplus X^0$ and identify elements of $\mathrm{GL}(X)$ with formal 2×2-matrices. Then an easy computation shows that $g \in \mathbf{G}(k)$ if and only if $g = \left(\begin{smallmatrix} 1 & 0 \\ w & h \end{smallmatrix}\right)$ where $w \in X^0$, $h \in \mathrm{O}_1(Q^0)$ and

$$Q(w) + B(u,w) = 0, \tag{3}$$

$$B(w,h(x)) + B(u,h(x) - x) = 0 \qquad (x \in X^0). \tag{4}$$

After replacing x by $h^{-1}(x)$, (4) is equivalent to

$$B^0(w,x) = B(u,x - h^{-1}(x)) \qquad (x \in X^0). \tag{5}$$

Now let $h \in \mathrm{O}_1(Q^0)$ be given. Then finding an element $g \in \mathbf{G}(k)$ with $\mathrm{res}(g) = h$ amounts to finding a solution $w \in X^0$ of (3) and (5). As a function of x, the right hand side of (5) is a linear form on X^0 which vanishes for $x = 1_X$. Hence it induces a linear form on $\overline{X}$ which is uniquely representable by $\overline{B}$. Lifting this back to X^0, there exists $w' \in X^0$, unique modulo $k \cdot 1_X$, such that (5) holds for all $w = \lambda \cdot 1_X + w'$. Then condition (3) becomes:

$$\begin{aligned} 0 &= Q(\lambda \cdot 1 + w') + B(u, \lambda \cdot 1 + w') \\ &= \binom{r}{2}\lambda^2 + (r-1)T(w') + Q(w') + (r-1)\lambda T(u) + B(u,w') \\ &= Q(w') - \lambda + B(u,w'), \end{aligned}$$

because $r = \binom{r}{2} = 0$ in k and $T(w') = 0$. This proves the existence of w, as desired. Since these arguments remain valid in all base extensions, we have (1).

It remains to show (2). Choose a decomposition $X^0 = k \cdot 1 \oplus M$ and let $Q' = Q^0 \big| M$. Then $Q^0 = \langle 0 \rangle \perp Q'$, and the canonical projection induces an isomorphism $(M,Q') \cong (\overline{X},\overline{Q})$. Writing the elements of $\mathrm{GL}(X^0)$ again as 2×2-matrices with respect to this decomposition, it is easy to see that $h \in \mathrm{O}_1(Q^0)$ if and only if $h = \left(\begin{smallmatrix} 1 & \alpha \\ 0 & h' \end{smallmatrix}\right)$ where $h' \in \mathrm{O}(Q')$ and $\alpha \in M^*$ are arbitrary. From this, the remaining assertions follow readily.

10.10. Corollary. *Let $\mathfrak{X}$ be a separable quadratic trace module of rank $r \geqslant 2$ and $\mathbf{G} = \mathbf{Aut}(\mathfrak{X})$. Then*

$$\mathbf{G} \text{ is reductive} \iff d(r) \in k^\times \iff Q^0 \text{ is separable.}$$

Proof. The equivalence of the second and third condition follows from Lemma 10.7. Suppose $d(r) \in k^\times$. If $r = 2n$ then by Theorem 10.8(a), $\mathbf{G} \cong \mathbf{SO}(Q^0) \times \mathbf{Z}_2$. After a faithfully flat base change, Q^0 becomes isomorphic to the standard quadratic form of rank $2n-1$ and therefore $\mathbf{SO}(Q^0)$ isomorphic to $\mathbf{SO}_{2n-1}$ which is known to be reductive. Hence so is $\mathbf{G}$. The proof in case r odd is similar, using Theorem 10.8(b) and reductivity of $\mathbf{O}_{2n}$. On the other hand, suppose $d(r) \notin k^\times$. Then there exists a prime ideal $\mathfrak{p}$ of k such that $d(r) = 0$ in $\boldsymbol{\kappa}(\mathfrak{p})$. As $d(2) = 1$, we have $r \geqslant 3$. Then the fibre $\mathbf{G} \otimes \boldsymbol{\kappa}(\mathfrak{p})$ has a unipotent radical of dimension $r-2$ by Theorem 10.9.

11. Appendix: Some determinant formulas

11.1. The half- and the quarter-determinant The $n \times n$ unit matrix is denoted I_n and the transpose of a matrix A is $A^\top$. If A is a matrix of odd order with indeterminate entries, the determinant of $A + A^\top$ is divisible by 2, so there is a well-defined integral polynomial $\operatorname{hdet}(A)$ in the entries of A such that

$$2\operatorname{hdet}(A) = \det(A + A^\top), \tag{1}$$

called the *half-determinant* of A. Similarly, if A is of even order $2n$, the *quarter-determinant* of A is the integral polynomial $\operatorname{qdet}(A)$ in the entries of A satisfying

$$4\operatorname{qdet}(A) = \det(A + A^\top) - (-1)^n \det(A - A^\top), \tag{2}$$

see [12, 1.1] for details. The Pfaffian of an alternating matrix of even order is denoted $\operatorname{Pf}(A)$.

11.2. Lemma. *Denote by U_n the $n \times n$-matrix which has zeros in and below the diagonal and all entries above the diagonal equal to* 1. *Then*

$$\det(U_n + U_n^\top) = (-1)^{n-1}(n-1), \tag{1}$$

$$\det(nI_n - U_n - U_n^\top) = (n+1)^{n-1}, \tag{2}$$

$$\operatorname{hdet}(U_{2n+1}) = n, \tag{3}$$

$$\operatorname{Pf}(U_{2n} - U_{2n}^\top) = 1, \tag{4}$$

$$\operatorname{qdet}(U_{2n}) = -\lfloor n/2 \rfloor, \tag{5}$$

$$\operatorname{qdet}(U_{2n} - nI_{2n}) = \varpi(n). \tag{6}$$

Proof. (1) and (2) are special cases of the formula

$$\det\big(aI_n + b(U_n + U_n^\top)\big) = (a-b)^{n-1}[a + (n-1)b] \tag{7}$$

which is easily proved by using the basis $e_1, e_2 - e_1, \dots, e_n - e_1$ of k^n. For the remaining formulas, we may assume $k = \mathbb{Z}$. Then (3) is clear from (1) and 11.1.1 since we can cancel a factor 2. Formula (4) follows by induction from the expansion formula for the Pfaffian given in [1, Exercise 5 for §5, p. 86]. Formula (5) is a consequence of 2.2.4, (1) and (4) and 11.1.2. Finally, by 11.1.2, (2), (4) and 2.3.3,

$$\begin{aligned} 4\,\mathrm{qdet}(U_{2n} - nI_{2n}) &= \det(U_{2n} + U_{2n}^\top - 2nI_{2n}) - (-1)^n \det(U_{2n} - U_{2n}^\top) \\ &= (2n+1)^{2n-1} - (-1)^n = 4\varpi(n), \end{aligned}$$

cf. 2.3.3.

11.3. Lemma. *Let A and D be matrices of size $l \times l$ and $m \times m$ with coefficients in k, let $x, u \in k^l$ and $y, v \in k^m$ be row vectors and let $\alpha \in k$. Then*

$$\det \begin{pmatrix} \alpha & v \\ y^\top & D \end{pmatrix} = \alpha \det(D) - vD^\dagger y^\top \tag{1}$$

$$= \alpha \det(D) + \det \begin{pmatrix} 0 & v \\ y^\top & D \end{pmatrix}, \tag{2}$$

$$\alpha^{m-1} \det \begin{pmatrix} \alpha & v \\ y^\top & D \end{pmatrix} = \det\left(\alpha D - y^\top v \right), \tag{3}$$

$$\det \begin{pmatrix} A & x^\top v \\ y^\top u & D \end{pmatrix} = \det(A)\det(D) - (uA^\dagger x^\top)(vD^\dagger y^\top). \tag{4}$$

If $m = 2n$ and U is an $m \times m$-matrix, then

$$\mathrm{hdet} \begin{pmatrix} \alpha & 2v \\ 0 & U \end{pmatrix} = \det \begin{pmatrix} \alpha & v \\ 2v^\top & U + U^\top \end{pmatrix}. \tag{5}$$

Here $A^\dagger$ denotes the adjoint matrix, so $AA^\dagger = \det(A)I_l$; in particular, $A^\dagger = 1$ if $l = 1$.

Proof. Formula (1) follows by expanding with respect to the first row and column, see also [3, p. 640, Exercise 13]. For (2), use (1) in the special case $\alpha = 0$. To prove the remaining formulae, we may by a standard density argument (or by working in the rational function field in the indeterminate entries of A, D, x, u, y, v over $\mathbb{Q}$) assume that A is invertible. Then, for rectangular matrices B, C of the appropriate size, a calculation shows

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I_l & 0 \\ CA^{-1} & I_m \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} I_l & A^{-1}B \\ 0 & I_m \end{pmatrix} \tag{6}$$

which implies

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(D - CA^{-1}B). \tag{7}$$

Let here in particular $l = 1$ and $A = \alpha \in k^\times$, so $B = v$ and $C^\top = y$ are in k^m. Then multiplying (7) by α^{m-1} yields

$$\alpha^{m-1} \det \begin{pmatrix} \alpha & v \\ y^\top & D \end{pmatrix} = \alpha^m \det(D - \alpha^{-1} y^\top v) = \det(\alpha D - y^\top v),$$

which is (3).

Now let $B = x^\top v$ and $C = y^\top u$. Then $CA^{-1}B = y^\top(uA^{-1}x^\top)v$. On the other hand, for $\lambda \in k$, by (3) and (1),

$$\det(D-\lambda y^\top v) = \det\begin{pmatrix} 1 & \lambda v \\ y^\top & D \end{pmatrix} = \det(D) - \lambda v D^\dagger y^\top. \tag{8}$$

Substituting (8) into (7) where $\lambda = uA^{-1}x^\top = \det(A)^{-1}(uA^\dagger x^\top)$ yields (4).

Finally, to prove (5), we may assume that α and the entries of v and U are indeterminates and work in the polynomial ring $\mathbb{Z}[\alpha, v, U]$. Then

$$2\mathrm{hdet}\begin{pmatrix} \alpha & 2v \\ 0 & U \end{pmatrix} = \det\begin{pmatrix} 2\alpha & 2v \\ 2v^\top & U+U^\top \end{pmatrix} = 2\det\begin{pmatrix} \alpha & v \\ 2v^\top & U+U^\top \end{pmatrix},$$

so the assertion follows by cancelling the factor 2.

11.4. Lemma. *Let $\alpha \in k$, $y, v \in k^m$ and $D \in \mathrm{Mat}_m(k)$. Then if $m = 2$,*

$$\mathrm{hdet}\begin{pmatrix} \alpha & v \\ y^\top & D \end{pmatrix} = \alpha\det(D+D^\top) + \det\begin{pmatrix} 0 & v+y \\ v^\top+y^\top & D \end{pmatrix}, \tag{1}$$

while for $m = 3$,

$$\mathrm{qdet}\begin{pmatrix} \alpha & v \\ y^\top & D \end{pmatrix} = \alpha\mathrm{hdet}(D) + \det\begin{pmatrix} 0 & v \\ v^\top & D \end{pmatrix} + \det\begin{pmatrix} 0 & y \\ y^\top & D \end{pmatrix} - v(D\times D^\top)y^\top. \tag{2}$$

Here $A \times B = (A+B)^\dagger - A^\dagger - B^\dagger$ is the bilinear map determined by the quadratic map $A \mapsto A^\dagger$.

Proof. We may assume that α and the entries of v, y and D are indeterminates and work in the ring $\mathbb{Z}[v_i, y_i, d_{ij}]$. Let first $m = 2$. Then $D \mapsto D^\dagger$ is linear and commutes with transposition. Hence by 11.1.1 and 11.3.1,

$$\begin{aligned} 2\mathrm{hdet}\begin{pmatrix} \alpha & v \\ y^\top & D \end{pmatrix} &= \det\begin{pmatrix} 2\alpha & v+y \\ v^\top+y^\top & D+D^\top \end{pmatrix} \\ &= 2\alpha\det(D+D^\top) - (v+y)(D+D^\top)^\dagger(v+y)^\top \\ &= 2\alpha\det(D+D^\top) - 2(v+y)D^\dagger(v+y)^\top \\ &= 2\alpha\det(D+D^\top) + 2\det\begin{pmatrix} 0 & v+y \\ v^\top+y^\top & D \end{pmatrix}. \end{aligned}$$

For $m = 3$, we use 11.1.2 and again 11.3.1:

$$\begin{aligned} 4\,\mathrm{qdet}\begin{pmatrix} \alpha & v \\ y^\top & D \end{pmatrix} &= \det\begin{pmatrix} 2\alpha & v+y \\ v^\top+y^\top & D+D^\top \end{pmatrix} - \det\begin{pmatrix} 0 & v-y \\ y^\top-v^\top & D-D^\top \end{pmatrix} \\ &= 2\alpha\det(D+D^\top) - (v+y)(D+D^\top)^\dagger(v+y)^\top \\ &\qquad + (v-y)(D-D^\top)^\dagger(y-v)^\top \\ &= 4\alpha\mathrm{hdet}(D) - 4vD^\dagger v^\top - 4yD^\dagger y^\top - 4v(D\times D^\top)y^\top. \end{aligned}$$

11.5. Lemma. *Let F_1 and F_2 be square matrices of size $2l$ and $2m$, respectively, and let $x \in k^{2l}$ and $y \in k^{2m}$ be row vectors. Then*

$$\operatorname{qdet}\begin{pmatrix} F_1 & x^\top y \\ 0 & F_2 \end{pmatrix} = \operatorname{qdet}\begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} - \operatorname{hdet}\begin{pmatrix} 0 & x \\ 0 & F_1 \end{pmatrix} \operatorname{hdet}\begin{pmatrix} 0 & y \\ 0 & F_2 \end{pmatrix}. \tag{1}$$

Proof. Since the asserted formula is a polynomial identity with integer coefficients in the entries of F_1, F_2, x, y, we may assume these entries to be indeterminates and work over the ring $\mathbb{Z}[F_1, F_2, x, y]$. Put $A = F_1 + F_1^\top$, $D = F_2 + F_2^\top$, $R = F_1 - F_1^\top$ and $S = F_2 - F_2^\top$. By Lemma 11.6, and since the square of the Pfaffian is the determinant, we have

$$\det\begin{pmatrix} R & x^\top y \\ -y^\top x & S \end{pmatrix} = \det\begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}.$$

Now 11.1.2 and 11.3.4 imply

$$\begin{aligned} 4\operatorname{qdet}\begin{pmatrix} F_1 & x^\top y \\ 0 & F_2 \end{pmatrix} - 4\operatorname{qdet}\begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} &= \det\begin{pmatrix} A & x^\top y \\ y^\top x & D \end{pmatrix} - \det\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \\ &= -(xA^\dagger x^\top)(yD^\dagger y^\top). \end{aligned} \tag{2}$$

On the other hand, by 11.1.1 and 11.3.1,

$$2\operatorname{hdet}\begin{pmatrix} 0 & x \\ 0 & F_1 \end{pmatrix} = \det\begin{pmatrix} 0 & x \\ x^\top & A \end{pmatrix} = -xA^\dagger x^\top \tag{3}$$

and similarly for the second factor. Now the assertion follows from (2) and (3) by cancelling the factor 4.

11.6. Lemma. (a) *Let R and S be alternating matrices with entries from k of even order $2l$ and $2m$, respectively, and let $x \in k^{2l}$ and $y \in k^{2m}$ be row vectors. Then*

$$\operatorname{Pf}\begin{pmatrix} R & x^\top y \\ -y^\top x & S \end{pmatrix} = \operatorname{Pf}\begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} = \operatorname{Pf}(R)\cdot\operatorname{Pf}(S). \tag{1}$$

(b) *Let R and S be alternating matrices of odd order $2l+1$ and $2m+1$, and let $x \in k^{2l+1}$ and $y \in k^{2m+1}$ be row vectors. Then*

$$\operatorname{Pf}\begin{pmatrix} R & x^\top y \\ -y^\top x & S \end{pmatrix} = \operatorname{Pf}\begin{pmatrix} 0 & x \\ -x^\top & R \end{pmatrix} \cdot \operatorname{Pf}\begin{pmatrix} 0 & y \\ -y^\top & S \end{pmatrix}. \tag{2}$$

Proof. (a) By a standard density argument it suffices to prove this in case R is invertible. A calculation shows that

$$\begin{pmatrix} I_{2l} & 0 \\ y^\top x & I_{2m} \end{pmatrix}\begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} I_{2l} & x^\top y \\ 0 & I_{2m} \end{pmatrix} = \begin{pmatrix} R & Rx^\top y \\ y^\top xR & S + y^\top xRx^\top y \end{pmatrix}.$$

Since R is alternating, we have $xRx^\top = 0$. Now the lemma follows from

$$\mathrm{Pf}(P^\top XP) = \det(P)\,\mathrm{Pf}(X), \quad \mathrm{Pf}\begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} = \mathrm{Pf}(R)\,\mathrm{Pf}(S) \tag{3}$$

(see [1, §5.2, Prop. 1]), and the fact that as x runs over k^{2l} so does xR, because R is invertible.

(b) Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -x & 0 \\ 0 & y \end{pmatrix}, \quad C = -B^\top, \quad D = \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}.$$

We compute the Pfaffian of $X := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in two ways. Since A is invertible and $(A^{-1}B)^\top = CA^{-1}$, we have by 11.3.6 and (3) that

$$\mathrm{Pf}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{Pf}\begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} = \mathrm{Pf}(D - CA^{-1}B),$$

and a computation shows $CA^{-1}B = \begin{pmatrix} 0 & -x^\top y \\ y^\top x & 0 \end{pmatrix}$. Hence $\mathrm{Pf}(X)$ equals the left hand side of (2). On the other hand, let

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -I_{2l+1} & 0 & 0 \\ 0 & 0 & 0 & I_{2m+1} \end{pmatrix}.$$

Then $\det(J) = 1$ and

$$J^\top XJ = \begin{pmatrix} 0 & x & 1 & 0 \\ -x^\top & R & 0 & 0 \\ -1 & 0 & 0 & y \\ 0 & 0 & -y^\top & S \end{pmatrix}.$$

Hence, by (3) and (1),

$$\mathrm{Pf}(X) = \mathrm{Pf}\begin{pmatrix} 0 & x & 1 & 0 \\ -x^\top & R & 0 & 0 \\ -1 & 0 & 0 & y \\ 0 & 0 & -y^\top & S \end{pmatrix} = \mathrm{Pf}\begin{pmatrix} 0 & x \\ -x^\top & R \end{pmatrix} \cdot \mathrm{Pf}\begin{pmatrix} 0 & y \\ -y^\top & S \end{pmatrix}.$$

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