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A SHORT PROOF OF THE WEDDERBURN-ARTIN THEOREM

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Abstract. The Wedderburn-Artin theorem is of fundamental importance in noncommutative ring theory. A short self-contained proof is given which requires only elementary facts about rings.

Throughout this note R will denote an associative ring with unity $1 \neq 0$. If X and Y are additive subgroups of R, define their product by

$$XY = \left\{ \sum_{i=1}^{n} x_i y_i \mid n \geq 1,\ \ x_i \in X,\ \ y_i \in Y \right\}.$$

This is an associative operation. An additive subgroup K is called a *left* (*right*) *ideal* of R if $RK \subseteq K$, and K is called an *ideal* if it is both a left and right ideal. The ring R is called *semiprime* if $A^2 \neq 0$ for every nonzero ideal A, and R is called *left artinian* if it satisfies the descending chain condition on left ideals (equivalently, every nonempty family of left ideals has a minimal member).

The following theorem is a landmark in the theory of noncommutative rings.

Wedderburn-Artin Theorem. *If R is a semiprime left artinian ring then*

$$R \cong M_{n_1}(D_1) \times M_{n_2}(D_2) \times \ldots \times M_{n_r}(D_r)$$

where each D_i is a division ring and $M_n(D)$ denotes the ring of $n \times n$ matrices over D.

In this form the theorem was proved [**1**] in 1927 by Emil Artin (1898–1962) generalizing the original 1908 result [**4**] of Joseph Henry Maclagan Wedderburn (1882–1948) who proved it for finitely generated algebras over a field. The purpose of this note is to give a quick, self-contained proof of this theorem. A key result is the following observation [**2**] of Richard Brauer (1902–1977). Call a left ideal K *minimal* if $K \neq 0$ and the only left ideals contained in K are 0 and K.

Brauer's Lemma. *Let K be a minimal left ideal of a ring R and assume $K^2 \neq 0$. Then $K = Re$ where $e^2 = e \in R$ and eRe is a division ring.*

Proof. Since $0 \neq K^2$, certainly $Ku \neq 0$ for some $u \in K$. Hence $Ku = K$ by minimality, so $eu = u$ for some $e \in K$. If $r \in K$, this implies re-$r \in L = \{a \in K \mid au = 0\}$. Now L is a left ideal, $L \subseteq K$, and $L \neq K$ because $eu \neq 0$. So $L = 0$ and it follows that $e^2 = e$ and $K = Re$.

Now let $0 \neq b \in eRe$. Then $0 \neq Rb \subseteq Re$ so $Rb = Re$ by minimality, say $e = rb$. Hence $(ere)b = er(eb) = erb = e^2 = e$, so b has a left inverse in eRe. It follows that eRe is a division ring. ∎

The following consequence will be needed later.

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Corollary. *Every nonzero left ideal in a semiprime, left artinian ring contains a nonzero idempotent.*

Proof. If $L \neq 0$ is a left ideal of R, the left artinian condition gives a minimal left ideal $K \subseteq L$. Now $(KR)^2 \neq 0$ because R is semiprime, so $(KR)^2 = KRKR \subseteq K^2R$ shows that $K^2 \neq 0$. Hence Brauer's lemma applies. ∎

A ring R is *simple* if R has no ideals other than 0 and R. Such a ring is necessarily semiprime. When R is simple the Wedderburn-Artin theorem is known as Wedderburn's Theorem and a short proof is well known (see Henderson [**3**]). Since this result is needed in the general case, we sketch the proof. The left artinian hypothesis is weakened to the existence of a minimal left ideal.

Wedderburn's Theorem. *If R is a simple ring with a minimal left ideal, then $R \cong M_n(D)$ for some $n \geq 1$ and some division ring D.*

Proof (Henderson). Let K be a minimal left ideal. Then $KR = R$ (it is a nonzero ideal) so $R = R^2 = (KR)^2 = KRKR \subseteq K^2R$. Hence $K^2 \neq 0$ so, by Brauer's lemma, $K = Re$ where $e^2 = e$ and $D = eRe$ is a division ring. Then K is a right vector space over D and, if $r \in R$, the map $\alpha_r : K \to K$ given by $\alpha_r(k) = rk$ is a D-linear transformation. Hence $r \to \alpha_r$ is a ring homomorphism $R \to \text{end}_D K$, and it is one-to-one because $\alpha_r = 0$ implies $rRe = 0$ so $0 = rReR = rR$ ($ReR = R$ because R is simple). To see that it is onto, write $1 \in ReR$ as $1 = \sum_{i=1}^n r_i e s_i$. Given $\alpha \in \text{end}_D K$, let $a = \sum_i \alpha(r_i e) e s_i$. Then the D-linearity of α gives

$$\alpha(re) = \alpha\left[\sum_i (r_i e s_i) re\right] = \sum_i \alpha(r_i e)(e s_i r e) = a \cdot re = \alpha_a(re)$$

for all $r \in R$, so $\alpha = \alpha_a$. Thus $R \cong \text{end}_D K$ and it remains to show that K_D is finite dimensional $\big($then $\text{end}_D K \cong M_n(D)$ where $n = \dim_D K\big)$. But if $\dim_D K$ is infinite, the set $A = \{\alpha \in \text{end}_D K \mid \alpha(K) \text{ has finite dimension}\}$ is a proper ideal of $\text{end}_D K$, contrary to the simplicity of R. ∎

It is worth noting that, if $e^2 = e \in R$ is such that $ReR = R$, the proof shows that $R \cong \text{end}_D K$ where $K = Re$ is regarded as a right module over $D = eRe$.

To prove the Wedderburn-Artin theorem, it is convenient to introduce a weak finiteness condition in a ring R. Let I denote the set of idempotents in R. Given e, f in I, write $e \leq f$ if $ef = e = fe$, that is if $eRe \subseteq fRf$. This is a partial ordering on I (with 0 and 1 as the least and greatest elements) and I is said to satisfy the *maximum condition* if every nonempty subset contains a maximal element, equivalently if $e_1 \leq e_2 \leq \dots$ in I implies $e_n = e_{n+1} = \dots$ for some $n \geq 1$. The *minimum condition* on I is defined analogously. A set of idempotents is called *orthogonal* if $ef = 0$ for all $e \neq f$ in the set.

Lemma 1. *The following are equivalent for a ring R:*

(1) *R has maximum condition on idempotents.*

(2) *R has minimum condition on idempotents.*

(3) *R has maximum condition on left ideals Re, $e^2 = e$ (on right ideals eR, $e^2 = e$).*

(4) *R has minimum condition on left ideals Re, $e^2 = e$ (on right ideals eR, $e^2 = e$).*

(5) *R contains no infinite orthogonal set of idempotents.*

Proof. The verification that (1) $\Leftrightarrow$ (2), (3) $\Leftrightarrow$ (4) and (3) $\Rightarrow$ (5) $\Rightarrow$ (1) are routine, so we prove that (1) $\Rightarrow$ (3). If $Re_1 \subseteq Re_2 \subseteq \dots$ where $e_i^2 = e_i$ for each i, then $e_ie_j = e_i$ for all $j \geq i$ so we inductively construct idempotents $f_1 \leq f_2 \leq \dots$ as follows: Take $f_1 = e_1$ and, if f_i has been specified, take $f_{i+1} = f_i + e_{i+1} - e_{i+1}f_i$. An induction shows that $f_i \in Re_i$ for each i, whence $f_ie_k = f_i$ for all $k \geq i$. Using this one verifies that $f_i^2 = f_i$ and $f_i \leq f_{i+1}$ hold for each $i \geq 1$. Thus (1) implies that $f_n = f_{n+1} = \dots$ for some n and hence that $e_{i+1} = e_{i+1}f_i \in Re_i$ for all $i \geq n$. It follows that $Re_n = Re_{n+1} = \dots$. The maximum condition on right ideals eR is proved similarly. ∎

Call a ring R *I-finite* if it satisfies the conditions in Lemma 1. It is clear that every left (or right) artinian or noetherian ring is I-finite.

Proof of the Wedderburn-Artin Theorem. Let R be a semiprime, left artinian ring, let K be a minimal left ideal, let $S = KR$, and let $M = \{a \in R \mid Sa = 0\}$. Then S and M are ideals of R and we claim that

$$R = S \oplus M. \tag{$*$}$$

First $S \cap M = 0$ because R is semiprime and $(S \cap M)^2 \subseteq SM = 0$. Since R is I-finite, let e be a maximal idempotent in S. To show that $R = S + M$, it suffices to show $1 - e \in M$. If not, then $S(1 - e) \neq 0$ so (by the Corollary to Brauer's lemma) let $f \in S(1 - e)$ be a nonzero idempotent. Then $fe = 0$ and one verifies that $g = e + f - ef$ is an idempotent in S and $e \leq g$. The maximality of e then gives $e = g$, so $f = ef$, whence $f = f^2 = fef = 0$, a contradiction. So $1 - e \in M$ and $R = S + M$, proving ($*$).

Hence S and M are rings (with unity) and they inherit the hypotheses on R because left ideals of S or M are left ideals of R by ($*$). Moreover, this shows that S is simple. Indeed, if $A \neq 0$ is an ideal of S then $A \cap K \neq 0$ (otherwise $A^2 \subseteq AKR \subseteq (A \cap K)R = 0$) so the minimality of K gives $K \subseteq A$, whence $S = KR \subseteq A$.

If $M = 0$ the proof is complete by Wedderburn's theorem. Otherwise, repeat the above with R replaced by M to get $R = S \oplus S_1 \oplus M_1$ where S_1 is simple. This cannot continue indefinitely by the artinian hypothesis (or I-finiteness), so Wedderburn's theorem completes the proof. ∎

Remark 1. The converse to both these theorems is true.

Remark 2. These proofs actually yield the following: A ring R is semiprime and left artinian if and only if it satisfies the following condition.

R is I-finite and every nonzero left ideal contains a nonzero idempotent. ($**$)

The necessity of (**) follows from Lemma 1 and the Corollary to Brauer's lemma. Conversely, if R satisfies (**) then the proofs of both theorems go through virtually as written once the following is established: If E is a minimal nonzero idempotent, then Re is a minimal left ideal. But if $L \subseteq Re$ is a left ideal and $L \neq 0$, let $0 \neq f^2 = f \in L$. Then $fe = f$ so $g = ef \in L$ is an idempotent, $g \neq 0$ (because $f = fg$) and $g \leq e$. Thus $g = e$ by the minimality of e, whence $L = Re$.

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