Proof. We first check N and c are admissible. For $y=(b_0, b_1, b_2)$ we have

$$\partial_y N|_x = t(a_0^\#, b_0)+\mu t(a_1^\#, b_1)+\mu^{-1}t(a_2^\#, b_2)-t(b_0a_1a_2)-t(a_0b_1a_2)-t(a_0a_1b_2)$$

since $\partial_b n|_a=t(a^\#, b)$ where $a^\#$ also denotes the adjoint in $\mathfrak{A}$. Then

$$\begin{aligned} T(x, y) &= \{\partial_x N|_c\}\{\partial_y N|_c\}-\partial_x\,\partial_y N|_c \\ &= \{t(a_0)\}\{t(b_0)\}-t(1\times a_0, b_0)-t(b_1, a_2)-t(a_1, b_2) \\ &= t(a_0, b_0)+t(a_1, b_2)+t(a_2, b_1) \end{aligned}$$

where $a\times b=(a+b)^\#-a^\#-b^\#$, $a\times 1=t(a)1-a$ in $\mathfrak{A}$ as usual. Clearly T is nondegenerate since t is. Also $\partial_y N|_x=T(x^\#, y)$ for

$$x^\# = (a_0^\#-a_1a_2, \mu^{-1}a_2^\#-a_0a_1, \mu a_1^\#-a_2a_0).$$

If we let $x=(a_0, a_1, a_2)$, $x^\#=(b_0, b_1, b_2)$, $x^{\#\#}=(c_0, c_1, c_2)$ then

$$\begin{aligned} c_0 &= b_0^\#-b_1b_2 = \{a_0^\#-a_1a_2\}^\#-\{\mu^{-1}a_2^\#-a_0a_1\}\{\mu a_1^\#-a_2a_0\} \\ &= a_0^{\#\#}-a_0^\#\times a_1a_2+(a_1a_2)^\#-a_2^\#a_1^\#-a_0(a_1a_2)a_0+\mu a_0(a_1a_1^\#)+\mu^{-1}(a_2^\#a_2)a_0 \\ &= \{n(a_0)+\mu n(a_1)+\mu^{-1}n(a_2)\}a_0-\{a_0^\#\times a_1a_2+a_0(a_1a_2)a_0\} \end{aligned}$$

since $a^{\#\#}=n(a)a$, $a^\#a=n(a)1=aa^\#$, $(ab)^\#=b^\#a^\#$ in $\mathfrak{A}$. Now since $\mathfrak{A}$ is of degree 3 it is easily verified that the analogue of (9) holds in $\mathfrak{A}$: $aba=t(a, b)a-a^\#\times b$. Hence the above becomes

$$= \{n(a_0)+\mu n(a_1)+\mu^{-1}n(a_2)-t(a_0, a_1a_2)\}a_0 = N(x)a_0.$$

Similar arguments show $c_i=N(x)a_i$ for $i=1, 2$, so $x^{\#\#}=N(x)x$. This holds for any extension of Φ, so N and c are admissible, and $\mathfrak{J}(\mathfrak{A}, \mu)$ is a Jordan algebra.

Suppose $\mathfrak{J}(\mathfrak{A}, \mu)$ is a Jordan division algebra; by Theorem 2 this is equivalent to $N(x)\neq 0$ for $x\neq 0$. Clearly $n(a)\neq 0$ for $a\neq 0$ or else $N(x)=0$ for $x=(a, 0, 0)\neq 0$. Thus $\mathfrak{A}$ is an associative division algebra. If $\mu=n(a)$ then $x=(-1, 0, a)$ has $N(x)=n(-1)+\mu^{-1}n(a)=0$, so μ can not be a generic norm.

Now suppose $\mathfrak{A}$ is a division algebra and μ is not a generic norm. If $N(x)=0$ for $x\neq 0$ then $x^{\#\#}=N(x)x=0$; either $x^\#=0$ for $x\neq 0$ or else $y=x^\#\neq 0$ has $y^\#=0$. In any case there exists $x=(a_0, a_1, a_2)\neq 0$ with $x^\#=0$. Then $a_0^\#=a_1a_2$, $\mu^{-1}a_2^\#=a_0a_1$, $\mu a_1^\#=a_2a_0$. Since $aa^\#=a^\#a=n(a)1$ in $\mathfrak{A}$, $n(a_0)1=a_0a_1a_2=\mu^{-1}n(a_2)1=a_2a_0a_1=\mu n(a_1)1$. Not all $a_i=0$, so not all $n(a_i)=0$ (since $\mathfrak{A}$ is a division algebra), so none are, and $\mu=n(a_0a_1^{-1})$ is a norm, contrary to hypothesis.

This completes the proof.

For Tit's second construction, let $\mathfrak{A}$ be a separable associative algebra with 1 of degree 3 over a field Ω, $*$ an involution of second kind on $\mathfrak{A}$ with fixed field Φ. Then the space $\mathfrak{A}_0=\mathfrak{H}(\mathfrak{A}, *)$ of symmetric elements contains an element u whose generic norm is of the form $n(u)=\mu\mu^*$ for some nonzero $\mu\in\Omega$. We set $\mathfrak{X}=\mathfrak{A}_0\oplus\mathfrak{A}$ as a vector space over Φ, let $c=(1, 0)$, and for $x=(a_0, a)$ define

$$N(x) = n(a_0)+\mu n(a)+\mu^* n(a^*)-t(a_0aua^*).$$

Note that the values of N actually lie in Φ. Indeed, if $\mathfrak{A}$ is any generically algebraic algebra over Ω and S a semilinear automorphism or antiautomorphism of $\mathfrak{A}$ with associated automorphism σ of Ω then the generic trace and norm satisfy $t(Sx)=t(x)^\sigma$ and $n(Sx)=n(x)^\sigma$. Taking $S=\sigma=*$ in our case we see that $t(a^*)=t(a)^*$ and $n(a^*)=n(a)^*$. Since a_0 and aua^* lie in $\mathfrak{H}(\mathfrak{A}, *)$ we see $n(a_0)$ and $t(a_0, aua^*)$ lie in $\mathfrak{H}(\Omega, *)=\Phi$, as does $\mu n(a)+\mu^* n(a^*)$, so $N(x)\in\Phi$. We denote $\mathfrak{J}(N, c)$ by

$$\mathfrak{J}(\mathfrak{A}, u, \mu, *).$$

THEOREM 7 (TITS' SECOND CONSTRUCTION). *If $\mathfrak{A}$ is a separable associative algebra with involution $*$ of the second kind, and u a symmetric element with norm $n(u)=\mu\mu^*$ for nonzero μ in Ω, then $\mathfrak{J}(\mathfrak{A}, u, \mu, *)$ is a Jordan algebra. It is a Jordan division algebra if and only if $\mathfrak{A}$ is an associative division algebra and μ is not the generic norm of an element of $\mathfrak{A}$.*

Proof. The fact that N and c are admissible can be proved as in Theorem 6, using the formulas

$$\begin{aligned} T(x, y) &= t(a_0, b_0)+t(ua^*, b)+t(au, b^*), \\ x^\# &= (a_0^\#-aua^*, \mu^*(a^*)^\# u^{-1}-a_0a) \end{aligned}$$

for $x=(a_0, a)$ and $y=(b_0, b)$ with a_0, b_0 in $\mathfrak{A}_0$ and a, b in $\mathfrak{A}$. The computations are much the same as in Theorem 6, though at one point we need the relation $\mu\mu^* u^{-1}=n(u)u^{-1}=u^\#$, which follows from our choice of u and μ. We will shortly given an alternate, and more illuminating, proof that $\mathfrak{J}(\mathfrak{A}, u, \mu, *)$ is a Jordan algebra.

If $\mathfrak{J}$ is a division algebra then so is $\mathfrak{A}$, since $n(a)=0$ for $a\neq 0$ would imply $N(x)=0$ for $x=(0, a)\neq 0$. Also, μ cannot be a generic norm, since $\mu=n(a)$ would imply $x=(a^*u^{-1}a, a^{-1})\neq 0$ has

$$\begin{aligned} N(x) &= n(a)^* n(u)^{-1}n(a)+\mu n(a)^{-1}+\mu^* n(a)^{*-1}-t(a^*u^{-1}a, a^{-1}ua^{-1*}) \\ &= \mu^*(\mu\mu^*)^{-1}\mu+\mu\mu^{-1}+\mu^*\mu^{*-1}-t(1) = 3-3 = 0. \end{aligned}$$

Conversely, if $\mathfrak{A}$ is a division algebra and μ is not a generic norm then $\mathfrak{J}$ is a division algebra, for if $N(x)=0$ for some $x\neq 0$ then as before $x^\#=0$ for some $x\neq 0$. For $x=(a_0, a)$ this means $a_0^\#=aua^*$, $\mu^*(a^*)^\# u^{-1}=a_0a$. Then

$$n(a_0)1 = a_0a_0^\# = a_0aua^* = \{\mu^*(a^*)^\# u^{-1}\}ua^* = \mu^*(a^*)^\# a^* = \mu^* n(a^*),$$

or $\mu n(a)=n(a_0)^*=n(a_0)$. If $n(a)=0$ then $n(a_0)=0$ too, so $a=a_0=0$ since $\mathfrak{A}$ is a division algebra, which contradicts $x\neq 0$. Thus $\mu=n(a_0a^{-1})$ is a generic norm, which is again a contradiction.

Another way to show that $\mathfrak{J}(\mathfrak{A}, u, \mu, *)$ is a Jordan algebra is to show it is imbedded in $\mathfrak{J}(\mathfrak{A}, \mu)$ as the subalgebra of fixed points relative to an automorphism. We claim the involution $*$ on $\mathfrak{A}$ extends to an involution (=automorphism of period 2)

$$x = (a_0, a_1, ua_2)\to x^* = (a_0^*, a_2^*, ua_1^*)$$