27-XI-20. "Teoría de repr."

Algunas propiedades de caract. complejos.

Si $\rho : G \to GL_n(\mathbb{C})$ repr. complejas. G grupo finito,

$\chi_\rho : G \to \mathbb{C}$; $\chi_\rho(g) = tr(\rho(g))$.

Podemos definir el prod. escalar de dos caracteres

$\rho : G \to GL_n(\mathbb{C})$, $\sigma : G \to GL_n(\mathbb{C})$ repr.

$$\langle \chi_\rho, \chi_\sigma \rangle := \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g) \overline{\chi_\sigma(g)}.$$

Es una forma sesquilineal

$\langle \chi_1 + \chi_2, \chi_3 \rangle = \langle \chi_1, \chi_3 \rangle + \langle \chi_2, \chi_3 \rangle$, χ_i = caract. de repres.

$\langle \lambda \chi_1, \chi_2 \rangle = \lambda \langle \chi_1, \chi_2 \rangle$, $\lambda \in \mathbb{Z}$

$\langle \chi_1, \chi_2 \rangle = \overline{\langle \chi_2, \chi_1 \rangle}$ $\Rightarrow$ $\langle \chi_1, \chi_2 + \chi_3 \rangle = \langle \chi_1, \chi_2 \rangle + \langle \chi_1, \chi_3 \rangle$

(a). Si ρ es irrep. compleja $\Leftrightarrow \langle \chi_\rho, \chi_\rho \rangle = 1$ (Apuntes)

Δ_3	1	g	g^2	s	sg	sg^2
χ_1	1	1	1	1	1	1
χ_u	1	1	1	-1	-1	-1
χ_d	2	-1	-1	0	0	0

$\langle \chi_d, \chi_d \rangle = \frac{1}{6}(2^2 + 1^2 + 1^2) = 1$

$\langle \chi_u, \chi_u \rangle = \frac{1}{6} \cdot 6 = 1 = \langle \chi_1, \chi_1 \rangle$

(b) Si ρ, σ son irreps complejas $\rho \neq \sigma$ $\langle \chi_\rho, \chi_\sigma \rangle = 0$

$\langle \chi_d, \chi_u \rangle = \frac{1}{6}(2 - 1 - 1) = 0 = \langle \chi_d, \chi_1 \rangle$

Los irreps son dos a dos $\perp$ y de módulo 1.

Habíamos visto: si G abeliano, finito, V espacio de una irrep $\Rightarrow$ $\dim(V) = 1$

Irreps complejas de $\mathbb{Z}_n = \{1, \pi, \pi^2, \ldots, \pi^{n-1}\}$ $\quad \rho : \mathbb{Z}_n \to \mathbb{C}^\times = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ hom. de grupos.

(R anillo, $R^\times$ = grupo de inversibles para $\cdot$ $\quad \mathbb{Z}^\times = \{\pm 1\}$

$R^* = R \setminus \{0\}$ $\quad \mathbb{Z}^* = \mathbb{Z} - \{0\}$

R es cuerpo $\quad R^\times = R^*$)

$\rho : \mathbb{Z}_n \to \mathbb{C}^\times$ $\quad \pi^n = 1$, $\omega^n = 1$ $\quad \omega$ = raíz n-ésima de 1.

$\pi \longmapsto \omega$

$\rho(\pi^i) = \omega^i$

Ejemplo. $\mathbb{Z}_2$ $\quad \omega = \sqrt{1} = \pm 1$

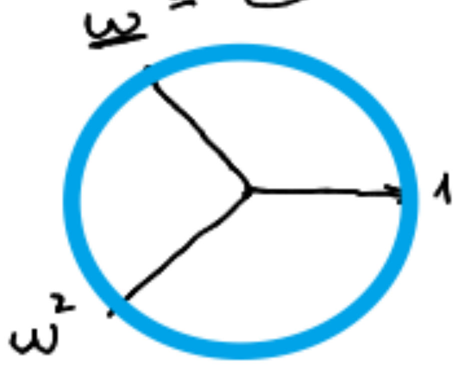
	1	π
ρ_1	1	1
ρ_2	1	-1

Tabla de caract. de $\mathbb{Z}_2$

G abeliano finito $\quad |G|$ = nº de clases de conjugación = nº de irreps complejas.

$\mathbb{Z}_3$ $\quad \omega$ = raíz cúbica.

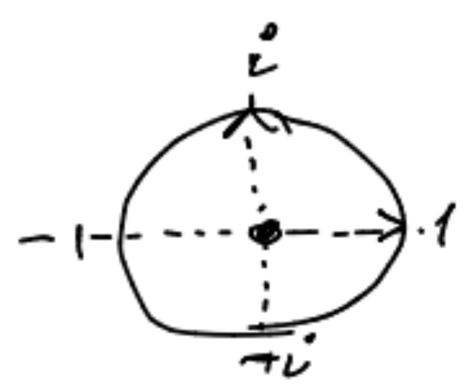
$\omega = e^{\frac{2\pi}{3} i}$



$e^{\frac{4\pi}{3} i} = \omega^2$

	1	π	π^2
ρ_1 =	1	1	1
ρ_2 =	1	ω	ω^2
ρ_3 =	1	ω^2	ω

$\mathbb{Z}_4$ $\quad \omega$ = raíz cuarta $\quad \pm 1, \pm i$



	1	π	π^2	π^3
ρ_1	1	1	1	1
ρ_2	1	-1	1	-1
ρ_3	1	i	-1	$-i$
ρ_4	1	$-i$	-1	i

$\rho(\pi) = -i$

	1	π	$\cdots$	π^{n-1}
ρ_1	1	1	$\cdots$	
ρ_2	1	ω	$\cdots$	
ρ_3	1	ω^2	$\cdots$	
$\vdots$				
ρ_n	1	ω^{n-1}	$\cdots$	

irreps de $\mathbb{Z}_n$

$\omega = \sqrt[n]{1} = e^{\frac{2\pi i}{n}}$ primitiva $(k=1)$

$\{1, \omega, \omega^2, \ldots, \omega^{n-1}\}$ Todas

Relación las irreps con idemp. del álgebra grupo.

Si ρ irrep (compleja) $\quad e$ = idemp. asociado a la irrep $\quad e_\rho = \frac{1}{n} \sum_{g \in \mathbb{Z}_n} \rho(g) g$ idempotente

$\mathbb{Z}_2$

$e_1 = \frac{1}{2}(1 + \pi)$

$e_2 = \frac{1}{2}(1 - \pi)$

$e_1 + e_2 = 1$

$e_1^2 = \frac{1}{4}(1 + \pi^2 + 2\pi) = \frac{1}{4}(2 + 2\pi) = \frac{1 + \pi}{2} = e_1$

$e_1 e_2 = \frac{1}{4}(1 + \pi)(1 - \pi) = \frac{1}{4}(1 - \pi^2) = 0$

Idemp. ortog. $e_1 + e_2 = 1$ en $\mathbb{C}\mathbb{Z}_2$

En $\mathbb{Z}_3$

$$\begin{array}{c|ccc} & 1 & \pi & \pi^2 \\ \hline e_1 & 1 & 1 & 1 \\ e_2 & 1 & \omega & \omega^2 \\ e_3 & 1 & \omega^2 & \omega \end{array}$$

$\omega = e^{\frac{2\pi}{3}i} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

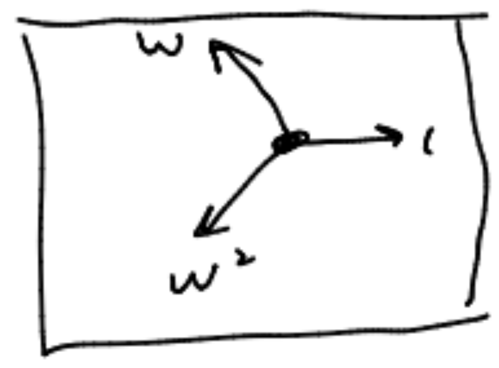
$\omega^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

Idemp. ortogonales

$e_1 + e_2 + e_3 = 1$

$e_1 = \frac{1}{3}(1 + \pi + \pi^2)$; $e_2 = \frac{1}{3}(1 + \omega\pi + \omega^2\pi^2)$; $e_3 = \frac{1}{3}(1 + \omega^2\pi + \omega\pi^2)$;

$\boxed{1 + \omega + \omega^2 = 0}$



$$\boxed{e_1 + e_2 + e_3 = 1 + \cancel{\frac{(1+\omega+\omega^2)\pi}{3}} + \cancel{\frac{(1+\omega+\omega^2)\pi^2}{3}} = 1}$$

en $\mathbb{C}\mathbb{Z}_3$

En $\mathbb{Z}_4$

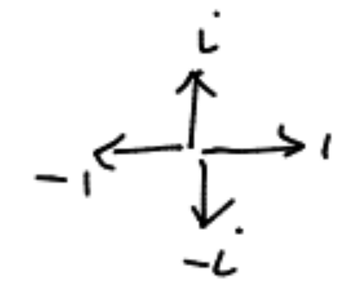
$$\begin{array}{c|cccc} & 1 & \pi & \pi^2 & \pi^3 \\ \hline e_1 & 1 & 1 & 1 & 1 \\ e_2 & 1 & -1 & 1 & -1 \\ e_3 & 1 & i & -1 & -i \\ e_4 & 1 & -i & -1 & i \end{array}$$

$e_1 = \frac{1}{4}(1 + \pi + \pi^2 + \pi^3)$

$e_2 = \frac{1}{4}(1 - \pi + \pi^2 - \pi^3)$

$e_3 = \frac{1}{4}(1 + i\pi - \pi^2 - i\pi^3)$

$e_4 = \frac{1}{4}(1 - i\pi - \pi^2 + i\pi^3)$



$\sum_1^4 e_i = 1 + \cancel{(1 - 1 + i - i)}\pi + \cancel{(1 + 1 - 1 - 1)}\pi^2 + \cancel{(1 - 1 - i + i)}\pi^3$

$\boxed{\sum_1^4 e_i = 1}$

Alguna aplicación.

$\mathbb{Z}_2 = \{1, \pi\}, \ \pi^2 = 1$

En $\mathbb{C}\mathbb{Z}_2$ $\quad e_1 = \frac{1}{2}(1 + \pi)$, $e_2 = \frac{1}{2}(1 - \pi)$

$1 = e_1 + e_2$

$\boxed{\mathbb{Z}_2 \times X \to X}$ acción de grupo

V = esp. vect complejo cuya base es $X = \{\sum_i \lambda_i x_i \ / \ \lambda_i \in \mathbb{C}, \ x_i \in X\}$

$\boxed{\mathbb{Z}_2 \times V \to V}$

g, $\quad g \cdot \sum_i \lambda_i x_i := \sum_i \lambda_i \, g \cdot x_i$,

$\mathbb{C}\mathbb{Z}_2 \times V \to V$

$\left(\sum_{g \in \mathbb{Z}_2} \lambda_g \cdot g\right) \cdot v := \sum_{g \in \mathbb{Z}_2} \lambda_g (g \cdot v)$

Acción del alg. grupo sobre el E.V. libre gen. por X

$\mathbb{C}\mathbb{Z}_2$

$1 = e_1 + e_2$

$\forall x \in X$

$\boxed{x = 1 \cdot x = e_1 \cdot x + e_2 x = \frac{1}{2}(1 + \pi) \cdot x + \frac{1}{2}(1 - \pi) x}$

Tomo $\quad x^2 + bx + c = 0, \quad b, c \in \mathbb{Q}$,

$\mathbb{Z}_2$ actuando sobre $X = \{x_1, x_2\}$

x_1, x_2 las soluciones de la ec.

$\mathbb{Z}_2 \times X \to X$

$1 \cdot x_i = x_i$

$\pi \cdot x_1 = x_2$

$\pi \cdot x_2 = x_1$

$x_1 = \frac{1}{2}(1 + \pi) x_1 + \frac{1}{2}(1 - \pi) x_1 = \boxed{\frac{1}{2}(x_1 + x_2) + \frac{1}{2}(x_1 - x_2)}$

$x_1 = -\frac{b}{2} + \frac{1}{2}\sqrt{b^2 - 4c}$

¿ Expr. $x_1 - x_2$ en func. de b c ?

Cardano - Vieta

$-b = x_1 + x_2$

$c = x_1 x_2$

$(x_1 - x_2)^2 = x_1^2 + x_2^2 - 2x_1x_2$

$(x_1 + x_2)^2 = x_1^2 + x_2^2 + 2x_1x_2$; $\quad b^2 = x_1^2 + x_2^2 + 2c$

$x_1^2 + x_2^2 = b^2 - 2c$

$(x_1 - x_2)^2 = b^2 - 2c - 2c = b^2 - 4c$

$\mathbb{C}\mathbb{Z}_3$

$x^3 + bx^2 + cx + d = \underbrace{(x - x_1)(x - x_2)}_{x^2 - xx_2 - xx_1 + x_1x_2}(x - x_3) = [x^2 - (x_1 + x_2)x + x_1x_2](x - x_3) =$

$= x^3 - x^2x_3 - (x_1 + x_2)x^2 + (x_1 + x_2)xx_3 + x_1x_2x - x_1x_2x_3 =$

$= x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_1x_3 + x_2x_3)x - x_1x_2x_3$.

$b = -(x_1 + x_2 + x_3)$

$c = x_1x_2 + x_1x_3 + x_2x_3$

$d = -x_1x_2x_3$

F. Cardano - Vietta

$x^2 + bx + c$; $\quad b = -(x_1 + x_2)$

$c = x_1x_2$

$x^4 + bx^3 + cx^2 + dx + e$, $\quad b = -\sum_1^4 x_i$; $\quad c = \sum_{i<j} x_i x_j$; $\quad d = -\sum_{i<j<k} x_i x_j x_k$; $\quad e = x_1x_2x_3x_4$

Cardano - Vietta

$x^3+bx^2+cx+d=0 \qquad b,c,d\in\mathbb{Q}.$

$\boxed{x=y-\frac{b}{3}}$

$y^3+py+q=0 \qquad p,q\in\mathbb{Q} \quad$ Incompleta en y^2

S.P.G $\quad \boxed{x^3+px+q=0.} \qquad X=\{x_1,x_2,x_3\}$

$\mathbb{Z}_3=\{1,\pi,\pi^2\} \qquad \mathbb{Z}_3\times X\to X$

$\mathbb{C}\mathbb{Z}_3\times V\to V$

$\underline{V}=$ e.v. libre generado por x_1,x_2,x_3

$1\cdot x_i=x_i$

$\pi\cdot x_1=x_2$

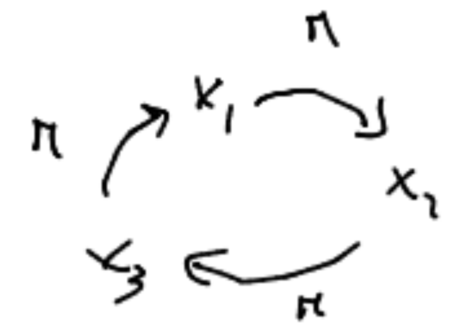
$\pi\cdot x_2=x_3$

$\pi\cdot x_3=x_1$

$\pi^2\cdot x_1=x_3$

$\pi^2\cdot x_2=x_1$

$\pi^2\cdot x_3=x_2$



$1=e_1+e_2+e_3$

$\boxed{e_1}=\frac{1}{3}(1+\pi+\pi^2)$

$e_2=\frac{1}{3}(1+\omega\pi+\omega^2\pi^2)$

$e_3=\frac{1}{3}(1+\omega^2\pi+\omega\pi^2)$

$(\omega=e^{\frac{2\pi}{3}i})$

$$x_1=1\cdot x_1=\underbrace{e_1\cdot x_1}+\underbrace{e_2x_1}+e_3x_1=\frac{1}{3}(\cancel{x_1+x_2+x_3})+\frac{1}{3}\underbrace{(x_1+\omega x_2+\omega^2x_3)}+\frac{1}{3}(x_1+\omega^2x_2+\omega x_3)=\boxed{\frac{1}{3}A+\frac{1}{3}B}$$

Podré expresar A,B en función de p,q

$\boxed{x^3+px+q=0}$

$$\begin{cases} x_1+x_2+x_3=0\\ p=x_1x_2+x_1x_3+x_2x_3\\ -q=x_1x_2x_3\end{cases}$$

$A=\frac{1}{3}(x_1+\omega x_2+\omega^2x_3)=e_2\cdot x_1$

$B=\frac{1}{3}(x_1+\omega^2x_2+\omega x_3)=e_3\cdot x_1$

$$A^3+B^3=\frac{1}{27}\Big(2\underbrace{(x_1^3+x_2^3+x_3^3)}_{\text{en func. de } p,q}-3\underbrace{(x_1^2x_2+x_1^2x_3+x_2^2x_1+x_2^2x_3+x_3^2x_1+x_3^2x_2)}_{\text{en func. de } p,q}+12\underbrace{x_1x_2x_3}_{-q}\Big)$$

$$\left.\begin{aligned} A^3+B^3&=-q\\ AB&=\tfrac{1}{3}p\end{aligned}\right\}$$

$B=\frac{p}{3A}$

Desp. A,B en f. de p,q

$$A=\sqrt[3]{-\frac{q}{2}\pm\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}\quad;\quad B=\frac{p}{3\sqrt[3]{-\frac{q}{2}\pm\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}}$$

$$\boxed{\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0}$$

$\boxed{f(x,y)\text{ sol.}}$

$\boxed{g(x,y)=f(x\cos t-y\,\mathrm{sen}\,t,\ x\,\mathrm{sen}\,t+y\cos t)}$

$\to$ Es también solución de la ec. dif. $\forall t$

$R=\begin{pmatrix}\cos t & \mathrm{sen}\,t\\ -\mathrm{sen}\,t & \cos t\end{pmatrix}$

$S^1\simeq SO(2)=\left\{\begin{pmatrix}\cos t & \mathrm{sen}\,t\\ -\mathrm{sen}\,t & \cos t\end{pmatrix}:\ t\in\mathbb{R}\right\}$ grup. esp. ortogonal

$SO(2)\times S\to S$

$SO(2)$ grupo de simetría

S es $SO(2)$-módulo.

$R\cdot f(x,y)=g(x,y)$

S irrep.

$\dim(S)=1$

$f(x,y)\in S\setminus\{0\}$

$\boxed{R\cdot f(x,y)=k\,f(x,y)}$

$f(x\cos t-y\,\mathrm{sen}\,t,\ x\,\mathrm{sen}\,t+y\cos t)=k_t f(x,y)$

$y=0 \qquad \Rightarrow f(x\cos t,\ x\,\mathrm{sen}\,t)=k_t\,f(x,0)\Leftarrow$

$x=r \qquad \to f(r\cos t,\ r\,\mathrm{sen}\,t)=\alpha(t)\,\beta(r)$

$x=r\cos t$

$y=r\,\mathrm{sen}\,t$

$\to f(x,y)=\boxed{\alpha(t)\cdot\beta(r)}$ Sep. de variables

Ec. dif en variables r,t se descomp. en $\underline{\text{ODE}}$

$$\begin{cases}u=r\cos t\\ v=r\,\mathrm{sen}\,t\end{cases}$$

$f(u,v)=\alpha(t)\beta(r)$

GELLMAN - NISHIJIMA

$SU(3)\times V\to V \qquad \dim V=10$

$T_i\cdot v_i=\lambda_i v_i$

T,Y energía hipercarga.

Ω^-

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$\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}=0$

$$\begin{pmatrix}\cos t & \mathrm{sen}\,t\\ -\mathrm{sen}\,t & \cos t\end{pmatrix}\cdot f(x,y)=\underbrace{f(x\cos t-y\,\mathrm{sen}\,t,\ x\,\mathrm{sen}\,t+y\cos t)}_{g(x,y)}$$

$R\cdot f(x,y)=k\,f(x,y)$

$\boxed{SO(2)\times S\to S}$

Para todo espacio de soluciones.