Teoría de Representaciones (20-XI-20)

Definición de traza. $A = (a_{ij})_{i,j=1}^{n}$, $\operatorname{tr}(A) = \sum_i^n a_{ii}$, $a_{ij} \in k$, cuerpo, $\operatorname{tr}: M_n(k) \longrightarrow k$, $A \longmapsto \operatorname{tr}(A)$ es lineal.

$\forall A, B \in M_n(k)$ $\quad \underline{\operatorname{tr}(A\cdot B) = \operatorname{tr}(B\cdot A)}$ "Prop. de conm. de la traza"

$A = (a_{ij})$, $\quad B = (b_{ij})$ $\qquad A\cdot B = (c_{ij})$ $\qquad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

$$\operatorname{tr}(AB) = \sum_{i=1}^{n} c_{ii} = \underbrace{\sum_{i,k=1}^{n} a_{ik} b_{ki}} \quad ; \quad \operatorname{tr}(BA) = \sum_{i,k=1}^{n} b_{ik} a_{ki} = \underbrace{\sum_{k,i=1}^{n} a_{ki} b_{ik}}$$

Definición de traza de una aplic. lineal

Sea V e.v de dim finita, $f: V \to V$ lineal, $\boxed{\operatorname{tr}(f) \overset{\text{def}}{:=} \operatorname{tr}(M_B(f))}$ donde B es base de V.

si B' es otra base $\quad \underline{\operatorname{tr}(M_{B'}(f)) \overset{?}{=} \operatorname{tr}(M_B(f))}$ $\qquad M' = M_{B'}(f)$, $\quad M = M_B(f)$

$\underline{M' = PMP^{-1}}$, $\quad$ P inversible

$$\operatorname{tr}(M') = \operatorname{tr}(\underbrace{P}_{A}\underbrace{MP^{-1}}_{B}) = \operatorname{tr}(MP^{-1}P) = \operatorname{tr}(M)$$

$$\det(M') = \cancel{\det(P)}\det(M)\cancel{\det(P)^{-1}} = \det(M) \qquad \boxed{\det(f) \overset{\text{def}}{:=} \det(M_B(f))}$$

Definir "caracter" de una repr. de un grupo G de grado finito $\qquad \rho: G \longrightarrow GL(V)$ represent. V ev dim finita

$$\chi_\rho : G \longrightarrow k, \quad g \longmapsto \operatorname{tr}(\rho(g)) \qquad \rho(g): V \to V \qquad \boxed{\chi_\rho(g) = \operatorname{tr}(\rho(g))}$$

$\rho(gg') = \rho(g)\cdot\rho(g')$

Ejemplo. $G = \Delta_3$, $\quad \rho: \Delta_3 \longrightarrow GL_2(k)$

$\Delta_3 = \{1, g, g^2, s, sg, sg^2\}$

$$\left.\begin{aligned} 1 &\longmapsto Id \\ g &\longmapsto \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} = G \\ g^2 &\longmapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = G^2 \\ s &\longmapsto \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} = S \\ sg &\longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = SG \\ sg^2 &\longmapsto \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \end{aligned}\right\}$$

	1	g	g^2	s	sg	sg^2
χ_ρ	2	-1	-1	0	0	0

Propiedad. ρ_1, ρ_2 dos repr. de G $\qquad \rho_i: G \to GL(V_i)$ $\quad i=1,2$ $\quad$ dim(V_i) finita

$\rho_i(g) \in GL(V_i)$ $\quad i = 1,2$

$\underline{\rho_1 \cong \rho_2} \iff \exists\, f: V_1 \to V_2$ isom de e.v.

$$\begin{tikzcd} V_1 \arrow[r, "\rho_1(g)"] \arrow[d, "f"'] & V_1 \arrow[d, "f"] \\ V_2 \arrow[r, "\rho_2(g)"'] & V_2 \end{tikzcd}$$

$f \circ \rho_1(g) = \rho_2(g) \circ f \quad \boxed{\forall g}$

$f \circ \rho_1(g) \circ f^{-1} = \rho_2(g)$

$\operatorname{tr}(\rho_1(g)) = \operatorname{tr}(\rho_2(g))$

$\chi_{\rho_1}(g) = \chi_{\rho_2}(g)$

$\boxed{\chi_{\rho_1} = \chi_{\rho_2}}$

El recíproco es cierto también

$$\boxed{\begin{array}{ll} \chi_{\rho_1} = \chi_{\rho_2} \Rightarrow \rho_1 \cong \rho_2 \\ \text{OJO}: k \text{ alg. cerrado, } \operatorname{char}(k) \nmid |G| & \\ G \text{ finito} \end{array}}$$

Aplic. Había otra repre. de Δ_3 $\qquad \tau: \Delta_3 \longrightarrow GL_2(k)$ $\qquad \operatorname{char}(k) \neq 2$, $\quad \sqrt{3} \in k$

$1 \longmapsto Id$

$s \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = S'$, $\qquad g^2 \longmapsto \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} = G'^2$

$g \longmapsto \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix} = G'$, $\qquad sg \longmapsto \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} = S'G'$

$sg^2 \longmapsto \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} = S'G'^2$

	1	g	g^2	s	sg	sg^2
χ_τ	2	-1	-1	0	0	0

$\chi_\tau = \chi_\rho \Rightarrow \tau \cong \rho$

Computo de las irreps complejas de Δ_3.

$$\text{Maschke} \qquad \mathbb{C}\Delta_3 \cong M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C} \qquad \dim(\mathbb{C}\Delta_3) = |\Delta_3| = 6$$

$\mathbb{C}^2 \quad \mathbb{C}^1 \quad \mathbb{C}^1$

Δ_3 tiene 3 irreps $\begin{cases} 1 \text{ de grado } 2 \\ 2 \text{ de grado } 1 \end{cases}$

Ejercicio. Comprobar que ρ (o τ) es irreducible.

$s^2 = g^3 = 1$
$sg = g^2 s$

$\Delta_3 \xrightarrow{1} GL_1(\mathbb{C}) \cong \mathbb{C}^\times$
$s \longmapsto 1$
$g \longmapsto 1$

$\Delta_3 \xrightarrow{u} \mathbb{C}$
$s \longmapsto -1$
$g \longmapsto 1$

	1	g	g^2	s	sg	sg^2
$\chi_1 = 1$	1	1	1	1	1	1
$\chi_u = u$	1	1	1	-1	-1	-1
χ_ρ	2	-1	-1	0	0	0

Tabla de caract. de Δ_3

Op. con repres. ρ_1, ρ_2 repr. de G $\quad \rho_i : G \to GL(V_i)$ $i=1,2$.

SUMA DIRECTA

$\rho_1(g) : V_1 \to V_1$
$\rho_2(g) : V_2 \to V_2$

$$\rho_1 \oplus \rho_2 : G \longrightarrow GL(V_1 \oplus V_2)$$
$$g \longmapsto (\rho_1 \oplus \rho_2)(g) : \quad V_1 \oplus V_2 \longrightarrow V_1 \oplus V_2$$
$$v_1 + v_2 \longmapsto \rho_1(g)(v_1) + \rho_2(g)(v_2).$$

$\rho_1 \oplus \rho_2$ es repre.

dim(V_i) finita $\quad \boxed{\chi_{\rho_1 \oplus \rho_2} \overset{?}{=} \chi_{\rho_1} + \chi_{\rho_2}}$

Ident. $GL(V_1) \cong \underset{n_1 \times n_1}{GL_{n_1}(k)}$ $\quad n_1 = \dim(V_1)$

$GL(V_2) \cong GL_{n_2}(k)$ $\quad n_2 = \dim(V_2)$

$$W = \{T \in GL(V_1 \oplus V_2) \ / \ T(V_i) \subset V_i, \ i=1,2\} \cong GL(V_1) \times GL(V_2)$$
$$T \longmapsto (T|_{V_1}, T|_{V_2})$$

Al tomar matrices

$\forall T \in W$, $\quad B$ base de $V_1 \oplus V_2$

$$M_B(T) = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

$\rho_1(g) : V_1 \longrightarrow V_1$

$M_{B_1}(\rho_1(g))$

$$\rho_1 \oplus \rho_2 : G \longrightarrow GL_{n_1+n_2}(k)$$
$$g \longmapsto \left(\begin{array}{c|c} M_{B_1}(\rho_1(g)) & 0 \\ \hline 0 & M_{B_2}(\rho_2(g)) \end{array}\right)$$

$$tr[(\rho_1 \oplus \rho_2)(g)] = tr(\rho_1(g)) + tr(\rho_2(g))$$
$$\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$$

Ejemplo.

$$\rho : \Delta_3 \to GL_n(k) \qquad s \mapsto \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \qquad g \mapsto \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$u : \Delta_3 \to GL_1(k) \qquad s \mapsto -1 \qquad g \mapsto 1$$

$$\rho \oplus u : \Delta_3 \longrightarrow GL_3(k)$$
$$s \longmapsto \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & -1 \end{array}\right)$$
$$g \longmapsto \left(\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

PRODUCTO TENSORIAL.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(k), \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in M_2(k) \qquad A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix} =$$

$$= \left(\begin{array}{cc|cc} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ \hline a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{array}\right).$$

$A = (a_{ij})_1^n \qquad B = (b_{ij})_1^m$

$$\Downarrow \qquad A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & & & \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{pmatrix}$$

$\rho_i : G \to GL(V_i)$ $\quad i=1,2$

$$\rho_1 \otimes \rho_2 : G \to GL(V_1 \otimes V_2)$$
$$g \longmapsto (\rho_1 \otimes \rho_2)(g) : V_1 \otimes V_2 \longrightarrow V_1 \otimes V_2$$
$$v_1 \otimes v_2 \longmapsto \underbrace{\rho_1(g)(v_1)}_{\in V_1} \otimes \underbrace{\rho_2(g)(v_2)}_{\in V_2}$$

Pasamos a matrices
dim(V_i) finita
$\rho_i : G \to GL_{n_i}(k)$
$n_i = \dim(V_i)$

$$\boxed{(\rho_1 \otimes \rho_2)(g) = \rho_1(g) \otimes \rho_2(g) \quad \forall g.}$$

El tensorial matricial es justamente el que da sentido a la fórmula.

$$\boxed{\begin{array}{l} \chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1} \cdot \chi_{\rho_2} \\ \chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2} \end{array}}$$

$$\chi_{\rho_1 \otimes \rho_2}(g) = \chi_{\rho_1}(g)\, \chi_{\rho_2}(g) \quad \forall g \in G$$

Anillo de representaciones.

Δ_3, $\quad \chi_\rho : G \to k \qquad +, \cdot$

$f, g : G \to k$

$(f+g)(x) = f(x) + g(x)$, $x \in G$

$(f \cdot g)(x) = f(x)\, g(x)$, $x \in G$

Anillo de representaciones de Δ_3.

$\mathbb{Z}$-módulo libre generado por $\underline{x_1}, \underline{x_u}, \underline{x_p}$

$\cdot$	x_1	x_u	x_p
x_1	x_1	x_u	x_p
x_u	$\cdot$	x_1	x_p
x_p	$\cdot$	$\cdot$	$x_1+x_u+x_p$

	1	g	g^2	s	sg	sg^2
$1 = x_1$	1	1	1	1	1	1
$u = x_u$	1	1	1	-1	-1	-1
x_p	2	-1	-1	0	0	0

Tabla de caract. de Δ_3

x_1^2	1	1	1	1	1	1
x_u^2	1	1	1	1	1	1
x_p^2	4	1	1	0	0	0
$x_u x_p$	2	-1	-1	0	0	0

$$\boxed{x_p^2 = x_1 + x_u + x_p}$$

$\mathbb{Z}^3$, $\{e_1, e_2, e_3\}$

$$(e_1+e_3)(e_2+e_3) = e_2 + e_3 + e_3 + e_1 + e_2 + e_3$$
$$\boxed{e_1 + 2e_2 + 3e_3}$$

$\cdot$	e_1	e_2	e_3
e_1	e_1	e_2	e_3
e_2		e_1	e_3
e_3			$e_1+e_2+e_3$

Tannaka Theory

$$\boxed{\mathbb{C}\Delta_3} \cong M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$$
$$\mathbb{C}^4 \quad \mathbb{C} \quad \mathbb{C}$$

$$6 = 2^2+1^2+1^2$$
$$= 1^2 + \cdots + 1^2 \qquad \boxed{\mathbb{C}^6} \text{ com.}$$

$6 = 2^2+1^2$

$\cancel{6 = 1^2 + \cdots + 1^2}$

Problema para el cálculo del álg. grupo

S_4 = grupo de permut. de $\{1,2,3,4\}$ $\quad |S_4| = 24$

A_4 = subgrupo de permut. pares $\quad |A_4| = 12$

$\mathbb{C}A_4 \leftarrow$ la quiero conocer para irreps de A_4

Maschke $\mathbb{C}A_4$ semisimple

$$\mathbb{C}A_4 = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$$
$$\boxed{12 = n_1^2 + n_2^2 + \cdots + n_k^2}$$

Como agilizar el cálculo del álg. grupo.

$$\boxed{A = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})} \qquad \dim(A) = n_1^2 + \cdots + n_k^2$$

$$Z(A) := \{x \in A \,/\, xy = yx \quad \forall y \in A\} \qquad Z(A) = Z(M_1(\mathbb{C})) \oplus \cdots \oplus Z(M_{n_k}(\mathbb{C}))$$

$$\boxed{Z(M_n(k)) = k\,\mathrm{Id} \cong k} \qquad \dim(Z(M_n(k))) = 1 \qquad A = \bigoplus_1^k M_{n_i}(\mathbb{C})$$

$$\boxed{\dim(Z(A)) = k} = \text{nº de sumandos directos}$$

Objetivo. hallar fórmula para nº de irreps complejas de un Gº finito

G finito $\qquad G \times G \longrightarrow G \qquad g \in G,\ h \in G \qquad g \cdot h := ghg^{-1}$ acción de G sobre sí mismo por conjugación.

$$\mathrm{Orb}(g) = \{hgh^{-1} \,/\, h \in G\} \quad \text{todos los conjugados de } g.$$

$$g \in Z(G) \qquad \boxed{\mathrm{Orb}(g) = \{g\}}$$
$$hgh^{-1} = hh^{-1}g = g$$

$G = \Delta_3$, $\quad \mathrm{orb}(1) = \{1\}$

$\mathrm{orb}(g) = \{g, g^2\}$

$$\boxed{g^i \cdot g \cdot g^{-i} = g^i g^{-i} g = g}$$

$\Delta_3 = \{1, g, g^2, s, sg, sg^2\}$

$s^2 = 1$

$$\boxed{sgs^{-1}} = (sg)s = (g^2 s)s = g^2 s^2 = g^2$$
$$(sg)g(sg)^{-1} = sg^2g^{-1}s^{-1} = sgs = g^2$$
$$(sg^2)g(sg^2)^{-1} = sg^3g^{-2}s^{-1} = sgs = g^2$$

$$\boxed{\begin{matrix} sg = g^2 s \\ s^2 = g^3 = 1 \end{matrix}}$$

$\mathrm{orb}(s) = \{s, sg, sg^2\}$

$$gsg^{-1} = g(sg^2) = g(sg)g = g(g^2 s)g = sg$$

$g^3 = 1$

$$(sg)s(sg)^{-1} = (sg)sg^2 s = g^2 s s g^2 s = g^4 s = gs = sg^2$$

Veremos que nº de irreps (salvo $\cong$) $=$ nº de órbitas de G actuando sobre G por conj.

$\overset{?}{=}$ " " clases de conjugación.

Sup. que G finito $\quad g \in G$, $\quad \mathrm{orb}(g) = \{hgh^{-1} \,/\, h \in G\} = \{g_1, g_2, \ldots, g_q\}$

$$\mathrm{orb}(g) \to \mathrm{orb}(g)$$
$$x \longmapsto hxh^{-1}$$
biyect

$$\bar{g} = \sum_1^q g_i \in kG \ ; \qquad h\bar{g}h^{-1} = \sum_1^q hg_ih^{-1} = \sum g_i = \bar{g}$$

aparecen todos los conj. de g

$$h\bar{g}h^{-1} = \bar{g}$$
$$h\bar{g} = \bar{g}h \quad \forall h$$
$$\bar{g} \in Z(kG)$$