ON NORMAL TERNARY WEAK AMENABILITY OF FACTORS

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Abstract. We show that the norm closure of the set of inner triple derivations from a factor von Neumann algebra into its predual, has codimension zero or one in the real vector space of all triple derivations. It is zero if and only if the factor is finite. We also prove that every Jordan derivation of a von Neumann algebra to itself is an inner Jordan derivation and give a new proof that every triple derivation of a von Neumann algebra to itself is an inner triple derivation.

1. Introduction

Every derivation of a von Neumann algebra into itself is inner (Sakai [14], Kadison [11]). Building on earlier work of Bunce and Paschke [2], Haagerup, on his way to proving that every C*-algebra is weakly amenable, showed in [5] that every derivation of a von Neumann algebra into its predual is inner. Thus the first Hochschild cohomology groups $H^1(M,M)$ and $H^1(M,M_*)$ vanish for any von Neumann algebra M.

It is also known that every triple derivation of a von Neumann algebra into itself is an inner triple derivation ([9, Theorem 2]). In [13], triple derivations and inner triple derivations into a Jordan triple module were defined, and in [10], the study of ternary weak amenability in operator algebras and triples was initiated. Triple derivations and inner triple derivations into a triple module are recalled in section 2.[1]

Among other things, it was shown in [10] that every commutative (real or complex) C*-algebra A is **ternary weakly amenable**, that is, every triple derivation of A into its dual A^* is an inner triple derivation, but the C*-algebras $K(H)$ of all compact operators and $B(H)$ of all bounded operators on an infinite dimensional Hilbert space H are **not** ternary weakly amenable.

Two consequences of [10] are that finite dimensional von Neumann algebras and abelian von Neumann algebras have the property that every triple derivation into the predual is an inner triple derivation,

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[1]In this paper, we use the terms 'triple' and 'ternary' interchangeably, while mindful that in some quarters 'triple' means 'Jordan triple' and 'ternary' refers to an associative triple setting, such as a TRO (ternary ring of operators)

analogous to the Haagerup result. We show that this rarely happens in a general von Neumann algebra, but it comes close.

Using ideas from [10], we prove a (triple) cohomological characterization of finite factors and a zero-one law for factors. Namely, we show that for any factor, the linear space of (automatically bounded by [13, Corollary 15]) triple derivations into the predual, modulo the norm closure of the inner triple derivations, has dimension 0 or 1: It is zero if and only if the factor is finite (Theorem 2); and it is 1 otherwise (Theorem 3).

We use similar ideas to show that every Jordan derivation of a von Neumann algebra into itself is an inner Jordan derivation (Theorem 1(b)), and to give an alternate proof of [9, Theorem 2], quoted above (Theorem 1(a)). The proof of Theorem 1 is also based on the deep result (also quoted above) that every derivation of a von Neumann algebra is inner, whereas the original proof in [9] depended on results of Upmeier [17] on derivations of Jordan operator algebras.

Inner triple derivations on a von Neumann algebra M into its predual M_* are closely related to the span of commutators of normal functionals with elements of M, denoted by $[M_*, M]$. In particular, a consequence of Proposition 4.5 is that in any factor of type II_1, not every normal functional which vanishes at the identity is a sum of such commutators (Corollary 4.6), although it is a norm limit of such commutators (Lemma 4.2).

2. Triple modules and triple derivations

In this section we recall the general context for triple derivations. In this note, we shall only use these concepts in the special case of a von Neumann algebra and its predual, described later in the section. For a more detailed discussion, the reader is referred to [13] and [10].

Varieties of modules. Let A be an associative algebra. Let us recall that an **A-bimodule** is a vector space X, equipped with two bilinear products $(a, x) \mapsto ax$ and $(a, x) \mapsto xa$ from $A \times X$ to X satisfying the following axioms:

$$a(bx) = (ab)x, \quad a(xb) = (ax)b, \text{ and, } (xa)b = x(ab),$$

for every $a, b \in A$ and $x \in X$. The space $A \oplus X$ becomes an associative algebra with respect to the product

$$(a, x)(b, y) := (ab, ay + xb).$$

Let A be a Jordan algebra. A **Jordan A-module** is a vector space X, equipped with two bilinear products $(a, x) \mapsto a \circ x$ and $(a, x) \mapsto x \circ a$ from $A \times X$ to X, satisfying:

$$a \circ x = x \circ a, \quad a^2 \circ (x \circ a) = (a^2 \circ x) \circ a, \text{ and,}$$

$$2((x \circ a) \circ b) \circ a + x \circ (a^2 \circ b) = 2(x \circ a) \circ (a \circ b) + (x \circ b) \circ a^2,$$

for every $a, b \in A$ and $x \in X$. The space $A \oplus X$ becomes a Jordan algebra with respect to the product

$$(a, x) \circ (b, y) := (a \circ b, a \circ y + x \circ b).$$

A complex (resp., real) **Jordan triple** is a complex (resp., real) vector space E equipped with a triple product

$$E \times E \times E \to E \qquad (xyz) \mapsto \{x, y, z\}$$

which is bilinear and symmetric in the outer variables and conjugate linear (resp., linear) in the middle one and satisfying the so-called *"Jordan Identity"*:

$$L(a, b)L(x, y) - L(x, y)L(a, b) = L(L(a, b)x, y) - L(x, L(b, a)y),$$

for all a, b, x, y in E, where $L(x, y)z := \{x, y, z\}$.

The Jordan identity is equivalent to

$$\{a, b, \{c, d, e\}\} = \{\{a, b, c\}, d, e\} - \{c, \{b, a, d\}, e\} + \{c, d, \{a, b, e\}\},$$

which asserts that the map $iL(a, a)$ is a *triple derivation* (to be defined shortly). It also shows that the span of the "multiplication" operators $L(x, y)$ is a Lie algebra.

Let E be a complex (resp. real) Jordan triple. A **Jordan triple E-module** is a vector space X equipped with three mappings

$$\{.,.,.\}_1 : X \times E \times E \to X, \quad \{.,.,.\}_2 : E \times X \times E \to X$$
$$\text{and } \{.,.,.\}_3 : E \times E \times X \to X$$

in such a way that the space $E \oplus X$ becomes a real Jordan triple with respect to the triple product $\{a_1 + x_1, a_2 + x_2, a_3 + x_3\} =$

$$\{a_1, a_2, a_3\}_E + \{x_1, a_2, a_3\}_1 + \{a_1, x_2, a_3\}_2 + \{a_1, a_2, x_3\}_3.$$

The Jordan identity

$$\{a, b, \{c, d, e\}\} = \{\{a, b, c\}, d, e\} - \{c, \{b, a, d\}, e\} + \{c, d, \{a, b, e\}\},$$

holds whenever exactly one of the elements belongs to X. (We shall suppress subscripts since it will always be clear which element is in the module.)

In the complex case we have the unfortunate technical requirement that $\{x, a, b\}_1$ (=$\{b, a, x\}_3$) is linear in a and x and conjugate linear in b; and $\{a, x, b\}_2$ is conjugate linear in each variable a, b, x. Therefore, according to this definition, a complex Jordan triple E is not a Jordan triple E-module. This anomaly, however, does not have any impact on our results, nor does it apply to associative or Jordan modules.

Every (associative) Banach A-bimodule (resp., Jordan Banach A-module) X over an associative Banach algebra A (resp., Jordan Banach algebra A) is a real Banach triple A-module (resp., A-module) with respect to the "*elementary*" product

$$\{a, b, c\} := \frac{1}{2}(abc + cba)$$

(resp., $\{a,b,c\} = (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b$), where one element of a, b, c is in X and the other two are in A.

In particular, any von Neumann algebra M is a complex Jordan triple under the triple product $\{abc\} = (ab^*c + cb^*a)/2$ and its predual M_* is a Jordan triple M-module, according to the following definition.

Definition 2.1. The dual space, E^*, of a complex (resp., real) Jordan Banach triple E is a complex (resp., real) triple E-module with respect to the products:

$$\{a,b,\varphi\}(x) = \{\varphi,b,a\}(x) := \varphi\{b,a,x\} \tag{2.1}$$

and

$$\{a,\varphi,b\}(x) := \overline{\varphi\{a,x,b\}}, \tag{2.2}$$

for every $x \in X, a, b \in E, \varphi \in E^*$.

Varieties of derivations. Let X be a Banach A-bimodule over an (associative) Banach algebra A. A linear mapping $D : A \to X$ is said to be a **derivation** if $D(ab) = D(a)b + aD(b)$, for every a, b in A. For emphasis we call this a **binary (or associative) derivation**. We denote the set of all continuous binary derivations from A to X by $\mathcal{D}_b(A,X)$.

When X is a Jordan Banach module over a Jordan Banach algebra A, a linear mapping $D : A \to X$ is said to be a **derivation** if $D(a \circ b) = D(a) \circ b + a \circ D(b)$, for every a, b in A. For emphasis we call this a **Jordan derivation**. We denote the set of continuous Jordan derivations from A to X by $\mathcal{D}_J(A,X)$.

In the setting of Jordan Banach triples, a **triple** or **ternary derivation** from a (real or complex) Jordan Banach triple, E, into a Banach triple E-module, X, is a *conjugate* linear mapping $\delta : E \to X$ satisfying

$$\delta\{a,b,c\} = \{\delta(a),b,c\} + \{a,\delta(b),c\} + \{a,b,\delta(c)\}, \tag{2.3}$$

for every a, b, c in E. We denote the set of all continuous ternary derivations from E to X by $\mathcal{D}_t(E,X)$. We remind the reader that ternary (or triple) derivations from E to E, such as in Theorem 1(a), are defined as linear maps.

The conjugate linearity (as opposed to linearity) of ternary derivations from a complex Jordan triple into a Jordan triple module is a reflection of the fact noted above that a complex Jordan triple is not a Jordan triple module over itself.

Let X be a Banach A-bimodule over an associative Banach algebra A. Given x_0 in X, the mapping $D_{x_0} : A \to X$, $D_{x_0}(a) = x_0 a - a x_0$ is a bounded (associative or binary) derivation. Derivations of this form are called **inner**. We shall use the customary notation $\operatorname{ad} x_0$ for these inner derivations. The set of all inner derivations from A to X will be denoted by $\mathcal{I}nn_b(A,X)$.

When x_0 is an element in a Jordan Banach A-module, X, over a Jordan Banach algebra, A, for each $b \in A$, the mapping

$$\delta_{x_0,b} = L(x_0)L(b) - L(b)L(x_0) : A \to X,$$

$$\delta_{x_0,b}(a) := (x_0 \circ a) \circ b - (b \circ a) \circ x_0, \ (a \in A),$$

is a bounded derivation. Here $L(x_0)$ (resp. $L(b)$) denotes the module action $a \mapsto x_0 \circ a$ (resp. multiplication $a \mapsto b \circ a$). Finite sums of derivations of this form are called **inner**. The set of all inner Jordan derivations from A to X is denoted by $\mathcal{I}nn_J(A, X)$.

Let E be a complex (resp., real) Jordan triple and let X be a triple E-module. For each $b \in E$ and each $x_0 \in X$, we conclude, via the main identity for Jordan triple modules, that the mapping

$$\delta = \delta(b, x_0) = L(b, x_0) - L(x_0, b) : E \to X,$$

defined by

$$\delta(a) = \delta(b, x_0)(a) := \{b, x_0, a\} - \{x_0, b, a\} \quad (a \in E), \tag{2.4}$$

is a ternary derivation from E into X. Finite sums of derivations of the form $\delta(b, x_0)$ are called **inner triple derivations**. The set of all inner ternary derivations from E to X is denoted by $\mathcal{I}nn_t(E, X)$.

3. Triple derivations on von Neumann algebras

In this section, since we only consider derivations with the same domain and range, we contract the notation from $\mathcal{D}_b(A, X)$ to $\mathcal{D}_b(A)$, *etc.*

Our proof of Theorem 1 below uses techniques from [10]. We summarize these tools in the following proposition, whose proof can be easily read off from the corresponding proofs in [10, Section 3].

Proposition 3.1. *Let A be a unital Banach *-algebra equipped with the ternary product given by $\{a, b, c\} = \frac{1}{2}\,(ab^*c + cb^*a)$ and the Jordan product $a \circ b = (ab + ba)/2$.*

- **(a):** ([10, Lemma 3.1]) *Let $\delta : A_{sa} \to A_{sa}$ be a (real) linear mapping. The following assertions are equivalent:*
  - *(i) δ is a ternary derivation and $\delta(1) = 0$.*
  - *(ii) δ is a Jordan derivation.*
- **(b):** ([10, Lemma 3.2(i)]) $\mathcal{I}nn_J(A) \subset \mathcal{I}nn_b(A)$.
- **(c):** ([10, Lemma 3.2(ii)]) *Let D be an element in $\mathcal{I}nn_b(A)$, that is, $D = ad\,a$ for some a in A. Then D is a *-derivation whenever $a^* = -a$. Conversely, if D is a *-derivation, then $a^* = -a + z$ for some z in the center of A.*
- **(d):** ([10, Lemma 3.4]) *Every ternary derivation δ in $\mathcal{D}_t(A)$ satisfies the identity $\delta(1)^* = -\delta(1)$.*

- **(e):** ([10, Lemma 3.5]) *Let* $\mathcal{D}_t^o(A)$ *be the set of all (continuous) ternary derivations from* A *to* A *vanishing at the unit element. Then* $\mathcal{D}_t(A) = \mathcal{D}_t^o(A) + \mathcal{I}nn_t(A)$. *More precisely, if* $\delta \in \mathcal{D}_t(A)$, *then* $\delta = \delta_0 + \delta_1$, *where* $\delta_0 \in \mathcal{D}_t^o(A)$ *and* δ_1, *defined by* $\delta_1(a) := \delta(1)\circ a = \frac{1}{2}(\delta(1)\, a + a\,\delta(1))$, *is the inner derivation* $-\frac{1}{2}L(1,\delta(1)) + \frac{1}{2}L(\delta(1),1)$.
- **(f):** ([10, Lemma 3.6]) *Let* $\delta : A \to A$ *be a linear mapping. Then* δ *lies in* $\mathcal{D}_J(A)$ *if, and only if,* $\delta\{a,1,b\} = \{\delta(a),1,b\} + \{a,1,\delta(b)\}$ *for all* $a,b \in A$. *Moreover,* $\mathcal{D}_t^o(A) = \mathcal{D}_J^*(A)$.
- **(g):** ([10, Prop. 3.7]) $\mathcal{D}_t(A) = \mathcal{D}_J^*(A) + \mathcal{I}nn_t(A)$.

Theorem 1. *Let* M *be any von Neumann algebra.*

- **(a):** *Every triple derivation of* M *is an inner triple derivation.*
- **(b):** *Every Jordan derivation of* M *is an inner Jordan derivation.*

Proof. To prove (a) it suffices, by Proposition 3.1(g), to show that $\mathcal{D}_J^*(M) \subset \mathcal{I}nn_t(M)$. Suppose δ is a self-adjoint Jordan derivation of M. By [15], δ is an associative derivation and by [11] and [14] and Proposition 3.1(b), $\delta(x) = ax - xa$ where $a^* + a = z$ is a self adjoint element of the center of M.

We shall use the fact that for every von Neumann algebra M, $M = Z(M) + [M,M]$, where $Z(M)$ denotes the center of M (see the beginning of the next section). Let us therefore write

$$a = z' + \sum_j [b_j + ic_j, b_j' + ic_j'],$$

where b_j, b_j', c_j, c_j' are self adjoint elements of M and $z' \in Z(M)$. It follows that

$$0 = a^* + a - z = (z')^* + z' - z + 2i\sum_j([c_j, b_j'] + [b_j, c_j'])$$

so that $\sum_j([c_j, b_j'] + [b_j, c_j'])$ belongs to the center of M. We now have

$$\delta = \operatorname{ad} a = \operatorname{ad} \sum_j ([b_j, b_j'] - [c_j, c_j']) \tag{3.1}$$

and therefore a direct calculation shows that δ is equal to the inner triple derivation

$$\sum_j \left(L(b_j, 2b_j') - L(2b_j', b_j) - L(c_j, 2c_j') + L(2c_j', c_j) \right),$$

which proves (a).

We have just shown that a self adjoint Jordan derivation δ of M has the form (3.1). Then another direct calculation shows that δ is equal to the inner Jordan derivation

$$\sum_j \left(L(b_j)L(b_j') - L(b_j')L(b_j) - L(c_j)L(c_j') + L(c_j')L(c_j) \right).$$

If δ is any Jordan derivation, so are δ^* and $i\delta$, so δ is an inner Jordan derivation. □

4. Normal ternary weak amenability for factors

We shall use the facts that if M is a finite von Neumann algebra, then every element of M of (central) trace zero is a finite sum of commutators ([4, Theoreme 3.2]), and if M is properly infinite (no finite central projections), then every element of M is a finite sum of commutators ([8],[6, Theorem 1],[7, Corollary to Theorem 8],[1, Lemma 3.1]).

Let M be a von Neumann algebra and consider the submodule $M_* \subset M^*$. Then

$$\mathcal{D}_t(M, M_*) = \mathcal{I}nn_b^*(M, M_*) \circ * + \mathcal{I}nn_t(M, M_*). \tag{4.1}$$

This was inadvertently stated and proved for M semifinite in [10, Cor. 3.10] but the same proof holds in general.

In particular, $\mathcal{D}_t(L^\infty, L^1) = \mathcal{I}nn_t(L^\infty, L^1)$, so that L^∞ is *normally ternary weakly amenable*, according to the following definition. (Recall from the introduction that a Jordan triple system is said to be *ternary weakly amenable* if $\mathcal{D}_t(E, E^*) = \mathcal{I}nn_t(E, E^*)$, and that L^∞ is ternary weakly amenable.)

Definition 4.1. A Jordan triple system E which is the dual space of a Banach space E_* is *normally ternary weakly amenable* if $\mathcal{D}_t(E, E_*) = \mathcal{I}nn_t(E, E_*)$.

Let M be any von Neumann algebra and let ϕ_0 be any fixed normal state. Then

$$M_* = \ker \hat{1} + \mathbb{C}\phi_0, \tag{4.2}$$

where

$$\ker \hat{1} = \{\psi \in M_* : \psi(1) = 0\}.$$

Lemma 4.2. *If M is a factor, then $\overline{[M_*, M]} = \ker \hat{1}$ (norm closure).*

Proof. It is clear that $[M_*, M] \subset \ker \hat{1}$. If $x \in M$ satisfies $\hat{x}([\varphi, b]) = 0$ for all $\varphi \in M_*$ and $b \in M$, then $\varphi(bx - xb) = 0$, and so x belongs to the center of M and is a scalar multiple of 1. Thus for any $\psi \in \ker \hat{1}$, $\psi(x) = 0$, proving the lemma. □

Lemma 4.3. *If M is a properly infinite von Neumann algebra, if $\psi \in M_*$ and if $D_\psi \circ *$ belongs to the norm closure of $\mathcal{I}nn_t(M, M_*)$, where $D_\psi = ad\,\psi$, then $\psi(1) = 0$.*

Proof. For $\epsilon > 0$, there exist $\varphi_j \in M_*$ and $b_j \in M$ such that

$$\|D_\psi \circ * - \sum_{j=1}^{n} (L(\varphi_j, b_j) - L(b_j, \varphi_j))\| < \epsilon.$$

For $x, a \in M$, direct calculations yield

$$\left|\psi(a^*x - xa^*) - \frac{1}{2}\sum_{j=1}^{n}(\varphi_j b_j - b_j^*\varphi_j^*)(a^*x) - \frac{1}{2}\sum_{j=1}^{n}(b_j\varphi_j - \varphi_j^* b_j^*)(xa^*)\right| < \epsilon\|a\|\|x\|.$$

We set $x = 1$ to get

$$\left|\frac{1}{2}\sum_{j=1}^{n}(\varphi_j b_j - b_j^*\varphi_j^*)(a^*) + \frac{1}{2}\sum_{j=1}^{n}(b_j\varphi_j - \varphi_j^* b_j^*)(a^*)\right| < \epsilon\|a\|,$$

and therefore

$$\left|\psi(a^*x - xa^*) - \frac{1}{2}\sum_{j=1}^{n}(\varphi_j b_j - b_j^*\varphi_j^*)(a^*x - xa^*)\right| < 2\epsilon\|a\|\|x\|, \tag{4.3}$$

for every $a, x \in M$, that is,

$$\left|\psi([a,x]) - \frac{1}{2}\sum_{j=1}^{n}(\varphi_j b_j - b_j^*\varphi_j^*)([a,x])\right| < 2\epsilon\|a\|\|x\|,$$

and therefore

$$\left|\psi([a,x]) - \frac{1}{2}\sum_{j=1}^{n}(\varphi_j^* b_j^* - b_j\varphi_j)([a,x])\right| < 3\epsilon\|a\|\|x\|.$$

Let us now write

$$\sum_j(\varphi_j b_j - b_j^*\varphi_j^*) = \sum_{j=1}^{n}(\varphi_j b_j - b_j\varphi_j + b_j\varphi_j - \varphi_j^* b_j^* + \varphi_j^* b_j^* - b_j^*\varphi_j^*)$$

so that

$$2\psi - \sum_j(\varphi_j b_j - b_j^*\varphi_j^*) =$$

$$2\psi - \sum_j[\varphi_j, b_j] - \sum(b_j\varphi_j - \varphi_j^* b_j^*) + 2\psi - 2\psi - \sum_j[\varphi_j^*, b_j^*]$$

and

$$4\psi - \sum_j[\varphi_j, b_j] - \sum_j[\varphi_j^*, b_j^*] =$$

$$2\psi - \sum_j(\varphi_j b_j - b_j^*\varphi_j^*) + 2\psi + \sum_j(b_j\varphi_j - \varphi_j^* b_j^*).$$

Thus

$$\left|4\psi([a,x]) - \sum_j[\varphi_j, b_j]([a,x]) - \sum_j[\varphi_j^*, b_j^*]([a,x])\right| \leq$$

$$\left|2\psi([a,x]) - \sum_j(\varphi_j b_j - b_j^*\varphi_j^*)([a,x])\right|$$

$$+\left|2\psi([a,x])+\sum_j(b_j\varphi_j)-\varphi_j^*b_j^*)([a,x])\right|<10\epsilon\|a\|\|x\|.$$

Since $[M,M]=M$, $1=\sum_k[a_k,x_k]$ and therefore

$$|4\psi(1)|\leq 10\epsilon\sum_k\|x_k\|\|a_k\|,$$

proving that $\psi(1)=0$. □

The proof of the following lemma is contained in [10, Lemma 3.2].

Lemma 4.4. *If M is a von Neumann algebra, and $\psi\in M_*$ satisfies $\psi^*=-\psi$, then D_ψ is a self-adjoint mapping. Conversely, if M is properly infinite and D_ψ is self-adjoint, then $\psi^*=-\psi$.*

Theorem 2. *Let M be a von Neumann algebra.*

- **(a):** *If every triple derivation of M into M_* is approximated in norm by inner triple derivations, then M is finite.*
- **(b):** *If M is a finite factor, then every triple derivation of M into M_* is approximated in norm by inner triple derivations.*
- **(c):** *If M is a factor, then M is finite if and only if every triple derivation of M into M_* is approximated in norm by inner triple derivations.*

Proof. (a) Assume that every triple derivation of M into M_* is a norm limit of inner such derivations and also assume for the moment that M is properly infinite. If $\psi\in M_*$ satisfies $\psi^*=-\psi$, then by Lemma 4.4, and (4.1), $D_\psi\circ *\in\mathcal{D}_t(M,M_*)$. Then by Lemma 4.3, $\psi(1)=0$. This is a contradiction if we take $\psi=i\phi_0$ where ϕ_0 is any normal state of M. This proves that M cannot be properly infinite.

If M is arbitrary, write $M=pM+(1-p)M$ for some central projection p, where pM is finite and $(1-p)M$ is properly infinite. It is easy to see that if $\delta\in\mathcal{D}_t(M,M_*)$, then $p\delta\in\mathcal{D}(pM,(pM)_*)$ and similarly for $(1-p)\delta$ and that if $\mathcal{I}nn_t(M,M_*)$ is norm dense in $\mathcal{D}_t(M,M_*)$, then $\mathcal{I}nn_t(pM,(pM)_*)$ is norm dense in $\mathcal{D}_t(pM,(pM)_*)$, and $\mathcal{I}nn_t((1-p)M,((1-p)M)_*)$ is norm dense in $\mathcal{D}_t((1-p)M,((1-p)M)_*)$. By the preceding paragraph, $1-p=0$, so that M is finite.

(b) Suppose that M is a finite factor. Let $\psi\in M_*$ be such that the inner derivation $D_\psi: x\mapsto\psi\cdot x-x\cdot\psi$, is self adjoint, that is, $D_\psi\in\mathcal{I}nn_b^*(M,M_*)$. By the proof of Lemma 4.4 (namely, [10, Lemma 3.2]), $\psi^*=-\psi$ on $[M,M]$. Let us assume temporarily that $\psi(1)\in i\mathbb{R}$, so that $\psi^*=-\psi$ on $M=\mathbb{C}1+[M,M]$. We also assume, temporarily, that $\psi=\hat{x}_\psi$ for some $x_\psi\in M$, that is, $\psi(y)=\mathrm{tr}\,(yx_\psi)$ for $y\in M$.

We then have

$$x_\psi=\mathrm{tr}\,(x_\psi)1+\sum_j[a_j+ib_j,c_j+id_j] \tag{4.4}$$

where a_j, b_j, c_j, d_j are self adjoint elements of M. Expanding the right side of (4.4) and using the fact that $x_\psi^* = -x_\psi$, we have

$$x_\psi = \mathrm{tr}\,(x_\psi)1 + \sum_j([a_j, c_j] - [b_j, d_j])$$

so that

$$\hat{x}_\psi = \mathrm{tr}\,(x_\psi)\mathrm{tr}\,(\cdot) + \sum_j([a_j, c_j]\hat{\ } - [b_j, d_j]\hat{\ }).$$

It is easy to check that for $a, b, x, y \in M$,

$$[a, b]\hat{\ }([x^*, y]) = \{\hat{a}, 2b, x\}\,(y) - \{2b, \hat{a}, x\}\,(y).$$

Thus

$$D_\psi(x^*)(y) = \psi(x^*y - yx^*) = \mathrm{tr}\,\left(\sum_j ([a_j, c_j] - [b_j, d_j])\,[x^*, y]\right)$$

so that

$$D_\psi \circ * = \sum_j \left(L(\hat{a}_j, 2c_j) - L(2c_j, \hat{a}_j) - L(\hat{b}_j, 2d_j) + L(2d_j, \hat{b}_j)\right) \tag{4.5}$$

belongs to $\mathcal{I}nn_t(M, M_*)$.

By replacing ψ by $\psi' = \psi - \Re\psi(1)\,\mathrm{tr}\,(\cdot)$, so that $D_\psi = D_{\psi'}$, we now have that if $\psi = \hat{x}_\psi$ for some $x_\psi \in M$, then $D_\psi \circ * \in \mathcal{I}nn_t(M, M_*)$. Since elements of the form $\hat{x}$ are dense in M_* and $\|D_\psi\| \le 2\,\|\psi\|$, it follows that for every $\psi \in M_*$, $D_\psi \circ *$ belongs to the norm closure of $\mathcal{I}nn_t(M, M_*)$. From (4.1), $\mathcal{I}nn_t(M, M_*)$ is norm dense in $\mathcal{D}_t(A, A^*)$.

(c) This is immediate from (a) and (b). □

Theorem 3. *If M is a properly infinite factor, then the real vector space of triple derivations of M into M_*, modulo the norm closure of the inner triple derivations, has dimension 1.*

Proof. Let $D_\psi \in \mathcal{I}nn_b^*(M, M_*)$ so that again by the proof of Lemma 4.4 (namely, [10, Lemma 3.2]), since $M = [M, M]$, we have $\psi^* = -\psi$ and so $\psi(1) = i\lambda$ for some $\lambda \in \mathbb{R}$. Write, by (4.2),

$$\psi = \varphi + i\lambda\phi_0 \tag{4.6}$$

with $\varphi(1) = 0$. By Lemma 4.2, for every $\epsilon > 0$, there exist $\varphi_j \in M_*$ and $b_j \in M$, such that with $\varphi_\epsilon = \sum_j[\varphi_j, b_j]$, we have $\|\varphi - \varphi_\epsilon\| < \epsilon$. Since $\varphi^* = -\varphi$ we may assume $\varphi_\epsilon^* = -\varphi_\epsilon$.

If we write $\varphi_j = \xi_j + i\eta_j$ and $b_j = c_j + id_j$ where $\xi_,, \eta_j, c_j, d_j$ are selfadjoint, then it follows from $\varphi_\epsilon^* = -\varphi_\epsilon$ that

$$\varphi_\epsilon = \sum_j([\xi_j, c_j] - [\eta_j, d_j]).$$

Further calculation shows that for all $x \in M$,

$$D_{\varphi_\epsilon}(x^*) = \sum_j(\{\xi_j, 2c_j, x\} - \{2c_j, \xi_j, x\} - \{\eta_j, 2d_j, x\} + \{2d_j, \eta_j, x\}).$$

This shows that $D_{\varphi_\epsilon} \circ * \in \mathcal{I}nn_t(M, M_*)$ so that $D_\varphi \circ *$ belongs to the norm closure of $\mathcal{I}nn_t(M, M_*)$.

According to (4.1), every $\delta \in \mathcal{D}_t(M, M_*)$ has the form $\delta = \delta_0 + \delta_1$, where $\delta_0 = D_\psi \circ *$ is selfadjoint, and $\delta_1 \in \mathcal{I}nn_t(M, M_*)$ is the inner triple derivation $\frac{1}{2}L(\delta(1), 1) - \frac{1}{2}L(1, \delta(1))$. Lemma 4.3 shows now that the map

$$\delta + \overline{\mathcal{I}nn_t(M, M_*)} \mapsto \lambda$$

is an isomorphism

$$\mathcal{D}_t(M, M_*)/\overline{\mathcal{I}nn_t(M, M_*)} \sim \mathbb{R},$$

where λ is defined by (4.6).

Explicitly, we define a map $\Phi : \mathcal{D}_t(M, M_*)/\overline{\mathcal{I}nn_t(M, M_*)} \to \mathbb{R}$ as follows. If $\delta \in \mathcal{D}_t(M, M_*)$, say $\delta = D_\psi \circ * + \delta_1$ as above, and $[\delta] = \delta + \overline{\mathcal{I}nn_t(M, M_*)}$, let $\Phi([\delta]) = -i\psi(1) \in \mathbb{R}$. It follows from Lemmas 4.4 and 4.3 that Φ is well defined, and it is easily seen to be linear, onto and one to one.

Explicitly, if $\lambda \in \mathbb{R}$ and we let $\psi = i\lambda\phi_0$ where ϕ_0 is any normal state, then $\Phi([D_\psi \circ *]) = \lambda$. Also, if $\Phi([\delta]) = 0$ where $\delta = D_\psi \circ * + \delta_1$, then $\psi(1) = 0$ and by the first part of the proof, $D_\psi \circ * \in \overline{\mathcal{I}nn(M, M_*)}$, so that $\delta \in \overline{\mathcal{I}nn(M, M_*)}$. □

In the following proposition we shall identify the predual M_* of a finite von Neumann algebra with the non-commutative L^1-space with respect to a fixed faithful normal finite trace, which we denote by tr. See, for example [16, Ch. IX.2] or [12]. We shall write $M_* = L^1(M, \mathrm{tr})$. For every $\psi \in M_*$, $\psi = \hat{T}$ for some $T \in L^1(M, \mathrm{tr})$, that is, $\psi(y) = \mathrm{tr}(yT)$ for $y \in M$. We shall write $(M_*)_0$ for the set of elements $\psi \in M_*$ such that $\psi(1) = 0$ (in (4.1) we called this space $\ker \hat{1}$).

Proposition 4.5. *Let M be a finite factor. Then M is normally ternary weakly amenable if and only if $(M_*)_0 = [M_*, M]$.*

Proof. The first part of this proof is similar to the proofs of Theorems 2 and 3. Suppose that M is finite, and that $(M_*)_0 = [M_*, M]$. Let $\psi \in M_*$ be such that the inner derivation $D_\psi : x \mapsto \psi \cdot x - x \cdot \psi$, is self adjoint, that is, $D_\psi \in \mathcal{I}nn_b^*(M, M_*)$. By Lemma 4.4, $\psi^* = -\psi$ on $[M, M]$. Let us assume temporarily that $\psi(1) \in i\mathbb{R}$, so that $\psi^* = -\psi$ on $M = \mathbb{C}1 + [M, M]$. We know that $\psi = \hat{T}$ for some $T \in L^1(M, \mathrm{tr})$.

By our assumption, we then have

$$T = \mathrm{tr}(T)1 + \sum_j [S_j + iT_j, c_j + id_j] \tag{4.7}$$

where S_j, T_j are self adjoint elements of $L^1(M, \mathrm{tr})$ and c_j, d_j are self adjoint elements of M. Expanding the right side of (4.7) and using the fact that $T^* = -T$, we have

$$T = \mathrm{tr}\,(T)1 + \sum_j([S_j, c_j] - [T_j, d_j])$$

so that

$$\hat{T} = \mathrm{tr}\,(T)\mathrm{tr}\,(\cdot) + \sum_j([S_j, c_j]\widehat{\ } - [T_j, d_j]\widehat{\ }).$$

It is easy to check that for $S \in L^1(M, \mathrm{tr}\,)$ and $c, x, y \in M$,

$$[S, c]\widehat{\ }([x^*, y]) = \left\{\hat{S}, 2c, x\right\}(y) - \left\{2c, \hat{S}, x\right\}(y).$$

Thus

$$D_\psi \circ * = \sum_j \left(L(\hat{S}_j, 2c_j) - L(2c_j, \hat{S}) - L(\hat{T}_j, 2d_j) + L(2d_j, \hat{T}_j)\right), \tag{4.8}$$

which belongs to $\mathcal{I}nn_t(M, M_*)$.

By replacing ψ by $\psi' = \psi - \Re\psi(1)\,\mathrm{tr}\,(\cdot)$, so that $D_\psi = D_{\psi'}$, we now have that for every ψ, $D_\psi \circ * \in \mathcal{I}nn_t(M, M_*)$. From (4.1), $\mathcal{D}_t(A, A^*) = \mathcal{I}nn_t(M, M_*)$ proving that M is normally ternary weakly amenable.

Conversely, suppose that M is a finite factor and that M is normally ternary weakly amenable. Let $\psi \in M_*$ with $\mathrm{tr}\,(\psi) = \psi(1) = 0$. Suppose first that $\psi^* = -\psi$ so that D_ψ is self adjoint and therefore $D_\psi \circ *$ belongs to $\mathcal{D}_t(M, M)$. By our assumption, there exist $\varphi_j \in M_*$ and $b_j \in M$ such that $D_\psi \circ * = \sum_{j=1}^n (L(\varphi_j, b_j) - L(b_j, \varphi_j))$ on M.

For $x, a \in M$, direct calculations yield

$$\psi(a^*x - xa^*) = \frac{1}{2}\sum_{j=1}^n (\varphi_j b_j - b_j^*\varphi_j^*)(a^*x) + \frac{1}{2}\sum_{j=1}^n (b_j\varphi_j - \varphi_j^* b_j^*)(xa^*).$$

We set $x = 1$ to get

$$0 = \frac{1}{2}\sum_{j=1}^n (\varphi_j b_j - b_j^*\varphi_j^*)(a^*) + \frac{1}{2}\sum_{j=1}^n (b_j\varphi_j - \varphi_j^* b_j^*)(a^*), \tag{4.9}$$

and therefore

$$\psi(a^*x - xa^*) = \frac{1}{2}\sum_{j=1}^n (\varphi_j b_j - b_j^*\varphi_j^*)(a^*x - xa^*), \tag{4.10}$$

for every $a, x \in M$.

Since $M = \mathbb{C}1 + [M, M]$ and $\psi(1) = 0$ it follows that

$$\psi = \frac{1}{2}\sum_{j=1}^n (\varphi_j b_j - b_j^*\varphi_j^*) = \frac{1}{2}\sum_{j=1}^n (\varphi_j^* b_j^* - b_j\varphi_j).$$

Hence

$$\begin{aligned} 2\psi &= \sum_{j=1}^{n}(\varphi_j b_j - b_j\varphi_j + b_j\varphi_j - \varphi_j^* b_j^* + \varphi_j^* b_j^* - b_j^*\varphi_j^*) \\ &= \sum_{j=1}^{n}[\varphi_j, b_j] - 2\psi + \sum_{j=1}^{n}[\varphi_j^*, b_j^*], \end{aligned}$$

which shows that $\psi \in [M_*, M]$.

Now let $\psi \in (M_*)_0$ and write $\psi = \psi_1 + \psi_2$, where $\psi_1^* = \psi_1$ and $\psi_2^* = -\psi_2$. Since $0 = \mathrm{tr}\,(\psi) = \mathrm{tr}\,(\psi_1) + \mathrm{tr}\,(\psi_2)$ and $\mathrm{tr}\,(\psi_1) = \psi_1(1)$ is real and $\mathrm{tr}\,(\psi_2) = \psi_2(1)$ is purely imaginary, $\mathrm{tr}\,(\psi_1) = 0 = \mathrm{tr}\,(\psi_2)$. By the previous paragraph, $i\psi_1, \psi_2 \in [M_*, M]$ and so $\psi = -i(i\psi_1) + \psi_2 \in [M_*, M]$, completing the proof. □

For a finite factor of type I, both statements in Proposition 4.5 are known to be true. For a finite factor of type II, the corresponding statements with $[M_*, M]$ and $\mathcal{I}nn_t(M, M_*)$ replaced by their norm closures are also true. No infinite factor can be approximately normally ternary weakly amenable by Theorem 2. So the analog of Proposition 4.5 involving norm closures is false for all infinite factors by Lemma 4.2.

Corollary 4.6. *For any factor of type II_1, $(M_*)_0 \neq [M_*, M]$.*

Proof. Suppose that $(M_*)_0 = [M_*, M]$. Let $\psi = i\phi_0 \in M_*$, where ϕ_0 is any normal state. Then $\psi^* = -\psi$, $D_\psi \in \mathcal{I}nn_b^*(M, M_*)$ so that $D_\psi \circ * = \sum_j (L(\varphi_, b_j) - L(b_j, \varphi_j))$ which implies that $\psi \in [M_*, M]$ and $\psi(1) = 0$, a contradiction. □

After proving Corollory 4.6, we learned from Ken Dykema that it can be obtained from [3, Theorem 4.6], which states a necessary and sufficient condition, in terms of its spectral decomposition, for a normal operator in $L^1(M)$ (where M a II_1 factor) to belong to $[L^1(M), M]$, and that the same holds for a factor of type II_∞ by using [3, Theorem 4.7].

As for the case of a factor of type I_∞, it is shown in [19, Main Theorem] that if $d_n \downarrow 0$, $\sum d_n < \infty$ and $T = \mathrm{diag}\,(-\sum d_n, d_1, d_2, \ldots) \in B(H)$, then $\sum d_n \log n < \infty$ implies $T \in [C_2(H), C_2(H)]$. Conversely, it was shown in [18, Theorem 10] that if $\sum d_n \log n = \infty$, then $T \notin [C_2(H), C_2(H)]$, so that $(M_*)_0 = C_1(H)_0 \neq [C_2(H), C_2(H)]$. Finally, it is stated in [20, Theorem 2.1] that $\sum d_n \log n < \infty$ if and only if $T \in [C_1(H), B(H)]$, thus, $(M_*)_0 \neq [M_*, M]$ for M an infinite factor of type I.

Problem 1. Do Theorem 3 and part (b) of Theorem 2 hold for general von Neumann algebras? (Direct integral theory, as used in [4], has resisted so far.)

Problem 2. Characterize those von Neumann algebras which are normally ternary weakly amenable. (Conjecture: finite of type I.)

Problem 3. Does Corollary 4.6 hold for factors of type III? (The techniques of this paper, as well as those of [3] are not available.)

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