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THE KANTOR–KOECHER–TITS CONSTRUCTION FOR JORDAN COALGEBRAS

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The relationship between Jordan and Lie coalgebras is established. We prove that from any Jordan coalgebra $\langle L(A), \Delta\rangle$, it is possible to construct a Lie coalgebra $\langle L(A), \Delta_L\rangle$. Moreover, any dual algebra of the coalgebra $\langle L(A), \Delta_L\rangle$ corresponds to a Lie algebra that can be determined from the dual algebra for $\langle A, \Delta\rangle$, following the Kantor–Koecher–Tits process. The structure of subcoalgebras and coideals of the coalgebra $\langle L(A), \Delta_L\rangle$ is characterized.

The notion of an (associative) coalgebra has long been traditionally associated with the theory of Hopf algebras. The sharp increase of interest in Hopf algebras was provoked by bringing into the focus quantum groups, as well as the notion of a Lie bialgebra introduced by Drinfeld in [1]. The definition of a coalgebra given in [2] is related to a certain variety of algebras; in particular, a Jordan coalgebra is defined as a coalgebra whose dual algebra is Jordan.

It is well known that to any Jordan algebra one can apply the Kantor–Koecher–Tits (KKT) construction to produce a Lie algebra, whose structure is closely connected with that of the initial Jordan algebra. The natural question arises of whether an analog of the KKT process is available for Jordan coalgebras. In this article we provide an account that answers this question.

Let k be a field. For linear spaces U and V over k, $U \otimes V$ denotes their tensor product. By V^* we denote the space of all linear functionals given on V. For elements $f \in V^*$ and $v \in V$, write $\langle f, v\rangle$ for a value of the linear functional f at v. If X is a subset of V (V^*), then, as usual, $X^\perp$ is an orthogonal complement of X in the space V^* (V).

The pair $\langle A, \Delta\rangle$, where A is a linear space, and $\Delta\colon A \to A \otimes A$ is a linear mapping, is called a *coalgebra.* The map Δ, in this case, is referred as *comultiplication.* For an element a of the coalgebra $\langle A, \Delta\rangle$, if $\Delta(a) = \sum a_{1i} \otimes a_{2i}$, we write $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$, following Sweedler [3].

Consider the dual space A^*. By the standard embedding $\rho\colon A^* \otimes A^* \to (A \otimes A)^*$ we mean, as usual, a linear map given by the following rule: $\langle \rho(f \otimes g), \sum a_i \otimes b_i\rangle = \sum \langle f, a_i\rangle\langle g, b_i\rangle$. Let $\Delta^*\colon (A \otimes A)^* \to A^*$ be the dual of the map Δ. On the space A^*, we can then define multiplication by setting $fg = \Delta^*\rho(f \otimes g)$ for any elements f and g in A^*. The space A^* with given multiplication is an ordinary algebra over a field k, which we call the *dual algebra* of the coalgebra A. It is easy to see that for any f, g in A^* and a in A, the equality $\langle fg, a\rangle = \langle \rho(f \otimes g), \Delta(a)\rangle$ holds. This immediately implies the following: $\langle fg, a\rangle = \sum \langle f, a_{(1)}\rangle\langle g, a_{(2)}\rangle$, where $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$.

The linear subspace B of $\langle A, \Delta\rangle$ is called a *subcoalgebra* (*coideal*) if $\Delta(B) \subseteq B \otimes B$ ($\Delta(B) \subseteq B \otimes A + A \otimes B$). The coalgebra $\langle A, \Delta\rangle$ is said to be *locally finite-dimensional* if each finitely generated subcoalgebra is finite-dimensional. If A^* is an algebra with unity 1, we assume that $1 \in B^\perp$ for any coideal B of $\langle A, \Delta\rangle$. We know from [2] that B is a subcoalgebra (coideal) in $\langle A, \Delta\rangle$ if and only if $B^\perp$ is an ideal (subalgebra) of A^*. The dual algebra A^* of $\langle A, \Delta\rangle$ induces

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a bimodule action $(\cdot)$ on the space A, defined as follows:

$$f \cdot a = \sum a_{(1)} \langle f, a_{(2)} \rangle \text{ and } a \cdot f = \sum \langle f, a_{(1)} \rangle a_{(2)},$$

where $f \in A^*$ and $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$. It is easy to see that for any f, $g \in A^*$ and $a \in A$, $\langle fg, a \rangle = \langle f, g \cdot a \rangle = \langle g, a \cdot f \rangle$. It is not hard to show that B is a subcoalgebra in $\langle A, \Delta \rangle$ if and only if B is an A^*-subbimodule of the bimodule of A.

Now let M be some variety of algebras over a field k. Then $\langle A, \Delta \rangle$ will be called an *M-coalgebra* if its dual algebra A^* belongs to M. Specifically, if the dual algebra A^* is Jordan, then $\langle A, \Delta \rangle$ is a *Jordan coalgebra*. In [2], it was shown that every Jordan coalgebra is locally finite-dimensional and is defined as a Jordan bimodule over its dual algebra.

1. WEAK INNER DERIVATIONS

Let J be a Jordan algebra over a field k (of characteristic not 2). The map $\mathrm{d} : J \to J$ is called a *derivation* if $(ab)\mathrm{d} = a\mathrm{d}b + b\mathrm{d}a$ for any elements a and b in J. The linear space $\mathrm{Der}(J)$ of all derivations of J is a Lie algebra w.r.t. commutation. If a is an element of J, a' stands for the operator of right multiplication by a. Let X be a subset of J. We then put $X' = \{a' | a \in X\}$. For elements a and b in J, write $[a, b] = a'b' - b'a'$. For subsets X and Y in J, write $[X, Y]$ to denote the linear space generated by elements $[x, y]$, where $x \in X$ and $y \in Y$. It is easy to see that for any a, b, c, and d in J, the following relations hold:

$$[ab, c] + [ac, b] + [bc, a] = 0 \tag{1}$$

and

$$(cd)[a, b] = c[a, b]d' + d[a, b]c'. \tag{2}$$

By (2), the operator $[a, b]$ is a derivation of J. Derivations of J of the form $\sum [a_i, b_i]$ are said to be *inner*. Denote by $\mathrm{IntDer}(J)$ the linear space generated by all inner derivations. The space $\mathrm{IntDer}(J)$ is then an ideal in the Lie algebra $\mathrm{Der}(J)$. Suppose that the characteristic of the field k is not 2; $\langle A, \Delta \rangle$ is a Jordan coalgebra and A^* is its dual algebra.

Definition. A derivation d of A^* is called *weakly inner* if, for any finite-dimensional subcoalgebra B in $\langle A, \Delta \rangle$, there exists a derivation i from $\mathrm{IntDer}(A^*)$ such that $\langle f\mathrm{d}, b \rangle = \langle f\mathrm{i}, b \rangle$ for any $f \in A^*$ and $b \in B$.

Denote by $\mathrm{WIntDer}(A^*)$ the linear space of all weakly inner derivations of A^*. Obviously, $\mathrm{IntDer}(A^*) \subseteq \mathrm{WIntDer}(A^*)$. It is easy to see that a derivation d of A^* belongs to $\mathrm{WIntDer}(A^*)$ if and only if, for any finite-dimensional subcoalgebra B in $\langle A, \Delta \rangle$, there is a derivation i of $\mathrm{IntDer}(A^*)$ such that $A^*(\mathrm{d} - \mathrm{i}) \subseteq B^\perp$.

Suppose i is an inner derivation of A^*. Then i naturally induces some linear map of the space A into itself, which we denote also by i. If $\mathrm{i} = \sum [f_i, g_i]$ and $a \in A$, put $a\mathrm{i} = \sum (f_i, a, g_i)$. It is not hard to see that for any elements x, y, $z \in A^*$ and $a \in A$, the equality $\langle (x, y, z), a \rangle = -\langle y, (x, a, z) \rangle$ holds. Therefore, for any elements f in A^* and a in A, we obtain $\langle f\mathrm{i}, a \rangle = -\langle f, a\mathrm{i} \rangle$.

LEMMA 1. The linear space $\mathrm{WIntDer}(A^*)$ is a subalgebra of the Lie algebra $\mathrm{Der}(A^*)$.

Proof. Suppose that elements d_1 and d_2 belong to $\mathrm{WIntDer}(A^*)$. In virtue of the definition of $\mathrm{WIntDer}(A^*)$, then, for any finite-dimensional subcoalgebra B in A, there are i_1 and i_2 from $\mathrm{IntDer}(A^*)$ such that for any $f \in A^*$ and $b \in B$ we have $\langle f\mathrm{d}_1, b \rangle = \langle f\mathrm{i}_1, b \rangle$ and $\langle f\mathrm{d}_2, b \rangle = \langle f\mathrm{i}_2, b \rangle$.

Now consider $\langle f(\mathrm{d}_1\mathrm{d}_2-\mathrm{d}_2\mathrm{d}_1), b\rangle$. We can show that $\langle f(\mathrm{d}_1\mathrm{d}_2-\mathrm{d}_2\mathrm{d}_1), b\rangle = -\langle f\mathrm{d}_1, b\mathrm{i}_2\rangle+\langle f\mathrm{d}_2, b\mathrm{i}_1\rangle$. Since B is a subcoalgebra, $b\mathrm{i}_1$, $b\mathrm{i}_2 \in B$. Therefore,

$$\langle f(\mathrm{d}_1\mathrm{d}_1 - \mathrm{d}_2\mathrm{d}_1), b\rangle = -\langle f\mathrm{i}_1, b\mathrm{i}_2\rangle + \langle f\mathrm{i}_1, b\mathrm{i}_2\rangle = \langle f(\mathrm{i}_1\mathrm{i}_2 - \mathrm{i}_2\mathrm{i}_1), b\rangle.$$

But $\mathrm{i}_1\mathrm{i}_2 - \mathrm{i}_2\mathrm{i}_1$ is an inner derivation, whence $\mathrm{d}_1\mathrm{d}_2 - \mathrm{d}_2\mathrm{d}_1 \in \mathrm{WIntDer}(A^*)$. Thus, $\mathrm{WIntDer}(A^*)$ is a subalgebra of the Lie algebra $\mathrm{Der}(A^*)$, as was to be proved.

Below we assume that the algebra A^* contains unity, and that this is equivalent to the possession of counity by the coalgebra $\langle A, \Delta\rangle$. In the space of linear endomorphisms of A^*, consider a subspace $R(A^*) = (A^*)' \oplus \mathrm{WIntDer}(A^*)$. The space $R(A^*)$ is a Lie algebra with respect to commutation $[\,,\,]$. Denote by ε the mapping from $R(A^*)$ into $R(A^*)$, given by the formula $\varepsilon(f' + \mathrm{d}) = -f' + \mathrm{d}$. Consider the linear space $L(A^*) = A^* \oplus R(A^*) \oplus \bar{A}^*$, where $\bar{A}^*$ is an isomorphic copy of the space A^*. Define multiplication $[\,,\,]$ on $L(A^*)$ by setting

$$[a_1 + \mathrm{d}_1 + \bar{b}_1, a_2 + \mathrm{d}_2 + \bar{b}_2] = a_1\mathrm{d}_2 - a_2\mathrm{d}_1 + a_1\nabla b_2 - a_2\nabla b_1 + [\mathrm{d}_1, \mathrm{d}_2] + \overline{b_1\varepsilon(\mathrm{d}_2)} - \overline{b_2\varepsilon(\mathrm{d}_1)},$$

where a_1, a_2, b_1, $b_2 \in A$, d_1, $\mathrm{d}_2 \in R(A^*)$, and $a\nabla b = (ab)' - [a, b]$. We know that the algebra thus obtained is a Lie algebra. The space $I(A^*) = A^* \oplus (A^*)' \oplus \mathrm{IntDer}(A^*) \oplus \bar{A}^*$ is an ideal in $L(A^*)$. Extend the map ε to the whole algebra $L(A^*)$ by setting $\varepsilon(a+c'+\mathrm{d}+\bar{b}) = b-c'+\mathrm{d}+\bar{a}$.

Consider the algebra $C = A^* + A$, the null extension of A^*. Let X be a subset of A. We can then put $\mathrm{Ann}(X) = \{f \in A^* | f \cdot X = 0\}$. Recall that for subsets X and Y in C, $[X, Y]$ is the linear space generated by operators $[x, y]$, where $x \in X$ and $y \in Y$.

Proposition 1. Let B be a subcoalgebra of (A, Δ). Then $\mathrm{Ann}(B) = B^\perp$ and $[B^\perp, B] = 0$. If B is a finite-dimensional space, the space $[A^*, B]$ is also finite-dimensional.

Proof. Since $B^\perp$ is an ideal in A^* (see [2]), we have $\langle f, B^\perp \cdot B\rangle = \langle fB^\perp, B\rangle = 0$. Consequently, $B^\perp \cdot B = 0$, and so $B^\perp \subseteq \mathrm{Ann}(B)$. Since $\langle \mathrm{Ann}(B), B\rangle = \langle 1, \mathrm{Ann}(B) \cdot B\rangle = 0$, it follows that $\mathrm{Ann}(B) \subseteq B^\perp$. Thus, $B^\perp = \mathrm{Ann}(B)$. Hence, $C[B^\perp, B] \subseteq B^\perp \cdot B = 0$, that is, $[B^\perp, B] = 0$. Let B be a finite-dimensional space. Then $B^\perp$ has a finite codimension. Therefore, $A^* = V + B^\perp$, where V is a finite-dimensional subspace, whence $[A^*, B] = [V, B]$. Thus, the space $[A^*, B]$ is finite-dimensional, as desired.

Proposition 2. Every derivation d of $\mathrm{WIntDer}(A^*)$ can be extended to a derivation D in C, in which case for any subcoalgebra B of $\langle A, \Delta\rangle$, the inclusion $BD \subseteq B$ holds, and we have $\langle f\mathrm{d}, a\rangle = -\langle f, aD\rangle$ for any $f \in A^*$ and $a \in A$. If the subcoalgebra B is finite-dimensional, there exists an element $\mathrm{i} \in \mathrm{IntDer}(A^*)$ such that $bD = b\mathrm{i}$ and $\langle fD, b\rangle = \langle f\mathrm{i}, b\rangle$ for any $b \in B$ and $f \in A^*$.

Proof. Let $d \in \mathrm{WIntDer}(A^*)$, $a \in A$, and B be a finite-dimensional subcoalgebra, containing a. There then exists $\mathrm{i} \in \mathrm{IntDer}(A^*)$ such that $\langle f\mathrm{d}, b\rangle = \langle f\mathrm{i}, b\rangle$ for any $f \in A^*$ and $b \in B$. Since $a \in B$, we put $aD = a\mathrm{i}$ and $fD = f\mathrm{d}$ for all $f \in A^*$. The map D does not depend on the choice of B. Indeed, let B_1 be another finite-dimensional subcoalgebra containing a. Then there exists $\mathrm{i}_1 \in \mathrm{IntDer}(A^*)$ such that $\langle f\mathrm{d}, b\rangle = \langle f\mathrm{i}_1, b\rangle$ for any $f \in A^*$ and $b \in B_1$. Therefore, for all f in A^* we have $\langle f\mathrm{d}, a\rangle = \langle f\mathrm{i}_1, a\rangle$. This implies that $\langle f, a(\mathrm{i} - \mathrm{i}_1)\rangle = -\langle f(\mathrm{i} - \mathrm{i}_1), a\rangle = 0$ for any $f \in A^*$, that is, $a\mathrm{i} = a\mathrm{i}_1$.

Obviously, $\langle f\mathrm{d}, b\rangle = -\langle f, bD\rangle$ for any $f \in A^*$ and $b \in A$, and $BD \subseteq B$ for any subcoalgebra B in $\langle A, \Delta\rangle$. Suppose that a subalgebra B in A is finite-dimensional. There then exists an element $\mathrm{i} \in \mathrm{IntDer}(A^*)$ such that $\langle f\mathrm{d}, b\rangle = \langle f\mathrm{i}, b\rangle$ for any $f \in A^*$ and $b \in B$. Therefore, $\langle f, bD\rangle = -\langle f\mathrm{d}, b\rangle = \langle f, b\mathrm{i}\rangle$, that is, $bD = b\mathrm{i}$ for any b in B. We show that D is a derivation of the algebra C. Let f, $g \in A^*$ and $a \in A$. Since $g\mathrm{d} = gD$ and $\langle f, (a \cdot g)D\rangle = -\langle f\mathrm{d}, (a \cdot g)\rangle$, we have

$$\langle f, (a\cdot g)D\rangle = -\langle f\mathrm{d}g, a\rangle = -\langle (fg)\mathrm{d}, a\rangle + \langle g\mathrm{d}f, a\rangle = \langle f, aD\cdot g\rangle + \langle f, gD\cdot a\rangle.$$

Hence, $(a\cdot g)D = aD\cdot g + gD\cdot a$, and so D is a derivation of C, as desired.

LEMMA 2. There exists an isomorphism ϕ between the space $\mathrm{WIntDer}(A^*)$ and the space of all linear functionals, defined on the space $[A^*, A]$, such that $\langle \phi(\mathrm{d}), [f, a]\rangle = \langle f\mathrm{d}, a\rangle$ for any $f \in A^*$, $a \in A$, and $\mathrm{d} \in \mathrm{WIntDer}(A^*)$.

We divide the proof of the lemma into a series of propositions.

Proposition 3. There exists an isomorphic embedding ϕ of the space $\mathrm{WIntDer}(A^*)$ into the space of all linear functionals, given on the space $[A^*, A]$, such that $\langle \phi(\mathrm{d}), [f, a]\rangle = \langle f\mathrm{d}, a\rangle$ for any $f \in A^*$, $a \in A$, and $d \in \mathrm{WIntDer}(A^*)$. If the coalgebra $\langle A, \Delta\rangle$ is finite-dimensional, then ϕ is an isomorphism.

Proof. Let d be an element in $\mathrm{WIntDer}(A^*)$ and v an arbitrary element from $[A^*, A]$. If $v = \sum\limits_i [f_i, a_i]$, the functional $\hat{\mathrm{d}}$ on $[A^*, A]$ is then defined by setting $\langle \hat{\mathrm{d}}, v\rangle = \sum\limits_i \langle f_i\mathrm{d}, a_i\rangle$. We show that $\hat{\mathrm{d}}$ is well defined. Indeed, let u be an element in $[A^*, A]$ such that $u = v$. Assume $u = \sum\limits_j [g_j, b_j]$. Suppose that B is a subcoalgebra of (A, Δ) generated by all elements a_i and b_j. Then B is finite-dimensional. Since $\mathrm{d} \in \mathrm{WIntDer}(A^*)$, there exists an element i in $\mathrm{IntDer}(A^*)$ such that $\langle f\mathrm{d}, b\rangle = \langle f\mathrm{i}, b\rangle$ for any $f \in A^*$ and $b \in B$. Since $\mathrm{i} = \sum\limits_k [h_k, e_k]$, where h_k, $e_k \in A^*$, we have

$$\langle \hat{\mathrm{d}}, v\rangle = \sum_i \langle f_i\mathrm{d}, a_i\rangle = \sum_i\sum_k \langle f_i[h_k, e_k], a_i\rangle = \sum_i\sum_k \langle e_k, h_k[f_i, a_i]\rangle =$$
$$\sum_k \langle e_k, h_k v\rangle = \sum_k \langle e_k, h_k u\rangle = \sum_j\sum_k \langle g_j[h_k, e_k], b_j\rangle = \sum_j \langle g_j\mathrm{d}, b_j\rangle = \langle \hat{\mathrm{d}}, u\rangle.$$

This means that the functional $\hat{\mathrm{d}}$ is well defined. It is now evident that the map $\phi\colon \mathrm{WIntDer}(A^*) \to [A^*, A]^*$, given by $\phi(\mathrm{d}) = \hat{\mathrm{d}}$, is the desired isomorphic embedding.

Let $\langle A, \Delta\rangle$ be a finite-dimensional coalgebra. Clearly, $\mathrm{WIntDer}(A^*) = \mathrm{IntDer}(A^*)$. Consider a nonzero element u from $[A^*, A]$. Since $(A^*)u \neq 0$, it follows that $fu \neq 0$ for some element f in A^*. Consequently, $\langle g, fu\rangle \neq 0$ for some functional g from A^*, and it is easy to see that $\langle \phi([f, g]), u\rangle = \langle g, fu\rangle$. Therefore, u determines the nonzero functional on the space $[A^*, A^*]$. Hence, ϕ is an isomorphism, as desired.

Now we let B be a finite-dimensional subcoalgebra of $\langle A, \Delta\rangle$. Then $A^* = V \oplus B^{\perp}$, where V is a finite-dimensional subspace in A^*.

Proposition 4. The space V can be given the structure of an algebra with multiplication $*$, so that the resulting algebra V is the dual of the coalgebra $\langle B, \Delta\rangle$, and for any elements f, $g \in V$ and $b \in B$, the equality $\langle f * g, b\rangle = \langle fg, b\rangle$ holds.

Proof. Let ρ be the standard embedding of the space $A^* \otimes A^*$ into $(A \otimes A)^*$. Obviously, $A = B \oplus W$ and $V = B^*$. For the elements v in $A^* \otimes A^*$ and f in A^*, denote by $\rho(v)_0$ and f_V their projections onto the spaces $(B \otimes B)^*$ and V, respectively. Define the map $\rho_B\colon B^* \otimes B^* \to (B \otimes B)^*$ by setting $\rho_B(f \otimes g) = \rho(f \otimes g)_0$. Clearly, ρ_B is a linear map. Since Δ is a linear map from B into $B \otimes B$, it possesses the dual map Δ^* from $(B \otimes B)^*$ into B^*, which is also linear. And $\Delta^*\rho_B$ is then a linear map from $V \otimes V$ into V.

First, let f and g be in V. We can then put $f * g = \Delta^*\rho_B(f \otimes g)$. It is not hard to see that the space V with $*$ is the dual algebra of the coalgebra $\langle B, \Delta\rangle$. Moreover, for any element b in B, we have

$$\langle f * g, b\rangle = \langle \Delta^*\rho_B(f \otimes g), b\rangle = \langle \rho_B(f \otimes g), \Delta(b)\rangle = \langle \rho(f \otimes g)_0, \Delta(b)\rangle.$$

Since $\rho(V \otimes B^\perp + B^\perp \otimes V + B^\perp \otimes B^\perp) \subseteq (B \otimes B)^\perp$, it follows that $\langle \rho(f \otimes g)_0, \Delta(b)\rangle = \langle \rho(f \otimes g), \Delta(b)\rangle$. Consequently, $\langle f * g, b\rangle = \langle fg, b\rangle$.

Next, let $D = V + B$ be the null extension of V. Denote multiplication in the algebra D by $*$. For elements a and b in D, put $\{a, b\} = a'b' - b'a'$, where a' and b' are operators of right multiplication in D. If X and Y are subsets of D, $\{X, Y\}$ is the linear space generated by operators $\{x, y\}$, where $x \in X$ and $y \in Y$. It is easy to see, then, that $f * a = f \cdot a$ for any elements f in V and a in B.

Further, let $f \in V$ and $b \in B$. Then, for any element g in V, the equality $g\{f, b\} = g[f, b]$ holds. Indeed, $g\{f, b\} = (g * f) * b - g * (f * b) = \sum b_{(1)}\langle g * f, b_{(2)}\rangle - g \cdot (f \cdot b) = \sum b_{(1)}\langle gf, b_{(2)}\rangle - g \cdot (f \cdot b) = (gf) \cdot b - g \cdot (f \cdot b) = g[f, b]$. Hence, $\sum\limits_{i=1} \{f_i, b_i\} = 0$ if and only if $\sum\limits_{i=1} [f_i, b_i] = 0$, where $f_i \in V$, $b_i \in B$, and $i = 1, \ldots, n$. In this case, however, it is not hard to see that the spaces $\{V, B\}$ and $[V, B]$ are isomorphic. By Proposition 3, we can therefore assume that $\{V, V\}$ is the space of all linear functionals on $[V, B]$. In addition, for any elements v in $\{V, V\}$, f in V, and b in B, we have $\langle v, [f, b]\rangle = \langle fv, b\rangle$.

Suppose $v \in \{V, V\}$ and $v \neq 0$. Then the functional v can be extended to the whole space $[A^*, A]$. Consider a nonzero element u in $[V, V]$. By Proposition 3, we can assume that $u \in [A^*, A]^*$. Consequently, for any $f \in V$ and $b \in B$ we have $\langle v - u, [f, b]\rangle = \langle f(v - u), b\rangle$. The elements v and u can be chosen in such a way that $f(v - u) = 0$. Therefore, $[V, V] \subseteq [V, B]^* + [V, B]^\perp$. We thus arrive at the following:

Proposition 5. Let B be a finite-dimensional subcoalgebra of the coalgebra $\langle A, \triangle\rangle$ and let V be the subspace in A^* such that $A^* = V \oplus B^\perp$. Then the space $\{V, B\}$ is isomorphic to $[V, B]$, and the following inclusion holds: $[V, V] \subseteq [V, B]^* + [V, B]^\perp$, where $[V, B]^\perp$ is an orthogonal complement of $[V, B]$ in the space $[A^*, A]^*$.

We are now in a position to proceed to the

Proof of Lemma 5. Let $\hat{\mathrm{d}}$ be a linear functional on $[A^*, A]$. Then we define the map $\mathrm{d} : A^* \longrightarrow A^*$ by setting $\langle f\mathrm{d}, a\rangle = \langle \hat{\mathrm{d}}, [f, a]\rangle$ for any f in A^* and a in A. We show that $\mathrm{d} \in \mathrm{WIntDer}(A^*)$. By the definition of d, in view of (1) we have

$$\langle (fg)\mathrm{d}, a\rangle = \langle \hat{\mathrm{d}}, [fg, a]\rangle = \langle \hat{\mathrm{d}}, [f, ga]\rangle + \langle \hat{\mathrm{d}}, [g, fa]\rangle = \langle f\mathrm{d}, g \cdot a\rangle + \langle g\mathrm{d}, f \cdot a\rangle = \langle f\mathrm{d}g + g\mathrm{d}f, a\rangle.$$

Consequently, d is a derivation of A^*. Let B be a finite-dimensional subcoalgebra of $\langle A, \Delta\rangle$ such that $\langle (A^*)\mathrm{d}, B\rangle \neq 0$ and let $U = [A^*, B]$. Then $A^* = V \oplus B^\perp$, where V is a finite-dimensional subspace, and $U = [V, B]$ by Proposition 1. Since $\langle (A^*)\mathrm{d}, B\rangle \neq 0$, we have $\langle \hat{\mathrm{d}}, U\rangle \neq 0$.

First consider the space $\{V, B\}$, introduced above. By Proposition 5, $\hat{\mathrm{d}}$ can be assumed to be a nonzero functional, given on the space $\{V, B\}$. In addition, for any elements f in V and b in B, the equalities $\langle \hat{\mathrm{d}}, \{f, b\}\rangle = \langle \hat{\mathrm{d}}, [f, b]\rangle = \langle f\mathrm{d}, b\rangle$ hold. By Proposition 3, there then exists an inner derivation $\sum\{f_i, g_i\}$ of V such that $\langle \hat{\mathrm{d}}, \{f, b\}\rangle = \langle f \sum\{f_i, g_i\}, b\rangle$. By Proposition 4, $\langle f \sum\{f_i, g_i\}, b\rangle = \langle f \sum[f_i, g_i], b\rangle$. Therefore, $\langle f\mathrm{d}, b\rangle = \langle \hat{\mathrm{d}}, [f, b]\rangle = \langle f \sum[f_i, g_i], b\rangle$, that is, $\mathrm{d} \in \mathrm{WIntDer}(A^*)$. It is now evident that the map $\phi : \mathrm{WIntDer}(A^*) \to [A^*, A^*]$, given by $\phi(\mathrm{d}) = \hat{\mathrm{d}}$, is the desired isomorphism.

We continue to consider the space $L(A) = A \oplus R(A) \oplus \bar{A}$, where $\bar{A}$ is a an isomorphic copy of the space A and $R(A) = A' \oplus [A^*, A]$. By Lemma 2, we can assume that $\mathrm{WIntDer}(A^*) = [A^*, A]^*$. Then the following is valid:

COROLLARY. $L(A^*)$ is the space of all linear functionals given on the space $L(A)$.

By Proposition 2, every derivation d from $\mathrm{WIntDer}(A^*)$ can be extended to a derivation in the algebra C. Denote the extended derivation also by d. The Lie algebra $\mathrm{Hom}_k(C, C)^{(-)}$ then satisfies the equalities $[a', \mathrm{d}] = (a\mathrm{d})'$ and $[[f, a], \mathrm{d}] = [[f\mathrm{d}, a] + [f, a\mathrm{d}]$, where $f \in A^*$ and

$a \in A$. Let π be the map from $L(A)$ into $L(A)$, defined by $\pi(a + b' + v + \bar{c}) = c - b' + v + \bar{a}$. Also let $a,\ c \in A$, $\bar{b} \in \bar{A}$, $w \in A' \oplus [A^*, A]$, $f \in A^*$, $\bar{g} \in (\bar{A}^*)$, $u \in (A^*)' \oplus \mathrm{WIntDer}(A^*)$, and $a\nabla g = (ag)' + [g, a]$. On the space $L(A)$, define the structure of an $L(A^*)$-module by setting

$$[a + w + \bar{b}, f + u + \bar{g}] = au - fw + a\nabla g - f\nabla b + [w, u] + \overline{b\varepsilon(u)} - \overline{g\pi(w)}.$$

It is well known that the action of the Lie algebra $L(A^*)$ on $L(A)$, thus defined, transforms the space $L(A)$ into a Lie $L(A^*)$-bimodule.

LEMMA 3. Let $l_1^*,\ l_2^*$ be elements in $L(A^*)$ and let $l \in L(A)$. Then $\langle [l_1^*, l_2^*],\ l\rangle = \langle l_2^*, [l, \varepsilon(l_1^*)]\rangle$.

Proof. Suppose $l_1^* = f_1 + g_1' + \mathrm{d}_1 + \bar{h}_1$, $l_2^* = f_2 + g_2' + \mathrm{d}_2 + \bar{h}_2$, and $l = a + b' + v + \bar{c}$, where $f_i,\ g_i \in A^*$, $d_i \in \mathrm{WIntDer}(A^*)$, $\bar{h}_i \in (A^*)$ and $a,\ b \in A$, $v \in [A^*, A]$, $\bar{c} \in \bar{A}$. Then

$$[l_1^*, l_2^*] = f_1(g_2' + \mathrm{d}_2) - f_2(g_1'\mathrm{d}_1) + f\nabla h_2 - f_2 h_1 +$$

$$[g_1' + \mathrm{d}_1, g_2' + \mathrm{d}_2] - \overline{h_1 g_2} + \overline{h_2 g_1} + \overline{h_1 \mathrm{d}_2} - \overline{h_2 \mathrm{d}_1}.$$

Hence, $\langle [l_1^*, l_2^*], l\rangle = \langle f_1(g_2' + \mathrm{d}_2) - f_2(g_1' + \mathrm{d}_1), a\rangle + \langle -\overline{h_1 g_2} + \overline{h_2 g_1} + \overline{h_1 \mathrm{d}_2} - \overline{h_2 \mathrm{d}_1}, \bar{c}\rangle + \langle f_1\nabla h_2 - f_2\nabla h_1 + [g_1' + \mathrm{d}_1, g_1'\mathrm{d}_2], b' + v\rangle$. By Lemma 2 and Proposition 2, we have $\langle f_1(g_2' + \mathrm{d}_2) - f_2(g_1' + \mathrm{d}_1), a\rangle = \langle g_2,\ f_1 \cdot a\rangle + \langle \mathrm{d}_2,\ [f_1, a]\rangle + \langle f_2,\ a\mathrm{d}_1\rangle$. Similarly, $\langle -\overline{h_1 g_2} + \overline{h_2 g_1} + \overline{h_1 \mathrm{d}_2} - \overline{h_2 \mathrm{d}_1},\ \bar{c}\rangle = -\langle g_2, h_1 \cdot c\rangle + \langle h_2,\ g_1 \cdot c\rangle + \langle \mathrm{d}_2, [h_1, c]\rangle + \langle h_2, c\mathrm{d}_1\rangle$.

Let $v = \sum [e_i, a_i]$. By Lemma 2, we obtain

$$\langle -[f_1, h_2] + [f_2, h_1] + [g_1, g_2] + [\mathrm{d}_1, \mathrm{d}_2], v\rangle =$$

$$\sum \langle -[f_1, h_2] + [f_2, h_1] + [g_1, g_2] + [\mathrm{d}_1, \mathrm{d}_2], [e_i, a_i]\rangle =$$

$$\sum \langle -(f_1, e_i, h_2) + (f_2, e_i, h_1) + (g_1, e_i, g_2), a\rangle + \sum \langle e_i[\mathrm{d}_1, \mathrm{d}_2], a_i\rangle.$$

Again,

$$\sum \langle -(f_1, e_i, h_2) + (f_2, e_i, h_1) + (g_1, e_i, g_2), a_i\rangle =$$

$$\sum -\langle h_2, (e_i, f_1, a_i)\rangle - \langle f_2, (e_i, h_1, a_i)\rangle + \langle g_2, (e_i, g_1, a_i)\rangle =$$

$$-\langle h_2, f_1 v\rangle - \langle f_2, h_1 v\rangle + \langle g_2, g_1 v\rangle.$$

Lemma 2 and Proposition 2 yield

$$\sum \langle e_i[\mathrm{d}_1, \mathrm{d}_2], a_i\rangle = \sum \langle \mathrm{d}_2, [e_i \mathrm{d}_1, a_i] + [e_i, a_i \mathrm{d}_1]\rangle = \sum \langle \mathrm{d}_2, [[e_i, a_i], d_1]\rangle = \langle \mathrm{d}_2, [v, d_1]\rangle,$$

from which we obtain

$$\langle f_1\nabla h_2 - f_2\nabla h_1 + [g_1' + \mathrm{d}_1, g_2' + \mathrm{d}_2], b' + v\rangle = \langle (f_1 h_2)' - (f_2 h_1)', b'\rangle +$$

$$\langle (g_1 \mathrm{d}_2)' - (g_2 \mathrm{d}_1)', b'\rangle + \langle -[f_1, h_2] + [f_2, h_1] + [g_1, g_2] + [\mathrm{d}_1, \mathrm{d}_2], v\rangle =$$

$$\langle f_1 h_2 - f_2 h_1 + g_1 \mathrm{d}_2 - g_2 \mathrm{d}_1, b\rangle - \langle h_2, f_1 v\rangle - \langle f_2, h_i v\rangle + \langle g_2, g_1 v\rangle + \langle \mathrm{d}_2, [v, \mathrm{d}_1]\rangle.$$

Since $\langle f_1 h_2 - f_2 h_1 + g_1 \mathrm{d}_2 - g_2 \mathrm{d}_1, b\rangle = \langle h_2, f_1 \cdot b\rangle - \langle f_2, h_1 \cdot b\rangle + \langle d_2, [g_1, b]\rangle + \langle g_2, b\mathrm{d}_1\rangle$, we have $\langle f_1\nabla h_2 - f_2\nabla h_1 + [g_1' + \mathrm{d}_1, g_2' + \mathrm{d}_2], b' + v\rangle = \langle h_2, f_1 \cdot b\rangle - \langle f_2, h_1 \cdot b\rangle + \langle \mathrm{d}_2, [g_1, b]\rangle + \langle g_2, b\mathrm{d}_1\rangle - \langle h_2, f_1 v\rangle - \langle f_2, h_1 v\rangle + \langle g_2, g_1 v\rangle + \langle \mathrm{d}_2, [v, \mathrm{d}_1]\rangle$. Thus $\langle [l_1^*, l_2^*], l\rangle = \langle f_2, -g_1 \cdot a + a\mathrm{d}_1 - h_1 \cdot b - h_1 \cdot v\rangle + \langle g_2, f_1 \cdot a - h_1 \cdot c + g_1 v + b\mathrm{d}_1\rangle + \langle \mathrm{d}_2, [f_1, a] + [h_1, c] + [g_1, b] + [v, \mathrm{d}_1]\rangle + \langle h_2, g_1 \cdot c + c\mathrm{d}_1 + f_1 \cdot b - f_1 v\rangle$. It is easy to verify that the expression on the right-hand side of the equality is equal to $\langle l_2^*, [l, \varepsilon(l_1^*)]\rangle$.

LEMMA 4. Let l^* be from $L(A^*)$ and $l \in L(A)$. Then there exists an element $\mathrm{i} \in I(A^*)$ satisfying the equalities $[l, l^*] = [l, \mathrm{i}]$ and $\langle l^*, l\rangle = \langle \mathrm{i}, l\rangle$.

Proof. It suffices to prove the equalities for the case when l^* is in $\mathrm{WIntDer}(A^*)$. Let $l = \sum[f_i, a_i]$ and g be an arbitrary element from A^*. Take the subcoalgebra B in $\langle A, \triangle\rangle$ generated by elements a_i. By Proposition 2, then, $Bl^* = B\mathrm{i}$ and $\langle gl^*, B\rangle = \langle g\mathrm{i}, B\rangle$ for some i in $\mathrm{IntDer}(A^*)$. Since $gl \in B$, we have $gll^* = gl\mathrm{i}$. On the other hand, $gl^*l = \sum\limits_i (f_i, gl^*, a_i)$. Let $\Delta(a_i) = \sum b^i_{(1)} \otimes b^i_{(2)}$, where $b^i_{(1)},\ b^i_{(2)} \in B$. Then

$$\sum_i (f_i, gl^*, a_i) = \sum_i \sum (b^i_{(1)}\langle f_i(gl^*), b^i_{(2)}\rangle - (f_i \cdot b^i_{(1)})\langle gl^*, b^i_{(2)}\rangle) =$$

$$\sum_i \sum (b^i_{(1)}\langle gl^*, (f_i \cdot b^i_{(2)})\rangle - (f_i \cdot b^i_{(1)})\langle gl^*, b^i_{(2)}\rangle) =$$

$$\sum_i \sum (b^i_{(1)}\langle g\mathrm{i}, (f_i \cdot b^i_{(2)})\rangle - (f_i \cdot b^i_{(1)})\langle g\mathrm{i}, b^i_{(2)}\rangle) = \sum_i (f_i, g\mathrm{i}, a_i) = gl\mathrm{i}.$$

Consequently, $g[l, l^*] = g[l, \mathrm{i}]$, thus proving the first equality. The second equality is evident.

2. THE KKT CONSTRUCTION FOR JORDAN ALGEBRAS

Let ρ be a standard embedding of the space $L(A^*) \otimes L(A^*)$ into $(L(A) \otimes L(A))^*$. Consider the finite-dimensional subcoalgebra B of $\langle A, \Delta\rangle$ and the linear space $L = B \oplus B' \oplus [A^*, B] \oplus \bar{B}$. Now if d is an element in $\mathrm{WIntDer}(A^*)$, then $A^*(\mathrm{d} - \mathrm{i}) \subseteq B^\perp$ for some inner derivation i. Therefore, it is not hard to see that $L(A^*) = I(A^*) + L^\perp$.

LEMMA 5. There exists a map Δ_B from L into $L(A) \otimes L(A)$ such that for any $f, g \in L(A^*)$ and $l \in L$,

$$\langle [f, g], l\rangle = \langle \rho(f \otimes g), \Delta_B(l)\rangle.$$

Proof. Since B is a finite-dimensional space, it follows that $B^\perp$, the orthogonal complement of B in A^*, has finite codimension. Therefore, $A^* = V \oplus B^\perp$, where V is a finite-dimensional subspace. The space L has finite dimension by Proposition 1. Obviously, $L(A) = L \oplus L_1$. It follows from Lemma 4 that L is an $L(A^*)$-subbimodule of $L(A)$. By Lemma 3, $L^\perp$, the orthogonal complement of L, is an ideal in $L(A^*)$. The ideal generated in $L(A^*)$ by the set $B^\perp$ is obviously contained in $L^\perp$. Therefore, $L(A^*) = V \oplus V' \oplus [V, V] \oplus \bar{V} \oplus L^\perp$. By Proposition 5, we can suppose the space $V \oplus V' \oplus [V, V] \oplus \bar{V}$ to be contained in $L^* + [A^*, B]^\perp$. And it is now possible to assume that $L(A^*) = L^* \oplus L^\perp$.

Let $\rho(L(A^*) \otimes L(A^*))_0$ be the projection of the space $\rho(L(A^*) \otimes L(A^*))$ onto $(L \otimes L)^*$ and let f_L be the projection of an element f from $L(A^*)$ onto the subspace L^*. It is then easy to see that there exists an injective map $\rho_B\colon L^* \otimes L^* \to (L \otimes L)^*$, in which case $\rho_B(f_L \otimes g_L) = \rho(f \otimes g)_0$ for any f and g in $L(A^*)$. Consequently, the dual map $\rho_B^*\colon L \otimes L \to (L^* \otimes L^*)^*$ is invertible. Consider the linear map $M\colon L^* \otimes L^* \to L^*$, defined by $M(f_L \otimes g_L) = [f, g]_L$. Since $L^\perp$ is an ideal in $L(A^*)$, and $L^* \cap L^\perp = 0$, the M is well defined. Suppose M^* is the dual of the map M. Then M^* is the map from L into $(L^* \otimes L^*)^*$, in which case $\langle f_L \otimes g_L,\ M^*(n)\rangle = \langle [f, g]_L,\ n\rangle$ for any elements $f,\ g \in L(A^*)$ and $n \in L$.

Put $\Delta_B = (\rho_B^*)^{-1}M^*$. Clearly, Δ_B is a map from L into $L \otimes L$, and $\langle [f, g]_L, l\rangle = \langle \rho_B(f_L \otimes g_L), \Delta_B(l)\rangle$ for any elements $f,\ g$ in $L(A^*)$ and $l \in L$. Consider elements f and g from $L(A^*)$. Then $f = f_L + l_1$ and $g = g_L + l_2$, where $f_L,\ g_L \in L^*$, $l_1,\ l_2 \in L^\perp$. Since $\rho(L^* \otimes L^\perp + L^\perp \otimes L^* + L^\perp \otimes L^\perp) \subseteq$

$(L \otimes L)^{\perp}$ and $L^{\perp}$ is an ideal in $L(A^*)$, for any elements f, g in $L(A^*)$ and l in L we obtain $\langle \rho(f \otimes g),\, \Delta_B(l)\rangle = \langle \rho(f \otimes g)_0,\, \Delta_B(l)\rangle = \langle \rho_B(f_L \otimes g_L),\, \Delta_B(l)\rangle = \langle [f, g]_L,\, l\rangle = \langle [f, g], l\rangle$.

We proceed to the main theorem.

THEOREM 1. On the linear space $L(A)$, comultiplication Δ_L can be defined in such a way that the coalgebra $\langle L(A), \Delta_L\rangle$ is a Lie coalgebra with the dual algebra $L(A^*)$.

Proof. Let $l \in L(A)$ and $l = a + b' + \sum [f_i, a_i] + \bar{c}$, where a, b, c, $a_i \in A$ and $f_i \in A^*$. Take the subcoalgebra B of $\langle A, \Delta\rangle$, generated by elements a, b, c and by all a_i. Then B is a finite-dimensional subspace. Let $L = B \oplus B' \oplus [A^*, B] \oplus \bar{B}$. Obviously, $l \in L$. Define comultiplication Δ_L by setting $\Delta_L(l) = \Delta_B(l)$, where Δ_B is the map specified in Lemma 5. The map Δ_L is well defined. Indeed, let $l_1 \in L(A)$ and $l = l_1$. It is then easy to see that $l_1 = a + b' + \sum [g_i, e_i] + \bar{c}$, where $g_i \in L(A^*)$ and $e_i \in L(A)$. Take the subcoalgebra D of $\langle A, \Delta\rangle$, generated by elements a, b, c and by all e_i. Let $L_1 = D \oplus D' \oplus [A^*, D] \oplus \bar{D}$. By Lemma 5, we obtain $\langle \rho(f \otimes g),\, \Delta_B(l)\rangle = \langle [f, g],\, l\rangle = \langle \rho(f \otimes g),\, \Delta_D(l)\rangle$ for any f, $g \in L(A^*)$. Since the space $\rho(L(A^*) \otimes L(A^*))$ is dense in $(L(A) \otimes L(A))^*$, the latter equality yields $\Delta_B(l) = \Delta_D(l)$. This means that Δ_L is well defined, and by Lemma 5, $\langle [f, g],\, l\rangle = \langle \rho(f \otimes g), \Delta_L(l)\rangle$ for any f, $g \in L(A^*)$ and $l \in L(A)$. Now it is not hard to see that Δ_L is the desired comultiplication.

3. SUBCOALGEBRAS AND COIDEALS OF $\langle L(A), \Delta_L\rangle$

In this section we reason to establish the relationship between coideals (subcoalgebras) of the Jordan algebra $\langle A, \Delta\rangle$ and the coalgebra $\langle L(A), \Delta_L\rangle$.

Let B be a subalgebra of A^* such that $1 \in B$. Then, as is known (see [4]), the spaces $L_1(B) = B \oplus B' \oplus [B, B] \oplus \bar{B}$ and $L_2(B) = B \oplus B' \oplus \mathrm{d}(B) \oplus \bar{B}$, where $\mathrm{d}(B) = \{\mathrm{d} \in \mathrm{WIntDer}(A^*) |\ B\mathrm{d} \subseteq B\}$, are subalgebras in $L(A^*)$. Either subalgebra contains a three-dimensional simple Lie algebra U generated in $L(A^*)$ by the elements 1, $(1)'$, and $\bar{1}$. Conversely, if L is a subalgebra in $L(A^*)$ that is closed under the map ε and contains a subalgebra U, then $B = A^* \cap L$ is a subalgebra in A^*, $1 \in B$, and $L_1(B) \subseteq L \subseteq L_2(B)$. Obviously, $L_1(B)$ is an ideal in $L_2(B)$.

First let V be a coideal of the coalgebra $\langle A, \Delta\rangle$. Then the orthogonal complement $V^{\perp}$ of the space V in A^* is a subalgebra, and $1 \in V^{\perp}$. It is not hard to note that $V^{\perp} \cdot V \subseteq V$. Consequently, $V^{\perp}[V^{\perp}, V] \subseteq V$. Consider the spaces $L_1^*(V) = V \oplus V' \oplus [V^{\perp}, V] \oplus \bar{V}$ and $L_2^*(V) = V \oplus V' \oplus \mathrm{d}^*(V) \oplus \bar{V}$, where $\mathrm{d}^*(V) = \{v \in [A^*, A] |\ V^{\perp}v \subseteq V\}$. Obviously, $L_1^*(V) \subseteq L_2^*(V)$.

We show that $L_1^*(V)^{\perp} = L_2(V^{\perp})$ and $L_2^*(V)^{\perp} = L_1(V^{\perp})$. Indeed, by Lemma 3, $\langle \mathrm{d}(V^{\perp}),\, [V^{\perp}, V]\rangle \subseteq \langle [V^{\perp}, \mathrm{d}(V^{\perp})],\, V'\rangle \subseteq \langle V^{\perp}\mathrm{d}(V^{\perp}),\, V\rangle \subseteq \langle V^{\perp},\, V\rangle = 0$. Therefore, $[V^{\perp}, V] \subseteq \mathrm{d}(V^{\perp})^{\perp}$. Again, if $\mathrm{d} \in \mathrm{WIntDer}(A^*)$ and $\mathrm{d} \in [V^{\perp}, V]^{\perp}$, by Lemma 3 we have $\langle V^{\perp}\mathrm{d},\, V\rangle \subseteq \langle [V^{\perp}, \mathrm{d}],\, V\rangle \subseteq \langle \mathrm{d},\, [V^{\perp}, V]\rangle = 0$. Therefore, $V^{\perp}\mathrm{d} \subseteq V^{\perp}$, that is, $\mathrm{d} \in \mathrm{d}(V^{\perp})$. This implies that $[V^{\perp}, V]^{\perp} \subseteq \mathrm{d}(V^{\perp})$, where $[V^{\perp}, V]^{\perp}$ is an orthogonal complement of the space $[V^{\perp}, V]$ in $\mathrm{WIntDer}(A^*)$. Consequently, $[V^{\perp}, V] = \mathrm{d}(V^{\perp})^{\perp}$, and it is now easy to see that $L_1^*(V)^{\perp} = L_2(V^{\perp})$. The second equality is proved similarly.

Since $L_1(V^{\perp})$ and $L_2(V^{\perp})$ are subalgebras of the Lie algebra $L(A^*)$, $L_1^*(V)$ and $L_2^*(V)$ are coideals in $L(A)$. The coideals are obviously closed under the map π. Therefore, the coalgebras $L_1(V^{\perp})$ and $L_2(V^{\perp})$ are closed under ε, and since $1 \in V^{\perp}$, the subalgebra U is contained in $L_1(V^{\perp})$ and $L_2(V^{\perp})$. Consequently, the coideals $L_1^*(V)$ and $L_2^*(V)$ are contained in $U^{\perp}$.

Next let L be a coideal in $\langle L(A), \Delta_L\rangle$ which is closed under the map π and is such that $L \subseteq U^{\perp}$. Then $L^{\perp}$ is a subalgebra in $L(A)$ that is closed with respect to ε, and $U \subseteq L^{\perp}$. Consequently, for the subalgebra $B = A^* \cap L^{\perp}$ we have $L_1(B) \subseteq L^{\perp} \subseteq L_2(B)$. Since $B = V^{\perp}$ for some coideal V of $\langle A,\, \Delta\rangle$, we obtain $L_1^*(V) \subseteq L \subseteq L_2^*(V)$ by the above argument. Obviously, $A \cap L = V$. Thus, the following is valid:

THEOREM 2. Let V be a coideal of the coalgebra $\langle A, \Delta\rangle$. Then the spaces $L_1^*(V)$ and $L_2^*(V)$ are coideals of the coalgebra $\langle L(A),\ \Delta_L\rangle$, in which case also $L_1^*(V)^\perp = L_2(V^\perp)$ and $L_2^*(V)^\perp = L_1(V^\perp)$. Conversely, if L is a coideal in $\langle L(A),\ \Delta_L\rangle$ that is closed under π and is such that $L \subseteq U^\perp$, then $V = A \cap L$ is a coideal in $\langle A,\ \Delta\rangle$, and $L_1^*(V) \subseteq L \subseteq L_2^*(V)$.

Assume that B is an ideal of the algebra A^*. Take the set $p(B) = \{\mathrm{d} \in \mathrm{WIntDer}(A^*) | (A^*)\mathrm{d} \subseteq B\}$. We show that $p(B)$ is an ideal in $\mathrm{WIntDer}(A^*)$. In fact, let d and d_1 be arbitrary elements from $p(B)$ and $\mathrm{WIntDer}(A)$, respectively. By Proposition 2, we have $(B^\perp)\mathrm{d}_1 \subseteq B^\perp$ and $\langle B\mathrm{d}_1,\ B^\perp\rangle \subseteq \langle B,\ (B^\perp)\mathrm{d}_1\rangle = 0$. Therefore, $\langle (A^*)[\mathrm{d}, \mathrm{d}_1],\ B^\perp\rangle \subseteq \langle B\mathrm{d}_1,\ B^\perp\rangle = 0$. This implies that $(A^*)[\mathrm{d}, \mathrm{d}_1] \subseteq B$, that is, $p(B)$ is an ideal in $\mathrm{WIntDer}(A^*)$. It is then not difficult to see that the spaces $I_1(B) = B \oplus B' \oplus [A^*, B] \oplus \bar{B}$ and $I_2(B) = B \oplus B' \oplus p(B) \oplus \bar{B}$ are ideals in $L(A^*)$. Conversely, if I is an ideal in $L(A^*)$, and $B = A^* \cap I$, appealing to the argument in [4, p. 331], it is not hard to prove that B is an ideal of A^*, and $I_1(B) \subseteq I \subseteq I_2(B)$.

We turn to a subcoalgebra B of the coalgebra $\langle A, \Delta\rangle$. Clearly, the orthogonal complement $B^\perp$ of the space B in A^* is an ideal in A^*. By Proposition 2, $B\mathrm{d} \subseteq B$ for any d from $\mathrm{WIntDer}(A^*)$. Consider the spaces

$$I_1^*(B) = B \oplus B' \oplus [A^*, B] \oplus \bar{B} \ \text{ and } \ I_2^*(B) = B \oplus B' \oplus p^*(B) \oplus \bar{B},$$

where $p^*(B) = \{v \in [A^*, A] |\ A^*v \subseteq B\}$. It is not hard to see that $p^*(B)$ is a $\mathrm{WIntDer}(A)$-subbimodule of the Lie $\mathrm{WIntDer}(A^*)$-bimodule $[A^*, A]$. It is also evident that $I_1^*(B) \subseteq I_2^*(B)$.

We show that $I_1^*(B)^\perp = I_2(B^\perp)$ and $I_2^*(B)^\perp = I_1(B^\perp)$. Indeed, by Lemma 3, $\langle p(B^\perp),\ [A^*, B]\rangle \subseteq \langle [A^*, p(B^\perp)],\ B'\rangle \subseteq \langle A^*p(B^\perp),\ B\rangle \subseteq \langle B^\perp,\ B\rangle = 0$. Therefore, $[A^*, B] \subseteq p(B^\perp)^\perp$. Again, if $\mathrm{d} \in \mathrm{WIntDer}(A^*)$ and $\mathrm{d} \in [A^*, B]^\perp$, by Lemma 3 we have $\langle A^*\mathrm{d},\ B\rangle \subseteq \langle [A^*, \mathrm{d}],\ B\rangle \subseteq \langle \mathrm{d},\ [A^*, B]\rangle = 0$. Hence, $A^*\mathrm{d} \subseteq B^\perp$, that is, $\mathrm{d} \in p(B^\perp)$. This implies that $[A^*, B]^\perp \subseteq p(B^\perp)$, where $[A^*, B]^\perp$ is an orthogonal complement of the space $[A^*, B]$ in $\mathrm{WIntDer}(A^*)$. Consequently, $[A^*, B]^\perp = p(B^\perp)$, and then it is easy to see that $I_1^*(B)^\perp = I_2(B^\perp)$. The second equality is proved similarly. Since $I_1(B^\perp)$ and $I_2(B^\perp)$ are ideals of the Lie algebra $L(A^*)$, it follows that $I_1^*(B)$ and $I_2^*(B)$ are subcoalgebras in $L(A)$.

Now let I be a subcoalgebra of $\langle L(A), \Delta_L\rangle$. Then $I^\perp$ is an ideal in $L(A^*)$. Consequently, for the ideal $B = A^* \cap I^\perp$ in A^* we have $I_1(B) \subseteq I^\perp \subseteq I_2(B)$. Since $B = V^\perp$ for some subcoalgebra V of $\langle A, \Delta\rangle$, the above argument implies $I_1^*(V) \subseteq I \subseteq I_2^*(V)$. Obviously, $A \cap I = V$. The following is thus valid:

THEOREM 3. Let B be a subcoalgebra of the coalgebra $\langle A, \Delta\rangle$. Then the spaces $I_1^*(B)$ and $I_2^*(B)$ are subcoalgebras of $\langle L(A),\ \Delta_L\rangle$, in which case also $I_1^*(B)^\perp = I_2(B^\perp)$ and $I_2^*(B)^\perp = I_1(B^\perp)$. Conversely, if I is a subcoalgebra in $\langle L(A),\ \Delta_L\rangle$, then $B = A \cap I$ is a subcoalgebra of A, and $I_1^*(B) \subseteq I \subseteq I_2^*(B)$.

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