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JORDAN SUPERALGEBRAS WITH A SOLVABLE EVEN PART

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In the article we show that a Jordan superalgebra over a ring with unity, containing an element 1/3, is solvable whenever its even part is solvable.

Let Φ be an associative commutative ring with unity and let Z_2 be a two-element Abelian group. An algebra A over Φ is called Z_2-*graded* if A is a direct sum of Φ-modules, i.e., $A = A_0 \oplus A_1$, $A_0A_0 + A_1A_1 \subseteq A_0$, and $A_0A_1 + A_1A_0 \subseteq A_1$. It is easy to note that A_0 is a subalgebra in A, which we call an *even part* of A, and that A_1 is an A_0-bimodule, an *odd part* of A. For A an arbitrary Φ-algebra, for τ an automorphism of A of order 2, and for $1/2 \in \Phi$, we may consider A as a Z_2-graded algebra if $A_0 = \{a \in A | \tau(a) = a\}$ and $A_1 = \{a \in A | \tau(a) = -a\}$. On the other hand, if A is Z_2-graded, it is easy to see that the map τ given by the rule $\tau(a_0 + a_1) = a_0 - a_1$, where $a_0 \in A_0$ and $a_1 \in A_1$, is an automorphism of order 2. Another example of a Z_2-graded algebra is represented by a Grassmann algebra $G = G_0 \oplus G_1$. For the Z_2-graded algebra A, we define $G(A)$, its Grassmann envelope, by setting $G(A) = A_0 \otimes G_0 \oplus A_1 \otimes G_1$.

Let μ be some variety of Φ-algebras. If $G(A) \in \mu$, we say that a Z_2-graded algebra A is a μ-*superalgebra*.

The main result of the article is the following:

THEOREM A. Let J be a Jordan superalgebra with a solvable even part and let $1/3 \in \Phi$. Then the superalgebra J is solvable.

It is not hard to state that an arbitrary superalgebra A is solvable iff its Grassmann envelope is solvable. Therefore, it suffices to prove the theorem for ordinary Z_2-graded Jordan Φ-algebras. Note that in the case where Φ is a field of characteristic 0, the required result follows from [1]. We also observe that an analog of Theorem A for alternative superalgebras was obtained in [2]. For J a Z_2-graded Jordan algebra, the even and odd parts will be denoted by, respectively, A and M.

1. JORDAN Z_2-GRADED ALGEBRAS WITH A TRIVIAL EVEN PART

Let $1/2 \in \Phi$ and let J be a Jordan Φ-algebra. For an element x of J, denote by x' the operator of right multiplication by x, i.e., $ax' = ax$ for any a in J. For A a subalgebra of J, $R(A)$ denotes the subalgebra in $\mathrm{End}_\Phi(J)$ generated by the set $\{x' | x \in J\}$. If N is a subset in J, by N' we denote a Φ-module generated by the set $\{x' | x \in N\}$. Recall that every Jordan algebra satisfies the following identity:

$$[(xy)z]v + [(xv)z]y + [(yv)z]x = (xy)(vz) + (xv)(yz) + (xz)(yv). \qquad (1)$$

Let V be a Φ-module and V_1 its submodule. Then for the elements x and y in V we will write $x \equiv y \pmod{V_1}$ if $x - y \in V_1$. In this section we argue to show that every Jordan Z_2-graded algebra A with a trivial even part (i.e., $A^2 = 0$) is solvable, and that $R(J)$, which is the multiplication algebra for A, satisfies the following equalities:

$$x'y'z' + z'y'x' + ((xz)y)' = 0; \tag{2}$$
$$(xy)'z' + (zy)'x' + (zy)'x' = 0; \tag{3}$$
$$x'(yz)' + y'(xz)' + z'(yx)' = 0. \tag{4}$$

LEMMA 1. Let $J = A + M$ be an arbitrary Jordan Z_2-graded algebra with a trivial even part. Then J obeys the identity $x^4 = 0$, and $[(JA)M]A = 0$.

Proof. Let x, $y \in M$ and $a \in A$. Then $(xa)x^2 = x(ax^2) = 0$; similarly, $xyx^2 = 0$. Therefore, for every u in J we have $xux^2 = 0$. Let b be an element in A. It is easy to see that $[(Ja)x]b = 0$. Consequently, by (1) we obtain $2(xa)^2 = -x^2a^2 + (a^2x)x + 2[(ax)x]a = 0$. Now let u be an arbitrary element in J. Then $u = a_0 + a_1$, where $a_0 \in A$ and $a_1 \in M$. It follows that $u^4 = 4(a_0a_1)^2 + 4a_1a_0a_1^2 + a_1^4 = 0$.

COROLLARY 1. Let $J = A + M$ be an arbitrary Jordan Z_2-graded algebra with a trivial even part. Then equalities (2), (3), and (4) are satisfied in $R(J)$, a multiplication algebra.

Now let $J = A + M$ be an arbitrary Jordan Z_2-graded algebra whose multiplication algebra satisfies (2), (3), and (4). We proceed to prove two lemmas which will be needed for our further reasoning.

LEMMA 2. Let $L = (A^2)' + (A^2)'M'M'$. Then, for any elements a, b, and c in A and for all x, y, v, and u in M, the following relations hold:

(a) $(ax)'c'(by)' \equiv -(by)'c'(ax)' \pmod{L}$;

(b) $(ax)'c'(by)' \equiv (ay)'c'(bx)' \pmod{L}$;

(c) $2(ax)'(uv)'(by)' \equiv v'b'(xy)'a'u' + u'b'(xy)'a'v' \pmod{L}$.

Let w_1 and w_2 be elements of $R(J)$. Suppose that w_2 is obtained from w_1 by transposing any two of its elements entered from J. Then w_1 is said to be *symmetric* (*skew-symmetric*) modulo L with respect to these elements if $w_1 \equiv w_2 \pmod{L}$ $[w_1 \equiv -w_2 \pmod{L}]$. Note that by (2),

$$M'M'(A^2)' \subseteq ((A^2M)M)' + (A^2)'M'M' \text{ and } (A^2M)M \subseteq ((AM)M)A \subseteq A^2.$$

Therefore, $M'M'(A^2)' \subseteq L$.

Let x and y be arbitrary elements in M. Then, in view of equalities (2), (3), and (4), we obtain $x'a'y' \equiv -y'a'x' \pmod{L}$, $(ax)'c'(by)' \equiv -(cx)'a'(by)' \pmod{L}$, and $(ax)'c'(by)' = -(ax)'b'(cy)' - (ax)'y'(cb)' \equiv -(ax)'b'(cy)' \pmod{L}$. Then the expression $(ax)'c'(by)'$ is skew-symmetric modulo L with respect to any two elements from the set $\{a, b, c\}$. Since

$$(ax)'c'(by)' \equiv -(by)'c'(ax)' \equiv (ay)'c'(bx)' \pmod{L},$$

the element $(ax)'c'(by)'$ is symmetric modulo L with respect to x and y. We prove that $(ax)'(v^2)'(by)' \equiv v'b'(xy)'a'v' \pmod{L}$. Indeed, in view of (4) we obtain

$$(ax)'(v^2)'(by)' = -(ax)'(b)'(yv^2)' - (ax)'(y)'(bv^2)'.$$

Since $bv^2 \in A^2$, we have $(ax)'(v^2)'(by)' \equiv -(ax)'(b)'(yv^2)' \pmod{L}$. Applying (2) to the expression $(ax)'(b)'(yv^2)'$, we obtain

$$(ax)'(v^2)'(by)' \equiv 2(ax)'(b)'v'y'v' \pmod{L}.$$

Again by (2), $(ax)'b'v'y'v' = -v'b'(ax)'y'v' - [((ax)v)b]'y'v'$, and since $((ax)v)b \in A^2$, we have $(ax)'(v^2)'(by)' \equiv -2v'b'(ax)'y'v' \pmod{L}$. On the other hand, $(ay)'(v^2)'(bx)' \equiv -2v'b'(ay)'x'v'$

$(\mathrm{mod}\, L)$. This implies that $(ax)'(v^2)'(by)' \equiv -v'b'(ax)'y'v' - v'b'(ay)'x'v' \ (\mathrm{mod}\, L)$, and by (3), we have $(ax)'(v^2)'(by)' \equiv v'b'(xy)'a'v' \ (\mathrm{mod}\, L)$. Linearizing the latter with respect to v yields

$$2(ax)'(vu)'(by)' \equiv v'b'(xy)'a'u' + u'b'(xy)'a'v' \ (\mathrm{mod}\, L).$$

LEMMA 3. Let N be a Φ-submodule of a Φ-module M and put

$$L = (A^2)' + (A^2)'M'M'(\mathrm{Id} + M'M') + (N^2)'A'M'A'M' + M'A'M'A'(N^2)'.$$

Then, for any elements a and b in A and for x, y, z, and v in N, we have $(ax)'((cy)(dv))'(bz)' \in L$.

Proof. In this lemma, $w_1 \equiv w_2$ is an abbreviation for $w_1 \equiv w_2 \ (\mathrm{mod}\, L)$. Consider the operator $(ax)'((cu)(dv))'(by)'$. By Lemma 2, we conclude that

$$2(ax)'((cu)(dv))'(by)' \equiv (cu)'b'(xy)'a'(dv)' + (dv)'b'(xy)'a'(cu)'.$$

In view of (2), we have

$$(cu)'b'(xy)'a'(dv)' = -(cu)'a'(xy)'b'(dv)' - (cu)'((xy)(ab))'(dv)',$$

by (4),

$$(cu)'((xy)(ab))'(dv)' = -(xy)'((cu)(ab))'(dv)' - (ab)'((xy)(cu))'(dv)' \equiv -(xy)'((cu)(ab))'(dv)',$$

and again by (2),

$$(cu)'((xy)(ab))'(dv)' \equiv (xy)'a'(cu)'b'(dv)' + (xy)'b'(cu)'a'(dv)' \equiv 0.$$

Consequently,

$$(cu)'b'(xy)'a'(dv)' \equiv -(cu)'a'(xy)'b'(dv)'. \tag{$*$}$$

On the other hand, by (3),

$$(dv)'b'(xy)'a'(cu)' = -(dv)'b'(xy)'c'(au)' - (dv)'b'(xy)'u'(ac)';$$

in view of (2) and (3),

$$(dv)'b'(xy)'u'(ac)' = -(dv)'u'(xy)'b'(ac)' - (dv)'((bu)(xy))'(ac)' \equiv$$

$$-(dv)'u'(xy)'b'(ac)' = (dv)'u'(ac)'b'(xy)' + (dv)'u'[((xy)(ac))b]' \equiv 0.$$

Consequently,

$$2(ax)'((cu)(dv))'(by)' \equiv -(cu)'a'(xy)'b'(dv)' - (dv)'b'(xy)'c'(au)';$$

however, by (3),

$$(cu)'a'(xy)'b'(dv)' = -(ac)'u'(xy)'b'(dv)' - (au)'c'(xy)'b'(dv)' =$$

$$(ac)'u'(bx)'y'(dv)' + (ac)'u'(by)'x'(dv)' - (au)'c'(xy)'b'(dv)' \equiv -(au)'c'(xy)'b'(dv)',$$

and in view of $(*)$, we have

$$(au)'c'(xy)'b'(dv)' \equiv -(au)'b'(xy)'c'(dv)'.$$

The following relation is therefore valid:

$$2(ax)'((cu)(dv))'(by)' \equiv -(au)'b'(xy)'c'(dv)' - (dv)'b'(xy)'c'(au)'.$$

This implies that

$$(ax)'((cu)(dv))'(by)' \equiv -(cx)'((au)(dv))'(by)' \pmod{L}.$$

Thus, it follows by Lemma 2 that $(ax)'((cu)(dv))'(by)'$ is skew-symmetric modulo L with respect to any two elements from $\{a, b, c, d\}$.

Now we argue that $(ax)'((by)(dv))'(cx)' \equiv 0$. Indeed, by Lemma 1

$$2(ax)'((cx)(dv))'(by)' \equiv (cx)'b'(xy)'a'(dv)' + (dv)'b'(xy)'a'(cx)'.$$

From this and (2), we obtain

$$(cx)'b'(xy)'a'(dv)' = -(xy)'b'(cx)'a'(dv)' - [((cx)(xy))b]'a'(dv)'.$$

Consequently,

$$(cx)'b'(xy)'a'(dv))' \equiv -[((cx)(xy))b]'a'(dv)'.$$

On the other hand, by $(*)$ and (2) we have

$$(dv)'b'(xy)'a'(cx)' \equiv -(dv)'a'(xy)'b'(cx)' =$$

$$(dv)'a'(cx)'b'(xy)' + (dv)'a'[((xy)(cx))b] \equiv (dv)'a'[((xy)(cx))b]'.$$

Hence

$$2(ax)'((cx)(dv))'(by)' \equiv -[((cx)(xy))b]'a'(dv)' + (dv)'a'[((cx)(xy))b]'.$$

Since $2(cx)(xy) = -x^2(cy)$ in view of (3), it follows that

$$2(ax)'((cx)(dv))'(by)' \equiv 1/2([(x^2(cy))b]'a'(dv)' - (dv)'a'[(x^2(cy))b]'),$$

and (2) implies that

$$2(ax)'((cx)(dv))'(by)' \equiv 1/2(-(cy)'b'(x^2)'a'(dv) + (dv)'a'(x^2)'b'(cy)').$$

Now, using $(*)$ and Lemma 2, we obtain

$$2(ax)'((cx)(dv))'(by)' \equiv -(ax)'((cy)(dv))'(bx)'; \tag{5}$$

since $(ax)'((cy)(dv))'(bx)'$ is skew-symmetric modulo L with respect to b, c and c, d, it follows that

$$2(ax)'((cx)(dv))'(by)' \equiv (ax)'((by)(dv))'(cx)' \tag{6}$$

and

$$2(ax)'((cx)(dv))'(by)' \equiv (ax)'((dy)(cv))'(bx)'. \tag{7}$$

Therefore, relations (5) and (7) imply

$$(ax)'((cx)(dv))'(by)' \equiv -(ax)'((cx)(dy))'(bv)';$$

since $(ax)'((cx)(dy))'(bv)'$ is skew-symmetric with respect to b and d, we conclude that $(ax)'((cx)(dv))'(by)' \equiv (ax)'((cx)(by))'(dv)'$.

On the other hand, from relation (6) and equality (3) it follows that

$$3(ax)'((cx)(dv))'(by)' \equiv -(ax)'((cx)(by))'(dv)'.$$

From the last two relations and (6) we obtain

$$(ax)'((by)(dv))'(cx)' \equiv 0.$$

By Lemma 2, the expression $(az)'((by)(dv))'(cw)'$, where z and $w \in M$, is symmetric modulo L with respect to these elements; therefore,

$$(ax)'((by)(dv))'(cz)' \equiv 0$$

for any elements x, y, z, and v in N. The lemma is proved.

Let B be an arbitrary Φ-algebra. Put $B^{\langle 1\rangle} = B^2$, and for a natural number n, $B^{\langle n+1\rangle} = (B^{\langle n\rangle})^2$. An algebra B is called *solvable of index n* if $B^{\langle n\rangle} = 0$.

THEOREM 1. Let $J = A + M$ be a Jordan Z_2-graded algebra and let $A^2 = 0$. Then $J^{\langle 6\rangle} = 0$.

Proof. Let a, b, c, and d be arbitrary elements in A, let x, y, z, and u be arbitrary elements in M, and let $N = AM$. Then, in view of Lemma 1 and Corollary 1, the algebra $R(J)$ satisfies equalities (2), (3), and (4). By Lemma 3 and the equality $A^2 = 0$, we obtain

$$(ax)'((by)(cz))'(du)' \in (N^2)'A'M'A'M' + M'A'M'A'(N^2)';$$

but $A'M'A' = 0$ by Lemma 1. Consequently, $(ax)'((by)(cz))'(du)' = 0$. Now let $L = (AN)N^2$. Then $LN = 0$. It is not hard to see that

$$J^{\langle 5\rangle} \subseteq N^2 + (LN^2)N^2,$$

and in view of (4), we have $(LN^2)N^2 \subset (LN)N^3 = 0$. It follows that $J^{\langle 5\rangle} \subseteq N^2 \subseteq A$. Thus, $J^{\langle 6\rangle} \subseteq A^2 = 0$.

2. PROOF OF THE MAIN THEOREM

In this section, we show that the Jordan Z_2-graded algebra $J = A+M$ with a solvable even part is solvable. Unless specified otherwise, assume that $R(J)$ satisfies equalities (2), (3), and (4).

Proposition 1. Let I be an A-subbimodule of an A-bimodule J such that $IM'M' \subseteq I$ and $N = AM$. Then the following inclusions hold:

(a) $IN'(N^2)'N'A'A' \subseteq IN'(N^2)'N'$;

(b) $IN'M'M' \subseteq IN$;

(c) $IN'(N^2)'N'M'M' \subseteq IN'(N^2)'N'$.

Proof. First we note that $N^2A'A' \subseteq (NA)A'N'$ and $N^2M'M' \subseteq (NM)M'N'$ hold by (2). Since $NA \subseteq N$ and $NM \subseteq A$, we have $N^2A'A' \subseteq N^2$ and $N^2M'M' \subseteq N^2$. Therefore, by (2) we obtain $IN'(N^2)'N'A'A' \subseteq IN'(N^2)'((NA)A)' + IN'(N^2)'A'A'N' \subseteq IN'(N^2)'N' + IN'((N^2A)A)'N' + IN'A'A'(N^2)'N' \subseteq IN'(N^2)'N' + IN'A'A'(N^2)'N' \subseteq IN'(N^2)'N' + IA'A'N'(N^2)'N' \subseteq IN'(N^2)'N'$. Inclusions (b) and (c) are proved similarly.

Now suppose that I and N are specified as above. Fix elements a, b, c, d, and e in N^2 and an element i in I. Define a function F of three arguments from M to J, setting $F(x,y,z) = i(ax)'b'(cy)'d'(ez)'$. We have

LEMMA 4. The following statements are valid:

(a) the function F is symmetric modulo IN with respect to any two of its arguments;

(b) if $1/3 \in \Phi$, then, for any x, y, and z in M, the element $F(x,y,z)$ is in IN.

Proof. Let $L = (A^2)' + (A^2)'M'M'$. Then $F(x,y,z) \in F(y,x,z) + IL(N^2)'N'$ by item (b) in Lemma 2. Since $IL \subseteq I$, we have $F(x,y,z) \equiv F(y,x,z) \pmod{IN}$. By analogy, we obtain $F(x,y,z) \in F(x,z,y) + IN'(N^2)'L$. Using (2) and Proposition 1, we conclude that

$$IN'(N^2)'(A^2)' \subseteq IN \text{ and } IN'(N^2)'L' \subseteq IN.$$

Thus, $F(x,y,z) \equiv F(x,z,y) \pmod{IN}$, proving (a).

We argue for (b). Since $1/3 \in \Phi$, equality (3) implies that $3(u^2)'u' = 0$ for any u in J. Consequently, the identity $u^2vu = 0$ is satisfied in J. Consider $F(x,x,x) = i(ax)'b'(cx)'d'(ex)'$. Without loss of generality, we may assume that $b = y^2$ and $d = z^2$, where y and z are elements from N. By Lemma 2, we then have

$$F(x,x,x) \in iy'c'(x^2)'a'y'(z^2)'(ex)' + ILd'(ex)'.$$

Therefore, $F(x,x,x) \equiv iy'c'(x^2)'a'y'(z^2)'(ex)' \pmod{IN}$. Applying (2) to the latter, we obtain

$$iy'c'(x^2)'a'y'(z^2)'(ex)' = -iy'c'y''a'(x^2)'(z^2)'(ex)' - iy'c'((x^2y)a)'(z^2)'(ex)'.$$

Since, by (2), $2y'c'y' = -(y^2c)'$, it follows that

$$F(x,x,x) \equiv -iy'c'((x^2y)a)'(z^2)'(ex)' \pmod{IN},$$

and again by Lemma 2,

$$F(x,x,x) \equiv -iy'c'z'e'((x^2y)x)'a'z' - iy'c'L \pmod{IN}.$$

Consequently, $F(x,x,x) \in Iy'c'L + IN$, but $Iy'c'(A^2)' \subseteq IN$ by virtue of (2). Therefore, it follows from (a) of Proposition 1 that the inclusion $Iy'c'L \subseteq IN + IN'M'M' \subseteq IN$ holds, i.e., $F(x,x,x) \in IN$. This implies that $F(x,y,z) \in IN$ is valid for all x, y, and z in M.

Proposition 2. Let $N = AM$ and $J_1 = A^2 + N^2 + N$ and suppose that I is an A-bimodule in J_1 such that $IM'M' \subseteq I$. Suppose that $K = \mathrm{Id}_{J_1}(I)$ is an ideal in J_1 generated by the set I, and $L = IN'(N^2)'$. Then we have $K = I + IN + L + LN + LN'(N^2)'$ if $1/3 \in \Phi$.

Proof. Let $K_1 = I + IN + L + LN + LN'(N^2)'$. Then by (4), $INA^2 \subseteq IA'(AN)' \subseteq IN$, and by (2) and Proposition 1, we have

$$LA \subseteq IN'(N^2)A' \subseteq IN + IA'(N^2)'N' \subseteq IN \text{ and } LN'(N^2)'A \subseteq LN.$$

In view of (4), we eventually have

$$LN'(A^2)' \subseteq LA'(AN)' \subseteq IN'N' \subseteq I.$$

Therefore, $K_1(A^2 + N^2) \subseteq K_1$. By (2), the following inclusion holds:

$$LN'N' \subseteq IN'(N^2)N'N' \subseteq IN'((N^2N)N)' + IN'N'N'(N^2)' \subseteq L.$$

Finally, by (b) of Lemma 4, we conclude that $L'N'(N^2)'N' \subseteq IN$. It is now easy to see that the ideal K has the desired form.

LEMMA 5. Let $J = A + M$ be a Jordan Z_2-graded algebra and let $A^{\langle 2\rangle} = 0$. Then $J^{\langle n\rangle} = 0$ for some n.

Proof. Let $I = B + P$, where $B = A^2 + A^2M'A'M'$ and $P = A^2M + (A^2M)A$. Then I is a Z_2-graded ideal in J. Indeed, by equality (2), we have $A^2M'M' \subseteq AM'M'A' \subseteq A^2$ and $A^2M'A'M'M' \subseteq A^2M'A' + A^2M'M'M'A' \subseteq (A^2M)A$. Thus, $BM \subseteq P$. Similarly, $PA \subseteq P$ and $PM \subseteq B$, and so I is an ideal in J.

Let $N = A^2M$. We show that $B^2P'B'P' \subseteq A^2 + B^2(NA)'B'(NA)'$. It follows from (2) that $B^2N'B'P' \subseteq B^2A + B^2P'B'N' \subseteq A^2 + B^2P'B'A'M'A' \subseteq A^2$. Similarly, $B^2P'B'N' \subseteq A^2$, whence the desired inclusion.

Now let $K = C + Q$, where $C = B^2 + B^2P'B'P'$ and $Q = B^2P + (B^2P)B$. By analogy with the latter, we then state that K is a Z_2-graded ideal in I, and the inclusion $C \subseteq A^2 + B^2(NA)'B'(NA)' \subseteq A^2 + ((A^2M)A)^2$ holds. Repeating the above argument, we obtain

$$C^2Q'C'Q' \subseteq A^2 + B^2((B^2P)B)'C'((B^2P)B)'.$$

Since, by (3), $A^2M'(A^2)'M' \subseteq A^2M'(AM)'A' \subseteq A^2$, it follows that

$$C^2Q'C'Q' \subseteq A^2 + B^2((B^2P)B)'((NA)^2)'((B^2P)B)' \subseteq A^2 + A^2(NA)'((NA)^2)'(NA)'.$$

By Lemma 3 and the equality $A^{\langle 2\rangle} = 0$, we conclude that

$$A^2(NA)'((NA)^2)'(NA)' \subseteq A^2(N^2)'A'M'A'M' + A^2M'A'M'A'(N^2)'.$$

Since $N^2 = (A^2M)^2 \subseteq A^2$ and $A^2M'A'M'A' \subseteq A^2$, we have $C^2Q'C'Q' \subseteq A^2$. By analogy with the previous derivation, we conclude that $K_1 = C^2 + C^2Q'C'Q' + C^2Q + (C^2Q)C$ is a Z_2-graded ideal in K, and $(C^2 + C^2Q'C'Q')^2 = 0$, in which case $K_1^{\langle 6\rangle} = 0$ by Theorem 1. Since the even parts of the factor algebras J/I, I/K, and K/K_1 are trivial, it is not hard to state that $J^{\langle n\rangle} = 0$ for some n.

LEMMA 6. Let $J = A + M$ be a Jordan Z_2-graded algebra, let $(A^2)^3 = 0$, and let $1/3 \in \Phi$. Then $J^{\langle n\rangle} = 0$ for some n.

The **proof** proceeds in two stages as follows:

1. Let $J_1 = A + AM$, $B = A^2 + (AM)^2$, $N = AM$, and $I = B + N$. Obviously, J_1 is an ideal in J, and I is an ideal in J_1. It is enough to show that I is solvable.

Since $A^2M'M' \subseteq A^2$ and $NM'M' \subseteq N$, by (2) we have $A^{\langle 2\rangle}M'M' \subseteq A^{\langle 2\rangle}$, $N^2M'M' \subseteq N^2$, and so $BM'M' \subseteq B$. Consequently, by (2) we have $B^2M'M' \subseteq B^2$ and $BN^2M'M' \subseteq BN^2$. By (3) we obtain

$$(A^2M^2)(A^{\langle 2\rangle}M) \subseteq (A^2((A^{\langle 2\rangle}M)M)M \subseteq ((A^2)^3)M = 0.$$

Let

$$L = (N(B^2N^2))B^{\langle 2\rangle}.$$

Then by (2),

$$LA \subseteq M(A^2)^3 + ((MA)A^{\langle 2\rangle})(B^2N^2) \subseteq (A^{\langle 2\rangle}M)(A^2M) = 0$$

and

$$LM'M' \subseteq N(B^2N^2)'(B^{\langle 2\rangle})'M'M' \subseteq N(B^2N^2)'((B^{\langle 2\rangle}M)M)'+$$

$$N(B^2N^2)'M'M'(B^{\langle 2\rangle})' \subseteq L + N((B^2N^2))M'M')'(B^{\langle 2\rangle})' + NM'M'(B^2N^2)'(B^{\langle 2\rangle})' \subseteq L.$$

Hence, $LN'(N^2)'N'(N^2)'N' \subseteq LM \subseteq A^{\langle 2\rangle}$ by Lemma 4.

Now let $P = LN$ and $K = P + PN^2$. By (3), $PA^2 \subseteq LA'(AN)' = 0$, and in view of (2), we have $P(N^2)'A' \subseteq L((NA)N^2) + LA'(N^2)'N' \subseteq P$. Therefore, K is an ideal in the algebra B; moreover, $KN'N' \subseteq K$. Since $P \subseteq A^{\langle 2\rangle}$, by (2) and (3) we have

$$PN'(N^2)'N' \subseteq A^{\langle 2\rangle}M'A'M' \subseteq A^2M'M' + A^2A'M'(A^2)'M' \subseteq A^2 + A^2M'(AM)'A' \subseteq A^2.$$

This implies that $KN'(N^2)'N' \equiv LN'(N^2)'N'(N^2)'N' \pmod{A^2}$, and it follows from the above that $KN'(N^2)'N' \subseteq A^2$. Next, let $R = \mathrm{Id}_I(K)$. From the latter inclusion and Proposition 2, it is then easy to see that the even part of the ideal R belongs to A^2. Therefore, R is a solvable ideal by Lemma 5. Hence, factoring out I by R, we can assume that the algebra J satisfies the equality $[(M(A^2M^2))A^{\langle 2\rangle}]M = 0$. Since $[(M(A^2M^2))A^{\langle 2\rangle}]A = 0$, we have $[(M(A^2M^2))A^{\langle 2\rangle}]J = 0$. Therefore, factoring out the algebra J by the ideal $[(M(A^2M^2))A^{\langle 2\rangle}]$, we can assume that J satisfies the equality $(M(A^2M^2))A^{\langle 2\rangle} = 0$.

2. Now let $K = A^{\langle 2\rangle} + A^{\langle 2\rangle}A$, $K_1 = A^{\langle 2\rangle} + A^{\langle 2\rangle}N^2$, and $K_2 = A^{\langle 2\rangle} + A^{\langle 2\rangle}(BN)^2$. Obviously, K is an ideal in A and B, and K_1 is an ideal in B; moreover, $(KM)M \subseteq K$ and $(K_1M)M \subseteq K_1$ by (2). By (3) and item 1, we conclude that $M(A^{\langle 2\rangle})(A^{\langle 2\rangle})' \subseteq M((A^2)^3)'(A^2)' = 0$,

$$(A^{\langle 2\rangle}M^2)^2 \subseteq A^{\langle 2\rangle}(M(A^{\langle 2\rangle}M^2))'M' = 0,$$

and $M(A^{\langle 2\rangle}M^2)'(A^{\langle 2\rangle}M^2)' \subseteq M((A^{\langle 2\rangle}M^2)M^2)'(A^{\langle 2\rangle})' + M((A^2)^3)'M^2 = 0$. Consequently, $K_1^2 = (K_1M)K_1 = 0$. Therefore, by (3) we have

$$(K_1M)^2B \subseteq (K_1M)(K_1B)'M' + (K_1M)(MB)'K_1' \subseteq (K_1M)K_1'M' + K_1^2 = 0.$$

Consider $C = B^2 + (BN)^2$, $L = BN$, and $I_1 = C + L$. Obviously, I_1 is an ideal in I, and K_2 is an ideal in I_1. Let $R = \{a \in C | Ba = 0\}$ and $K_3 = \mathrm{Id}_{I_1}(R)$. Then $(K_2L)^2 \subseteq K_3$. Clearly, $RN'N' \subseteq R$. By Proposition 2, the even part $(K_3)_0$ of the ideal K_3 has the form $R + RL'(L^2)'L' + RL'(L^2)'L'(L^2)'$, and by Lemma 2 and the equality $R(A^2 + N^2) = RB = 0$, we have

$$RL'(L^2)'L' \subseteq RN'B'N'B'(L^2)' \subseteq A^2.$$

Consequently, $(K_3)_0 \subseteq R + A^2$, and so $(K_3)_0$ is solvable of index 2. Then, by Lemma 5, the ideal K_3 is solvable. Factoring out I_1 by K_3, we can assume that I_1 satisfies $(K_2L)^2 = 0$. Since $K_2 \subseteq K_1$ and $C^{\langle 2\rangle} + C^{\langle 2\rangle}C \subseteq K_2$, there is no loss of generality in assuming that $(KM)^2 = K^2 = (KM)K = 0$. Therefore, in view of (2) and (3) we obtain

$$KN'(N^2)'N'K' \subseteq (KN)^2 + KN'K'N'(N)' = 0$$

and

$$KN'(N^2)'K' \subseteq KN'(NK)'N' = 0.$$

Now let $R = \mathrm{Id}_I(K)$ and let R_0 be the even part of the ideal R. Then, by Proposition 2, $R_0 = K + KN'(N^2)'N' + KN'(N^2)'N'(N^2)'$. We argue that $R_0^2 \subseteq K$. It suffices to show that $KN'(N^2)'N'(A^2)' \subseteq K$ and $[KN'(N^2)'N']^2 \subseteq K$. By (2) and (3), $KN'(N^2)'N'(A^2)' \subseteq KN'N' + KN'(A^2)'N'(N^2)' \subseteq K + KA'N'N'(N^2)' \subseteq K$. We will prove the second inclusion. By (3) we obtain

$$KN'(N^2)'(AM)'(KN)' \subseteq KN'(N^2)'(KNM)'A' + KN'(N^2)'((KN)A)'M' \subseteq$$

$$KN'(NK)'N'A' + KN'(N^2)'((KN)A)'M' \subseteq KN'(N^2)'((KN)A)'M',$$

whereas by (2), we have

$$KN'(N^2)'((KN)A)'M' \subseteq KN'(N^2)'K'A'N'M' + KN'(N^2)'N'A'K'M' \subseteq KM.$$

Therefore, $KN'(N^2)'N'(KN)' \subseteq KM$. This, in view of (2), yields

$$[KN'(N^2)'N']^2 \subseteq KN'(N^2)'N'(N^2)'N'(KN)'+$$
$$KN'(N^2)'N'(KN)'N'(N^2)' \subseteq K + KM'N'(N^2) \subseteq K.$$

Consequently, $R_0^{\langle 2\rangle} = 0$, and by Lemma 5, the ideal R, the factor algebra I/R, and hence the algebra J, are solvable.

LEMMA 7. Let $J = A+M$ be a Jordan Z_2-graded algebra, let A be a solvable subalgebra, and let $1/3 \in \Phi$. Then $J^{\langle n\rangle} = 0$ for some number n.

The **proof** proceeds by induction on the solvability index of the algebra A. Let $A^{\langle k\rangle} = 0$. If $k = 1$, then $J^{\langle 6\rangle} = 0$ by Theorem 1. Suppose that the lemma is proven to hold for every algebra whose even part has solvability index less than k. Consider algebras $J_1 = A + AM$ and $J_2 = A^2 + (AM)^2 + AM$. It is enough to prove that J_2 is solvable. Obviously, $(A^2)^3$ is an ideal in $A^2 + (AM)^2$, and by equality (2), $(A^2)^3M'M' \subseteq (A^2)^3$. By (2) and (3),

$$(A^2)^3M'A'M' \subseteq A^{\langle 2\rangle}M'M' + A^{\langle 2\rangle}A'M'(A^2)'M' \subseteq A^2 + A^2M'(AM)'A' \subseteq A^2.$$

It follows, then, by Proposition 2 that the even part of the ideal K in J_2 generated by $(A^2)^3$ belongs to A^2. By the induction hypothesis, K is a solvable ideal; the factor algebra J_2/K is solvable by Lemma 6. This means that J_2 is solvable, and so therefore is J.

THEOREM 2. Let $J = A+M$ be an arbitrary Z_2-graded Jordan Φ-algebra with a solvable even part and let $1/3 \in \Phi$. Then J is solvable.

Proof. Let B be some subset in J. Denote by $I_4(B)$ the Φ-module generated by elements of the form b^4, where $b \in B$. Put $A_1 = \{a \in A | a + m \in I_4(J)$ for some m in $M\}$ and $M_1 = \{m \in M | a + m \in I_4(J)$ for some a in $A\}$. Since $I_4(J)$ is an ideal in J, it is not hard to see that A_1 is an ideal in A, and M_1 is an A-bimodule. Therefore, $J_1 = A_1 + M_1$ is a Z_2-graded ideal in J. Repeating this argument for the algebra J_1, we infer that J_1 has J_2, a Z_2-graded ideal, which contains $I_4(J_1)$. By Lemma 13 in [3], the even part $(J_2)_0$ is contained in A^2. Thus, we can assume that J_2 is a solvable algebra. By Lemma 7, J/J_1 and J_1/J_2 are solvable. Hence, J is also a solvable algebra.

Theorem A follows from Theorem 2; the results of [4] and [5] can be combined with Theorem A to give the following:

THEOREM 3. Let J be a Jordan superalgebra whose even part is a special or a weakly special Jordan nil-algebra of bounded index n. Suppose that the additive group of the ring Φ contains no elements of order less than $2n$. Then J is solvable.

We continue to extend Theorem A to color Jordan superalgebras. Let Z_2^∞ be a direct sum of countably many copies of Z_2 and let Φ be a field of characteristic not 2, 3. By a *color superalgebra* we mean the usual Z_2^∞-graded Φ-algebra $A = \bigoplus_\alpha A_\alpha$, $\alpha \in Z_2^\infty$, $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$.

Let V be a countably dimensional vector space with basis v_i, $i \geq 0$, and with symmetric bilinear form $\langle\ ,\ \rangle$ such that $\langle v_i, v_j\rangle = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. Consider $Cl(V)$, a Clifford algebra that relates to this form. Then, for any i and j, in the algebra $Cl(V)$ we have $v_i \circ v_j = \delta_{ij}$, where $\circ$ is multiplication in $Cl(V)$, and the elements $v_{i_1} \ldots v_{i_k}$, $1 \leq i_1 < \ldots < i_k$, constitute its basis. To each such basis element v we assign an element $(\sigma_1, \sigma_2, ...) \in Z_2^\infty$, setting $\sigma_i = 1$ if v_i has an entry in v; if not, set $\sigma_i = 0$. It is easy to see that in this case, $Cl(V)$ is a Z_2^∞-graded algebra. Therefore, $Cl(A) = \oplus(A_\alpha \otimes Cl(V)_\alpha)$, where $\alpha \in Z_2^\infty$, is a Z_2^∞-graded subalgebra of the tensor product $A \otimes Cl(V)$.

Consider the homomorphism $s\colon Z_2^\infty \to Z_2$, setting

$$s\colon (\sigma_1, \sigma_2, \ldots) \to \sum_i \sigma_i.$$

Let A be a color superalgebra. It is then easy to see that the superalgebra $A = A_0 + A_1$, where $A_i = \sum\{A_\alpha | s(\alpha) = i\}$ and $i = 0, 1$, is ordinary. Therefore, $Cl(A) = Cl(A)_0 + Cl(A)_1$ is also an ordinary superalgebra, which we call the *Clifford envelope* of A. According to Proposition 2 in [6], a color superalgebra A is a *color Jordan superalgebra* if its Clifford envelope $Cl(A)$ is a Jordan superalgebra.

THEOREM 4. Let $A = \oplus A_\alpha$, $\alpha \in Z_2^\infty$, be a color Jordan superalgebra over a field Φ. Suppose that the subalgebra $\oplus\{A_\alpha | s(\alpha) = 0\}$ is solvable. Then A is a solvable algebra.

Proof. Consider the Clifford envelope $Cl(A)$ of the color superalgebra A. The hypothesis of the theorem specifies that the even part of $Cl(A)$ is the solvable subalgebra for $Cl(A)$, a Jordan superalgebra. By Theorem 2, the algebra $Cl(A)$ is then solvable, and it is easy to see that A is also.

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