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ON A CLASS OF JORDAN D-BIALGEBRAS

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ABSTRACT. A class of Jordan bialgebras related to the "Jordan analog" of the classical Yang–Baxter equation is considered. It is proved that every Jordan bialgebra defined on a finite-dimensional semisimple Jordan algebra belongs to that class.

The Lie bialgebras were introduced by Drinfel′d in the paper [1], where they served as a tool for the study of the solutions of the classical Yang–Baxter equation (CYBE). In contrast to the associative bialgebras, the Lie bialgebras are Lie algebras with a Lie comultiplication that is a 1-cocycle. Interest in the Lie bialgebras is motivated by the quantization problem for the solutions of CYBE (see [2, 3]). In [4, 5] the author introduced the concept of a Jordan bialgebra in the sense of Drinfel′d (a D-bialgebra). In the same papers, the relationship was studied between some types of Jordan bialgebras and the Lie bialgebras. In particular, in [4] it was shown that if $L(J)$ in the Lie algebra obtained from a Jordan algebra J in accordance with the Kantor–Keher–Tits construction and admitting the structure of a Lie bialgebra, then, under some natural restrictions, the algebra J admits the structure of a Jordan bialgebra. In the paper [6], the triangle and the quasitriangle Jordan bialgebras were singled out in agreement with the corresponding concepts for the Lie bialgebras, and a characterization was obtained of the finite-dimensional Jordan algebras admitting a nontrivial structure that makes it a quasitriangle Jordan bialgebra. In [7] a class of Jordan bialgebras related to the "Jordan analog" of CYBE was introduced, and it was proved that, over an algebraically closed field, every Jordan bialgebra defined on a finite-dimensional semisimple Jordan algebra belongs to that class.

In the present paper we prove an analog of the latter result in the case of an arbitrary field.

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Let Φ be a field, and let U, V be linear spaces over Φ. We denote by $U \otimes V$ the tensor product of U and V over Φ. On the space $V \otimes V$ we define a linear mapping τ by

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putting $\tau(\sum_i a_i \otimes b_i) = \sum_i b_i \otimes a_i$. Let ξ be the linear mapping $V \otimes V \otimes V \to V \otimes V \otimes V$ defined by $\xi(x \otimes y \otimes z) = y \otimes z \otimes x$. As usial, id stands for the identity mapping. We denote by V^* the space of all linear functionals on V. For $f \in V^*$ and $v \in V$, the denotes the linear functional f evaluated at v symbol $\langle f, v\rangle$. The space $V^* \otimes V^*$ is well known to be embeddable in the space $(V \otimes V)^*$, in accordance with the rule $\langle f \otimes g, \sum_i a_i \otimes b_i\rangle = \sum_i \langle f, a_i\rangle\langle g, b_i\rangle$.

Definition. Let A be a linear space over Φ, and let $\Delta : A \to A \otimes A$ be a linear mapping. The couple (A, Δ) is called a *coalgebra*; the mapping Δ is a *comultiplication*.

As in Sweedler's paper [8], for $a \in A$ we write $\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$.

On the space A^* we define multiplication by putting

$$\langle fg, a\rangle = \sum_a \langle f, a_{(1)}\rangle\langle g, a_{(2)}\rangle,$$

where $f, g \in A^*$, $a \in A$ and $\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$. The space A^* equipped with this multiplication is an ordinary algebra over Φ; this algebra is called the *dual algebra* of the coalgebra (A, Δ). Obviously, $\langle fg, a\rangle = \langle f \otimes g, \Delta(a)\rangle$.

The dual algebra A^* of the coalgebra (A, Δ) gives rise to the following bimodule action on the space A:

$$f \cdot a = \sum_a a_{(1)}\langle f, a_{(2)}\rangle, \quad a \cdot f = \sum_a \langle f, a_{(1)}\rangle a_{(2)},$$

where $f \in A^*$ and $\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$. It is easy to check that for any $f, g \in A^*$ and any $a \in A$ we have $\langle fg, a\rangle = \langle f, g \cdot a\rangle = \langle g, a \cdot f\rangle$.

Now, let A be an arbitrary algebra endowed with a comultiplication Δ, and let A^* be the dual algebra for (A, Δ). The algebra A determines the following bimodule action $(\bullet)$ on $A^* : \langle f \bullet a, b\rangle = \langle f, ab\rangle$, $\langle b \bullet f, a\rangle = \langle f, ab\rangle$. On the space $D(A) = A \oplus A^*$ we define multiplication by putting

$$(a + f)(b + g) = (ab + f \cdot b + a \cdot g) + (fg + f \bullet b + a \bullet g).$$

Then $D(A)$ turns into an ordinary algebra over Φ, and A and A^* become subalgebras of $D(A)$. The algebra $D(A)$ will be called the *Drinfel′d double* of the bialgebra (A, Δ).

On the algebra $D(A)$ we define a form Q as follows:

$$Q(a + f, b + g) = \langle g, a\rangle + \langle f, b\rangle$$

for any $f, g \in A^*$ and any $a, b \in A$. The form Q is nondegenerate and symmetric. We show that Q is associative. Indeed,

$$\begin{aligned} &Q((a + f)(b + g), c + h) \\ &\quad = \langle h, ab + f \cdot b + a \cdot g\rangle + \langle fg + f \bullet b + a \bullet g, c\rangle \\ &\quad = \langle f, b \cdot h + g \cdot c + bc\rangle + \langle b \bullet h + gh + g \bullet c, a\rangle \\ &\quad = Q(a + f, (b + g)(c + h)) \end{aligned}$$

for any $f, g, h \in A^*$ and any $a, b, c \in A$.

Definition. Let M be an arbitrary manifold of Φ-algebras, and let A be an algebra in M equipped with a supplementary comultiplication Δ. The couple (A, Δ) is called an M-bialgebra in the sense of Drinfel′d if the algebra $D(A)$ is in M.

In what follows, the Jordan bialgebras in the sense of Drinfel′d will be called the D-bialgebras.

Suppose that $1/2 \in \Phi$. If (J, Δ) is a Jordan D-bialgebra over Φ, then for any $a, b, c \in J$ we have

$$\begin{aligned}(1)\qquad &\Delta((ab)c) - \Delta(ab)(c \otimes 1) - \Delta(c)(1 \otimes ab) + \Delta(c)(a \otimes b + b \otimes a)\\ &- \Delta(ac)(b \otimes 1) - \Delta(bc)(a \otimes 1) - \Delta(a)(1 \otimes bc) - \Delta(b)(1 \otimes ac)\\ &+ (\Delta(a)(c \otimes 1))(b \otimes 1) + (\Delta(b)(c \otimes 1))(a \otimes 1)\\ &+ (\Delta(a)(1 \otimes c))(1 \otimes b) + (\Delta(b)(1 \otimes c))(1 \otimes a) = 0.\end{aligned}$$

This relation can be obtained from formula (6) in [4] by linearization. If J has unit 1, formula (1) shows that $\Delta(1) = 0$.

We recall the definition of a Lie bialgebra. For a Lie algebra L, a couple (L, Δ) is a Lie bialgebra if and only if (L, Δ) is a Lie coalgebra, and the comultiplication Δ is a 1-cocycle (see [1]), i.e., it satisfies the identity

$$\Delta([a, b]) = \sum_a ([a_{(1)}, b] \otimes a_{(2)} + a_{(1)} \otimes [a_{(2)}, b]) + \sum_b ([a, b_{(1)}] \otimes b_{(2)} + b_{(1)} \otimes [a, b_{(2)}]).$$

Among the Lie bialgebras, the triangle and the quasitriangle ones play an important role. Let L be a Lie algebra over Φ, and let $R = \sum_i a_i \otimes b_i$ be an element of $(\mathrm{id} - \tau)(L \otimes L)$ (i.e., $\tau(R) = -R$). We define a comultiplication Δ_R on L by putting $\Delta_R(a) = \sum_i [a_i, a] \otimes b_i + a_i \otimes [b_i, a]$ for any $a \in L$. Then the comultiplication Δ_R is anticommutative and is a 1-cocycle. Drinfel′d showed (see [9]) that the couple (L, Δ_R) is a Lie coalgebra if and only if the element

$$C_L(R) = [R_{12}, R_{13}] + [R_{12}, R_{23}] + [R_{13}, R_{23}]$$

is ad-invariant (L-invariant). Here $[R_{12}, R_{13}] = \sum_{ij} [a_i, a_j] \otimes b_i \otimes b_j$, $[R_{12}, R_{23}] = \sum_{ij} a_i \otimes [b_i, a_j] \otimes b_j$, and $[R_{13}, R_{23}] = \sum_{ij} a_i \otimes a_j \otimes [b_i, b_j]$. In particular, if

$$(2)\qquad [R_{12}, R_{13}] + [R_{12}, R_{23}] + [R_{13}, R_{23}] = 0,$$

then (L, Δ_R) is a Lie coalgebra. Relation (2) is called the classical Yang–Baxter equation, or the triangle equation. Since $\tau(R) = -R$, (2) can be written in the form

$$\sum_{ij} ([a_i, a_j] \otimes b_i \otimes b_j + b_i \otimes b_j \otimes [a_i, a_j] + b_j \otimes [a_i, a_j] \otimes b_i) = 0.$$

In the class of Jordan bialgebras we can also distinguish those related to the "Jordan analog" of (2).

Let J be a Jordan Φ-algebra. We take an element $r = \sum_i a_i \otimes b_i$ of $(\mathrm{id} - \tau)(J \otimes J)$ and define a comultiplication Δ_r on J via the formula $\Delta_r(a) = \sum_i a_i a \otimes b_i - a_i \otimes ab_i$. We put $r_{12} = \sum_i a_i \otimes b_i \otimes 1, r_{13} = \sum_i a_i \otimes 1 \otimes b_i$, $r_{23} = \sum_i 1 \otimes a_i \otimes b_i$, and

$$C_J(r) = r_{12}r_{13} + r_{13}r_{23} - r_{12}r_{23}.$$

Here $r_{12}r_{13}, r_{13}r_{23}$ and $r_{12}r_{23}$ are the products of the corresponding elements in the algebra $J \otimes J \otimes J$. Calculating $C_J(r)$, we obtain

$$C_J(r) = \sum_{ij}(a_i a_j \otimes b_i \otimes b_j + a_i \otimes a_j \otimes b_i b_j - a_i \otimes b_i a_j \otimes b_j).$$

Since $\tau(r) = -r$, we have

$$C_J(r) = \sum_{ij}(a_i a_j \otimes b_i \otimes b_j + b_i \otimes b_j \otimes a_i a_j + b_j \otimes a_i a_j \otimes b_i).$$

For an element a of a Jordan algebra J, we denote by a' the operator of right multiplication by a, and for $a, b \in J$ we put $[a', b'] = a'b' - b'a'$. It is well known that any linear combination of operators of the form $[a', b']$ is a differentiation of the algebra J, and the associator (a, c, b) coincides with $c[a', b']$ for any $c \in J$. The following operator identities are valid for any $x, y, z \in J$:

$$x'y'z' + z'y'x' + ((xz)y)' = (xy)'z' + (xz)'y' + (yz)'x', \tag{3}$$

$$x'y'z' + z'y'x' + ((xz)y)' = z'(xy)' + y'(xz)' + x'(yz)', \tag{4}$$

$$[(xy)', z'] + [(xz)', y'] + [(yz)', x'] = 0. \tag{5}$$

§1. Coboundary Jordan bialgebras

In this section, our aim is to show that the couple (J, Δ_r), where J is a Jordan algebra and $C_J(r) = 0$, is a Jordan D-bialgebra.

Proposition 1. *If A is a commutative Φ-algebra equipped with a comultiplication Δ satisfying* (1), *then for any $a, b, c \in A$ we have*

$$\begin{aligned}\Delta((a,b,c)) &\\ = \sum_b &(a, b_{(1)}, c) \otimes b_{(2)} + b_{(1)} \otimes (a, b_{(2)}, c)\\ &+ (\Delta(c)(a \otimes 1 - 1 \otimes a) + \Delta(a)(1 \otimes c - c \otimes 1))(b \otimes 1 - 1 \otimes b).\end{aligned}$$

Proof. Using (1) and the commutativity of A, we can write

$$
\begin{aligned}
\Delta((a,b,c)) &= \Delta((ab)c) - \Delta(a(bc)) \\
&= -(\Delta(b)(c\otimes 1))(a\otimes 1) \\
&\quad - (\Delta(b)(1\otimes c))(1\otimes a) + (\Delta(b)(a\otimes 1))(c\otimes 1) + (\Delta(b)(1\otimes a))(1\otimes c) \\
&\quad - \Delta(c)(a\otimes b + b\otimes a) + (\Delta(c)(a\otimes 1))(b\otimes 1) + (\Delta(c)(1\otimes a))(1\otimes b) \\
&\quad + \Delta(a)(c\otimes b + b\otimes c) - (\Delta(a)(c\otimes 1))(b\otimes 1) - (\Delta(a)(1\otimes c))(1\otimes b) \\
&= \sum_b ((ab_{(1)})c - a(b_{(1)}c))\otimes b_{(2)} + b_{(1)}\otimes((ab_{(2)})c - a(b_{(2)}c)) \\
&\qquad + (\Delta(c)(a\otimes 1 - 1\otimes a)(b\otimes 1 - 1\otimes b) \\
&\qquad + \Delta(a)(1\otimes c - c\otimes 1))(b\otimes 1 - 1\otimes b) \\
&= \sum_b (a,b_{(1)},c)\otimes b_{(2)} + b_{(1)}\otimes(a,b_{(2)},c) \\
&\qquad + (\Delta(c)(a\otimes 1 - 1\otimes a) + \Delta(a)(1\otimes c - c\otimes 1))(b\otimes 1 - 1\otimes b). \square
\end{aligned}
$$

Let (A,Δ) be an arbitrary Φ-bialgebra. Consider the Drinfel'd double $D(A)$ of the bialgebra (A,Δ). We denote by λ_1 and λ_2 the projections of $D(A)$ to A and to A^*, respectively.

Proposition 2. *If A is a commutative Φ-algebra equipped with a comultiplication Δ satisfying* (1), *then for any $a,b\in A$ and any $f\in A^*$ in the algebra $D(A)$ we have*

$$\lambda_2((f^2,b,a) + 2(fa,b,f)) = 0.$$

Proof. Let $a,b\in A$, $f\in A^*$. Since

$$
\begin{aligned}
&\lambda_2((fa,b,f)) \\
&\quad = ((f\bullet a)\bullet b)f + ((f\bullet a)\cdot b)\bullet f - (f\bullet a)(b\bullet f) \\
&\qquad - (f\bullet a)\bullet(b\cdot f) + ((f\cdot a)b)\bullet f - (f\cdot a)\bullet(b\bullet f),
\end{aligned}
$$

for any $c\in A$ we have

$$
\begin{aligned}
&\langle \lambda_2((fa,b,f)), c\rangle \\
&\quad = \sum_c (\langle f,(c_{(1)}b)a\rangle\langle f,c_{(2)}\rangle - \langle f,c_{(1)}a\rangle\langle f,bc_{(2)}\rangle) \\
&\qquad + \sum_b (\langle f,b_{(1)}a\rangle\langle f,b_{(2)}c\rangle - \langle f,b_{(1)}\rangle\langle f,(b_{(2)}c)a\rangle) \\
&\qquad + \sum_a (\langle f,a_{(1)}\rangle\langle f,(a_{(2)}b)c\rangle - \langle f,a_{(1)}\rangle\langle f,(a_{(2)}c)b\rangle),
\end{aligned}
$$

and since Δ is commutative, we can write

$$
\begin{aligned}
&2\langle \lambda_2((fa,b,f)),c\rangle \\
&\quad= \sum_b \langle f\otimes f, b_{(1)}a\otimes b_{(2)}c + b_{(1)}c\otimes b_{(2)}a - b_{(1)}\otimes (b_{(2)}c)a - (b_{(1)}c)a\otimes b_{(2)}\rangle \\
&\qquad+ \sum_c \langle f\otimes f, -c_{(1)}a\otimes c_{(2)}b - c_{(1)}b\otimes c_{(2)}a + c_{(1)}\otimes (c_{(2)}b)a + (c_{(1)}b)a\otimes c_{(2)}\rangle \\
&\qquad+ \sum_a \langle f\otimes f, a_{(1)}\otimes (a_{(2)}b)c + a_{(2)}\otimes (a_{(1)}b)c - a_{(1)}\otimes (a_{(2)}c)b - a_{(2)}\otimes (a_{(1)}c)b\rangle \\
&\quad= \langle f\otimes f, (\Delta(b)(1\otimes c - c\otimes 1) + \Delta(c)(b\otimes 1 - 1\otimes b))(a\otimes 1 - 1\otimes a)\rangle \\
&\qquad+ \langle f\otimes f, (b,a_{(1)},c)\otimes a_{(2)} + a_{(1)}\otimes (b,a_{(2)},c)\rangle.
\end{aligned}
$$

Proposition 1 yields

$$2\langle \lambda_2((fa,b,f)),c\rangle = \langle f\otimes f, \Delta((b,a,c))\rangle = \langle f^2,(b,a,c)\rangle.$$

But $\langle \lambda_2((f^2,b,a)),c\rangle = \langle f^2,(c,a,b)\rangle$, so that the required identity is a consequence of the two relations obtained above. $\square$

Let J be a Jordan Φ-algebra. We fix $r = \sum_i a_i\otimes b_i \in (\mathrm{id}-\tau)(J\otimes J)$. On J, the element r induces a comultiplication Δ_r. Following the terminology of Lie algebras, we call (J,Δ_r) the coboundary bialgebra. The Drinfel′d double of the bialgebra (J,Δ_r) will be denoted by $D(J,r)$.

Proposition 3. *The comultiplication Δ_r is cocommutative and satisfies* (1).

Proof. Let $a\in J$. Since $\Delta_r(a) = \sum_i a_ia\otimes b_i - a_i\otimes ab_i$ and $\tau(r) = -r$, we have $\tau(\Delta_r(a)) = \sum_i b_i\otimes aa_i - ab_i\otimes a_i = \Delta_r(a)$.

We prove that Δ_r satisfies (1). Let $a,b,c\in J$. The definition of Δ_r implies that the left-hand side of (1) looks like this:

$$
\begin{aligned}
&\sum_i ([(ab)c]a_i - [(ab)a_i]c - [(ac)a_i]b - [(bc)a_i]c + [(a_ia)c]b + [(a_ib)c]a)\otimes b_i \\
&-\sum_i a_i\otimes ([(ab)c]b_i - (cb_i)(ab) - (ab_i)(cb) - (bb_i)(ca) + [(bb_i)c]a + [(ab_i)c]b).
\end{aligned}
$$

Identities (3) and (4) show that the expression obtained is equal to zero, and the result follows. $\square$

We recall that

$$C_J(r) = \sum_{ij} (a_ia_j\otimes b_i\otimes b_j + b_i\otimes b_j\otimes a_ia_j + b_j\otimes a_ia_j\otimes b_i).$$

Proposition 4. *For any $a \in J$ we have*

$$\begin{aligned}
&(\Delta_r \otimes \mathrm{id} - \mathrm{id} \otimes \Delta_r)\Delta_r(a) \\
&\quad = C_J(r)(a \otimes 1 \otimes 1 - 1 \otimes 1 \otimes a) \\
&\qquad + \sum_{ij}((a, a_i, a_j) \otimes b_j \otimes b_i + a_i \otimes (b_i, a, a_j) \otimes b_j + b_i \otimes b_j \otimes (a_j, a_i.a)).
\end{aligned}$$

Proof. The definition of the comultiplication Δ_r implies that

$$\begin{aligned}
&(\Delta_r \otimes \mathrm{id} - \mathrm{id} \otimes \Delta_r)\Delta_r(a) \\
&\quad = \sum_{ij}((a_i a)a_j \otimes b_j \otimes b_i - a_i a \otimes a_j b_i \otimes b_j + a_i a \otimes a_j \otimes b_i b_j) \\
&\qquad + \sum_{ij}(-a_j \otimes (a_i a)b_j \otimes b_i + a_i \otimes (ab_i)a_j \otimes b_j) \\
&\qquad + \sum_{ij}(-a_i a_j \otimes b_j \otimes ab_i + a_j \otimes (a_i b_j) \otimes ab_i - a_i \otimes a_j \otimes (ab_i)b_j).
\end{aligned}$$

Since $\sum_{ij} a_j \otimes (aa_i)b_j \otimes b_i = \sum_{ij} a_i \otimes (aa_j)b_i \otimes b_j$ and $\tau(r) = -r$, we have

$$\begin{aligned}
&(\Delta_r \otimes \mathrm{id} - id \otimes \Delta_r)\Delta_r(a) \\
&\quad = \sum_{ij}(((a, a_i, a_j) + (a_i a_j)a) \otimes b_j \otimes b_i + b_i a \otimes a_j a_i \otimes b_j + b_i a \otimes b_j \otimes a_i a_j) \\
&\qquad + \sum_{ij}(-a_i \otimes (a_j a)b_i \otimes b_j + a_i \otimes (b_i a)a_j \otimes b_j) \\
&\qquad + \sum_{ij}(-a_i a_j \otimes b_j \otimes ab_i - b_j \otimes a_i a_j \otimes ab_i - b_i \otimes b_j \otimes (a(a_i a_j) + (a_j, a_i, a))),
\end{aligned}$$

which proves the claim. □

Lemma 1. *For any $a, b \in J$ and any $f \in J^*$ we have*

$$Q((f^2, a, f), b) = \langle f \otimes f \otimes f, C_J(r)([b', a'] \otimes 1 \otimes 1)\rangle.$$

Proof. The definition of multiplication in $D(J, r)$ implies that

$$\lambda_2((f^2, a, f)) = (f^2 \bullet a)f + (f^2 \cdot a) \bullet f - f^2(a \bullet f) - f^2 \bullet (a \cdot f).$$

Since the comultiplication Δ_r is cocommutative, we have

$$\begin{aligned}
&\langle (f^2 \bullet a)f + (f^2 \cdot a) \bullet f, b\rangle \\
&\quad = 2\sum_{ij}\langle f \otimes f \otimes f, [(a_i b)a]a_j \otimes b_j \otimes b_i - (b_i a)b \otimes a_i a_j \otimes b_j\rangle, \\
&\langle f^2(a \bullet f) + f^2 \bullet (a \cdot f), b\rangle \\
&\quad = 2\sum_{ij}\langle f \otimes f \otimes f, [(a_i a)b]a_j \otimes b_j \otimes b_i - (b_i b)a \otimes a_i a_j \otimes b_j\rangle.
\end{aligned}$$

Consequently,

$$Q((f^2,a,f),b)=2\sum_{ij}\langle f\otimes f\otimes f,(b,a_i,a)a_j\otimes b_j\otimes b_i+(b,b_i,a)\otimes a_ia_j\otimes b_j\rangle.$$

Putting $S=\sum_{ij}((b,a_i,a)a_j\otimes b_j\otimes b_i+(b,b_i,a)\otimes a_ia_j\otimes b_j)$, we see that

$$\begin{aligned}S=&\sum_{ij}((b,a_i,a)a_j\otimes b_j\otimes b_i-(b,a_ia_j,a)\otimes b_j\otimes b_i\\&-(b,b_j,a)\otimes b_i\otimes a_ia_j)+C_J(r)([b',a']\otimes 1\otimes 1)\\=&-\sum_{ij}((b,a_j,a)a_i\otimes b_j\otimes b_i+(b,b_j,a)\otimes b_i\otimes a_ia_j)+C_J(r)([b',a']\otimes 1\otimes 1).\end{aligned}$$

Interchanging the indices i and j in the latter sum, we obtain

$$S=-\sum_{ij}((b,a_i,a)a_j\otimes b_i\otimes b_j+(b,b_i,a)\otimes b_j\otimes a_ia_j)+C_J(r)([b',a']\otimes 1\otimes 1).$$

This results in the relation

$$\begin{aligned}&\sum_{ij}\langle f\otimes f\otimes f,(b,a_i,a)a_j\otimes b_j\otimes b_i+(b,b_i,a)\otimes a_ia_j\otimes b_j\rangle\\&\quad=-\sum_{ij}\langle f\otimes f\otimes f,(b,a_i,a)a_j\otimes b_j\otimes b_i+(b,b_i,a)\otimes b_j\otimes a_ia_j\rangle\\&\quad\quad+\langle f\otimes f\otimes f,C_J(r)([b',a']\otimes 1\otimes 1)\rangle.\end{aligned}$$

Thus, $Q((f^2,a,f),b)=\langle f\otimes f\otimes f,C_J(r)([b',a']\otimes 1\otimes 1)\rangle$. □

Lemma 2. *For any $a,b\in J$ and any $f,g\in J^*$, in the algebra $D(J,r)$ we have*

$$\begin{aligned}&(fg,b,a)+(fa,b,g)+(ga,b,f)\\&\quad=(f\otimes g\otimes \mathrm{id})\\&\quad\quad\times(C_J(r)(b\otimes a\otimes 1+a\otimes b\otimes 1+1\otimes 1\otimes ab\\&\quad\quad\quad-b'a'\otimes 1\otimes 1-1\otimes b'a'\otimes 1-1\otimes 1\otimes b'a')).\end{aligned}$$

Proof. First, we prove this identity for $f=g$. Propositions 3 and 2 show that $(f^2,b,a)+2(fa,b,f)=\lambda_1((f^2,b,a)+2(fa,b,f))$.

Since the comultiplication Δ_r is cocommutative and $\lambda_1((f^2,b,a))=(f^2\bullet b)\cdot a+(f^2\cdot b)a-f^2\cdot(ab)$, we obtain

$$\begin{aligned}&\lambda_1((f^2,b,a))\\&\quad=2\sum_{ij}(\langle f,[(a_ia)b]a_j\rangle\langle f,b_j\rangle b_i-\langle f,[(ab)a_i]a_j\rangle\langle f,b_j\rangle b_i\\&\quad\quad-\langle f,a_ia_j\rangle\langle f,b_j\rangle(b_ib)a+\langle f,a_ia_j\rangle\langle f,b_j\rangle(ab)b_i).\end{aligned}$$

In a similar way, we employ Proposition 4 to show that

$$
\begin{aligned}
&\lambda_1((fa,b,f))\\
&\quad=(f\otimes \mathrm{id}\otimes f)(C_J(r)(b\otimes 1\otimes a-1\otimes 1\otimes b'a'))\\
&\qquad+\sum_{ij}(\langle f,(b,a_i,a_j)\rangle\langle f,ab_i\rangle+\langle f,aa_i\rangle\langle f,(b_ib)a_j\rangle-\langle f,aa_i,\rangle\langle f,(a_jb_i)b\rangle)b_j\\
&\qquad+\sum_{ij}(\langle f,a_i\rangle\langle f,ab_j\rangle(b_i,b,a_j)-\langle f,a_ia\rangle\langle f,a_j\rangle(b_ib)b_j+\langle f,aa_i\rangle\langle f,a_j\rangle(bb_j)b_i)\\
&\qquad+\sum_{ij}(\langle f,b_i\rangle\langle f,(a_j,a_i,b)a\rangle-\langle f,a_i\rangle\langle f,[(ab_i)b]a_j\rangle+\langle f,a_i\rangle\langle f,[(ab_i)a_j]b\rangle)b_j\\
&\qquad+\sum_{ij}(\langle f,a_i\rangle\langle f,a_j\rangle[(ab_i)b]b_j-\langle f,a_i\rangle\langle f,a_j\rangle(ab_i)(bb_j)).
\end{aligned}
$$

The first sum on the right, say S, looks like this:

$$S=\sum_{ij}(\langle f,(b,a_i,a_j)\rangle\langle f,ab_i\rangle b_j+\langle f,aa_i\rangle\langle f,(b,b_i,a_j)\rangle b_j).$$

Interchanging the elements a_i and b_i in the first summand of the above sum, we obtain $S=0$. Similarly,

$$\sum_{ij}(\langle f,a_i\rangle\langle f,ab_j\rangle(b_i,b,a_j)-\langle f,a_ia\rangle\langle f,a_j\rangle(b_ib)b_j+\langle f,aa_i\rangle\langle f,a_j\rangle(bb_j)b_i)=0.$$

Now, let

$$S=\sum_{ij}(\langle f,b_i\rangle\langle f,(a_j,a_i,b)a\rangle-\langle f,a_i\rangle\langle f,[(ab_i)b]a_j\rangle+\langle f,a_i\rangle\langle f,[(ab_i)a_j]b\rangle)b_j.$$

Here we interchange the elements a_i and b_i in the last two summands, obtaining

$$S=\sum_{ij}(\langle f,b_i\rangle\langle f,(a_j,a_i,b)a\rangle+\langle f,b_i\rangle\langle f,[(aa_i)b]a_j\rangle-\langle f,b_i\rangle\langle f,[(aa_i)a_j]b\rangle)b_j.$$

Since

$$(a_j,a_i,b)a+[(aa_i)b]a_j-[(aa_i)a_j]b=(a_j,a_i,b)a-(a_j,aa_i,b)=-(a_j,a,b)a_i,$$

we have $S=-\sum_{ij}\langle f,b_i\rangle\langle f,(a_j,a,b)a_i\rangle b_j=-\sum_{ij}\langle f,b_j\rangle\langle f,(a_i,a,b)a_j\rangle b_i$. Therefore,

$$
\begin{aligned}
&(f^2,b,a)+2(fa,b,f)\\
&\quad=2(f\otimes \mathrm{id}\otimes f)(C_J(r)(b\otimes 1\otimes a-1\otimes 1\otimes b'a'))\\
&\qquad+2\sum_{ij}(-\langle f,a_ia_j\rangle\langle f,b_j\rangle(b_ib)a+\langle f,a_ia_j\rangle\langle f,b_j\rangle(ab)b_i)\\
&\qquad+2\sum_{ij}(\langle f,a_i\rangle\langle f,a_j\rangle[(ab_i)b]b_j-\langle f,a_i\rangle\langle f,a_j\rangle(ab_i)(bb_j)).
\end{aligned}
$$

Let $S = \sum_{ij}(-\langle f, a_i a_j\rangle\langle f, b_j\rangle(b_i b)a + \langle f, a_i a_j\rangle\langle f, b_j\rangle(ab)b_i)$. Then

$$\begin{aligned} S &= (f \otimes f \otimes \mathrm{id})(C_J(r)(1 \otimes 1 \otimes ab - 1 \otimes 1 \otimes b'a')) \\ &\quad + \sum_{ij}(\langle f, b_j\rangle\langle f, b_i\rangle[(a_i a_j)b]a + \langle f, b_i\rangle\langle f, a_i a_j\rangle(b_j b)a \\ &\qquad - \langle f, b_j\rangle\langle f, b_i\rangle(ab)(a_i a_j) - \langle f, b_i\rangle\langle f, a_i a_j\rangle(ab)b_j). \end{aligned}$$

In the second and forth summands we interchange i and j, while in the first and third we interchange a_i and b_i, and also a_j and b_j. This results in the relation

$$\begin{aligned} 2S &= (f \otimes f \otimes \mathrm{id})(C_J(r)(1 \otimes 1 \otimes ab - 1 \otimes 1 \otimes b'a')) \\ &\quad + \sum_{ij}(\langle f, a_j\rangle\langle f, a_i\rangle[(b_i b_j)b]a - \langle f, a_i\rangle\langle f, a_j\rangle(ab)(b_i b_j)). \end{aligned}$$

From (4) it follows that

$$\begin{aligned} &\sum_{ij}\langle f, a_j\rangle\langle f, a_i\rangle([(b_i b_j)b]a - (ab)(b_i b_j)) \\ &\quad = \sum_{ij}\langle f, a_j\rangle\langle f, a_i\rangle(-[(b_i a)b]b_j - [(b_j a)b]b_i + (ab_i)(bb_j) + (ab_j)(bb_i)). \end{aligned}$$

Interchanging i and j in the second and the forth summand, we obtain

$$\begin{aligned} 2S &= (f \otimes f \otimes \mathrm{id})(C_J(r)(1 \otimes 1 \otimes ab - 1 \otimes 1 \otimes b'a')) \\ &\quad + 2\sum_{ij}\langle f, a_j\rangle\langle f, a_i\rangle(-[(b_i a)b]b_j + (ab_i)(bb_j)). \end{aligned}$$

Consequently,

$$\begin{aligned} &(f^2, b, a) + 2(fa, b, f) \\ &\quad = 2(f \otimes \mathrm{id} \otimes f)(C_J(r)(b \otimes 1 \otimes a - 1 \otimes 1 \otimes b'a')) \\ &\qquad + (f \otimes f \otimes \mathrm{id})(C_J(r)(1 \otimes 1 \otimes ab - 1 \otimes 1 \otimes b'a')). \end{aligned}$$

Since

$$\begin{aligned} &(\mathrm{id} \otimes \tau)(C_J(r)) \\ &\quad = (\mathrm{id} \otimes \tau)\Big(\sum_{ij}(a_i a_j \otimes b_i \otimes b_j + b_i \otimes b_j \otimes a_i a_j + b_j \otimes a_i a_j \otimes b_i)\Big) \\ &\quad = \sum_{ij}(a_i a_j \otimes b_j \otimes b_i + b_i \otimes a_i a_j \otimes b_j + b_j \otimes b_i \otimes a_i a_j) \\ &\quad = C_J(r), \end{aligned}$$

we have

$$(\mathrm{id}\otimes\tau)(C_J(r)(b\otimes 1\otimes a-1\otimes 1\otimes b'a'))=C_J(r)(b\otimes a\otimes 1-1\otimes b'a'\otimes 1).$$

Thus,

$$\begin{aligned}&(f\otimes\mathrm{id}\otimes f)(C_J(r)(b\otimes 1\otimes a-1\otimes 1\otimes b'a'))\\&\quad=(f\otimes f\otimes\mathrm{id})(\mathrm{id}\otimes\tau)(C_J(r)(b\otimes 1\otimes a-1\otimes 1\otimes b'a'))\\&\quad=(f\otimes f\otimes\mathrm{id})(C_J(r)(b\otimes a\otimes 1-1\otimes b'a'\otimes 1)).\end{aligned}$$

This shows that

$$\begin{aligned}&(f^2,b,a)+2(fa,b,f)\\&\quad=2(f\otimes f\otimes\mathrm{id})(C_J(r)(b\otimes a\otimes 1-1\otimes b'a'\otimes 1))\\&\qquad+(f\otimes f\otimes\mathrm{id})(C_J(r)(1\otimes 1\otimes ab-1\otimes 1\otimes b'a')).\end{aligned}$$

Linearizing the latter identity with respect to f, we obtain

$$\begin{aligned}&2(fg,b,a)+2(fa,b,g)+2(ga,b,f)\\&\quad=(f\otimes g\otimes\mathrm{id})(\mathrm{id}+\tau\otimes\mathrm{id})\\&\qquad\times(C_J(r)(2(b\otimes a\otimes 1-1\otimes b'a'\otimes 1)+1\otimes 1\otimes ab-1\otimes 1\otimes b'a')),\end{aligned}$$

and since

$$\begin{aligned}&(\tau\otimes\mathrm{id})(C_J(r)(2(b\otimes a\otimes 1-1\otimes b'a'\otimes 1)+1\otimes 1\otimes ab-1\otimes 1\otimes b'a'))\\&\quad=C_J(r)(2(a\otimes b\otimes 1-b'a'\otimes 1\otimes 1)+1\otimes 1\otimes ab-1\otimes 1\otimes b'a'),\end{aligned}$$

we arrive at the relation

$$\begin{aligned}&(fg,b,a)+(fa,b,g)+(ga,b,f)\\&\quad=(f\otimes g\otimes\mathrm{id})\\&\qquad\times(C_J(r)(b\otimes a\otimes 1+a\otimes b\otimes 1+1\otimes 1\otimes ab\\&\qquad\qquad-b'a'\otimes 1\otimes 1-1\otimes b'a'\otimes 1-1\otimes 1\otimes b'a')),\end{aligned}$$

completing the proof of the lemma. □

Proposition 5. *Let (J,Δ_r) be a Jordan coalgebra. If in the algebra $D(J,r)$ we have*

$$(f^2,b,a)+2(fa,b,f)=0\quad\textit{and}\quad(f^2,a,f)=0$$

for any $a, b \in J$ and any $f \in J^*$, then the couple (J, Δ_r) is a Jordan D-bialgebra.

Proof. Clearly, the algebra $D(J, r)$ is commutative. Since (J, Δ_r) is a Jordan coalgebra, the dual algebra J^* is Jordan. Therefore, it suffices to show that the following relations are true in the algebra $D(J, r)$ for any $a, b \in J$ and any $f, g \in J^*$:

$$(a^2, b, f) + 2(fa, b, a) = 0, \tag{6}$$

$$(f^2, g, a) + 2(fa, g, f) = 0, \tag{7}$$

$$(a^2, g, f) + 2(fa, g, a) = 0, \tag{8}$$

$$(a^2, f, a) = 0. \tag{9}$$

We prove (6). Let $S(a, b, c)$ denote the expression on the left in (1). Since

$$\begin{aligned}
&\lambda_1((a^2, b, f) + 2(fa, b, a)) \\
&\quad = f \cdot (a^2 b) - a^2 \cdot (b \bullet f) - a^2 \bullet (b \cdot f) \\
&\qquad + 2([(f \bullet a) \bullet b] \cdot a + [(f \bullet a) \cdot b]a \\
&\qquad - (f \bullet a) \cdot (ba) + [(f \cdot a)b]a - (f \cdot a)(ba)),
\end{aligned}$$

by direct calculation we obtain

$$\langle g, \lambda_1((a^2, b, f) + 2(fa, b, a))\rangle = \langle g \otimes f, S(a, a, b)\rangle.$$

Therefore, $\lambda_1((a^2, b, f) + 2(fa, b, a)) = 0$ by Proposition 3.

Let c be an arbitrary element of J. We have

$$Q((a^2, b, f) + 2(fa, b, a), c) = Q(f, (b, a^2, c) - 2(b, a, c)a) = 0,$$

which yields (6).

Since $Q((f^2, g, a) + 2(fa, g, f), b) = -Q(g, (f^2, b, a) + 2(fa, b, f))$, from the assumptions of the proposition it follows that $Q((f^2, g, a) + 2(fa, g, f), b) = 0$. Let $h \in J^*$. Then

$$Q(h, (f^2, g, a) + 2(fa, g, f)) = Q(-(h, f^2, g) + 2(h, f, g)f, a) = 0,$$

and we arrive at (7).

We pass to the proof of (8). Since

$$Q((a^2, g, f) + 2(fa, g, a), c) = -Q(g, (a^2, c, f) + 2(fa, c, a)),$$

we can use (6) we deduce the formula $Q((a^2, g, f) + 2(fa, g, a), c) = 0$. On the other hand,

$$\begin{aligned}
&Q(h, (a^2, g, f) + 2(fa, g, a)) \\
&\quad = Q((h, f, g), a^2) - 2Q(h, f \bullet a, g), a) + 2Q(h, (f \cdot a, g, a)).
\end{aligned}$$

Putting $e = h \otimes f \otimes g$ and applying Proposition 4, we obtain

$$\begin{aligned}
&Q((h,f,g),a^2) - Q(2(h,f\bullet a,g),a)\\
&\quad= \langle e, C_J(r)(a^2\otimes 1\otimes 1 + 2(1\otimes a\otimes a)) - C_J(r)(1\otimes 1\otimes a^2 + 2(a\otimes a\otimes 1))\rangle\\
&\qquad+\sum_{ij}\langle e,(a^2,a_i,a_j)\otimes b_j\otimes b_i + a_i\otimes(b_i,a^2,a_j)\otimes b_j + b_i\otimes b_j\otimes(a_j,a_i.a^2)\rangle\\
&\qquad-2\sum_{ij}\langle e,(a,a_i,a_j)\otimes ab_j\otimes b_i + a_i\otimes(b_i,a,a_j)a\otimes b_j + b_i\otimes b_ja\otimes(a_j,a_i.a)\rangle.
\end{aligned}$$

Let τ_{13} denote the linear mapping of the space $J\otimes J\otimes J$ into itself that satisfies $\tau_{13}(x\otimes y\otimes z) = (z\otimes y\otimes x)$. Then

$$\tau_{13}(C_J(r)(1\otimes 1\otimes a^2 + 2(a\otimes a\otimes 1))) = C_J(r)(a^2\otimes 1\otimes 1 + 2(1\otimes a\otimes a)).$$

The assumptions of the proposition and Lemma 2 imply the relation

$$C_J(r)(1\otimes 1\otimes a^2 + 2(a\otimes a\otimes 1)) = C_J(r)(a'a'\otimes 1\otimes 1 + 1\otimes a'a'\otimes 1 + 1\otimes 1\otimes a'a').$$

Applying τ_{13} to both sides of this identity, we get

$$C_J(r)(a^2\otimes 1\otimes 1 + 2(1\otimes a\otimes a)) = C_J(r)(a'a'\otimes 1\otimes 1 + 1\otimes a'a'\otimes 1 + 1\otimes 1\otimes a'a').$$

Consequently,

$$C_J(r)(a^2\otimes 1\otimes 1 + 2(1\otimes a\otimes a)) - C_J(r)(1\otimes 1\otimes a^2 + 2(a\otimes a\otimes 1)) = 0.$$

Since $(b_i,a^2,a_j) = 2(b_i,a,a_j)a$, we have

$$\begin{aligned}
&Q((h,f,g),a^2) - Q(2(h,f\bullet a,g),a)\\
&\quad=\sum_{ij}\langle e,(a^2,a_i,a_j)\otimes b_j\otimes b_i + b_i\otimes b_j\otimes(a_j,a_i,a^2)\rangle\\
&\qquad-2\sum_{ij}\langle e,(a,a_i,a_j)\otimes ab_j\otimes b_i + b_i\otimes b_ja\otimes(a_j,a_i.a)\rangle.
\end{aligned}$$

The relation $\lambda_1(f\cdot a,g,a) = \sum_j(\langle f,b_j\rangle\lambda_1((a_ja,g,a)) - \langle f,ab_j\rangle\lambda_1((a_j,g,a)))$ allows us to calculate the element $\lambda_1((a_ja,g,a))$ as follows:

$$\begin{aligned}
&\lambda_1((a_ja,g,a)) = [(a_ja)\bullet g]\cdot a + [(a_ja)\cdot g]a - (a_ja)\cdot(g\bullet a) - (a_ja)(g\cdot a)\\
&\quad=\sum_i(\langle g,(a_ia)(a_ja)\rangle b_i - \langle g,a_i(a_ja)\rangle ab_i + \langle g,a_i(a_ja)\rangle ab_i - \langle g,a_i\rangle[(a_ja)b_i]a\\
&\qquad\quad-\langle g,[(a_ja)a_i]a\rangle b_i + \langle g,a_ia\rangle(aa_j)b_i - \langle g,a_ia\rangle(aa_j)b_i + \langle g,a_i\rangle(aa_j)(ab_i))\\
&\quad=\sum_i(\langle g,(a_ja,a_i,a)\rangle b_i - \langle g,a_i\rangle(a,b_i,aa_j)).
\end{aligned}$$

Recalling the identity $\tau(r) = -r$, we see that

$$\lambda_1((a_j a, g, a)) = \sum_i (\langle g, (a_j a, a_i, a)\rangle b_i + \langle g, b_i\rangle (a, a_i, aa_j)).$$

In a similar way we get $\lambda_1((a_j, g, a)) = \sum_i (\langle g, (a_j, a_i, a)\rangle b_i + \langle g, b_i\rangle (a, a_i, a_j)).$, which yields

$$\begin{aligned} &Q(h, (f \cdot a, g, a)) \\ &\quad = \sum_{ij} \langle e, (a, a_i, a_j) \otimes ab_j \otimes b_i + b_i \otimes ab_j \otimes (a_j, a_i, a) \\ &\qquad\qquad - (a, a_i, aa_j) \otimes b_j \otimes b_i - b_i \otimes b_j \otimes (aa_j, a_i, a)\rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} &Q((h, f, g), a^2) - Q(2(h, f \bullet a, g), a) + 2Q(h, (f \cdot a, g, a)) \\ &\quad = \sum_{ij} \langle e, (a^2, a_i, a_j) \otimes b_j \otimes b_i - 2(a, a_i, aa_j) \otimes b_j \otimes b_i \\ &\qquad\qquad + b_i \otimes b_j \otimes (a_j, a_i, a^2) - 2b_i \otimes b_j \otimes (aa_j, a_i, a)\rangle \\ &\quad = 0 \end{aligned}$$

by (5). Consequently, $Q(h, (a^2, g, f) + 2(fa, g, a)) = 0$. Thus, $(a^2, g, f) + 2(fa, g, a)) = 0$, which proves (8).

Finally, we prove (9). For any $b \in J$ we have $Q((a^2, f, a), b) = -Q(f, (a^2, b, a)) = 0$. Therefore,

$$(a^2, f, a) = \lambda_1((a^2, f, a)) = (a^2 \bullet f) \cdot a + (a^2 \cdot f)a - a^2 \cdot (f \bullet a) - a^2(f \cdot a),$$

whence

$$Q(g, (a^2, f, a)) = \langle f \otimes g, \Delta_r(a)(a^2 \otimes 1 - 1 \otimes a^2) + \Delta_r(a^2)(1 \otimes a - a \otimes 1)\rangle$$

for any $g \in J^*$. The definition of the comultiplication Δ_r shows that

$$\begin{aligned} &\Delta_r(a)(a^2 \otimes 1 - 1 \otimes a^2) + \Delta_r(a^2)(1 \otimes a - a \otimes 1) \\ &\quad = \sum_i ((a_i a)a^2 \otimes b_i + a_i \otimes (ab_i)a^2 - a_i \otimes (a^2 b_i)a - (a_i a^2)a \otimes b_i) \\ &\quad = 0. \end{aligned}$$

As a consequence, we obtain $(a^2, f, a) = 0$. Thus, $D(J, r)$ is a Jordan algebra, and the couple (J, Δ_r) is a Jordan D-bialgebra. □

Proposition 6. *Let (J, Δ_r) be a Jordan coalgebra, and let $C_J(r) = 0$. Then the couple (J, Δ_r) is a Jordan D-bialgebra.*

Proof. Since (J, Δ_r) is a Jordan coalgebra, the dual algebra J^* is Jordan. Let $a, b \in J$ and let $f, g \in J^*$. We have $C_J(r) = 0$ by assumption. Lemma 1, shows that $Q((f^2, a, f), b) = 0$. On the other hand, $Q(g, (f^2, a.f)) = -Q((f^2, g, f), a) = 0$, whence $(f^2, a, f) = 0$. Since $C_J(r) = 0$, Lemma 2 shows that $(f^2, b, a) + 2(fa, b, f) = 0$. By Proposition 5, the couple (J, Δ_r) is a Jordan D-bialgebra. □

Lemma 3. *Suppose that $C_J(r) = 0$ and that the characteristic of the field Φ is different from 3. Then the couple (J, Δ_r) is a Jordan coalgebra.*

Proof. By Proposition 3, the dual algebra J^* of the coalgebra (J, Δ_r) is commutative. Let $f, g \in J^*$, $a \in J$. Since Δ_r is cocommutative, Proposition 4 implies that

$$\begin{aligned}
&\langle (f^2, g, f), a\rangle \\
&\quad = 2\sum_{ijk} \langle f, (a, a_i, a_j)a_k\rangle\langle f, b_k\rangle\langle g, b_j\rangle\langle f, b_i\rangle \\
&\qquad + \langle f, (a_j, a_i, a)\rangle\langle f, b_i a_k\rangle\langle g, b_j\rangle\langle f, b_k\rangle + \langle f, a_i a_k\rangle\langle f, b_k\rangle\langle g, (b_i, a, a_j)\rangle\langle f, b_j\rangle).
\end{aligned}$$

We put $e = f \otimes f \otimes g \otimes f$. Since $C_J(r) = 0$, we have

$$\begin{aligned}
\sum_{ijk} \langle f, a_i a_k\rangle\langle f, b_k\rangle\langle g, (b_i, a, a_j)\rangle\langle f, b_j\rangle &= \sum_{ijk} \langle e, a_i a_k \otimes b_k \otimes (b_i, a, a_j) \otimes b_j\rangle \\
&= -\sum_{ijk} \langle e, b_k \otimes b_i \otimes (a_i a_k, a, a_j) \otimes b_j + b_i \otimes a_k a_i \otimes (b_k, a, a_j) \otimes b_j\rangle.
\end{aligned}$$

In the sum $\sum_{ijk} b_k \otimes b_i \otimes (a_i a_k, a, a_j) \otimes b_j$ we interchange the indices k and j and then i and k; this yields $\sum_{ijk} b_k \otimes b_i \otimes (a_i a_k, a, a_j) \otimes b_j = \sum_{ijk} b_j \otimes b_i \otimes (a_i a_j, a, a_k) \otimes b_k = \sum_{ijk} b_j \otimes b_k \otimes (a_j a_k, a, a_i) \otimes b_i$. Consequently,

$$\begin{aligned}
&3\sum_{ijk} \langle f \otimes f \otimes g \otimes f, b_k \otimes b_i \otimes (a_i a_k, a, a_j) \otimes b_j\rangle \\
&\qquad = \sum_{ijk} \langle f, b_k\rangle\langle f, b_i\rangle\langle f, b_j\rangle\langle g, (a_i a_k, a, a_j) + (a_i a_j, a, a_k) + (a_j a_k, a, a_i)\rangle \\
&\qquad = 0
\end{aligned}$$

by (5). Thus, we get

$$\sum_{ijk} \langle e, a_i a_k \otimes b_k \otimes (b_i, a, a_j) \otimes b_j\rangle = -\sum_{ijk} \langle e, b_i \otimes a_k a_i \otimes (b_k, a, a_j) \otimes b_j\rangle.$$

On the right, we interchange i and k, obtaining $\sum_{ijk}\langle f, a_i a_k\rangle\langle f, b_k\rangle\langle g,(b_i,a,a_j)\rangle\langle f, b_j\rangle = 0$. Consequently,

$$\begin{aligned}
&\langle (f^2, g, f), a\rangle \\
&\quad = 2\sum_{ijk}\langle f,(a,a_i,a_j)a_k\rangle\langle f,b_k\rangle\langle g,b_j\rangle\langle f,b_i\rangle \\
&\qquad + 2\sum_{ijk}\langle f,(a_j,a_i,a)\rangle\langle f,b_i a_k\rangle\langle g,b_j\rangle\langle f,b_k\rangle \\
&\quad = 2\sum_{ijk}\langle e,(a,a_i,a_j)a_k\otimes b_k\otimes b_j\otimes b_i + (a_j,a_i,a)\otimes b_i a_k\otimes b_j\otimes b_k\rangle.
\end{aligned}$$

We put $S = \sum_{ijk}(a_j,a_i,a)\otimes b_i a_k\otimes b_j\otimes b_k$. Interchanging a_i and b_i shows that $S = -\sum_{ijk}(a_j,b_i,a)\otimes a_i a_k\otimes b_j\otimes b_k$. Since $C_J(r)=0$, we obtain

$$S = \sum_{ijk}((a_j,a_i a_k,a)\otimes b_k\otimes b_j\otimes b_i + (a_j,b_k,a)\otimes b_i\otimes b_j\otimes a_i a_k).$$

Therefore,

$$\begin{aligned}
&\sum_{ijk}\langle e,(a,a_i,a_j)a_k\otimes b_k\otimes b_j\otimes b_i + (a_j,a_i,a)\otimes b_i a_k\otimes b_j\otimes b_k\rangle \\
&\quad = \sum_{ijk}\langle e,((a,a_i,a_j)a_k + (a_j,a_i a_k,a))\otimes b_k\otimes b_j\otimes b_i + (a_j,b_k,a)\otimes b_i\otimes b_j\otimes a_i a_k\rangle \\
&\quad = \sum_{ijk}\langle e,(a_j,a_k,a)a_i\otimes b_k\otimes b_j\otimes b_i + (a_j,b_k,a)\otimes b_i\otimes b_j\otimes a_i a_k\rangle.
\end{aligned}$$

In the resulting sum we first interchange i and k and then, in the second summand, a_i and b_i. Recalling that $\tau(r) = -r$, we get

$$\begin{aligned}
&\sum_{ijk}\langle e,(a,a_i,a_j)a_k\otimes b_k\otimes b_j\otimes b_i + (a_j,a_i,a)\otimes b_i a_k\otimes b_j\otimes b_k\rangle \\
&\qquad = \sum_{ijk}\langle e,(a_j,a_i,a)a_k\otimes b_i\otimes b_j\otimes b_k - (a_j,a_i,a)\otimes b_k\otimes b_j\otimes b_i a_k\rangle.
\end{aligned}$$

This proves the identity

$$\sum_{ijk}\langle e,(a,a_i,a_j)a_k\otimes b_k\otimes b_j\otimes b_i + (a_j,a_i,a)\otimes b_i a_k\otimes b_j\otimes b_k\rangle = 0.$$

Thus, $\langle (f^2,g,f),a\rangle = 0$, whence $(f^2,g,f)=0$. $\square$

The following statement is a consequence of Proposition 6 and Lemma 3.

Theorem 1. *Let J be a Jordan algebra over a field Φ of characteristic different from 2 and 3, let $r = \sum_i a_i \otimes b_i$, and let $\tau(r) = -r$. If $C_J(r) = 0$, then the couple (J, Δ_r) is a Jordan D-bialgebra.*

Now we present examples of coboundary Jordan D-bialgebras (J, Δ_r) for which $C_J(r) = 0$.

Proposition 7. *Let J be a Jordan Φ-algebra, let A be a subalgebra generated by $a, b \in J$, and let $r = a \otimes b - b \otimes a$. Suppose that $A^2 = 0$. Then the couple (J, Δ_r) is a Jordan D-bialgebra.*

Proof. Since A is an algebra with zero multiplication, we have $C_J(r) = 0$. We show that (J, Δ_r) is a Jordan coalgebra. Indeed, by (3), $a'a'a' = b'b'b' = 0$ and $a'b'a' = b'a'b' = 0$. Proposition 4 implies that

$$\begin{aligned}
&\langle (f^2, g, f), c\rangle \\
&\quad = \langle f, -2c(a'a'b' + b'a'a')\rangle \langle f, a\rangle \langle g, b\rangle \langle f, b\rangle \\
&\qquad + \langle f, 2c(a'b'b' + b'b'a')\rangle \langle f, a\rangle \langle g, a\rangle \langle f, b\rangle
\end{aligned}$$

for any $c \in J$ and any $f, g \in J^*$. Consequently, $\langle (f^2, g, f), c\rangle = 0$ by (3), which shows that (J, Δ_r) is a Jordan coalgebra. Since $C_J(r) = 0$, to prove that the couple (J, Δ_r) is a Jordan D-bialgebra it suffices to refer to Proposition 6. $\square$

Proposition 8. *Let J be a Jordan Φ-algebra with a nontrivial idempotent e. Let b be a nonzero element of $J_{\frac{1}{2}}(e)$ such that $b^2 = 0$, and let $r = e \otimes b - b \otimes e$. Then the couple (J, Δ_r) is a Jordan D-bialgebra with nonzero comultiplication.*

Proof. Since $eb = \frac{1}{2} b$ and $b^2 = 0$, we have

$$C_J(r) = (\mathrm{id} + \xi + \xi^2)\Big(e \otimes b \otimes b - \frac{1}{2} b \otimes b \otimes e - \frac{1}{2} b \otimes e \otimes b \Big).$$

It is easy to check that $C_J(r) = 0$. We prove that the couple (J, Δ_r) is a Jordan coalgebra. Let $f, g \in J^*$, $a \in J$. Since $\Delta_r(b) = b \otimes b$, $\Delta_r(e) = \frac{1}{2}(e \otimes b + b \otimes e)$, and, by (3), $b'b'b' = 0$, we can apply Proposition 4 to obtain

$$\begin{aligned}
&\langle (f^2, g, f), a\rangle \\
&\quad = \langle f, a(2(e')^3 + e' - 3e'e')\rangle \langle f, b\rangle \langle g, b\rangle \langle f, b\rangle \\
&\qquad + \Big\langle f, 2a\Big(- e'e'b' - b'e'e' - \frac{1}{4} b' + e'b' + b'e'\Big) \Big\rangle \langle f, e\rangle \langle g, b\rangle \langle f, b\rangle \\
&\qquad + \langle f, 2a(e'b'b' + b'b'e' - b'b')\rangle \langle f, e\rangle \langle g, e\rangle \langle f, b\rangle \\
&\qquad + \langle f, a(2(b'e'b' - b'b'))\rangle \langle f, e\rangle \langle g, b\rangle \langle f, e\rangle \\
&\qquad + \Big\langle f, a\Big(- 2e'b'e' - \frac{1}{2} b' + e'b' + b'e'\Big) \Big\rangle \langle f, b\rangle \langle g, e\rangle \langle f, b\rangle.
\end{aligned}$$

Therefore, from (3) it follows that $\langle (f^2, g, f), a\rangle = 0$, so that (J, Δ_r) is a Jordan coalgebra. Since $C_J(r) = 0$, Proposition 6 shows that (J, Δ_r) is a Jordan D-bialgebra. $\square$

§2. The bialgebra structure on semisimple Jordan algebras

In this section we show that every Jordan bialgebra (J, Δ), where J is a semisimple finite-dimensional Jordan algebra, is a bialgebra of the form (J, Δ_r) with $C_J(r) = 0$.

Let D be an associative Φ-algebra with unit 1. We introduce the matrix algebra D_n and denote by e_{ij} the matrix units of D_n.

Proposition 9. *Let a nondegenerate bilinear associative form f be defined on the algebra D. Then, on the algebra D_n, a nondegenerate bilinear associative form F can be defined in such a way that $F(xe_{ij}, ye_{ji}) = f(x, y)$ for any $x, y \in D$ and any i, j.*

Proof. Consider two matrices $a = \sum_{ij} \alpha_{ij} e_{ij}$ and $b = \sum_{ij} \beta_{ij} e_{ij}$. We introduce F by putting

$$F(a, b) = \sum_i f(\alpha_{ii}, \beta_{ii}) + \sum_{ij\neq} f(\alpha_{ij}, \beta_{ji}).$$

Obviously, F is a nondegenerate bilinear form, and $F(xe_{ij}, ye_{ji}) = f(x, y)$ for any $x, y \in D$ and any i, j. We check that F is associative. Let $c = \sum_{ij} \gamma_{ij} e_{ij}$ be an element of D_n. We have

$$\begin{aligned}
&F(ac, b)\\
&\quad= \sum_i \sum_k f(\alpha_{ik}\gamma_{ki}, \beta_{ii}) + \sum_{ij\neq} \sum_k f(\alpha_{ik}\gamma_{kj}, \beta_{ji})\\
&\quad= \sum_i f(\alpha_{ii}\gamma_{ii}, \beta_{ii}) + \sum_i \sum_{k\neq i} f(\alpha_{ik}\gamma_{ki}, \beta_{ii})\\
&\qquad+ \sum_{ij\neq} f(\alpha_{ii}\gamma_{ij}, \beta_{ji}) + \sum_{ij\neq} \sum_{k\neq i} f(\alpha_{ik}\gamma_{kj}, \beta_{ji})\\
&\quad= \sum_i f(\alpha_{ii}, \gamma_{ii}\beta_{ii}) + \sum_i \sum_{j\neq i} f(\alpha_{ii}, \gamma_{ij}\beta_{ji})\\
&\qquad+ \sum_i \sum_{k\neq i} f(\alpha_{ik}, \gamma_{ki}\beta_{ii}) + \sum_{ij\neq} \sum_{k\neq i} f(\alpha_{ik}, \gamma_{kj}\beta_{ji})\\
&\quad= \sum_i \sum_j f(\alpha_{ii}, \gamma_{ij}\beta_{ji}) + \sum_{ik\neq} \sum_j f(\alpha_{ik}, \gamma_{kj}\beta_{ji})\\
&\quad= F(a, cb).
\end{aligned}$$

In a similar way, $F(ac, b) = F(c, ba)$. This proves the proposition. □

Suppose that D is endowed with an involution $d \mapsto \overline{d}$. In the algebra D_n we take a diagonal invertible matrix $a = \operatorname{diag}\{a_1, \dots, a_n\}$ with symmetric elements on the diagonal. Then the mapping $S_a : x \mapsto a^{-1}\overline{x}^t a$ is an involution of the algebra D_n. Let $H(D_n, S_a)$ denote the space of matrices fixed by the involution S_a. The space $J = H(D_n, S_a)$ is known to be a Jordan algebra with respect to the symmetrized matrix multiplication

$a \odot b = \frac{1}{2}(ab + ba)$. For $x \in D$ we put $x[ij] = xe_{ij} + a_j^{-1}\overline{x}a_ie_{ji}$. Then $J = \sum_{i,j=1}^{n} \oplus J_{ij}$, where $J_{ij} = \{x[ij] \mid x \in D\}$. In the algebra J we have the following identities:

$$2x[ii] \odot y[ij] = (xy + a_i^{-1}\overline{x}a_iy)[ij], \quad i \neq j; \tag{10}$$

$$2x[ij] \odot y[ji] = xy[ii] + yx[jj], \quad i \neq j; \tag{11}$$

$$x[ij] = a_j^{-1}\overline{x}a_i[ji]. \tag{12}$$

Lemma 4. *Let a nondegenerate bilinear associative form Q be defined on the Jordan algebra $J = H(D_n, S_a)$. Then, on the algebra D_n, a nondegenerate bilinear associative form F can be defined in such a way that $F(h, 1) = 2Q(h, 1)$ for any $h \in J$.*

Proof. Consider J_{ij} and J_{kl} with $i \neq k, l$. Since $J_{ij} \odot 1[ii] = J_{ij}$ and $J_{kl} \odot 1[ii] = 0$, the associativity of Q shows that

$$Q(J_{ij}, J_{kl}) = Q(J_{ij} \odot 1[ii], J_{kl}) = Q(J_{ij}, 1[ii] \odot J_{kl}) = 0.$$

Consequently, $Q(J_{ij}, J_{kl}) = 0$ for any couple of indices i, j different from the couple k, l. Since the form Q is nondegenerate, so it is on every space J_{ij}.

By (11), we have $2(1[ij] \odot y[ij]) = y[ii] + y[jj]$ for $(i \neq j)$. Since Q is associative, we can use (10) to deduce that

$$\begin{aligned} Q(x[ii], y[ii]) &= 2Q(x[ii], 1[ij] \odot y[ji]) = 2Q(x[ii] \odot 1[ij], y[ji]) \\ &= Q((x + a_i^{-1}\overline{x}a_i)[ij], y[ji]). \end{aligned}$$

On the other hand, using (11) and, once again, the associativity of Q, we can write

$$Q(x[ii], y[ii]) = 2Q(x[ij] \odot 1[ji], y[ii]) = 2Q(x[ij], y[ii] \odot 1[ji]).$$

By (10) and (12), we have

$$\begin{aligned} y[ii] \odot 1[ji] &= y[ii] \odot a_i^{-1}a_j[ij] = (ya_i^{-1}a_j + a_i^{-1}\overline{y}a_j)[ij] \\ &= a_j^{-1}\overline{(ya_i^{-1}a_j + a_i^{-1}\overline{y}a_j)}a_i[ji] = (y + a_i^{-1}\overline{y}a_i)[ji], \end{aligned}$$

whence $Q(x[ii], y[ii]) = Q(x[ij], (y + a_i^{-1}\overline{y}a_i)[ji])$. This yields

$$Q((x + a_i^{-1}\overline{x}a_i)[ij], y[ji]) = Q(x[ij], (y + a_i^{-1}\overline{y}a_i)[ji]).$$

Consequently,

$$Q((a_i^{-1}\overline{x}a_i)[ij], y[ji]) = Q(x[ij], (a_i^{-1}\overline{y}a_i)[ji]). \tag{13}$$

We prove that

$$Q(x[ii], 1[ii]) = Q(x[jj], 1[jj]) \tag{14}$$

for any i, j. Indeed, (10), (11), and the associativity of Q show that

$$\begin{aligned} Q(x[jj],1[jj]) &= 2Q(x[jj],1[ij]\odot 1[ji]) = 2Q(x[jj]\odot 1[ji],1[ij]) \\ &= Q((x+a_j^{-1}\overline{x}a_j)[ji],1[ij]). \end{aligned}$$

Using (13) and (10), we obtain

$$\begin{aligned} &Q((x+a_j^{-1}\overline{x}a_j)[ji],1[ij]) \\ &\quad = 2Q(x[ji],1[ij]) = 2Q(x[ji],1[ii]\odot 1[ij]) = 2Q(x[ji]\odot 1[ij],1[ii]) \\ &\quad = Q(x[ii],1[ii]), \end{aligned}$$

which implies (14).

On the algebra D we define a form f as follows:

$$f(x,y) = Q(x[ij],y[ji])$$

for any $x,y\in D$; the indices i and j $(i\neq j)$ are assumed to be fixed.

The facts proved above yield

$$(*) \qquad Q(x[ii],y[ii]) = f(x+a_i^{-1}\overline{x}a_i,y) = f(x,y+a_i^{-1}\overline{y}a_i),$$

whence

$$(15) \qquad f(a_i^{-1}\overline{x}a_i,y) = f(x,a_i^{-1}\overline{y}a_i).$$

Let $x,y,z\in D$. Using (10), (11), and the associativity of Q, we obtain

$$\begin{aligned} f(y,z) &= Q(y[ij],z[ji]) = Q(1[ii]\odot y[ij],z[ji]) \\ &= Q(1[ii],y[ij]\odot z[ji]) = \frac{1}{2}Q(1[ii],yz[ii]+zy[jj]) \\ &= \frac{1}{2}Q(1[ii],yz[ii]). \end{aligned}$$

Therefore, from (*) it follows that $f(y,z)=f(1,yz)$. Consequently, $f(xy,z)=f(1,xyz)=$ $f(x,yz)$. On the other hand, by (15),

$$\begin{aligned} f(xy,z) &= f(xy,a_i^{-1}\overline{a_i^{-1}\overline{z}a_i}a_i) = f(a_i^{-1}\overline{xy}a_i,a_i^{-1}\overline{z}a_i) \\ &= f(a_i^{-1}\overline{y}a_ia_i^{-1}\overline{x}a_i,a_i^{-1}\overline{z}a_i) = f(a_i^{-1}\overline{y}a_i,a_i^{-1}\overline{x}a_ia_i^{-1}\overline{z}a_i) \\ &= f(a_i^{-1}\overline{y}a_i,a_i^{-1}\overline{zx}a_i) = f(y,a_i^{-1}\overline{a_i^{-1}\overline{zx}a_ia_i}) \\ &= f(y,zx). \end{aligned}$$

Thus, f is a bilinear associative form on D. Suppose that $f(x, D) = 0$ for some element x. Then, by (*), we have $Q(x[ii], y[ii]) = 0$ for any $y \in D$, and the form Q turns out to be degenerate. This shows that f is nondegenerate.

By Proposition 9, on the algebra D_n we can define a nondegenerate bilinear associative form F such that $F(xe_{ij}, ye_{ji}) = f(x, y)$ for any $x, y \in D$ and any i, j. Let $h \in J$. Then $h = \sum_{ij} h_{ij}[ij]$ and $Q(h, 1) = \sum_i Q(h_{ii}[ii], \frac{1}{2}[ii])$. Relations (14) and (*) imply that $2Q(h, 1) = \sum_i f(h_{ii} + a_i^{-1}\overline{h}_{ii}a_i, 1)$. Consequently,

$$2Q(h, 1) = \sum_i F((h_{ii} + a_i^{-1}\overline{h}_{ii}a_i)e_{ii}, e_{ii}) = F(h, 1).$$

□

Let J be a finite-dimensional simple Jordan Φ-algebra with unit 1; we assume that J admits the structure of a Jordan bialgebra with nontrivial comultiplication Δ. Let $D(J)$ be the Drinfel′d double of the bialgebra (J, Δ), and let Q be a nondegenerate symmetric associative bilinear form on the algebra $D(J)$. By (1), we have $\Delta(1) = 0$. Therefore, 1 is the unit of the algebra $D(J)$.

Lemma 5. *If U is a nonzero proper ideal of the algebra $D(J)$, then the dimensions of J^* and U coincide.*

Proof. First, we note that J^* is an irreducible J-bimodule. Indeed, let N be a J-subbimodule in J^*, and let $N^\perp$ be the orthogonal complement of the space N in J. Then $N^\perp$ is an ideal of J. Since J is a simple algebra, we have either $N = 0$, or $N = J^*$. In order to show that J^* and U are of the same dimension, we consider the space $W = \{f \in J^* \mid f + a \in U \text{ for some } a \in J\}$. Since $D(J) = J + J^*$, the space W is nonzero. Let $f \in W$, and let b be an arbitrary element of J. For some $a \in J$ we have $f + a \in U$, and $f \bullet b \in W$ because $(f + a)b = f \bullet b + f \cdot b + ab \in U$. Consequently, W is a J-subbimodule of the J-bimodule J^*, whence $W = J^*$. This yields $\dim_\Phi U \geq \dim_\Phi J^*$. Since J is a simple algebra, we have $U \cap J = 0$, which shows that $\dim_\Phi(U + J) \geq \dim_\Phi D(J)$. Consequently, $\dim_\Phi U = \dim_\Phi J^*$. □

Theorem 2. *Let J be a finite-dimensional simple Jordan Φ-algebra with unit 1 admitting the structure of a Jordan bialgebra with nontrivial comultiplication Δ. Let N be the nil-radical of the algebra $D(J)$. Then $N \neq 0$, $N^2 = 0$. Moreover, if $N^\perp$ is the orthogonal complement of N in $D(J)$ relative to the form Q, then $N = N^\perp$.*

Proof. Suppose that $N = 0$. Then $D(J) = B_1 \oplus \ldots \oplus B_n$ is a direct sum of simple algebras. By Lemma 5, $\dim_\Phi B_i = \dim_\Phi J$. Since $\dim_\Phi D(J) = 2\dim_\Phi J$, we have $D(J) = B_1 \oplus B_2$. Also, $J \cap B_i = 0$ for $i = 1, 2$, because $1 \in J$. Therefore, each algebra B_i is isomorphic to J. Clearly, J^* is not an ideal of the algebra $D(J)$ (because otherwise the comultiplication Δ is zero). If J is a division algebra, then so is B_i, $i = 1, 2$. Suppose that for some $f \in J^*$ we have $f = b_1 + b_2$, where b_1 and b_2 are nonzero elements of B_1 and B_2, respectively. Then f is invertible in $D(J)$. Since J^* is a finite-dimensional subalgebra of $D(J)$, the unit of $D(J)$ belongs to J^*. It follows that for any $f \in J^*$ we have either $f \in B_1$ or $f \in B_2$, whence $J^* = (J^* \cap B_1) \cup (J^* \cap B_2)$. In this case, either $J^* = B_1$ or $J^* = B_2$, so that J, B_1, and B_2 are not division algebras. Consequently, J is an algebra having a nontrivial idempotent; J is either the algebra of a nondegenerate symmetric bilinear form over the center, or the exceptional algebra $H(C_3, S_a)$ over the center, or the matrix algebra $H(D_n, S_a)$ (see [10]). In the last-mentioned case, D is an associative algebra with involution $^-$, and either D is a division algebra, or $D = A \oplus A^{op}$, where A is a division algebra and $^-$ is the involution permuting the components.

Let e_i be the unit of the algebra B_i, $i = 1, 2$. Since Q is a nondegenerate bilinear associative form on $D(J)$, we have $Q(B_1, B_2) = 0$. Hence, Q is a nondegenerate bilinear associative form on B_1 and on B_2.

We show that if f is a nonzero nilpotent element of J^*, then $Q(f, 1) = 0$.

Consider the case where $B_1 = H(C_3, S_a)$. We put $u_i = \frac{1}{2}[ii]$. Then the elements u_1, u_2, u_3 are orthogonal idempotents, and the sum of them is e_1. Let B_{1ij} denote the components of the Albert decomposition of the algebra B_1 relative to the idempotents u_1, u_2, u_3. For any two different couples of indices i, j and k, l we have $Q(B_{1ij}, B_{1kl}) = Q(B_{1ij}B_{1ii}, B_{1kl}) = Q(B_{1ij}, B_{1ii}B_{1kl}) = 0$ because the form Q is associative. In the space B_{1ij} there exists a nonzero element u_{ij} such that $u_{ij}^2 = \alpha(u_i + u_j)$, where α is an element of the center of the algebra B_1. For any central element z of B_1 we have

$$\begin{aligned} Q(u_i, zu_i) &= Q(\alpha^{-1}u_{ij}^2, zu_i) = Q(\alpha^{-1}u_{ij}, zu_iu_{ij}) \\ &= Q(\alpha^{-1}u_{ij}, zu_ju_{ij}) = Q(\alpha^{-1}u_{ij}^2, zu_j) \\ &= Q(u_j, zu_j). \end{aligned}$$

We can write $fe_1 = \sum_{i=1}^{3} z_iu_i + w$, where the z_i are central elements, and $w \in \sum_{ij} B_{1ij}$, $i \neq j$. Consequently,

$$Q(fe_1, e_1) = \sum_i Q(z_iu_i, u_i) = Q(\sum_i z_iu_1, u_1).$$

Since the element fe_1 is nilpotent, we have $\sum_i z_i = 0$, whence $Q(fe_1, e_1) = 0$. In a similar way we obtain $Q(fe_2, e_2) = 0$. Thus, $Q(f, 1) = 0$.

The case where B_1 is the algebra of a symmetric nondegenerate bilinear form can be analyzed similarly.

Let $B_1 = H(D_n, S_a)$. By Lemma 4, on the algebra D_n we can define a nondegenerate bilinear associative form F in such a way that $F(fe_1, e_1) = 2Q(fe_1, e_1)$. Since D is either a division ring, or the direct sum of two division rings, and the element fe_1 is nilpotent, there exists an invertible matrix $X \in D_n$ such that $X(fe_1)X^{-1}$ is an upper triangular matrix with zero entries on the diagonal (here $X(fe_1)X^{-1}$ is a product in the algebra D_n). By the associativity of F, we have $F(X(fe_1)X^{-1}, e_1) = 0$. Consequently, $F(fe_1, e_1) = F(fe_1, X^{-1}X) = F(X(fe_1)X^{-1}, e_1) = 0$. Then $Q(fe_1, e_1) = 0$ and, similarly, $Q(fe_2, e_2) = 0$. Thus, $Q(f, 1) = 0$.

Now, let $f \in J^*$ be an element for which $Q(f,1) \neq 0$. We may assume that f is nilpotent. Indeed, we have $J^* = (J^*)^2 + N(J^*)$, where $N(J^*)$ is the nil-radical of the algebra J^*. Hence, $f = c + r$ with $c \in (J^*)^2$, $r \in N(J^*)$. Since $Q((J^*)^2, 1) = \langle J^* \otimes J^*, \Delta(1)\rangle = 0$, we see that $Q(c,1) = 0$, whence $Q(r,1) \neq 0$. This shows that as f we can take the element r. As was proved above, we have $Q(f,1) = 0$, which implies that $D(J)$ is not semisimple.

Thus, $N \neq 0$. By Lemma 5, $\dim_\Phi N = \dim_\Phi J$. Let $N^\perp$ be the orthogonal complement of the ideal N in $D(J)$ relative to the form Q. Then $N^\perp$ is an ideal of $D(J)$, and $\dim_\Phi N^\perp = \dim_\Phi N$. Since Q is nondegenerate, we have $N^\perp \neq D(J)$. Clearly, $D(J) = J \oplus N^\perp$. Consider the subspace $W = \{a \in J \mid a + u \in N \text{ for some } u \in N^\perp\}$. Suppose that $N \not\subseteq N^\perp$. Then $W \neq 0$. Obviously, W is an ideal of J. Consequently, $W = J$. Therefore, there is an element $u \in N^\perp$ such that $1 + u \in N$. However, this implies that $N^\perp = D(J)$, so that the form Q is degenerate. Thus, $N = N^\perp$. Since $Q(N^2, D(J)) = Q(N, ND(J)) = 0$, we get $N^2 = 0$. □

Lemma 6. *Let J be a simple finite-dimensional Jordan algebra over Φ. Suppose that J is endowed with the structure of a Jordan D-bialgebra with comultiplication Δ. Then there exists an element $r \in (\mathrm{id} - \tau)(J \otimes J)$ such that $C_J(r) = 0$ and $\Delta = \Delta_r$.*

Proof. By Theorem 2, the algebra $D(J)$ can be written as $J + N$, where N is the nil-radical of $D(J)$, and $N^2 = 0$. Consequently, for any $f \in J^*$ there exists $a \in J$ such that $f = a + u$ with $u \in N$. We define a mapping $\phi : J^* \mapsto J$ by putting $\phi(f) = a$. The mapping ϕ is well defined, and ϕ is an algebra homomorphism. Since J^* is the space dual to J, there is an element $r = \sum_i a_i \otimes b_i \in J \otimes J$ such that $\phi(f) = \sum_i \langle f, b_i\rangle a_i$. The relation $u^2 = 0$ implies that $f^2 - 2f \bullet a = 0$. Since $\Delta(1) = 0$, we have $2\langle f, a\rangle = -\langle f^2 - 2f \bullet a, 1\rangle = 0$. Therefore, $\sum_i \langle f \otimes f, a_i \otimes b_i\rangle = 0$ for any $f \in J^*$, which shows that

$$\sum_i \langle h \otimes g, a_i \otimes b_i + b_i \otimes a_i\rangle = \sum_i \langle h \otimes g + g \otimes h, a_i \otimes b_i\rangle = 0$$

for any $h, g \in J^*$. Thus, $\tau(r) = -r$.

Let $g = b + v$. Since $uv = 0$, we have $fg = f \bullet b + g \bullet a$. Recalling that $a = \sum_i \langle f, b_i\rangle a_i$ and $b = \sum_i \langle g, b_i\rangle a_i$, we obtain

$$fg = \sum_i (\langle g, b_i\rangle f \bullet a_i + \langle f, b_i\rangle g \bullet a_i).$$

Hence,

$$\langle fg, a\rangle = \sum_i \langle f \otimes g, a_i a \otimes b_i + b_i \otimes a a_i\rangle$$

for any $a \in J$. Consequently, $\Delta(a) = \sum_i (a_i a \otimes b_i + b_i \otimes a a_i)$; since $\tau(r) = -r$, we see that $\Delta(a) = \sum_i (a_i a \otimes b_i - a_i \otimes a b_i)$, whence $\Delta = \Delta_r$.

We prove that $C_J(r)=0$. Since $\phi(f)\phi(g)=\phi(fg)$, we can write

$$\sum_{ij}\langle f,b_i\rangle\langle g,b_j\rangle a_ia_j=\sum_i\langle fg,b_i\rangle a_i=\sum_{ij}\langle f,a_jb_i\rangle\langle g,b_j\rangle a_i-\sum_{ij}\langle f,a_j\rangle\langle g,b_ib_j\rangle a_i.$$

In the first sum on the right we interchange a_i and b_i, and in the second we interchange a_i and b_i and also a_j and b_j; this yields

$$\sum_{ij}\langle f,b_i\rangle\langle g,b_j\rangle a_ia_j=-\sum_{ij}\langle f,a_ja_i\rangle\langle g,b_j\rangle b_i-\sum_{ij}\langle f,b_j\rangle\langle g,a_ia_j\rangle b_i.$$

Consequently, $\langle f\otimes g\otimes h,C_J(r)\rangle=0$ for any $f,g,h\in J^*$, which proves that $C_J(r)=0$.□

Theorem 3. *Let J be a semisimple finite-dimensional Jordan algebra over Φ. Suppose that J is endowed with the structure of a Jordan D-bialgebra with comultiplication Δ. Then there exists an element $r\in(\mathrm{id}-\tau)(J\otimes J)$ such that $C_J(r)=0$ and $\Delta=\Delta_r$.*

Proof. By assumption, $J=A_1\oplus\ldots\oplus A_n$ is a direct sum of simple algebras. We put $J_1=A_1$ and $J_2=A_2\oplus\ldots\oplus A_n$; then $J=J_1\oplus J_2$. Let e_i denote the unit of J_i, $i=1,2$. For $i=1,2$, the index i of an element a_{ij} indicates that $a_{ij}\in J_i$.

Let $\Delta(e_1)=\sum_i(u_{1i}+v_{2i})\otimes(v_{1i}+u_{2i})$. By (1), we have

$$\begin{aligned}\Delta(e_1)&=3\Delta(e_1)(e_1\otimes 1)+3\Delta(e_1)(1\otimes e_1)\\&\quad-2\Delta(e_1)(e_1\otimes e_1)-2\Delta(e_1)(e_1\otimes 1)(e_1\otimes 1)-2\Delta(e_1)(1\otimes e_1)(1\otimes e_1)\\&=\sum_i u_{1i}\otimes(v_{1i}+u_{2i})+\sum_i(u_{1i}+v_{2i})\otimes v_{1i}-2\sum_i u_{1i}\otimes v_{1i},\end{aligned}$$

whence $\Delta(e_1)=\sum_i(u_{1i}\otimes u_{2i}+v_{2i}\otimes v_{1i})$. Since $\tau(\Delta(e_1))=\Delta(e_1)$, we obtain

$$\Delta(e_1)=\sum_i(u_{1i}\otimes u_{2i}+u_{2i}\otimes u_{1i}).$$

Suppose that $a\in J_1$, $\Delta(a)=\sum_i(a_{1i}+a_{2i})\otimes(b_{1i}+b_{2i})$. Using (1), we can write

$$\begin{aligned}\Delta(a)&=\Delta(e_1)(a\otimes 1)+\Delta(a)(1\otimes e_1)-2\Delta(a)(e_1\otimes e_1)+2\Delta(a)(e_1\otimes 1)\\&\quad+\Delta(e_1)(1\otimes a)-2(\Delta(e_1)(a\otimes 1))(e_1\otimes 1)-2(\Delta(e_1)(1\otimes a))(1\otimes e_1)\\&=\sum_i(-u_{1i}a\otimes u_{2i}+a_{1i}\otimes b_{1i}+a_{2i}\otimes b_{1i}+2(a_{1i}\otimes b_{2i})).\end{aligned}$$

Consequently, $\sum_i a_{2i}\otimes b_{2i}=0$ and $\sum_i a_{1i}\otimes b_{2i}=u_{1i}a\otimes u_{2i}$. This yields

$$\begin{aligned}\Delta(a)&=\sum_i(a_{1i}\otimes b_{1i}+u_{1i}a\otimes u_{2i}+u_{2i}\otimes au_{1i})\\&=\sum_i a_{1i}\otimes b_{1i}+\Delta(e_1)(a\otimes 1+1\otimes a).\end{aligned}$$

Similarly, $\Delta(b) = \Delta(e_2)(b \otimes 1 + 1 \otimes b) + \sum_i a_{2i} \otimes b_{2i}$ for any $b \in J_2$. Since $\Delta(1) = 0$, we have $\Delta(e_2) = -\Delta(e_1)$. Thus, $\Delta(b) = -\Delta(e_1)(b \otimes 1 + 1 \otimes b) + \sum_i a_{2i} \otimes b_{2i}$.

Let $r_0 = \sum_i (u_{1i} \otimes u_{2i} - u_{2i} \otimes u_{1i})$. Then, for any $a \in J_1$ and any $b \in J_2$,

$$\Delta(a) = \sum_i a_{1i} \otimes b_{1i} + \Delta_{r_0}(a) \quad \text{and} \quad \Delta(b) = \sum_i a_{2i} \otimes b_{2i} + \Delta_{r_0}(b).$$

On the algebra J_i we define a comultiplication Δ_i as follows: $\Delta_1(a) = \sum_i a_{1i} \otimes b_{1i}$, $\Delta_2(b) = \sum_i a_{2i} \otimes b_{2i}$. Clearly, $J^* = J_1^* \oplus J_2^*$, and J_i^* is a subalgebra of J^*; moreover, J_i^* is the dual algebra for the coalgebra (J_i, Δ_i). Consider the subspace $B = J_1^* \oplus J_1 \oplus J_2$ of the algebra $D(J)$. The definition of multiplication in $D(J)$ implies that for any $f \in J_1^*$ and any $a \in J_1$, $b \in J_2$ we have $fb = -\sum_i \langle f, u_{1i}\rangle u_{2i} b$ and $f \cdot a = \sum_i (\langle f, a_{1i}\rangle b_{2i} + \langle f, u_{1i} a\rangle u_{2i})$. Consequently, B is a subalgebra of $D(J)$, and J_2 is an ideal of B. Clearly, the Drinfel′d double $D(J_1)$ of the bialgebra (J_1, Δ_1) is the factor B over the ideal J_2. Thus, $D(J_1)$ is a Jordan algebra, and the couple (J_1, Δ_1) is a Jordan bialgebra. In a similar way we show that (J_2, Δ_2) is a Jordan bialgebra. Since the algebra J_1 is simple, Lemma 6 yields the existence of an element $r_1 \in (\mathrm{id} - \tau)(J_1 \otimes J_1)$ such that $\Delta_1 = \Delta_{r_1}$ and $C_J(r_1) = 0$. Since the number of direct summands of the algebra J_2 is smaller than n, we may assume that $\Delta_2 = \Delta_{r_2}$ and that $C_J(r_2) = 0$ for some $r_2 \in (\mathrm{id} - \tau)(J_2 \otimes J_2)$.

Let $r_i = \sum_j a_{ij} \otimes b_{ij}$, where $i = 1, 2$ and $r = r_0 + r_1 + r_2$. Then $\tau(r) = -r$, and $\Delta(a) = \Delta_r(a)$ for any $a \in J$. We prove that $C_J(r) = 0$. Since $C_J(r_1) = 0$ and $C_J(r_2) = 0$, we have

$$\begin{aligned} C_J(r) = \Xi\Big(\sum_{ij} (u_{2i}u_{2j} \otimes u_{1i} \otimes u_{1j} + u_{1i}a_{1j} \otimes u_{2i} \otimes b_{1j} + a_{1j}u_{1i} \otimes b_{1j} \otimes u_{2i})\Big) \\ + \Xi\Big(\sum_{ij} (u_{1i}u_{1j} \otimes u_{2i} \otimes u_{2j} - u_{2i}a_{2j} \otimes u_{1i} \otimes b_{2j} - a_{2j}u_{2i} \otimes b_{2j} \otimes u_{1i})\Big), \end{aligned}$$

where $\Xi = \mathrm{id} + \xi + \xi^2$. Since the bialgebra (J, Δ) is Jordan, Lemma 2 shows that

$$\begin{aligned} C_J(r)(b \otimes a \otimes 1 + a \otimes b \otimes 1 + 1 \otimes 1 \otimes ab \\ - b'a' \otimes 1 \otimes 1 - 1 \otimes b'a' \otimes 1 - 1 \otimes 1 \otimes b'a') = 0 \end{aligned}$$

for any $a, b \in J$. In this identity we substitute $a = e_1, b = e_2$, obtaining $C_J(r)(e_1 \otimes e_2 \otimes 1 + e_2 \otimes e_1 \otimes 1) = 0$. Observe that in the expression for $C_J(r)$ the first term of the type $\Xi(\cdots)$ belongs to $J_2 \otimes J_1 \otimes J_1 + J_1 \otimes J_2 \otimes J_1 + J_1 \otimes J_1 \otimes J_2$, and the second term belongs to $J_2 \otimes J_2 \otimes J_1 + J_2 \otimes J_1 \otimes J_2 + J_1 \otimes J_2 \otimes J_2$. Therefore,

$$\begin{aligned} &\Xi\Big(\sum_{ij} (u_{2i}u_{2j} \otimes u_{1i} \otimes u_{1j} + u_{1i}a_{1j} \otimes u_{2i} \otimes b_{1j} + a_{1j}u_{1i} \otimes b_{1j} \otimes u_{2i})\Big) \\ &\times (e_1 \otimes e_2 \otimes 1 + e_2 \otimes e_1 \otimes 1) = 0. \end{aligned}$$

This implies the relations

$$\sum_{ij}(u_{2i}u_{2j} \otimes u_{1i} \otimes u_{1j} + u_{2i} \otimes b_{1j} \otimes u_{1i}a_{1j} + u_{2i} \otimes a_{1j}u_{1i} \otimes b_{1j}) = 0,$$

$$\sum_{ij}(u_{1i}a_{1j} \otimes u_{2i} \otimes b_{1j} + u_{1j} \otimes u_{2i}u_{2j} \otimes u_{1i} + b_{1j} \otimes u_{2i} \otimes a_{1j}u_{1i}) = 0,$$

$$\sum_{ij}(a_{1j}u_{1i} \otimes b_{1j} \otimes u_{2i} + u_{1i} \otimes u_{1j} \otimes u_{2i}u_{2j} + b_{1j} \otimes u_{1i}a_{1j} \otimes u_{2i}) = 0,$$

whence

$$\Xi(\sum_{ij}(u_{2i}u_{2j} \otimes u_{1i} \otimes u_{1j} + u_{1i}a_{1j} \otimes u_{2i} \otimes b_{1j} + a_{1j}u_{1i} \otimes b_{1j} \otimes u_{2i})) = 0.$$

In a similar way we get

$$\Xi(\sum_{ij}(u_{1i}u_{1j} \otimes u_{2i} \otimes u_{2j} - u_{2i}a_{2j} \otimes u_{1i} \otimes b_{2j} - a_{2j}u_{2i} \otimes b_{2j} \otimes u_{1i})) = 0.$$

Thus, $C_J(r) = 0$, which proves the theorem. $\square$

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