INVOLUTIONS ON COMPOSITION ALGEBRAS

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ABSTRACT. Involutions on composition algebras over rings where 2 is invertible are investigated. It is proved that there is a one-one correspondence between non-standard involutions of the first kind, and composition subalgebras of half rank. Every non-standard involution of the first kind is isomorphic to the natural generalization of Lewis's hat-involution [L]. Any involution of the second kind on a composition algebra C over a quadratic etale R-algebra S can be written as the tensor product of the standard involution of a unique R-composition subalgebra of C and the standard involution of S/R. The latter generalizes a well-known theorem of Albert on quaternion algebras with unitary involutions.

INTRODUCTION

It is common knowledge that octonion algebras possess a unique scalar involution which is usually called the *standard* (or *canonical*) involution. However, there seems to be nothing explicit in the literature about involutions on octonion algebras in general. The aim of this paper is to study these involutions.

We define involutions of the first and second kind on composition algebras. For rings where 2 is an invertible element, it turns out that non-standard involutions of the first kind are in one-one correspondence with the composition subalgebras of half rank. This observation, which implies that there exist composition algebras over rings which do not have any involution of the first kind other than the standard one, is a direct consequence of the fact that the reflections on composition algebras (i.e. the automorphisms of order two) are in one-one correspondence with the composition subalgebras of half the rank of the

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algebra. This is proved by Jacobson [J] in the special case where the base ring is a field. We conclude that every non-standard involution of the first kind on a composition algebra of rank 4 or 8 is isomorphic to the hat-involution.

Albert [A] proved that involutions of the second kind on a quaternion division algebra over a field are of a very special type: Let C be a quaternion division algebra with center a separable quadratic field extension F/K. If C has an involution of the second kind τ, then $C = C_0 \otimes_K F$ with C_0 a quaternion algebra over K stabilized by τ. The involution τ is the tensor product of the standard involutions of C_0 and F/K.

A close look at the proof of this result as given in [KMRT, (2.22)] shows that the most relevant feature of a quaternion algebra used is the fact that it possesses a unique scalar involution. Thus the result can easily be generalized to arbitrary (unital) algebras over a quadratic etale R-algebra S, which are finitely generated projective as S-modules, and have such a unique scalar involution σ. We thus obtain an analogous result for involutions of the second kind on any such algebra, and in particular, on generalized Cayley-Dickson algebras, as well as composition algebras.

The investigation of the hermitian level of composition algebras was one of the motivations of this paper (see [PU]).

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1. Preliminaries

Let R be a unital commutative associative ring. The term "R-algebra" always refers to unital nonassociative algebras which are finitely generated projective as R-modules. Assume that $a1_A = 0$ implies $a = 0$ for every $a \in R$ (cf. [M]).

An R-algebra A is called *quadratic* in case there exists a quadratic form $n\colon A \to R$ such that $n(1_A) = 1$ and $x^2 - n(1_A, x)x + n(x)1_A = 0$ for all $x \in A$, where $n(x, y)$ denotes the induced symmetric bilinear form $n(x, y) := n(x + y) - n(x) - n(y)$. The form n is uniquely determined and called the *norm* of the quadratic algebra A.

A map $\tau : A \to A$ is called an *involution* on A if it is an anti-automorphism of period 2, i.e. if $\tau(x+y) = \tau(x) + \tau(y)$, $\tau(xy) = \tau(y)\tau(x)$, and $\tau^2 = id$, for all $x, y \in A$. It is called *scalar* if all the *norms* $\tau(x)x$ are scalars in R, and hence by linearization all *traces* $\tau(x) + x$ are scalars in R. In that case, $n_A(x) = \tau(x)x$ resp. $t_A(x) = \tau(x) + x$ is a quadratic, resp. a linear form on A [M, p.86]. An involution τ on A is called *of the first kind* if $\tau|_R = id$.

Let S be a quadratic etale R-algebra with standard involution s_0, and let A be an algebra over S. An involution on A whose restriction to S is the standard involution is called *of the second kind*.

An R-algebra C is called a *composition algebra*, if it carries a quadratic form $n\colon C \to R$ satisfying the following two conditions:

(i) Its induced symmetric bilinear form $n(x, y)$ is nondegenerate, i.e. determines an R-module isomorphism $C \xrightarrow{\sim} C^\vee = \mathrm{Hom}_R(C, R)$.

(ii) n permits composition, that is, $n(xy) = n(x)n(y)$ for all $x, y \in C$.

A non-associative algebra is called *alternative* if its associator $[x, y, z] = (xy)z - x(yz)$ is alternating. Composition algebras are quadratic alternative algebras. More precisely, a quadratic form n of the composition algebra satisfying (i) and (ii) above agrees with its norm as a quadratic algebra and thus is unique. It is called the *norm* of the composition algebra C and is sometimes also denoted by n_C. A quadratic alternative algebra is a composition algebra if and only if its norm is nondegenerate [M, 4.6]. Composition algebras only exist in ranks 1, 2, 4 or 8. Those of rank 2 are exactly the quadratic etale R-algebras, those of rank 4 exactly the well-known quaternion algebras. The ones of rank 8 are also called *octonion algebras*.

A composition algebra C has a *standard involution* $\overline{}$ given by $\overline{x} = t(x)1_C - x$, where $t\colon C \to R$ is the *trace map* given by $t(x) := n(1_C, x)$. This involution is scalar, and $x + \overline{x} = t(x)1 \in R1$.

The Cayley-Dickson doubling process is a well-known way to construct new algebras with scalar involution out of a given one. Let A be an R-algebra with involution $*$ and let $\mu \in R$ be an element such that $\mu x = 0$ implies $x = 0$ in A. Then the R-module $A \oplus A$ becomes a unital algebra via the multiplication

$$(u,v)(u',v') = (uu' + \mu v'^* v, v'u + vu'^*)$$

for $u, u', v, v' \in A$, with involution

$$(u,v)^* = (u^*, -v).$$

It is called the *(classical) Cayley-Dickson doubling* of A, and denoted by $\mathrm{Cay}(A,\mu)$. The new involution $*$ is a scalar involution on $\mathrm{Cay}(A,\mu)$ if and only if $*$ is a scalar involution on A, with norm $n_{\mathrm{Cay}(A,\mu)}((u,v)) = n_A(u) - \mu n_A(v)$. The Cayley-Dickson doubling process depends on the scalar μ only up to an invertible square. By repeated application of the Cayley-Dickson doubling starting from a composition algebra C over R we obtain either again a composition algebra (if the rank of the new algebra is at most 8), or a *generalized Cayley-Dickson algebra* of rank $2^m \mathrm{rank}(C) \geq 16$. The latter are no longer alternative, but still flexible with a scalar involution [M].

Over fields, the classical Cayley-Dickson process indeed generates all possible composition algebras. Over rings, a more general version is required, which still does not always yield all possible composition algebras, only those containing a composition subalgebra of half its rank. This *generalized Cayley-Dickson doubling process* is due to Petersson [P]:

Let D be a composition algebra of rank ≤ 4 over R with standard involution $\bar{\ }$. Let P be a finitely generated projective right D-module of rank one, carrying a nondegenerate $\bar{\ }$-hermitian form $h\colon P \times P \to D$ (i.e., a biadditive map $h\colon P \times P \to D$ with $h(ws, w't) = \bar{s}h(w,w')t$ and $h(w,w') = \overline{h(w',w)}$ for all $s, t \in D$, $w, w' \in P$, and where $P \to \overline{P}^\vee$, $w \mapsto h(w, \cdot)$ is an isomorphism of right D-modules). Define a *norm* $N\colon P \to D$ by $N(w) = h(w,w)$ for $w \in P$. The R-module $A = D \oplus P$ becomes a new R-algebra by the multiplication

$$(u,w)(u',w') = (uu' + h(w',w), w' \cdot u + w \cdot \overline{u}')$$

for $u, u' \in D$, $w, w' \in P$, with $\cdot$ denoting the right D-module structure of P.

The algebra constructed above is denoted by $\mathrm{Cay}(D,P,N)$

$= \mathrm{Cay}(D, P, h)$. Its norm is given by the form $n((u, w)) = n_D(u) - N(w)$. D itself is standardly a (free) right D-module of rank one and "norm one" (cf. [P]). Any norm on D is similar to n_D (resp., any nondegenerate hermitian form $h\colon D \times D \to D$ is similar to the standard form given by the involution, i.e. to $(w, w') \mapsto \overline{w}'w$). In this special case the "classical" doubling process $\mathrm{Cay}(D, \mu) = \mathrm{Cay}(D, D, \mu n_D)$ with $\mu \in R$ is obtained.

2. Involutions of the first kind

Following Jacobson [J] a non-trivial automorphism f of a composition algebra C over a ring R is called a *reflection* if $f^2 = id$, i.e. if it has order two. The observation in [J, p.66] can be easily adapted to our more general situation:

2.1 Theorem *Let R be a ring where 2 is invertible, and C a composition algebra over R of rank r. There is a one-one correspondence between reflections on C and composition subalgebras of C of rank $\frac{r}{2}$:*
(i) Given a reflection $f \in Aut(C)$, the subalgebra $B = \{x \in C | f(x) = x\}$ of fixed elements with respect to f is a composition subalgebra of rank $\frac{r}{2}$, and

$$C \cong \mathrm{Cay}(B, P, N)$$

with

$$P = \{x \in C | f(x) = -x\}.$$

(ii) Given a composition subalgebra B of C of rank $\frac{r}{2}$, let P be the orthogonal complement of B with respect to the norm of C. Then $C = B \oplus P$ as R-modules, and

$$f : \mathrm{Cay}(B, P, N) \to C,$$

$$f(u + w) = u - w$$

for $u \in B$, $w \in P$ is a reflection on C.

Proof (i) Clearly, the subalgebra B is a quadratic alternative R- subalgebra of C with norm $n_B = n_C|_B : B \to R$. In the field case it is known that n_B is nondegenerate [J]. This implies the nondegeneracy of n_B also in our setting, since we may simply pass to the localisations

and to the residue class algebras for all $P \in \mathrm{Spec}(R)$ to prove the assertion. Hence B is a composition algebra. Moreover, B locally has rank $\frac{r}{2}$, since again this is true for fields of characteristic not 2 by [J].
(ii) This is the generalized Cayley-Dickson doubling process [P, 2.5]. □

2.2 Corollary *Let* 2 *be an invertible element in* R*, and* C *a composition algebra over* R *of rank* r*, which does not contain a composition algebra of rank* $\frac{r}{2}$*. Then there are no reflections on* C*.*

From now on assume that 2 is an invertible element in R. Let τ be an involution on A of the first kind which is *not* the standard involution $^{-}$. Then $\tau \circ {}^{-} = {}^{-} \circ \tau$. Thus the assigments $\tau \to \tau \circ {}^{-}$ and $f \to f \circ {}^{-}$ give inverse bijections between the involutions of C of the first kind and the automorphisms of C of period 2. We obtain the following result as an immediate corollary from the above theorem:

2.3 Theorem *Let* R *be a ring where* 2 *is invertible, and* C *a composition algebra over* R *of rank* r*. There is a one-one correspondence between the non-standard involutions on* C *of the first kind and the composition subalgebras of rank* $\frac{r}{2}$*:*
(i) Given a non-standard involution τ *on* C *of the first kind, then*

$$C \cong \mathrm{Cay}(B, P, N)$$

is the Cayley-Dickson doubling of a composition algebra, with

$$B = \{x \in C | \tau(x) = \bar{x}\},$$

and

$$P = \{x \in C | \tau(x) = -\bar{x}\}.$$

(ii) Given a composition subalgebra B *of* C *of rank* $\frac{r}{2}$*, we obtain a reflection* $f : \mathrm{Cay}(B, P, N) \to C$ *and an involution* $\tau = f \circ {}^{-}$ *of the first kind on* C*.*

2.4 Corollary *Let* 2 *be an invertible element in* R*, and* C *a composition algebra over* R *of rank* r*, which does not contain a composition algebra of rank* $\frac{r}{2}$*. Then there are no involutions on* C *of the first kind other that the standard one.*

Examples of such algebras can be found in [KPS1], where octonion algebras over polynomial rings in two indeterminates are constructed, which only have the ring itself as a composition subalgebra. See [Pu] for examples on quaternion algebras.

The hat-involution is defined by Lewis [L], for quaternion algebras over fields of characteristic not two. This definition is generalized as follows.

2.5 Proposition *Let R be an arbitrary ring. Let C be a composition algebra over R of rank greater that 2 which is the Cayley-Dickson doubling of a composition algebra D of half its rank, i.e. $C = \mathrm{Cay}(D, P, N)$. Let $^{-}$ be the standard involution of D. Then*

$$(u, w)\hat{} = (\bar{u}, w)$$

is an involution on C, which is not scalar.

The proof is a straightforward computation which uses in particular that the multiplication on C is of the form

$$(u, w)(u', w') = (uu' + h(w', w), w' \cdot u + w \cdot \overline{u'})$$

for $u, u' \in D$ and $w, w' \in P$, and that $h : P \times P \to D$ satisfies the identity $h(wu, wv) = h(w, w)\bar{v}u$. We call this involution the *hat-involution.* If C is a quaternion algebra, then $Sym(C, \hat{}\,) = \{x \in C | \bar{x} = x\}$ is a projective R-module of constant rank 3, and $Skew(C, \hat{}\,) = \{x \in C | \bar{x} = -x\}$ is a projective R-module of constant rank 1, if C is an octonion algebra then $Sym(C, \hat{}\,)$ is projective of constant rank 5, and $Skew(C, \hat{}\,)$ projective of rank 3.

2.6 Corollary *Let R be a ring where 2 is invertible, and C a composition algebra over R with non-standard involution τ of the first kind. Then*

$$C \cong \mathrm{Cay}(B, P, N)$$

and

$$\tau((u, v)) = (\bar{u}, v)$$

is the hat-involution on C.

3. Involutions of the second kind

Let A be an S-algebra with a scalar involution σ of the first kind, with norm $n_A : A \to S$, $n_A(x) = \sigma(x)x$. This scalar involution is unique, since A is a quadratic S-algebra whose norm n_A is uniquely determined [P, 1.2]. Let τ be an involution of the second kind on A. Take an element $x \in A$. Let $y = \tau(x)$. Then $x(\tau \circ \sigma \circ \tau(x)) = \tau(\tau(x))\tau(\sigma(\tau(x))) = \tau(y)\tau(\sigma(y)) = \tau(\sigma(y)y) = s_0(n_A(y)) \in S \cdot 1$ with n_A the norm of A and hence also $x + \tau \circ \sigma \circ \tau(x) = s_0(t_A(y)) \in S \cdot 1$. Therefore, $\tau \circ \sigma \circ \tau$ is a scalar involution on C with $\tau \circ \sigma \circ \tau|_S = \mathrm{id}$. This implies [M, (1.1)]

$$(1) \qquad x^2 - t_1(x)x + n_1(x)1 = 0 \quad , \quad n_1(1_C) = 1,$$

with $t_1(x) := x + \tau \circ \sigma \circ \tau(x)$, $n_1(x) := x(\tau \circ \sigma \circ \tau(x))$, i.e., (A, n_1) is a quadratic algebra. Since the norm n_A of the quadratic algebra A is uniquely determined by condition (1), it must agree with the norm n_1 on A. It follows that $\tau \circ \sigma \circ \tau = \sigma$. so τ and σ commute, and $\tilde{\sigma} = \tau\sigma$ is a descent map over R, for the Galois group $Gal(S/R) = \mathbb{Z}/2\mathbb{Z}$ (viewed as group schemes over R). The R-subalgebra

$$A_0 = \{x \in A \mid \tau\sigma(x) = x\}$$

of A is the descent of $\tilde{\sigma}$, and the map

$$\phi : A_0 \otimes_R S \to A, \qquad x \otimes a \to ax$$

is an S-algebra isomorphism (see [KPS2] for a short description of descent, or [K]). A_0 is a quadratic R-algebra with unique norm $n_{A_0} = n_A|_{A_0}$ and unique scalar involution $\sigma_0 = \sigma|_{A_0}$. Obviously, we have $n_{A_0} \otimes id_S = n_A$. Since these conditions imply that every element in A_0 is invariant under $\tau \circ \sigma$, the algebra A_0 is uniquely determined by τ. Moreover, ${}^{\tau}A = A$.

A straightforward computation shows that, indeed, $\tau = \sigma_0 \otimes s_0$. We thus proved that involutions of the second kind on algebras with a scalar involution behave in a special way as follows:

3.1 Proposition *Let S be a quadratic etale R-algebra, and A a (quadratic) S-algebra with a scalar involution σ of the first kind. Let τ be any involution on A of the second kind. Then there exists a unique* *quadratic R-subalgebra A_0 of A with a unique scalar involution σ_0 such that*

$$A = A_0 \otimes_R S \quad \textit{and} \quad \tau = \sigma_0 \otimes s_0.$$

The algebra A_0 is uniquely determined by these conditions.

Note that the following converse of this result is obvious: If $A = A_0 \otimes_R S$, where A_0 is a quadratic R-algebra with a scalar involution σ_0 of the first kind, then there is exactly one involution τ on A of the second type such that $\tau|_{A_0} = \sigma_0$ is the scalar involution.

Moreover, the proof of the above proposition only requires the existence of an involution of the first and one of the second kind which commute.

As an immediate application of the above result it turns out that involutions of the second kind on composition algebras, and more generally, on generalized Cayley-Dickson algebras, are well behaved.

3.2 Theorem *Suppose 2 is invertible in R. Let S be a quadratic etale R-algebra, C a composition algebra over S of rank ≥ 4, and $A = \mathrm{Cay}(C, \mu_1, .., \mu_m)$ a generalized Cayley-Dickson algebra of* rank $A = 2^m$rank $C \geq 16$ *(so A is a noncommutative Jordan algebra). Let $\sigma = {}^-$ denote the scalar involution σ on A. Let τ be any involution on A of the second kind.*
(i) There exists a unique flexible quadratic R-subalgebra A_0 of A with a unique scalar involution σ_0 such that

$$A = A_0 \otimes_R S \quad \textit{and} \quad \tau = \sigma_0 \otimes s_0.$$

The algebra A_0 is uniquely determined by these conditions.
(ii) Let S/R be faithfully flat. There exists a unique composition R-subalgebra C_0 of C such that

$$C = C_0 \otimes_R S \quad \textit{and} \quad \tau = \sigma_0 \otimes s_0.$$

The algebra C_0 is uniquely determined by these conditions.

This generalizes a well-known theorem by Albert [KMRT, I.2.(2.22)] for quaternion algebras over fields.

Proof (i) is obvious, since the algebra A itself is flexible by [M, 6.8(iv)].

(ii) C_0 certainly is a quadratic alternative algebra with multiplicative norm n_{C_0}. The norm n_{C_0} satisfies $n_{C_0} \otimes_R id_S = n_C$ and is nondegenerate since S/R is faithfully flat. □

Note the following: If rank $C = 4$ then C_0 must be a quaternion algebra, for any quadratic etale algebra S/R (cf. [KPS, p.79]).

References

[A] Albert, A.A.,"Structure of algebras", AMS Colloq. Publ. 24, Providence, RI.

[J] Jacobson, N., *Composition algebras and their automorphisms*, Rend. Circ. Mat. Palermo **7**, 55–80 (1958).

[K] Knus, M.A., Ojanguren, M., Theorie de la descente et Algebres d'Azumaya, Lec. Notes in Math. 389, Springer-Verlag, Berlin-Heidelberg-New York.

[KPS1] Knus, M., Parimala, R., Srinivas, V., *On compositions and triality*, J. Reine Angew. Math. **457**, 45–70 (1994).

[KPS2] Knus, M, Parimala, R., Srinivas, V., *Azumaya Algebras with Involutions*, J. Algebra **130**, 65–82 (1990).

[L] Lewis, D., *Sums of hermitian squares*, J. Algebra **115**, 466–480 (1988).

[M] McCrimmon, K., *Nonassociative algebras with scalar involution*, Pacific J. of Math. **116** (1), 85–108 (1985).

[MPP] Morandi, P.J., Perez-Izquierdo, J.M., Pumpluen, S., *On the tensor product of composition algebras*, J. Algebra **243**, 41–68 (2001).

[KMRT] Knus, M.A., Merkurjev, A., Rost, M., Tignol, J.-P., "The Book of Involutions", AMS Coll. Publications, Vol.44 (1998).

[P] Petersson, H., *Composition algebras over algebraic curves of genus 0*, Trans. Amer. Math. Soc. **337**, 473–491 (1993).

[Pu] Pumpluen, S., *Quaternion algebras over elliptic curves*, Comm. Algebra **26** (12), 4357–4373 (1998).

[PU] Pumpluen, S., Unger, T., *The hermitian level of composition algebras*, submitted.

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