A CONSTRUCTION METHOD FOR ALBERT ALGEBRAS OVER ALGEBRAIC VARIETIES

S. PUMPLÜN

Abstract. Let X be an integral proper scheme such that $2, 3 \in H^0(X, \mathcal{O}_X)$. We construct cubic Jordan algebras and, in particular, Albert algebras, over X by providing the space of trace zero elements of a quartic Jordan algebra over X with a new multiplication, generalizing a construction by B. N. Allison and J. R. Faulkner. In the process, we construct admissible cubic algebras and pseudo-composition algebras over X. Results on the structure of these algebras are obtained. Examples of admissible cubic algebras, Albert algebras and pseudo-composition algebras are constructed over elliptic curves.

Introduction

In his PhD thesis [Ach1], Achhammer developed a generalized Tits process for algebras over locally ringed spaces which, specialized to algebras over rings, generalized the classical Tits process and, in particular, the classical first and second Tits constructions. The results of his thesis were only partly published in [Ach2].

Independently, Albert algebras over integral schemes were investigated by Parimala-Sridharan-Thakur [Pa-S-T1, 2]. In [Pa-S-T2], generalized first and second Tits constructions for Albert algebras over a domain R with $2, 3 \in R^\times$ were introduced. Furthermore, a family of non-isomorphic Albert algebras over $\mathbb{A}^2_k$ was constructed who arise neither from a first nor second Tits construction, proving that the situation over integral schemes (resp., domains) differs from the one over fields. Albert algebras over locally ringed spaces and in particular, over curves of genus zero and one, were further studied in [Pu3, 4], with an emphasis on those arising from a first Tits construction. In this paper we will introduce a different construction for Albert algebras over integral schemes (resp., over domains).

How to obtain separable cubic Jordan algebras over fields of characteristic not 2 or 3 as the space of trace zero elements of separable Jordan algebras of degree 4 was first described by Allison-Faulkner [A-F, 5.4]. Indeed, every separable cubic Jordan algebra over a field of characteristic not 2 or 3 can be obtained this way [A-F, 5.6]. More recently, this idea was discussed again in a paper by Elduque-Okubo [E-O] on admissible cubic algebras: over fields of characteristic not 2 or 3, admissible cubic algebras are related to cubic and quartic Jordan algebras, as well as to pseudo-composition algebras. Pseudo-composition algebras over fields were also investigated by Meyberg-Osborn [M-O] and Röhrl-Walcher [R-W]. A

Date: 11.5.2008.

2000 *Mathematics Subject Classification.* Primary: 17C50; Secondary: 17C40, 17A30.

Key words and phrases. Albert algebras, Jordan algebras.

variety containing both cubic Jordan algebras and pseudo-composition algebras was studied by Hentzel-Peresi [H-Pe].

In this paper, we generalize parts of the theory developed in [E-O] to algebras over integral schemes X with $2, 3 \in H^0(X, \mathcal{O}_X^\times)$, with a special emphasis on cubic and quartic Jordan algebras over X with separable, and in particular central simple, residue class algebras of degree 4 (although most results also hold in a more general setting). Our main interest is to construct examples of admissible cubic algebras, cubic Jordan algebras and pseudo-composition algebras with interesting underlying module structures. In particular, we obtain examples of Albert algebras out of quartic Jordan algebras over X, which are the symmetric elements of an Azumaya algebra of constant rank 64 over X, which carries a symplectic involution. In order to keep the paper within a reasonable length, we mainly generalize the parts of [E-O] needed for this endeavor, although most of the theory developed in [E-O] can be easily transferred to our more general setting.

The method used in this paper suggests how to achieve a list or even a partial classification of those admissible cubic algebras (resp., cubic Jordan algebras, pseudo-composition algebras) over integral proper schemes X, which can be obtained from quartic Jordan algebras. We give a series of examples. We restrict our investigation to $\mathcal{O}_X$-algebras which are locally free of (automatically constant) finite rank as an $\mathcal{O}_X$-module and put a special emphasis on algebras over elliptic curves.

One should point out that, despite the fact that cubic Jordan algebras over curves, and in particular the exceptional ones (the Albert algebras) among them, are certainly interesting in their own right, the methods developed here can be easily translated into the setting of Jordan algebras over base rings. The results on the module structures which appear for the cubic Jordan algebras over curves also serve as examples on what underlying module structures can appear when studying admissible cubic algebras, cubic Jordan algebras, and pseudo-composition algebras over domains. While it is unlikely that all Albert algebras over X can be constructed with the method employed here, it is nonetheless a good way to find many interesting examples of these algebras and their underlying vector bundles, with relatively little computational effort.

The contents of the paper are as follows. After a preliminary section, where the terminology and some basic facts are collected, we develop some general theory for admissible algebras and quartic Jordan algebras over locally ringed spaces in Section 2, explain the relation between them and admissible cubic algebras, and deal with the existence of idempotents in the algebras we study. In Section 3, pseudo-composition algebras and cubic Jordan algebras are constructed from the trace zero elements of a quartic Jordan algebra over an integral scheme. Azumaya algebras over X of rank 16 (with orthogonal involution) are used to construct examples of quartic Jordan algebras, admissible cubic algebras pseudo-composition algebras and cubic Jordan algebras in Section 4 and 5. The resulting cubic Jordan algebras have central simple residue class algebras. In Section 6, we construct Albert algebras out of certain Azumaya algebras of rank 64 with symplectic involutions.

We then apply our results to algebras over elliptic curves, since the vector bundles of elliptic curves are well-known, interesting, and display a particularly nice behaviour. In

Section 7 we construct examples of admissible cubic algebras and cubic Jordan algebras (whose residue class algebras are central simple) of rank 9 and 15 and pseudo-composition algebras of ranks 8 and 14 over an elliptic curve. In the process, we find admissible cubic algebras over elliptic curves whose underlying $\mathcal{O}_X$-module is indecomposable (Theorem 9). Given a quartic Jordan algebra of rank 28 of the type $\mathcal{J} = H(\mathcal{A}, \tau)$ with τ a symplectic involution on an Azumaya algebra $\mathcal{A}$ of rank 64, we construct examples of admissible cubic algebras of rank 27, Albert algebras and pseudo-composition algebras of rank 26 over an elliptic curve in Section 8.

We use the standard terminology from algebraic geometry, see Hartshorne's book [H] and the one for algebras developed in [P1]. For the standard terminology on Jordan algebras, the reader is referred to the books by McCrimmon [McC], Jacobson [J] and Schafer [S]. In the following, let $(X, \mathcal{O}_X)$ be a locally ringed space such that $2, 3 \in H^0(X, \mathcal{O}_X^\times)$, let R be a ring such that $2, 3 \in R^\times$ and k a field of characteristic not 2 or 3.

1. Preliminaries

1.1. **Algebras over X.** For $P \in X$ let $\mathcal{O}_{P,X}$ be the local ring of $\mathcal{O}_X$ at P and m_P the maximal ideal of $\mathcal{O}_{P,X}$. The corresponding residue class field is denoted by $k(P) = \mathcal{O}_{P,X}/m_P$. For an $\mathcal{O}_X$-module $\mathcal{F}$ the stalk of $\mathcal{F}$ at P is denoted by $\mathcal{F}_P$. $\mathcal{F}$ is said to have *full support*, if $\operatorname{Supp}\mathcal{F} = X$; i.e., if $\mathcal{F}_P \neq 0$ for all $P \in X$. We call $\mathcal{F}$ *locally free of finite rank* if for each $P \in X$ there is an open neighborhood $U \subset X$ of P such that $\mathcal{F}|_U = \mathcal{O}_U^r$ for some integer $r \geq 0$. The *rank* of $\mathcal{F}$ is defined to be $\sup\{\operatorname{rank}_{\mathcal{O}_{P,X}} \mathcal{F}_P \,|\, P \in X\}$.

In the following, the term "$\mathcal{O}_X$-algebra" (or "algebra over X") refers to a not necessarily associative $\mathcal{O}_X$-algebra $\mathcal{A}$ which is locally free of finite rank as $\mathcal{O}_X$-module. A unital $\mathcal{O}_X$-algebra $\mathcal{A}$ is called *alternative* if $x^2y = x(xy)$ and $yx^2 = (yx)x$ for all sections x, y of $\mathcal{A}$ over the same open subset of X. A unital algebra $\mathcal{A}$ over $\mathcal{O}_X$ is called *separable* if $\mathcal{A}(P)$ is a separable $k(P)$-algebra for all $P \in X$. A global section $f \in H^0(X, \mathcal{A})$ such that $f^2 = f$ is called an *idempotent* of $\mathcal{A}$. Every unital algebra A over a base ring R which is finitely generated projective with full support is faithful; i.e., for $r \in R$, $rA = 0$ implies $r = 0$. If $\mathcal{A}$ has full support, then $\mathcal{A}(U)$ is finitely generated projective and faithful, for each open set $U \subset X$.

An associative unital $\mathcal{O}_X$-algebra $\mathcal{A}$ is called an *Azumaya algebra* if $\mathcal{A}_P \otimes_{\mathcal{O}_{P,X}} k(P)$ is a central simple algebra over $k(P)$ for all $P \in X$ [Kn].

1.2. **Involutions.** Let $\mathcal{A}$ be a unital, not necessarily associative, $\mathcal{O}_X$-algebra. An anti-automorphism $\sigma : \mathcal{A} \to \mathcal{A}$ of order 2 is called an *involution* on $\mathcal{A}$. Define $H(\mathcal{A}, \sigma) = \{a \in \mathcal{A} \,|\, \sigma(a) = a\}$ and $\operatorname{Skew}(\mathcal{A}, \sigma) = \{a \in \mathcal{A} \,|\, \sigma(a) = -a\}$. Then $\mathcal{A} = H(\mathcal{A}, \sigma) \oplus \operatorname{Skew}(\mathcal{A}, \sigma)$. If $\mathcal{A}$ is an Azumaya algebra, σ is called *of the first kind*, if $\sigma|_{\mathcal{O}_X} = id$. An involution σ of the first kind is called an *orthogonal* involution if the induced involutions $\sigma(P)$ on the residue class algebras $\mathcal{A}(P)$ are all orthogonal, and a *symplectic* involution if they are all of symplectic type.

Let X be a k-scheme and l/k a separable quadratic field extension. Let $\mathcal{A}$ be an Azumaya algebra over $X' = X \times_k l$ together with an involution σ. Then σ is called *of the second kind* if $\sigma|_{\mathcal{O}_{X'}} = \iota$, where $\iota : \mathcal{O}_{X'} \to \mathcal{O}_{X'}$ is the canonical extension of the non-trivial Galois automorphism ι of l/k.

1.3. **Forms of higher degree over X.** Let d be a positive integer. Let $\mathcal{M}$, $\mathcal{N}$ be two $\mathcal{O}_X$-modules which are locally free of finite rank. When talking about maps of degree d, we will always assume that $d! \in H^0(X, \mathcal{O}_X^\times)$. A *map of degree* d over X is a map $N : \mathcal{M} \to \mathcal{N}$ such that $N(ax) = a^d N(x)$ for all sections a in $\mathcal{O}_X$, x in $\mathcal{M}$, where the map

$$\theta : \mathcal{M} \times \cdots \times \mathcal{M} \to \mathcal{N}$$

defined by

$$\theta(x_1, \ldots, x_d) = \frac{1}{d!} \sum_{1 \le i_1 < \cdots < i_l \le d} (-1)^{d-l} N(x_{i_1} + \cdots + x_{i_l})$$

is a d-linear map over $\mathcal{O}_X$ (the range of summation of l being $1 \le l \le d$). Obviously, $N(x) = \theta(x, \ldots, x)$ for all sections x of $\mathcal{M}$ over the same open subset of X. We canonically identify a map of degree d and its associated symmetric d-linear map θ.

If $\mathcal{N} = \mathcal{O}_X$, then a map of degree d is called a *form of degree* d, θ is called the *symmetric* d*-linear form* associated with N and $(\mathcal{M}, \theta)$ a d*-linear space.*

A form $N : \mathcal{M} \to \mathcal{O}_X$ of degree d on a locally free $\mathcal{O}_X$-module of finite rank with full support is called *nondegenerate* if $N(P) : \mathcal{M}(P) \to k(P)$ is nondegenerate in the classical sense for all $P \in X$. This means that the residue maps $\theta' \otimes k(P)$ of the map

$$\theta' : \mathcal{M} \to \mathcal{H}om_X(\mathcal{M} \otimes \cdots \otimes \mathcal{M}, \mathcal{O}_X)$$

($(d-1)$-copies of $\mathcal{M}$) defined by

$$x_1 \to \theta_{x_1}(x_2 \otimes \cdots \otimes x_d) = \theta(x_1, x_2, \ldots, x_d).$$

are injective for all $P \in X$. This is equivalent to saying that θ' is an isomorphism of $\mathcal{M}$ onto a direct sumand of $\mathcal{H}om_X(\mathcal{M} \otimes \cdots \otimes \mathcal{M}, \mathcal{O}_X)$. This concept of nondegeneracy is invariant under base change.

Two d-linear spaces $(\mathcal{M}_i, \theta_i)$, $i = 1, 2$ are called *isomorphic* (written $(\mathcal{M}_1, \theta_1) \cong (\mathcal{M}_2, \theta_2)$ or just $\theta_1 \cong \theta_2$) if there exists an $\mathcal{O}_X$-module isomorphism $f : \mathcal{M}_1 \to \mathcal{M}_2$ such that $\theta_2(f(x_1), \ldots, f(x_d)) = \theta_1(x_1, \ldots, x_d)$ for all sections $x_1, \ldots, x_d$ of $\mathcal{M}_1$ over the same open subset of X.

1.4. **Azumaya algebras over X.** Let $d! \in H^0(X, \mathcal{O}_X^\times)$. Let $\mathcal{A}$ be an Azumaya algebra over X of constant rank d^2. Then there exists a nondegenerate form $n_{\mathcal{A}} : \mathcal{A} \to \mathcal{O}_X$ of degree d on $\mathcal{A}$, such that $n_{\mathcal{A}}(xy) = n_{\mathcal{A}}(x) n_{\mathcal{A}}(y)$ for all sections x, y over the same open set of X and $n_{\mathcal{A}}(1) = 1$. Let θ be the d-linear form associated to $n_{\mathcal{A}}$. For $i = 1, \ldots, d$ define the form $t_i : \mathcal{A} \to \mathcal{O}_X$ of degree i via

$$t_i(x) = \binom{d}{i} \theta(x, \ldots, x, 1, \ldots, 1) \quad (i\text{-times } x).$$

Then $n_{\mathcal{A}}(x) = t_d(x)$ for all sections x, y of $\mathcal{A}$ over the same open subset of X.

The linear form $t_{\mathcal{A}} : \mathcal{A} \to \mathcal{O}_X$, $t_{\mathcal{A}}(x) = t_1(x) = d\theta(x, 1, \dots, 1)$ is called the *trace*. Put $s_{\mathcal{A}} : \mathcal{A} \to \mathcal{O}_X$, $s_{\mathcal{A}}(x) = t_2(x)$. Obviously, $t_{\mathcal{A}}(a1) = da$ for all a in $\mathcal{O}_X$. Define $\mathcal{A}_0 = \ker t_{\mathcal{A}}$. We have $\mathcal{A} = \mathcal{O}_X\, 1 \oplus \mathcal{A}_0$. Every section x of $\mathcal{A}$ over the same open subset of X satisfies

$$x^d - t_{\mathcal{A}}(x)x^{d-1} + s_{\mathcal{A}}(x)x^{d-2} - t_3(x)x^{d-3} + \cdots + (-1)^d n_{\mathcal{A}}(x)1 = 0.$$

The symmetric bilinear form $t_{\mathcal{A}}(x, y) = t_{\mathcal{A}}(xy)$ on $\mathcal{A}$ is associative and nondegenerate [Pu5].

1.5. **Composition algebras over X.** Following [P1], a unital $\mathcal{O}_X$-algebra $\mathcal{C}$ is called a *composition algebra* over X, if it has full support and if there exists a quadratic form $N : \mathcal{C} \to \mathcal{O}_X$ such that the induced symmetric bilinear form $N(u, v) = N(u + v) - N(u) - N(v)$ is *nondegenerate*; i.e., it determines a module isomorphism $\mathcal{C} \xrightarrow{\sim} \mathcal{C}^\vee = \mathcal{H}om(\mathcal{C}, \mathcal{O}_X)$, and such that $N(uv) = N(u)N(v)$ for all sections u, v of $\mathcal{C}$ over the same open subset of X. (This definition of associated bilinear form is equivalent to the one introduced in 1.3 since $2 \in H^0(X, \mathcal{O}_X^\times)$.)

N is uniquely determined, called the *norm* of $\mathcal{C}$ and is denoted by $N_{\mathcal{C}}$. Composition algebras over X exist only in ranks 1, 2, 4 or 8. A composition algebra of constant rank 2 (resp. 4 or 8) is called a *quadratic étale algebra* (resp. *quaternion algebra* or *octonion algebra*). Every composition algebra has a *trace* $T_{\mathcal{C}}(x) = N_{\mathcal{C}}(x, 1)$ and a *canonical involution* $\bar{x} = N_{\mathcal{C}}(x, 1)1 - x$. This involution is of the first kind, since $\bar{\ }|_{\mathcal{O}_X} = id$. A composition algebra over X of constant rank is called *split* if it contains a composition subalgebra isomorphic to $\mathcal{O}_X \oplus \mathcal{O}_X$. There are several construction methods for composition algebras over locally ringed spaces, e.g., the Cayley-Dickson process $\mathrm{Cay}(\mathcal{D}, \mathcal{P}, N_{\mathcal{P}})$ introduced in [P1]. Every quaternion algebra $\mathcal{D}$ over X can be built out of a locally free $\mathcal{O}_X$-module $\mathcal{M}$ of constant rank 3 carrying a nondegenerate quadratic form q with trivial determinant. We write $\mathcal{D} = \mathrm{Quat}(\mathcal{M}, q)$ and have $\mathcal{D} = \mathcal{O}_X \oplus \mathcal{M}$ as $\mathcal{O}_X$-module, cf. [Pu1, 2.7].

Let $\mathcal{D}$ be a quaternion algebra over X. Its canonical involution is symplectic. If $\sigma \neq \bar{\ }$ is another involution of the first kind then $\mathcal{D} \cong \mathrm{Cay}(\mathcal{T}, \mathcal{P}, N)$, where $\mathcal{T} = \{x \in \mathcal{D} \,|\, \sigma(x) = \bar{x}\}$, is a quadratic étale algebra over X. Thus, if $\mathcal{D}$ does not contain a quadratic étale subalgebra, the canonical involution is its only involution of the first kind. Suppose $\mathcal{D} = \mathrm{Cay}(\mathcal{T}, \mathcal{P}, N)$ is a Cayley-Dickson doubling. Then the hat-involution $(u, v)\hat{\ } = (\bar{u}, v)$ with $u \in \mathcal{T}$, $v \in \mathcal{P}$ is an orthogonal involution on $\mathcal{D}$ and $H(\mathcal{D}, \hat{\ }) = \mathrm{Sym}(\mathcal{D}, \hat{\ }) = \mathcal{O}_X \oplus \mathcal{P}$. Up to isomorphism, this is the only orthogonal involution on $\mathcal{D}$ [Pu2].

1.6. **Jordan algebras over X.** (cf. [Ach1, 1.7 ff.] or [Pu3, 4])

Let $\mathcal{J}$ be an $\mathcal{O}_X$-module. $(\mathcal{J}, U, 1)$ with $1 \in H^0(X, \mathcal{J})$ is a *(unital quadratic) Jordan algebra* over X if:

(1) The U-operator $U : \mathcal{J} \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{J}), x \to U_x$ is a quadratic map;
(2) $U_1 = id_{\mathcal{J}}$;
(3) $U_{U_x(y)} = U_x \circ U_y \circ U_x$ for all sections x, y in $\mathcal{J}$;
(4) $U_x \circ U_{y,z}(x) = U_{x, U_x(z)}(y)$ for all sections x, y, z in $\mathcal{J}$;
(5) for every commutative associative $\mathcal{O}_X$-algebra $\mathcal{O}'_X$, $\mathcal{J} \otimes \mathcal{O}'_X$ satisfies (3) and (4).

We write $\mathcal{J}$ instead of $(\mathcal{J}, U, 1)$.

Jordan algebras are invariant under base change: If $\sigma : X' \to X$ is a morphism of locally ringed spaces and $(\mathcal{J}, U, 1)$ a Jordan algebra over X then $\sigma^*(\mathcal{J}, U, 1) = (\sigma^*\mathcal{J}, \sigma^*U, 1)$ is a Jordan algebra over X'.

Let $\mathcal{A}$ be a unital associative algebra over X. Then $\mathcal{A}^+ = (\mathcal{A}, U, 1)$ with $U : \mathcal{A} \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{A}), x \to U_x(y) = xyx$, $1 = 1_{\mathcal{A}}$, is a Jordan algebra over X.

An $\mathcal{O}_X$-algebra $\mathcal{J}$ is called an *Albert algebra* if $\mathcal{J}(P) = J_P \otimes k(P)$ is an Albert algebra over $k(P)$ for all $P \in X$.

If $\mathcal{J}$ is a Jordan algebra over a scheme $(X, \mathcal{O}_X)$, then $\mathcal{J}$ is an Albert algebra if and only if there is a covering $V_i \to X$ in the flat topology on X such that $\mathcal{J} \otimes \mathcal{O}_{V_i} \cong H_3(\mathrm{Zor}(\mathcal{O}_{V_i}))$, where $\mathrm{Zor}(\mathcal{O}_{V_i})$ denotes the octonion algebra of Zorn vector matrices over $\mathcal{O}_{V_i}$ and $H_3(\mathrm{Zor}(\mathcal{O}_{V_i}))$ is the reduced Jordan algebra of 3-by-3 hermitian matrices with entries in $\mathrm{Zor}(\mathcal{O}_{V_i})$ and scalars $\mathcal{O}_{V_i}$ on the diagonal [Ach1, 1.10]. This terminology is compatible with the one used in [Pa-S-T1], cf. [P2, Section 2].

With the usual definition of quadratic Jordan algebras over rings [J, 1.3.4] we have: let $\mathcal{J}$ be a locally free $\mathcal{O}_X$-module of finite rank, $1 \in H^0(X, \mathcal{J})$ and $U : \mathcal{J} \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{J}), x \to U_x$ a quadratic map. $(\mathcal{J}, U, 1)$ is a Jordan algebra over X if and only if $(\mathcal{J}(V), U(V), 1|_V)$ is a Jordan algebra over $\mathcal{O}_X(V)$ for all open subsets $V \subset X$, if and only if $(\mathcal{J}_P, U_P, 1_P)$ is a Jordan algebra over $\mathcal{O}_{P,X}$ for all $P \in X$.

There is a canonical equivalence between the category of Jordan algebras over the affine scheme $Z = \operatorname{Spec} R$ which are locally free as $\mathcal{O}_X$-modules and the category of Jordan algebras over R which are finitely generated projective as R-modules given by the global section functor $J \longrightarrow H^0(Z, J)$ and the functor $J \longrightarrow \widetilde{J}$.

Let X be a scheme over the affine scheme $Y = \operatorname{Spec} R$ and suppose $H^0(X, \mathcal{O}_X) = R$. Then a Jordan algebra (resp., an Azumaya algebra) $\mathcal{J}$ over X is defined over R provided it is globally free as an $\mathcal{O}_X$-module [Pu3, Lemma 2].

Let $\mathcal{A}$ be an Azumaya algebra over X of constant rank d^2 as in 1.4. If σ is an involution on $\mathcal{A}$ of orthogonal type, then every section x of the Jordan algebra $\mathcal{J} = H(\mathcal{A}, \sigma)$ over the same open subset of X satisfies

$$x^d - t_{\mathcal{A}}(x)x^{d-1} + s_{\mathcal{A}}(x)x^{d-2} - \cdots + (-1)^d n_{\mathcal{A}}(x)1 = 0$$

and $t = t_{\mathcal{A}}$ is the trace of $\mathcal{J}$, $n = n_{\mathcal{A}}$ the norm of $\mathcal{J}$.

If σ is an involution on $\mathcal{A}$ of symplectic type, then the Jordan algebra $\mathcal{J} = H(\mathcal{A}, \sigma)$ has trace $t = \frac{1}{2}t_{\mathcal{A}}$ and its norm satisfies $n(x)^2 = n_{\mathcal{A}}(x)$. In both cases, the symmetric bilinear trace form $t_{\mathcal{J}}(x, y) = t(x \cdot y)$ is associative.

$\mathcal{J}$ is called *separable*, if the residue class algebra $\mathcal{J}(P)$ over $k(P)$ is a separable algebra for every $P \in X$. We will eventually restrict our investigations to separable Jordan algebras which are locally free of finite constant rank over X.

Remark 1. If J is a Jordan algebra over R, finitely generated projective as R-module, such that there exist a cubic, a quadratic and a linear form n, s and t from J to R satisfying

$$x^3 - t(x)x^2 + s(x)x - n(x)1 = 0$$

for all $x \in J$, and such that for each $P \in \operatorname{Spec} R$, there exists an element $u \in J \otimes_R k(P)$ such that $1, u, u^2$ are linearly independent over $k(P)$, then the cubic, quadratic and linear form n, s and t are uniquely determined by the equation above [Ach1, 1.12].

Given a (cubic) Jordan algebra $\mathcal{J}$ over X which is locally free as $\mathcal{O}_X$-module, therefore the cubic, quadratic and linear maps n, r, s and t satisfying

$$x^3 - t(x)x^2 + s(x)x - n(x)1 = 0$$

for all sections $x \in \mathcal{J}$ are uniquely determined, if for every $P \in X$, there is an element $u \in \mathcal{J}(P)$ such that 1, u and u^2 are linearly independent over $k(P)$. Thus, if $\mathcal{J} = \mathcal{A}^+$ with $\mathcal{A}$ an Azumaya algebra of constant rank 9 over X, or if X is a k-scheme and k is an infinite base field, n, r, s and t are uniquely determined.

1.7. **Some facts on proper schemes.** Let X be a proper scheme over a perfect field k. Then the Theorem of Krull-Schmidt holds for vector bundles over X, i.e., every vector bundle on X can be decomposed as a direct sum of indecomposable vector bundles, unique up to isomorphisms and order of summands. Moreover, non-isomorphic vector bundles on X extend to non-isomorphic vector bundles on $X_l = X \times_k l$, for every separable algebraic field extension l/k [AEJ1, p. 1324 and p. 1325].

For a vector bundle $\mathcal{N}$ on X_l, the direct image $\pi_*\mathcal{N}$ of $\mathcal{N}$ under the projection morphism $\pi\colon X_l \to X$ is a vector bundle on X denoted by $tr_{l/k}(\mathcal{N})$.

1.8. **Elliptic curves.** Results and terminology from Atiyah [At], Arason, Elman and Jacob [AEJ1, 2] and [Pu1] are used. Let k be a perfect field of characteristic not 2 or 3 and $\overline{k}$ an algebraic closure of k. Put $\overline{X} = X \times_k \overline{k}$.

An elliptic curve X/k can be described by a Weierstraß equation of the form

$$y^2 = x^3 + b_2x^2 + b_1x + b_0 \qquad (b_i \in k)$$

with the infinite point as base point O. Let $q(x) = x^3 + b_2x^2 + b_1x + b_0$ be the defining polynomial in $k[x]$. The k-rational points of order 2 on X are the points $(a, 0)$, where $a \in k$ is a root of $q(x)$. Let $K = k(X) = k(x, \sqrt{q(x)})$ be the function field of X. We distinguish three different cases (see also [AEJ2]).

Case I. X has three k-rational points of order 2 which is equivalent to ${}_2\mathrm{Pic}(X) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Write $q(x) = (x-a_1)(x-a_2)(x-a_3)$ and ${}_2\mathrm{Pic}(X) = \{\mathcal{O}_X, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3\}$ where $\mathcal{L}_i$ corresponds with the point $(a_i, 0)$ for $i = 1, 2, 3$.

Case II. X has one k-rational point of order 2 which is equivalent to ${}_2\mathrm{Pic}(X) \cong \mathbb{Z}_2$. Write $q(x) = (x - a_1)q_1(x)$ and ${}_2\mathrm{Pic}(X) = \{\mathcal{O}_X, \mathcal{L}_1\}$ with $\mathcal{L}_1$ corresponding with $(a_1, 0)$. Define $l_2 = k(a_2)$ with a_2 a root of $q_1, K_2 = K \otimes_k l_2$. Let $\mathrm{Gal}(l_2/k) = \{id, \sigma\}$.

Case III. X has no k-rational point of order 2 which is equivalent to ${}_2\mathrm{Pic}(X) = \{\mathcal{O}_X\}$. Define $l_1 = k(a_1)$ with a_1 a root of the irreducible polynomial $q(x)$, $K_1 = K \otimes_k l_1$ and let $\Delta(q) = (a_1 - a_2)^2(a_1 - a_3)^2(a_2 - a_3)^2$ be the discriminant of q. If l_1/k is Galois, let $\mathrm{Gal}(l_1/k) = \{id, \sigma_1, \sigma_2\}$.

Correspondingly, X/k is called *of type I, II* or *III.*

We denote a line bundle of order 3 on X by $\mathcal{N}_i$ and a line bundle of order 4 on X by $\mathcal{H}_i$. Following [AEJ1], we denote the selfdual line bundles $\mathcal{L}_i \otimes_{\mathcal{O}_X} \mathcal{O}_{X_l}$ on $X_l = X \times_k l$ for any field extension l/k also by $\mathcal{L}_i$, $i = 1, 2, 3$, to avoid complicated terminology. We do the same for the line bundles $\mathcal{N}_i \otimes_{\mathcal{O}_X} \mathcal{O}_{X_l}$ of order 3 and $\mathcal{H}_i \otimes_{\mathcal{O}_X} \mathcal{O}_{X_l}$ of order 4. This abuse of notation is justified by the fact that the natural map $\mathrm{Pic}X \to \mathrm{Pic}X_l$ is injective [AEJ1, p. 1325].

If k has characteristic zero, ${}_3\mathrm{Pic}\,(\overline{X}) = \{\mathcal{N}_i \,|\, 0 \leq i \leq 8\}$ where $\mathcal{N}_0 = \mathcal{O}_{\overline{X}}$ [At, Lemma 22]. Hence ${}_3\mathrm{Pic}\,(X) = \{\mathcal{N}_i \,|\, 0 \leq i \leq m\}$ for some even integer m, $0 \leq m \leq 8$, where $\mathcal{N}_0 = \mathcal{O}_X$. Furthermore, then ${}_4\mathrm{Pic}(\overline{X}) = \{\mathcal{H}_i \,|\, 0 \leq i \leq 15\}$ where $\mathcal{H}_0 = \mathcal{O}_{\overline{X}}$ [At, Lemma 22] and hence ${}_4\mathrm{Pic}(X) = \{\mathcal{H}_i \,|\, 0 \leq i \leq n\}$, $0 \leq n \leq 15$ where $\mathcal{H}_0 = \mathcal{O}_X$.

For any integer r, there exists an absolutely indecomposable vector bundle of rank r and degree 0 on X we call $\mathcal{F}_r$, which is unique up to isomorphism, such that $\mathcal{F}_r$ has nontrivial global sections [At, Theorem 5]. Each $\mathcal{F}_r$ is selfdual. In particular, $\mathcal{F}_1 = \mathcal{O}_X$. Furthermore, if $\mathcal{M}$ is an absolutely indecomposable vector bundle of rank r and degree 0 on X, there is a line bundle $\mathcal{L} \in \mathrm{Pic}X$ of degree 0, such that $\mathcal{M} \cong \mathcal{L} \otimes \mathcal{F}_r$. This line bundle is unique up to isomorphism.

Recall also that for X of type III, the elliptic curve $X_1 = X \times_k l_1$ is of type I and the selfdual line bundle $\mathcal{L}_1$ over X_1 is not defined over X. The vector bundle $tr_{l_1/k}(\mathcal{L}_1)$ is indecomposable of rank 3 and $tr_{l_1/k}(\mathcal{L}_1) \otimes \mathcal{O}_{X_1} \cong \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$.

For X of type II, the elliptic curve $X_2 = X \times_k l_2$ is of type I and the selfdual line bundles $\mathcal{L}_2$ and $\mathcal{L}_3$ on X_2 are not defined over X. The vector bundle $tr_{l_2/k}(\mathcal{L}_2) \cong tr_{l_2/k}(\mathcal{L}_3)$ is indecomposable of rank 2 and $tr_{l_2/k}(\mathcal{L}_3) \otimes \mathcal{O}_{X_2} \cong \mathcal{L}_2 \oplus \mathcal{L}_3$.

2. From admissible cubic algebras to quartic Jordan algebras

2.1. **Admissible Cubic algebras.** A commutative $\mathcal{O}_X$-algebra $\mathcal{A}$, which is locally free of constant rank as $\mathcal{O}_X$-module, is called an *admissible cubic algebra* if $\mathcal{A}$ carries a cubic form $N : \mathcal{A} \to \mathcal{O}_X$, called the *norm* of $\mathcal{A}$, such that

$$(1) \qquad x^2x^2 = N(x)x$$

for all sections x in $\mathcal{A}$ over the same open subset of X. Let $N(x, y, z)$ denote the trilinear form associated with N.

Hence, in the setting of rings, an *admissible cubic algebra* A over R is a commutative R-algebra, finitely generated projective of constant rank as R-module, together with a cubic form $N : A \to R$, the *norm* of A, such that $x^2x^2 = N(x)x$ for all $x \in A$.

As in [E-O, p. 277] we obtain

$$(3) \qquad 4x^2(xy) = 3N(x, x, y) + N(x)y$$

by linearizing (1) and

$$(4) \qquad 4N(x)x(x^2y) = 3N(x^2, x^2, y)x^2 + N(x^2)y$$

by substituting x by x^2 in (3) and using (1).

Remark 2. (i) There is a one-one correspondence between admissible cubic algebras over the affine scheme $Z = \mathrm{Spec}\, R$ and admissible cubic algebras over R given by the global

section functor $\mathcal{A} \longrightarrow H^0(Z, \mathcal{A})$ and the functor $A \longrightarrow \widetilde{A}$.
(ii) If $\mathcal{A}$ is an admissible cubic algebra over X with norm N, then $\mathcal{A}_P$ is an admissible cubic algebra over $\mathcal{O}_{P,X}$ with norm N_P for all $P \in X$ and so is $\mathcal{A}(P)$ over $k(P)$, with norm $N(P)$.
(iii) Let $\mathcal{A}$ be an admissible cubic algebra over X with norm N. Equation (1) implies that

$$(2) \qquad N(x^2) = N(x)^2$$

for all sections x in the same open subset U of X with $x^2 \neq 0$, if $\mathcal{A}(U)$ is faithful as an $\mathcal{O}_X(U)$-module.
(iv) Let A be an admissible cubic algebra over a domain R. A is a projective R-module, thus torsion free as an R-module. Hence for all $x \in A$ with $x^2 \neq 0$, $N(x^2) = N(x)^2$.

2.2. **Quartic Jordan algebras.** Let $\mathcal{J}$ be a Jordan algebra over X, which is locally free of constant rank as $\mathcal{O}_X$-module. Following [E-O], we call $\mathcal{J}$ a *quartic Jordan algebra* (its multiplication will usually be denoted by a dot $\cdot$), if it is endowed with a linear form $t : \mathcal{J} \to \mathcal{O}_X$ (the *trace* of $\mathcal{J}$), a quadratic form $s : \mathcal{J} \to \mathcal{O}_X$, a cubic form $r : \mathcal{J} \to \mathcal{O}_X$ and a quartic form $n : \mathcal{J} \to \mathcal{O}_X$, such that the following holds:

(1) Every element x in $\mathcal{J}$ satisfies

$$x^{\cdot 4} - t(x)x^{\cdot 3} + s(x)x^{\cdot 2} - r(x)x + n(x)1 = 0.$$

(2) $t(1) = 4$.
(3) The Newton formulae hold for each x in $\mathcal{J}$:
$2s(x) = t(x)^2 - t(x^{\cdot 2})$,
$6r(x) = t(x)^3 - 3t(x)t(x^{\cdot 2}) + 2t(x^{\cdot 3})$.
$24n(x) = t(x)^4 - 6t(x)^2t(x^{\cdot 2}) + 8t(x)t(x^{\cdot 3}) + 3t(x^{\cdot 2})^2 - 6t(x^{\cdot 4})$.

Theorem 1. *Let J be a Jordan algebra over R, finitely generated projective as R-module, and let n, r, s and t be a quartic, cubic, quadratic and linear map from J to R such that*

$$(1) \qquad x^{\cdot\, 4} - t(x)x^{\cdot\, 3} + s(x)x^{\cdot\, 2} - r(x)x + n(x)1 = 0$$

for all $x \in A$. Suppose that for each $P \in \operatorname{Spec} R$ there exists an element $u \in J \otimes_R k(P)$ such that $1, u, u^{\cdot\, 2}, u^{\cdot\, 3}$ are linearly independent over $k(P)$. Now let N, R, S and T be another quartic, cubic, quadratic and linear map from J to R such that

$$x^{\cdot\, 4} - T(x)x^{\cdot\, 3} + S(x)x^{\cdot\, 2} - R(x)x + N(x)1 = 0$$

for all $x \in J$. Then $N = n$, $R = r$, $S = s$ and $T = t$.

This is proved analogously as [Ach1, 1.12].

Corollary 1. *(i) Let $\mathcal{J}$ be a quartic Jordan algebra over X. The maps n, r, s and t in (1) are uniquely determined, if for every $P \in X$, there is an element $u \in \mathcal{J}(P)$ such that $1, u, u^{\cdot\, 2}, u^{\cdot\, 3}$ are linearly independent over $k(P)$.*
(ii) Let $\mathcal{J} = \mathcal{A}^+$ with an Azumaya algebra $\mathcal{A}$ of constant rank 16 over X. If $\mathcal{A} \cong \mathcal{E}nd_X(\mathcal{E})$ is trivial, or if X is a k-scheme and k an infinite field, then the maps n, r, s and t in (1) are uniquely determined.

We will mostly restrict out investigation to quartic Jordan algebras $\mathcal{J}$ whose trace form t is nondegenerate over all residue class fields, i.e. whose residue class algebras $\mathcal{J}(P)$ are separable, and where all residue class algebras $\mathcal{J}(P)$ have degree 4.

2.3. Let $\mathcal{J}$ be a quartic Jordan algebra with trace t. Let $\mathcal{J}_0 = \ker t$. Then $\mathcal{J} = \mathcal{O}_X 1 \oplus \mathcal{J}_0$.

There is the following relation between admissible cubic algebras and quartic Jordan algebras:

Theorem 2. *$\mathcal{J}_0$ becomes an admissible cubic algebra over X with multiplication*

$$xy = x \cdot y - \frac{1}{4} t(x \cdot y)1$$

and norm

$$N(x) = r(x) = \frac{1}{3} t(x^{\cdot 3}).$$

The symmetric bilinear form on $\mathcal{J}_0$ defined via

$$\langle x|y \rangle = \frac{1}{3} t(x \cdot y)$$

is associative and satisfies

$$N(x) = \langle x|x^2 \rangle.$$

Proof. We proceed as in [E-O, p. 287]: the new multiplication xy is the projection $\mathrm{proj}_{\mathcal{J}_0}(x \cdot y)$ of the product $x \cdot y$ onto $\mathcal{J}_0$ relative to the decomposition $\mathcal{J} = \mathcal{O}_X 1 \oplus \mathcal{J}_0$. We get

$$x^2 x^2 = r(x)x = \frac{1}{3} t(x \cdot x^{\cdot 2})x$$

for any x in $\mathcal{J}_0$ and thus $\mathcal{J}_0$ is an admissible algebra over X with norm

$$N(x) = r(x) = \frac{1}{3} t(x \cdot y)$$

using the Newton formulae. $\langle x|y \rangle$ is associative, since the trace form $(x, y) \to t(x \cdot y)$ of a quartic Jordan algebra is associative.

□

Remark 3. (i) The norm N of the admissible algebra constructed above is non-zero if and only if there is an $x \in \mathcal{J}_0$ such that $t(x^{\cdot 3}) \neq 0$.

(ii) Let $\mathcal{J}$ be a separable quartic Jordan algebra. Suppose $N = 0$ in the above construction, then $N(P) = 0$, hence $r(P) = 0$ and $t(P)(\, , \,) = 0$ on $\mathcal{J}_0(P) = \mathcal{J}_0 \otimes k(P)$ for all $P \in X$, implying that the trace form $t(P) \cong t(P)|_{k(P)} \perp t(P)|_{\mathcal{J}_0}$ is degenerate, a contradiction. Thus $N \neq 0$.

However, it may happen that N is the zero map on the global sections, i.e. that $N(X) : H^0(X, \mathcal{J}_0) \to H^0(X, \mathcal{O}_X)$ is zero. Then both t and r are the zero maps on the global sections as well, see Theorem 10 (1) below.

(iii) For an admissible cubic algebra $\mathcal{A}$ with norm N, which does not arise as the trace zero elements of a quartic Jordan algebra as described above, it is not clear how to define an associative symmetric bilinear form $\langle \, | \, \rangle$ on $\mathcal{A}$ such that $N(x) = \langle x|x^2 \rangle$, as it was done in [E-O, Thm. 1] for cubic admissible algebras over base fields.

Example 1. As mentioned in [E-O, p. 305], the characteristic equation for the norm of admissible cubic algebras appears in the theory of cubic Jordan algebras: let $\mathcal{J}$ be an $\mathcal{O}_X$-module of constant rank and $(\mathcal{J}, \sharp, 1)$ a cubic form with adjoint and base point on $\mathcal{J}$ (cf. [Ach1], [P-R1]). Then, as over base fields, we have the adjoint identity

$$x^{\sharp\sharp} = N(x)x$$

for all sections x in $\mathcal{J}$ over the same open subset of X. Define a commutative multiplication on $\mathcal{J}$ using the adjoint map $\sharp$ via

$$x^2 = x^\sharp,$$

then $\mathcal{J}$ is an admissible cubic algebra over X with norm N and non-zero idempotent 1. Since $S(x) = T(x^\sharp)$ and $x^{\cdot 2} = x^\sharp + T(x)x - S(x)1$, $\mathcal{J}$ becomes a cubic Jordan algebra with unit 1, such that

$$x^{\cdot 3} - T(x)x^{\cdot 2} + S(x)x - N(x)1 = 0$$

for all sections x in $\mathcal{J}$ over the same open subset of X. We have

$$2S(x) = T(x)^2 - T(x^{\cdot 2})$$

and

$$6N(x) = 2T(x^{\cdot 3}) - 3T(x)T(x^{\cdot 2}) + T(x)^3.$$

Let $\widetilde{\mathcal{J}}$ be the quartic Jordan algebra $\widetilde{\mathcal{J}} = \mathcal{O}_X \oplus \mathcal{J}$, which is the direct sum of the Jordan algebras $\mathcal{O}_X$ and $\mathcal{J}$ with trace $t((a, x)) = a + T(x)$. An analogous argument as the one given in [E-O] now proves that the admissible cubic algebra $\mathcal{A}$ we obtain by providing the trace zero elements of $\widetilde{\mathcal{J}}$ with the multiplication of Theorem 2 is identical to the one just constructed.

The proof of [E-O, Theorem 5] easily adapts to our setting and yields:

Proposition 1. *Let X be an integral scheme. Let $\mathcal{J}$ be a quartic Jordan algebra. Let $\mathcal{A} = \mathcal{J}_0$ be the admissible algebra defined in Theorem 2.*
(i) Define $\mathcal{J}_{\mathcal{A}} = \mathcal{O}_X \oplus \mathcal{A}$ and let

(1) $t(z) = 4a$,
(2) $s(z) = 6a^2 - \frac{3}{2}\langle x|x\rangle$,
(3) $r(z) = 4a^3 - 3a\langle x|x\rangle + N(x)$,
(4) $n(z) = a^4 - \frac{3}{2}a^2\langle x|x\rangle + aN(x) + \frac{9}{16}\langle x|x\rangle^2 - \frac{3}{4}\langle x^2|x^2\rangle$

for any $z = a1 + x$ in $\mathcal{O}_X \oplus \mathcal{A}$. Then $\mathcal{J}_{\mathcal{A}}$, together with the commutative multiplication defined via $1 \cdot 1 = 1$, $1 \cdot x = x$ and

$$x \cdot y = xy + \frac{3}{4}\langle x|y\rangle 1$$

for x, y in $\mathcal{A}$, becomes a quartic Jordan algebra, where

$$z^{\cdot 4} - t(z)z^{\cdot 3} + s(z)z^{\cdot 2} - r(z)z + n(z)1 = 0$$

for any $z = a1 + x$ in $\mathcal{J}_{\mathcal{A}} = \mathcal{O}_X \oplus \mathcal{A}$.
(ii) The admissible cubic algebra $(\mathcal{J}_{\mathcal{A}})_0$ with norm N obtained using the method of Theorem 2, is the same one as $\mathcal{A}$.

2.4. **Idempotents.**

Lemma 1. *Let R be a domain. Let A be any unital algebra, which is finitely generated projective as R-module. Suppose that A contains an idempotent $f \neq 0, 1$.*
(i) 1 and f are linearly independent over R.
(ii) Let A be an octonion algebra, an Azumaya algebra of constant rank greater than 1, or a cubic or quartic Jordan algebra, with trace t and $t(f) = a$. If $(t(1) - a)a$ is invertible in R, then $A \cong R \oplus R \oplus A'$, where A' is the orthogonal complement of the subspace of A spanned by 1 and f with respect to the symmetric bilinear trace form $t(x, y) = t(xy)$ of A.

Proof. (i) Let $1a + fb = 0$ for $a, b \in R$. Multiplication with f implies $(a + b)f = 0$ and hence $a + b = 0$, since A is a projective R-module, thus torsion free as an R-module. Now $b = -a$, so $0 = 1a + fb = (1 - f)a$ yields $a = 0$ since $1 - f \neq 0$, hence also $b = 0$.
(ii) By assumption, $\det(t|_{R1 \oplus Rf}) = t(1)a - a^2$ is invertible in R. Hence $(R1 \oplus Rf, t)$ is nonsingular and $(A, t) \cong (R1 \oplus Rf, t|_{R1 \oplus Rf}) \perp (A, t|_A)$ [Kn, (3.6.2), p. 17]. □

Lemma 2. *Let $\mathcal{J}$ be a quartic Jordan algebra over X, which is locally free of finite constant rank as $\mathcal{O}_X$-module.*
(i) Let $f \in H^0(X, \mathcal{J})$ be a non-trivial idempotent. Then $\mathcal{O}_X\, f$ is a direct summand of $\mathcal{J}$.
(ii) If f has trace 1, then $e = 2f - \frac{1}{2}$ is a non-zero idempotent in the admissible cubic algebra $\mathcal{A} = \mathcal{J}_0$ over X.
(iii) Suppose X is an integral scheme. Then $\mathcal{O}_X\, e$ is a direct summand of the admissible cubic algebra $\mathcal{A} = \mathcal{J}_0$.

Proof. Let $H^0(X, \mathcal{O}_X) = R$.
(i) By [Lo, 0.5], $R\,f$ is a direct summand of $H^0(X, \mathcal{J})$ which implies the assertion.
(ii) Let $e = 2f - \frac{1}{2}$, then $t_{\mathcal{J}}(e) = 0$ and $e^2 = e$ in the admissible cubic algebra $\mathcal{A}$, thus e is a non-zero idempotent in $\mathcal{A}$, see [E-O, Lemma 9].
(iii) If R is a domain, then $e = e^2 e^2 = N(e)e$ implies $N(e) = 1$, hence $(Re, \langle | \rangle)$ is nonsingular and $(H^0(X, \mathcal{A}), \langle | \rangle) \cong (Re, \langle | \rangle|_{Re}) \perp (V, \langle | \rangle|_V)$ with $V = (Re)^\perp$ [Kn, p. 17, (3.6.2)]. Thus $R\,e$ is a direct summand of $H^0(X, \mathcal{A})$ and $\mathcal{O}_X\, e$ a direct summand of $\mathcal{A} = \mathcal{J}_0$. □

Corollary 2. *Let $\mathcal{J}$ be a quartic Jordan algebra over an integral scheme X, which is locally free of finite rank as $\mathcal{O}_X$-module. Let $f \in H^0(X, \mathcal{J})$ be a non-trivial idempotent of trace 1, 2 or 3, then*

$$\mathcal{J} \cong \mathcal{O}_X^2 \oplus \widetilde{\mathcal{J}}$$

for a suitable locally free $\mathcal{O}_X$-module $\widetilde{\mathcal{J}}$.

Proof. For $a = t(f) = 1$, 2 or 3, the element $t(1)a - a^2$ is invertible in R, hence $\mathcal{J} \cong \mathcal{O}_X \oplus \mathcal{O}_X \oplus \dots$ by Lemma 1. □

Remark 4. (i) If $H^0(X, \mathcal{O}_X)$ is a field of characteristic not 2 or 3, and f a non-trivial idempotent in $\mathcal{J}$, then $t(f) = 1$, 2 or 3 [E-O, p. 296], hence $\mathcal{J} \cong \mathcal{O}_X^2 \oplus \dots$ using Corollary 2.
Now suppose X is an integral scheme over k, with k a field of characteristic not 2 or 3. If $t(X)$ does not divide the cubic form $r(X)$ on $H^0(X, \mathcal{J})$, i.e., $H^0(X, \mathcal{J})$ is a Jordan algebra

whose reduced degree is 3 or 4 [E-O, p. 286], then there is a field extension l/k of degree 1, 2 or 4 such that $H^0(X_l, \mathcal{J}')$ contains idempotents of trace 1, with $X_l = X \times_k l$ and $\mathcal{J}' = \mathcal{J} \otimes_k l$. [E-O, Proposition 10].
(ii) Let e be a non-zero idempotent in an admissible cubic algebra A over a domain R. Then $N(e) = 1$. Hence, if $\mathcal{A}$ is an admissible cubic algebra over an integral scheme X and e a non-zero idempotent in $\mathcal{A}$, the existence of e forces the norm N of $\mathcal{A}$ to be non-zero.
(iii) Let X be an integral scheme and $\mathcal{D}$ a quaternion algebra over X. An element f is a non-trivial idempotent of $\mathcal{D}$ if and only if $n_{\mathcal{D}}(f) = 0$ and $t_{\mathcal{D}}(f) = 1$. Moreover, $\mathcal{D}$ contains a non-trivial idempotent if and only if it is split: If $\mathcal{D}$ is split then clearly $(0, 1)$ is a non-trivial idempotent in $\mathcal{D}$. Conversely, if $\mathcal{D}$ has a non-trivial idempotent f then f gives rise to a Peirce decomposition $\mathcal{D} = \mathcal{D}_{11} \oplus \mathcal{D}_{12} \oplus \mathcal{D}_{21} \oplus \mathcal{D}_{22}$ with $\mathcal{D}_{11} \cong \mathcal{O}_X$ and $\mathcal{D}_{22} \cong \mathcal{O}_X$, where the vector bundles $\mathcal{D}_{12}$ and $\mathcal{D}_{21}$ are dual to each other and of rank 1. Also, the system of idempotents $(f, 1 - f)$ spans a split quadratic étale subalgebra of $\mathcal{D}$.

Proposition 2. *Let X be a proper integral scheme over k.*
(i) If $\mathcal{D} = Quat(\mathcal{M}, N_{\mathcal{M}})$ and $\mathcal{M}$ does not contain $\mathcal{O}_X$ as a direct summand, then $\mathcal{D}$ does not contain any non-trivial idempotents.
(ii) If $\mathcal{D} = \mathrm{Cay}(k(\sqrt{c}) \otimes_k \mathcal{O}_X, \mathcal{P}, N_{\mathcal{P}})$ is the Cayley-Dickson doubling of a quadratic étale algebra which is defined over k and $k(\sqrt{c}) \otimes_k \mathcal{O}_X$ is not split, then $\mathcal{D}$ does not contain any non-trivial idempotents.

Proof. Let $\mathcal{D}$ be a quaternion algebra over X and f be a non-trivial idempotent of $\mathcal{D}$. Then $\mathcal{D}$ is split by Remark 4 (iii). By the Krull-Schmidt Theorem, this yields a contradiction in both cases. □

Proposition 3. *Let X be a proper integral scheme over k. Let $\mathcal{E}$ be an indecomposable vector bundle over X and $\mathcal{B} = \mathcal{E}nd_X(\mathcal{E})$ the trivial Azumaya algebra.*
(i) The global sections of the Jordan algebra $\mathcal{J} = \mathcal{B}^+$ contain only trivial idempotents.
(ii) If $\mathcal{B}$ carries an involution σ of the first kind, then the global sections of the Jordan algebra $\mathcal{J} = H(\mathcal{B}, \sigma)$ contain only trivial idempotents.

Proof. For a finitely generated projective R-module E of constant rank over a ring R, the Azumaya algebra $\mathrm{End}_R(E)$ does not contain non-trivial idempotents, if $\mathrm{End}_R(E)$ is a local ring [La, p. 443]. Since $\mathcal{E}$ is indecomposable, $\mathrm{End}(\mathcal{E})$ is a local ring [AEJ1, p. 1324] and so $H^0(X, \mathcal{E}nd_X(\mathcal{E})) \cong \mathrm{End}(\mathcal{E})$ contains only trivial idempotents. This implies (i). Part (ii) follows immediately observing that if there would be a non-trivial idempotent in $H(\mathcal{B}, \sigma)$ this also would be a non-trivial idempotent in $\mathcal{B}$. □

3. From admissible cubic algebras to pseudo-composition algebras and cubic Jordan algebras

Let $(\mathcal{V}, *)$ be a commutative algebra over X, which is locally free as $\mathcal{O}_X$-module, together with a symmetric bilinear form $(\,|\,)$ such that

$$(x * x) * x = (x|x)x$$

for all sections x in $\mathcal{V}$ over the same open subset of X. Then $(\mathcal{V},*)$ is called a *pseudo-composition algebra* over X. Correspondingly, a commutative algebra $(V,*)$ over a ring R, which is finitely generated projective as R-module and which carries a symmetric bilinear form $(\,|\,)$ such that $(x*x)*x=(x|x)x$ for all $x\in V$, is called a *pseudo-composition algebra* over R [M-O]. If V is a pseudo-composition algebra over a domain R, then

$$(x^2|x^2)=(x|x)^2$$

for all $x\in V$ with $x^2\neq 0$, since V is projective as R-module, thus torsion free.

Using [E-O, p. 291] or adjusting the proof of [W, (3.1)] we obtain:

Theorem 3. *Let $\mathcal{J}$ be a cubic Jordan algebra such that $\mathcal{J}(P)$ has degree 3 for each $P\in$* Spec R. *Then the set of trace zero elements $\mathcal{J}_0$ becomes a pseudo-composition algebra with multiplication given by*

$$x\star y=xy+\frac{2}{3}s_{\mathcal{J}}(x,y)1$$

with $s_{\mathcal{J}}(x,y)=\frac{1}{2}(s_{\mathcal{J}}(x,y)-s_{\mathcal{J}}(x)-s_{\mathcal{J}}(y))$ and associative symmetric bilinear form

$$(x|y)=\frac{1}{6}t_{\mathcal{J}}(x,y)$$

for all $x,y\in\mathcal{J}_0$. If $\mathcal{J}$ is separable, then the bilinear form $(\,|\,)$ is nondegenerate.

Example 2. Let $\mathcal{M}$ be a locally free $\mathcal{O}_X$-module of constant rank 3 and consider the cubic Jordan algebra $\mathcal{J}=\mathcal{E}nd_X(\mathcal{M})^+$. Since 3 is invertible in $H^0(X,\mathcal{O}_X)$, $\mathcal{E}nd(\mathcal{M})\cong\mathcal{O}_X\oplus\mathcal{M}'$ where $\mathcal{M}'$ is the subspace of the endomorphisms of trace 0 (for a proof see [At, Lemma 19], which holds more generally) and $\mathcal{M}'$ can be made into a pseudo-composition algebra.

Proposition 4. *Let $(\mathcal{V},*)$ be a pseudo-composition algebra over X of constant rank. Suppose that $(\,|\,)$ is associative.*
(i) The algebra

$$\mathcal{J}=\mathcal{O}_Xe\oplus\mathcal{V}$$

obtained by adjoining an identity element e with multiplication given by

$$(ae+x)\diamond(be+y)=(ab+2(x|y))e+(ay+bx+x*y) \tag{5}$$

for all a,b in $\mathcal{O}_X$ and x,y in $\mathcal{V}$, is a Jordan algebra of constant rank.
(ii) If for every $P\in X$, there is an element $u\in\mathcal{J}(P)$ such that $1,u,u^{\diamond 2}$ are linearly independent over $k(P)$, then $\mathcal{J}$ is a cubic Jordan algebra with norm

$$n(ae+x)=a^3-3a(x|x)+2(x|x*x)$$

and trace $t(ae+x)=3a$.

Proof. We proceed analogously as described in [E-O, p. 290] (cf. also [W, (3.1)]).
(i) Linearizing the defining equality of a pseudo-composition algebra yields the equation

$$(x*x)*y+2(x*y)*x=2(x|y)x+(x|x)y$$

which in turn implies

$$((x*x)*y)*x-(x*x)*(y*x)=2(x|y)x*x-2(x*x|y)x$$

for all $x, y \in \mathcal{V}$. We thus obtain a Jordan algebra $\mathcal{J} = \mathcal{O}_X e \oplus \mathcal{V}$ of constant rank by equipping $\mathcal{J}$ with the multiplication given by $e \diamond x = x$ and $x \diamond y = x * y + 2(x|y)e$ for all x, y in $\mathcal{V}$.
(ii) For all $z = ae + x$ in $\mathcal{J}$ we have

$$z^{\diamond 3} - 3az^{\diamond 2} + 3(a^2 - (x|x))z - (a^3 - 3a(x|x) + 2(x|x * x))e = 0.$$

Hence $\mathcal{J}$ is a cubic Jordan algebra with the claimed norm and trace by 1.6. □

Theorem 4. *Let X be an integral scheme. Let $\mathcal{J}$ be a quartic Jordan algebra. Suppose that there is a non-trivial idempotent $f \in H^0(X, \mathcal{J})$ of trace 1. Put $e = 2f - \frac{1}{2}$.*
(i) The orthogonal complement $\mathcal{V}$ of $e\mathcal{O}_X$ (relative to $\langle | \rangle$) in the admissible cubic algebra $\mathcal{J}_0$ can be made into a pseudo-composition algebra over X with an associative bilinear form $(|)$ which satisfies

$$(x \,|\, y) = -\langle e|xy\rangle.$$

(ii) The $\mathcal{O}_X$-linear endomorphism $\varphi : \mathcal{V} \to \mathcal{V}$, $\varphi(x) = -2ex$, is an automorphism of $(\mathcal{V}, *)$ with $\varphi^2 = 1$.

Proof. $\mathcal{J}_0$ can be made into an admissible cubic algebra with norm $N = r$ over X through the multiplication $xy = x \cdot y - \frac{1}{4}t(x \cdot y)1$ by Theorem 2.
(i) The element $e = 2f - \frac{1}{2}$ is a non-zero idempotent in the admissible cubic algebra $\mathcal{A} = \mathcal{J}_0$ such that $N(e) = 1$ (Lemma 2). Let $\mathcal{V}$ be the orthogonal complement to $\mathcal{O}_X\, e$ relative to $\langle | \rangle$. Since $N(x) = \langle x|x^2\rangle$, we get $N(x, y, z) = \langle x|yz\rangle$ by linearization. We now follow [E-O, p. 298]: Eq. (3) yields

$$(6) \qquad 4x^2(xy) = 3\langle x^2|y\rangle x + \langle x|x^2\rangle y$$

thus

$$(7) \qquad 4e(ex) = 3\langle e|x\rangle e + \langle e|e\rangle x$$

and

$$\mathcal{V} = \{x \in \mathcal{A} \,|\, 4e(ex) = v\} = \{x \in \mathcal{A} \,|\, (2e)((2e)x) = v\}.$$

Define

$$x * y = -2ex^2 + 2\langle e|x^2\rangle.$$

As in [E-O, p. 298], we obtain

$$(x * x) * x = -\langle e|x^2\rangle,$$

so that $(\mathcal{V}, *)$ together with $(x \,|\, x) = -\langle e|x^2\rangle$ is a pseudo-composition algebra over X. Moreover,

$$(x * y|z) = (x|y * z)$$

for all $x, y, z \in \mathcal{V}$.
(ii) By [E-O, Proposition 11], the map $\varphi(P) : \mathcal{V}(P) \to \mathcal{V}(P)$ is bijective for all $P \in X$, which implies the first assertion. $\varphi^2 = 1$ follows from Equation (7). □

Proposition 5. *Assume the situation of Proposition 1. Suppose that there is a non-trivial idempotent $f \in H^0(X, \mathcal{J})$ of trace 1. Put $e = 2f - \frac{1}{2}$.*
(i) The $\mathcal{O}_X$-linear endomorphism $\phi : \mathcal{A} \to \mathcal{A}$ defined by $\phi(e) = e$ and $\phi(x) = \varphi(x)$, is an automorphism of $(\mathcal{V}, *)$.
(ii) The extension $\Phi : \mathcal{O}_X \oplus \mathcal{A} \to \mathcal{O}_X \oplus \mathcal{A}$ of ϕ to the cubic Jordan algebra $(\mathcal{J}_{\mathcal{A}}, \cdot)$ defined in Proposition 1 satisfies $\Phi(1) = 1$ and is an automorphism of $(\mathcal{J}_{\mathcal{A}}, \cdot)$.

Proof. By [E-O, Proposition 11], both the map $\phi(P) : \mathcal{V}(P) \to \mathcal{V}(P)$ and the map $\Phi(P) : k(P) \oplus \mathcal{A}(P) \to k(P) \oplus \mathcal{A}(P)$ is bijective, for all $P \in X$, which implies the first assertion. The fact that $\Phi(1) = 1$ follows from Equation (7). $\square$

As in [E-O, p. 306 ff.], we can now construct a cubic Jordan algebra on of the trace zero elements of a quartic Jordan algebra:

Theorem 5. *Assume the situation of Theorem 4. Put $e = 2f - \frac{1}{2}$.*
(i) $\mathcal{J}_0 = \mathcal{O}_X\, e \oplus \mathcal{V}$ can be made into a Jordan algebra over X with unit element e via the multiplication $\diamond$ defined as $e \diamond x = x$ and

$$x \diamond y = -2e \cdot (x \cdot y) + \frac{1}{2} t(x \cdot y)e + t(e \cdot x \cdot y)(\frac{2}{3}e - \frac{1}{6})$$

for all x, y in $\mathcal{V}$.
(ii) If for every $P \in X$, there is an element $u \in \mathcal{J}(P)$ such that $1, u, u^{\diamond 2}$ are linearly independent over $k(P)$, then $\mathcal{J}$ is a cubic Jordan algebra with norm

$$N_{\mathcal{J}_0}(\alpha e + x) = \alpha^3 + \alpha t(e \cdot x^{\cdot 2}) + \frac{1}{3} t(x^{\cdot 3}),$$

for all α in $\mathcal{O}_X$, x in $\mathcal{V}$.

Proof. We use the same notation as in Theorem 4.
(i) Define a multiplication on $\mathcal{J}_0 = \mathcal{O}_X\, e \oplus \mathcal{V}$ via $e \diamond x = x$ for all x in $\mathcal{V}$ and via

$$x \diamond x = x * x + 2(x|x)e = x * x + \langle \phi(x)|x \rangle e.$$

The second equation holds because we have

$$\langle x|y \rangle = 2(\varphi(x)|y) \tag{8}$$

for all x, y in $\mathcal{V}$, which follows from the equation

$$(x|y) = -\langle e|xy \rangle = -\langle ex|y \rangle = \frac{1}{2} \langle \varphi(x)|y \rangle.$$

A straightforward calculation yields the claimed multiplication.
(ii) As in Proposition 4, we see that $\mathcal{J}_0$ is a cubic Jordan algebra over X with norm $N_{\mathcal{J}_0}(ae + x) = a^3 - 3a(x|x) + 2(x|x * x)$ which, expressed in terms of the multiplication of $\mathcal{J}$ is of the above form. $\square$

Every degree 3 separable Jordan algebra J over a field of characteristic not 2 or 3 can be obtained through this process starting with the non-simple algebra $B = k \oplus J$ [A-F, 5.6]. The proof from [A-F, 5.6] generalizes as follows:

Proposition 6. *Let X be an integral scheme. Let $\mathcal{J}$ be a separable cubic Jordan algebra with generic norm $n_{\mathcal{J}}$, generic trace $t_{\mathcal{J}}$ and identity $1_{\mathcal{J}}$ such that $\mathcal{J}(P)$ has degree 3 for all $P \in X$. Define $\mathcal{B} = \mathcal{O}_X \oplus \mathcal{J}$ as algebras, then there is a linear bijection from $\mathcal{J}$ to $\mathcal{B}_0$ such that $N_{\mathcal{B}_0}(x) = N_{\mathcal{J}}(x)$ for all $x \in \mathcal{J}$.*

Proof. The linear map can be globally defined as in [A-F, 5.6], the fact that it is an isomorphism follows from [A-F, 5.6], applied to the residue class algebras. □

Remark 5. Let X be an integral scheme over k and $\mathcal{J}$ a quartic Jordan algebra over X which contains an idempotent of trace 1. Let $X' = X \times_k l$ for a field extension l/k. Then $\mathcal{J}' = \mathcal{J} \otimes_X \mathcal{O}_{X'}$ contains an idempotent of trace 1 and the Jordan algebras constructed on the trace zero elements of J and $\mathcal{J}'$ as described in Theorem 5 satisfy $\mathcal{J}_0' \cong \mathcal{J}_0 \otimes_X \mathcal{O}_{X'}$.

Corollary 3. *Assume the situation of Theorem 4. If $\mathcal{J}$ is a separable Jordan algebra over X of degree 4 and rank 28 such that $\mathcal{J}(P)$ is central simple for all $P \in X$ then $\mathcal{J}_0$ can be made into an Albert algebra over X.*

Proof. By Theorem 5, $\mathcal{J}_0$ can be made into a cubic Jordan algebra over X. Now [A-F, 5.4, 5.5], together with [E-O, p. 306 ff.], applied to the residue class algebras, implies the assertion. □

It is not clear if all Albert algebras can be obtained as the trace zero elements out of a central simple separable Jordan algebra of rank 28.

Theorem 6. *Let $\mathcal{J}$ and $\widetilde{\mathcal{J}}$ be quartic Jordan algebras, which are both locally free of constant rank as $\mathcal{O}_X$-module. Suppose that $\widetilde{\mathcal{J}}$ is a Jordan subalgebra of $\mathcal{J}$.*
(a) By projecting the multiplication on the trace zero elements, both $\widetilde{\mathcal{J}}_0$ and $\mathcal{J}_0$ can be made into an admissible cubic algebra over X and $\widetilde{\mathcal{J}}_0$ is an admissible cubic subalgebra of $\mathcal{J}_0$.
(b) Suppose X is an integral scheme and that there is a non-trivial idempotent $f \in H^0(X, \widetilde{\mathcal{J}})$ of trace 1.
(i) The orthogonal complements $\widetilde{\mathcal{V}}$ of $f\mathcal{O}_X$ in $\widetilde{\mathcal{J}}_0$ (resp., $\mathcal{V}$ of $f\mathcal{O}_X$ in $\mathcal{J}_0$) can be made into pseudo-composition algebras and $\widetilde{\mathcal{V}}$ is a pseudo-composition subalgebra of $\mathcal{V}$.
(iii) If for every $P \in X$, there is an element $u \in \widetilde{\mathcal{J}}_0(P)$ such that $1, u, u^{\diamond 2}$ are linearly independent over $k(P)$, then $\widetilde{\mathcal{J}}_0$ and $\mathcal{J}_0$ can be made into cubic Jordan algebras and $\widetilde{\mathcal{J}}_0$ is a cubic Jordan subalgebra of $\mathcal{J}_0$.

Proof. (a) $\mathcal{J}_0$ can be made into an admissible cubic algebra over X through the multiplication $xy = x \cdot y - \frac{1}{4}t(x \cdot y)1$. Restricting this multiplication to $\widetilde{\mathcal{J}}_0$ yields the multiplication of the admissible cubic algebra $\widetilde{\mathcal{J}}_0$.
(b) follows analogously from Theorem 5. □

Using Proposition 3 we conclude:

Corollary 4. *Let X be a proper integral scheme over k.*
(a) Let $\mathcal{E}$ be an indecomposable vector bundle over X of constant rank 4 and $\mathcal{B} = \mathcal{E}nd_X(\mathcal{E})$.
(i) If $\mathcal{B}$ carries an orthogonal involution σ, then the elements of trace zero of the quartic Jordan algebra $\mathcal{J} = H(\mathcal{B}, \sigma)$ cannot be made into a cubic Jordan algebra using our construction, and there is no pseudo-composition algebra associated to $\mathcal{J}$.

(ii) The elements of trace zero of $\mathcal{J} = \mathcal{B}^+$ cannot be made into a cubic Jordan algebra using our construction, and there is no pseudo-composition algebra associated to $\mathcal{J}$.
(b) Let $\mathcal{E}$ be an indecomposable vector bundle over X of constant rank 8 and $\mathcal{B} = \mathcal{E}nd_X(\mathcal{E})$. If $\mathcal{B}$ carries a symplectic involution τ, then the elements of trace zero of the quartic Jordan algebra $\mathcal{J} = H(\mathcal{B}, \tau)$ cannot be made into a cubic Jordan algebra using our construction, and there is no pseudo-composition algebra associated to $\mathcal{J}$.

From now on, let k be an *infinite* field and X a proper integral scheme over k. In the following, we will restrict ourselves to separable quartic Jordan algebras $\mathcal{J}$ over X, which are locally free of constant rank as $\mathcal{O}_X$-module. We will consider the cases that $\mathcal{J} = \mathcal{A}^+$ is an Azumaya algebra of constant rank 16; that $\mathcal{J} = H(\mathcal{A}, \sigma)$ for a symplectic involution σ on an Azumaya algebra of constant rank 64, or for an orthogonal involution σ on an Azumaya algebra of constant rank 16. Then t does not divide r (or else $t(P)$ would also divide $r(P)$ for $P \in X$, hence the reduced degree of the Jordan residue class algebras $\mathcal{J}(P)$ would be 1 or 2 [E-O, p. 286], a contradiction).

4. Azumaya algebras of rank 16 with orthogonal involution

Let X be a proper integral scheme over k and let $R = H^0(X, \mathcal{O}_X)$. We write $\mathcal{M}^s$ for $\mathcal{M} \oplus \cdots \oplus \mathcal{M}$ (s-copies of the $\mathcal{O}_X$-module $\mathcal{M}$). Let $\mathcal{E}$ be a vector bundle over X of rank r. Recall that if r is invertible in $H^0(X, \mathcal{O}_X)$, $\mathcal{E}nd(\mathcal{E}) \cong \mathcal{O}_X \oplus \mathcal{E}'$ where $\mathcal{E}'$ is the subspace of the endomorphisms of trace 0 (for a proof see [At, Lemma 19], which holds more generally).

Let $\mathcal{A}$ be an Azumaya algebra over X of constant rank 16 with an orthogonal involution τ. Then $\mathcal{J} = H(\mathcal{A}, \tau)$ together with the multiplication $x \cdot y = \frac{1}{2}(xy + \tau(xy))$ is a quartic Jordan subalgebra of $\mathcal{A}^+$, which is locally free of constant rank 10 as an $\mathcal{O}_X$-module.

Let $(\mathcal{D}_i, \tau_i)$ be a quaternion algebra over X with an involution of the first kind, $i = 1, 2$. Let $\mathcal{A} = \mathcal{D}_1 \otimes \mathcal{D}_2$ with involution $\tau = \tau_1 \otimes \tau_2$. Then τ is orthogonal, if either both τ_i are canonical involutions, or if both of them are orthogonal, i.e. hat-involutions.

(A) Let both τ_i be canonical involutions. Write $\mathcal{D}_i = \mathrm{Quat}(\mathcal{M}_i, N_i)$ with $\mathcal{M}_i = \ker t_i = \mathrm{Skew}(\mathcal{D}_i, ^-)$. Then

$$\mathcal{J} = H(\mathcal{A}, \tau) \cong \mathcal{O}_X \oplus \mathcal{M}_1 \otimes \mathcal{M}_2.$$

The $\mathcal{O}_X$-module

$$\mathcal{M}_1 \otimes \mathcal{M}_2$$

can be made into an admissible cubic algebra over X by Theorem 2. It cannot always be made into a cubic Jordan algebra, though, see Theorem 9 below.

(A1) In particular, let $\mathcal{D} = \mathcal{D}_1 = \mathcal{D}_2 = \mathrm{Quat}(\mathcal{M}, N_{\mathcal{M}})$. Analogously as in [KMRT, (11.1)], we have

$$(\mathcal{D}, ^-) \otimes (\mathcal{D}, ^-) \cong (\mathcal{E}nd_X(\mathcal{D}), \sigma) \cong (\begin{bmatrix} \mathcal{O}_X & \mathcal{M} \\ \mathcal{M}^\vee & \mathcal{E}nd_X\mathcal{M} \end{bmatrix}, \sigma)$$

where σ is the adjoint involution with respect to the bilinear trace form $T_{(\mathcal{D},^-)}$ defined by the trace $T_{\mathcal{D}}$ of $\mathcal{D}$ via $T_{(\mathcal{D},^-)}(x,y) = T_{\mathcal{D}}(\bar{x}y)$, and

$$\mathcal{J} = H(\mathcal{A},\tau) \cong \begin{bmatrix} \mathcal{O}_X & 0 \\ 0 & \mathcal{E}nd_X\mathcal{M} \end{bmatrix}$$

contains the non-trivial idempotent $f = \mathrm{diag}(0, id)$ of trace 1. Use that $\mathcal{M}\otimes\mathcal{M} \cong \mathcal{M}\otimes\mathcal{M}^\vee \cong \mathcal{O}_X \oplus \mathcal{M}'$ for some $\mathcal{O}_X$-module $\mathcal{M}'$ of rank 8. Hence the $\mathcal{O}_X$-module

$$\mathcal{J}_0 \cong \mathcal{M} \otimes \mathcal{M} \cong \mathcal{O}_X \oplus \mathcal{M}'$$

can be made into a Jordan algebra and $\mathcal{M}'$ into a pseudo-composition algebra. It is straightforward to check that $\mathcal{J}_0 \cong \mathcal{E}nd_X(\mathcal{M})^+$.

(B) Let τ_1 and τ_2 be hat-involutions. Then $\mathcal{D}_i = \mathrm{Cay}(\mathcal{T}_i, \mathcal{P}_i, N_{\mathcal{P}_i})$ with $\mathcal{T}_i = \mathcal{O}_X \oplus \mathcal{L}_i$ a composition algebra of rank 2, $\mathcal{L}_i \in \mathrm{Pic}\, X$ of order 2, and $H(\mathcal{D}_i, \hat{\ }) = \mathcal{O}_X \oplus \mathcal{P}_i$ for $i = 2, 3$. We obtain

$$\mathcal{J} = H(\mathcal{A},\tau) \cong \mathcal{O}_X \oplus (\mathcal{L}_1 \otimes \mathcal{L}_2) \oplus \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus (\mathcal{P}_1 \otimes \mathcal{P}_2).$$

The $\mathcal{O}_X$-module

$$\mathcal{J}_0 \cong (\mathcal{L}_1 \otimes \mathcal{L}_2) \oplus \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus (\mathcal{P}_1 \otimes \mathcal{P}_2).$$

can be made into an admissible cubic algebra over X, but not always into a cubic Jordan algebra, see Example 5 below.

(B1) In particular, let $\mathcal{D} = \mathcal{D}_1 = \mathcal{D}_2 = \mathrm{Cay}(\mathcal{T}, \mathcal{P}, N_{\mathcal{P}})$ with $\mathcal{T} = \mathrm{Cay}(\mathcal{O}_X, \mathcal{L}, N_{\mathcal{L}})$. Then, analogously as in [KMRT, (11.1)], we have

$$(\mathcal{D}, \hat{\ }) \otimes (\mathcal{D}, \hat{\ }) \cong (\mathcal{E}nd_X(\mathcal{D}), \sigma) \cong (\begin{bmatrix} \mathcal{O}_X & \mathcal{L} & \mathcal{P} \\ \mathcal{L}^\vee & \mathcal{O}_X & \mathcal{H}om_X(\mathcal{L},\mathcal{P}) \\ \mathcal{P}^\vee & \mathcal{H}om_X(\mathcal{P},\mathcal{L}) & \mathcal{E}nd_X(\mathcal{P}) \end{bmatrix}, \sigma)$$

where σ is the adjoint involution with respect to the involution trace form $T_{(\mathcal{D},\hat{\ })}$ now. We obtain

$$\mathcal{J} = H(\mathcal{A},\tau) \cong \begin{bmatrix} \mathcal{O}_X & 0 & \mathcal{P} \\ 0 & \mathcal{O}_X & 0 \\ \mathcal{P}^\vee & 0 & \mathcal{E}nd_X(\mathcal{P}) \end{bmatrix}.$$

$f = \mathrm{diag}(0,1,0)$ is a non-trivial idempotent in $H^0(X, \mathcal{J})$ of trace 1. Thus

$$\mathcal{J}_0 \cong \mathcal{O}_X \oplus \mathcal{P} \oplus \mathcal{P}^\vee \oplus (\mathcal{P} \otimes \mathcal{P}^\vee)$$

can be made into a cubic Jordan algebra and there also exists a pseudo-composition algebra structure on the $\mathcal{O}_X$-module

$$\mathcal{P} \oplus \mathcal{P}^\vee \oplus (\mathcal{P} \otimes \mathcal{P}^\vee).$$

Question 1. Do we have $\mathcal{J}_0 \cong \mathcal{E}nd_X(\mathcal{O}_X \oplus \mathcal{P})^+$?

Cases (A) and (B) cover all possible orthogonal involutions on $\mathcal{A} = \mathcal{D}_1 \otimes \mathcal{D}_2$ which are tensor products of involutions on the $\mathcal{D}_i$'s.

5. Cubic Jordan algebras obtained from Azumaya algebras of rank 16

We now construct examples of admissible cubic algebras of rank 15, cubic Jordan algebras of rank 15, whose residue class algebras are central simple, and pseudo-composition algebras of rank 14:

(I) For every locally free $\mathcal{O}_X$-module $\mathcal{E}$ of constant rank 4, $\mathcal{B} = \mathcal{E}nd_X(\mathcal{E})$ is an Azumaya algebra of rank 16 and $\mathcal{J} = \mathcal{B}^+$ is a quartic Jordan algebra over X. We have $\mathcal{E}nd(\mathcal{E}) \cong \mathcal{O}_X \oplus \mathcal{E}'$, where $\mathcal{E}'$ is the subspace of the endomorphisms of trace 0, and $\mathcal{J}_0 \cong \mathcal{E}'$ can be made into an admissible cubic algebra over X.

(I.1) Let $\mathcal{E}$ be indecomposable. Then $\mathcal{J}$ only contains trivial idempotents by Proposition 3. Therefore $\mathcal{J}_0$ cannot be made into a cubic Jordan algebra. There also is no pseudo-composition algebra attached to it.

(I.2) Let $\mathcal{M}$ be a locally free $\mathcal{O}_X$-module of constant rank 3 over X. Let $\mathcal{E} = \mathcal{L} \oplus \mathcal{M}$ with $\mathcal{L} \in \operatorname{Pic} X$. We obtain

$$\mathcal{J} \cong \begin{bmatrix} \mathcal{O}_X & \mathcal{L}^\vee \otimes \mathcal{M} \\ \mathcal{L} \otimes \mathcal{M}^\vee & \mathcal{E}nd_X(\mathcal{M}) \end{bmatrix}.$$

Proposition 7. *Let $\mathcal{B} = \mathcal{E}nd_X(\mathcal{M} \oplus \mathcal{L})$ be an Azumaya algebra of rank 16 over X, with $\mathcal{M}$ an $\mathcal{O}_X$-module of rank 3 and $\mathcal{L}$ a line bundle on X. Consider the quartic Jordan algebra $\widetilde{\mathcal{J}} = \mathcal{A}^+ = \mathcal{O}_X \oplus \mathcal{E}nd_X(\mathcal{M})$. Then $\widetilde{\mathcal{J}}$ is a quartic Jordan subalgebra of $\mathcal{J} = \mathcal{B}^+$ of rank 10.*
(a) By projecting the multiplication on the trace zero elements, both $\widetilde{\mathcal{J}}_0$ and $\mathcal{J}_0$ can be made into an admissible cubic algebra over X and $\widetilde{\mathcal{J}}_0$ is an admissible cubic subalgebra of $\mathcal{J}_0$.
(b) Suppose X is an integral scheme.
(i) For a non-trivial idempotent $f \in H^0(X, \widetilde{\mathcal{J}})$, the orthogonal complements $\widetilde{\mathcal{V}}$ of $f\mathcal{O}_X$ in $\widetilde{\mathcal{J}}_0$ (resp., $\mathcal{V}$ of $f\mathcal{O}_X$ in $\mathcal{J}_0$) can be made into pseudo-composition algebras and $\widetilde{\mathcal{V}}$ is a pseudo-composition subalgebra of $\mathcal{V}$.
(ii) $\widetilde{\mathcal{J}}_0$ and $\mathcal{J}_0$ can be made into cubic Jordan algebras and $\widetilde{\mathcal{J}}_0$ is a cubic Jordan subalgebra of $\mathcal{J}_0$ which is isomorphic to $\mathcal{E}nd_X(\mathcal{M})^+$.

Proof. The algebra $\mathcal{A}^+ = \mathcal{O}_X \oplus \mathcal{E}nd(\mathcal{M})$ is a quartic Jordan algebra over X with nondegenerate quartic norm $N(a + x) = aN_0(x)$, where N_0 is the norm of $\mathcal{E}nd(\mathcal{M})$. $\widetilde{\mathcal{J}} = \mathcal{A}^+$ is a subalgebra of $\mathcal{B}^+$: the inclusion is given by

$$a + M \to \begin{bmatrix} M & & 0 \\ & & 0 \\ 0 & 0 & a \end{bmatrix},$$

for $a \in \mathcal{O}_X$, $M \in \mathcal{E}nd_X(\mathcal{M})$. There is a non-trivial idempotent $f \in H^0(X, \widetilde{\mathcal{J}})$, $f = \operatorname{diag}(1, 0, 0)$, of trace 1. Now use Theorem 6 and Proposition 6. □

Moreover, we have $\mathcal{E}nd_X(\mathcal{M}) \cong \mathcal{O}_X \oplus \mathcal{M}'$, where $\mathcal{M}'$ is the subspace of the endomorphisms of trace 0. The subalgebra $\widetilde{\mathcal{J}}_0$ contains the non-trivial idempotent $0 + id$ of trace 1. By Proposition 7, the $\mathcal{O}_X$-module

$$\mathcal{J}_0 \cong \mathcal{E}nd_X(\mathcal{M}) \oplus (\mathcal{L}^\vee \otimes \mathcal{M}) \oplus (\mathcal{L} \otimes \mathcal{M}^\vee)$$

can be made both into an admissible cubic algebra with an admissible cubic subalgebra defined on the vector bundle $\mathcal{O}_X \oplus \mathcal{M}'$ and into a cubic Jordan algebra with cubic Jordan subalgebra $\mathcal{E}nd_X(\mathcal{M})^+$. The $\mathcal{O}_X$-module

$$\mathcal{M}' \oplus (\mathcal{L} \otimes \mathcal{M}^\vee) \oplus (\mathcal{L}^\vee \otimes \mathcal{M})$$

can be made into a pseudo-composition algebra with a pseudo-composition subalgebra defined on the vector bundle $\mathcal{M}'$.

(II) Let $\mathcal{J} = \mathcal{D}_1 \otimes \mathcal{D}_2$ with $\mathcal{D}_i = \mathrm{Quat}(\mathcal{M}_i, N_i)$ $(i = 1, 2)$ two quaternion algebras over X. Then

$$\mathcal{J} \cong \mathcal{O}_X \oplus \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus (\mathcal{M}_1 \otimes \mathcal{M}_2)$$

and

$$\mathcal{J}_0 \cong \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus (\mathcal{M}_1 \otimes \mathcal{M}_2)$$

is an admissible cubic algebra.

(II.1) If $\mathcal{D}_1 = \mathcal{D}_2 = \mathrm{Quat}(\mathcal{M}, N_\mathcal{M})$ then

$$\mathcal{J} \cong \mathcal{E}nd_X(\mathcal{O}_X \oplus \mathcal{M}) \cong \begin{bmatrix} \mathcal{O}_X & \mathcal{M} \\ \mathcal{M}^\vee & \mathcal{E}nd(\mathcal{M}) \end{bmatrix}$$

is of type (I.2). Hence

$$\mathcal{J}_0 \cong \mathcal{M} \oplus \mathcal{M} \oplus \mathcal{E}nd(\mathcal{M})$$

is an admissible cubic algebra, can be made into a cubic Jordan algebra with Jordan subalgebra $\mathcal{E}nd_X(\mathcal{M})^+$ and

$$\mathcal{M}' \oplus \mathcal{M}^\vee \oplus \mathcal{M},$$

can be made into a pseudo-composition algebra with a pseudo-composition subalgebra defined on $\mathcal{M}'$.

6. Quartic Jordan algebras of rank 28

Let $\mathcal{A}$ be an Azumaya algebra over X of constant rank 64 with a symplectic involution τ. Then $\mathcal{J} = H(\mathcal{A}, \tau)$ together with the multiplication $x \cdot y = \frac{1}{2}(xy + \tau(xy))$ is a quartic Jordan subalgebra of $\mathcal{A}^+$ which is locally free of constant rank 28 as an $\mathcal{O}_X$-module.

(A*) Let $\mathcal{A} = \mathcal{E}nd_X(\mathcal{E})$ with $\mathcal{E}$ a locally free $\mathcal{O}_X$-module of rank 8. If $\mathcal{E}$ carries a nondegenerate skew-symmetric bilinear form $b : \mathcal{E} \to \mathcal{E}^\vee$, then b induces an involution σ_b on $\mathcal{A}$ which is symplectic [Kn, p. 172 ff.]. If $\mathcal{E}$ is indecomposable, then the trace zero elements of $\mathcal{J} = H(\mathcal{A}, \sigma_b)$ are an admissible cubic algebra, but $\mathcal{J}$ does not contain non-trivial idempotents (Proposition 3).

Question 2. If $\mathcal{E}$ is decomposable, how could the trace zero elements of $\mathcal{J} = H(\mathcal{A}, \sigma_b)$ look like?

Let $(\mathcal{B}, \tau_1)$ be an Azumaya algebra of constant rank 16 over X with an involution of the first kind and $\mathcal{D} = \mathrm{Quat}(\mathcal{M}, N)$ be a quaternion algebra over X with an involution of the first kind τ_2. Let $\mathcal{A} = \mathcal{B} \otimes \mathcal{D}$ with involution $\tau = \tau_1 \otimes \tau_2$. Then τ is symplectic if τ_1 is symplectic and τ_2 is orthogonal or vice versa.

(B*) Let τ_1 be symplectic and τ_2 orthogonal. Then $\mathcal{D} \cong \mathrm{Cay}(\mathcal{T}, \mathcal{P}, N)$ where $\mathcal{T} = \{x \in \mathcal{D} \,|\, \tau_2(x) = \bar{x}\} \cong \mathcal{O}_X \oplus \mathcal{N}$ and τ_2 is isomorphic to the hat-involution. We have

$$H(\mathcal{A}, \tau) \cong \mathrm{Skew}(\mathcal{B}, \tau_1) \otimes \mathcal{N} \oplus H(\mathcal{B}, \tau_1) \otimes (\mathcal{O}_X \oplus \mathcal{P}).$$

(C*) Let τ_1 be orthogonal and τ_2 symplectic. Then τ_2 is the canonical involution and

$$\mathcal{J} = H(\mathcal{A}, \tau) \cong \mathrm{Skew}(\mathcal{B}, \tau_1) \otimes \mathcal{M} \oplus H(\mathcal{B}, \tau_1) \otimes \mathcal{O}_X.$$

Now $H(\mathcal{B}, \tau_1)$ is a Jordan algebra of degree 4 and rank 10 which is a subalgebra of $\mathcal{J}$. If it has a non-trivial idempotent of trace 1, then our construction method yields an Albert algebra $\mathcal{J}_0$ and a cubic Jordan subalgebra $H(\mathcal{B}, \tau_1)_0$ of rank 9 inside $\mathcal{J}_0$. (The choice of the quaternion algebra $\mathcal{D}$ is not relevant here, only the fact that $H(\mathcal{B}, \tau_1)$ has a non-trivial idempotent of trace 1.) If this subalgebra is of the kind $\mathcal{E}^+$ for an Azumaya algebra $\mathcal{E}$, $\mathcal{J}_0$ is a first Tits construction starting with $\mathcal{E}$; if it is of the type $H(\mathcal{C}, *)$ with $\mathcal{C}$ an Azumaya algebra of rank 9 with an involution of the second kind as in [Pu4, 1.5], it is a Tits process.

We now take a closer look at tensor products of three quaternion algebras and their involutions: Let $(\mathcal{D}_i, \tau_i)$ be a quaternion algebra with an involution of the first kind over X, $1 \leq i \leq 3$. Let $\mathcal{A} = \mathcal{D}_1 \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$ with involution $\tau = \tau_1 \otimes \tau_2 \otimes \tau_3$. Then τ is symplectic if either all the τ_i are canonical involutions or if only one of them is, and the other two are orthogonal, i.e. hat-involutions.

(D*) Let all τ_i be canonical. Write $\mathcal{D}_i = \mathrm{Quat}(\mathcal{M}_i, N_i)$ with $\mathcal{M}_i = \ker t_i = \mathrm{Skew}(\mathcal{D}_i, {}^-)$. Then

$$\mathcal{J} = H(\mathcal{A}, \tau) \cong \mathcal{O}_X \oplus (\mathcal{M}_1 \otimes \mathcal{M}_2) \oplus (\mathcal{M}_2 \otimes \mathcal{M}_3) \oplus (\mathcal{M}_1 \otimes \mathcal{M}_3).$$

Thus the $\mathcal{O}_X$-module

$$(\mathcal{M}_1 \otimes \mathcal{M}_2) \oplus (\mathcal{M}_2 \otimes \mathcal{M}_3) \oplus (\mathcal{M}_1 \otimes \mathcal{M}_3)$$

can be made into an admissible cubic algebra over X. It cannot always be made into an Albert algebra, though:

Example 3. Let X be an elliptic curve of type III. Let $\mathcal{M}_1 = \mathcal{F}_3$, $\mathcal{M}_2 = tr_{l_1/k}(\mathcal{L}_1)$ and $\mathcal{M}_3 = \mathcal{O}_X^3$. Then

$$\mathcal{J}_0 \cong tr_{l_1/k}(\mathcal{L}_1 \otimes \mathcal{F}_3) \oplus tr_{l_1/k}(\mathcal{L}_1) \oplus tr_{l_1/k}(\mathcal{L}_1) \oplus tr_{l_1/k}(\mathcal{L}_1)$$

can be made into an admissible cubic algebra, but cannot be made into an Albert algebra.

Let w.l.o.g. $(\mathcal{B}, \tau_1) = (\mathcal{D}_1 \otimes \mathcal{D}_2, {}^- \otimes {}^-)$ as in (D*). If the Jordan algebra $H(\mathcal{B}, \tau_1) \cong \mathcal{O}_X \oplus \mathcal{M}_1 \otimes \mathcal{M}_2$ has a non-trivial idempotent of trace 1, then $H(\mathcal{B}, \tau_1)_0 \cong \mathcal{M}_1 \otimes \mathcal{M}_2$ is a cubic Jordan subalgebra of rank 9 in the Albert algebra $\mathcal{J}_0$. (Note that the third quaternion algebra $\mathcal{D}_3$ is not important in this argument, it can be any such algebra.)

Question 3. Does $H(\mathcal{B}, \tau_1) \cong \mathcal{O}_X \oplus (\mathcal{M}_1 \otimes \mathcal{M}_2)$ have a non-trivial idempotent of trace 1 only if $\mathcal{D}_1 \cong \mathcal{D}_2$?

(D1*) In particular, let $\mathcal{D} = \mathcal{D}_1 = \mathcal{D}_2 = \mathrm{Quat}(\mathcal{M}, N)$ in (D*).

Theorem 7. *In the setting of (D1*), the Albert algebra* $\mathcal{J}_0$ *is a first Tits construction* $\mathcal{J}(\mathcal{E},\mathcal{P}_0,N_{\mathcal{P}_0})$ *starting with the Azumaya algebra* $\mathcal{E}nd_X(\mathcal{M})$.

Moreover,

$$(\mathcal{M}\otimes\mathcal{M}_3)\oplus(\mathcal{M}\otimes\mathcal{M}_3)\oplus\mathcal{M}'$$

can be made into a pseudo-composition algebra, where $\mathcal{E}nd_X\mathcal{M}\cong\mathcal{O}_X\oplus\mathcal{M}'$.

Proof. Obviously,

$$(\mathcal{D},{}^-)\otimes(\mathcal{D},{}^-)\cong(\mathcal{E}nd_X(\mathcal{D}),\sigma)$$

as in (A1) and

$$(\mathcal{A},\tau)\cong(\mathcal{E}nd_X(\mathcal{D}),\sigma)\otimes(\mathcal{D}_3,{}^-).$$

By (C*),

$$\mathcal{J}=H(\mathcal{A},\tau)\cong\mathrm{Skew}(\mathcal{E}nd_X(\mathcal{D}),\sigma_{N_\mathcal{D}})\otimes\mathcal{M}_3\oplus H(\mathcal{E}nd_X(\mathcal{D}),\sigma)\otimes\mathcal{O}_X$$

with

$$H(\mathcal{E}nd_X(\mathcal{D}),\sigma)\cong H(\mathcal{D}\otimes\mathcal{D},{}^-\otimes{}^-)\cong\begin{bmatrix}\mathcal{O}_X & 0\\ 0 & \mathcal{E}nd_X\mathcal{M}\end{bmatrix}.$$

Now $f=\mathrm{diag}(0,1)\otimes 1$ is a non-trivial idempotent in $H^0(X,\mathcal{J})$ of trace $t_\mathcal{J}(f)=\frac{1}{2}t_\mathcal{A}(f)=1$ which lies in the quartic Jordan subalgebra $H(\mathcal{E}nd_X(\mathcal{D}),\sigma)$ of $\mathcal{J}$. By Theorem 6, hence $H(\mathcal{E}nd_X(\mathcal{D}),\sigma)_0\cong\mathcal{E}nd_X(\mathcal{M})^+$ (by (A1)) is a cubic Jordan subalgebra of the Albert algebra $\mathcal{J}_0$ through our construction. Thus we obtain $\mathcal{J}_0$ as a first Tits construction starting with $\mathcal{E}nd_X(\mathcal{M})$ and

$$\mathcal{J}_0\cong\mathcal{E}nd_X\mathcal{M}\oplus(\mathcal{M}\otimes\mathcal{M}_3)\oplus(\mathcal{M}\otimes\mathcal{M}_3)$$

can be made into an Albert algebra which contains the Jordan subalgebra $\mathcal{E}nd_X\mathcal{M}^+$ as well as

$$(\mathcal{M}\otimes\mathcal{M}_3)\oplus(\mathcal{M}\otimes\mathcal{M}_3)\oplus\mathcal{M}'$$

into a pseudo-composition algebra, where $\mathcal{E}nd_X\mathcal{M}\cong\mathcal{M}\otimes\mathcal{M}=\mathcal{O}_X\oplus\mathcal{M}'$.

□

Remark 6. In (D1*), the Albert algebra $\mathcal{J}_0$ is a first Tits construction $\mathcal{J}_0\cong\mathcal{J}(\mathcal{E},\mathcal{P}_0,N_{\mathcal{P}_0})$ starting with $\mathcal{E}=\mathcal{E}nd_X(\mathcal{M})$, where $\mathcal{M}$ has trivial determinant. However, not all possible first Tits constructions starting with this algebra are obtained in (D1*). (D1*) only covers the locally free left $\mathcal{E}$-modules $\mathcal{P}_0$ of the kind $\mathcal{P}_0\cong\mathcal{M}\otimes\mathcal{F}$, for some *selfdual* $\mathcal{F}$ of rank 3 with trivial determinant. That means, those with $\mathcal{P}_0\cong\mathcal{M}\otimes\mathcal{F}$, for some $\mathcal{F}$ of rank 3 with trivial determinant which are not selfdual, are not obtained.

(E*) Let τ_1 be canonical and τ_2, τ_3 be hat-involutions. Then $\mathcal{D}_i=\mathrm{Cay}(\mathcal{T}_i,\mathcal{P}_i,n_i)$ with $\mathcal{T}_i=\mathcal{O}_X\oplus\mathcal{L}_i$, $\mathcal{L}_i\in\mathrm{Pic}\,X$ of order 2 and $H(\mathcal{D}_i,\hat{\ })=\mathcal{O}_X\oplus\mathcal{P}_i$ for $i=2,3$. Let $\mathcal{D}_1=\mathrm{Quat}(\mathcal{M}_1,N_1)$. We obtain

$$H(\mathcal{A},\tau)\cong\mathcal{O}_X\oplus\mathcal{P}_2\oplus\mathcal{P}_3\oplus(\mathcal{P}_2\otimes\mathcal{P}_3)\oplus(\mathcal{M}_1\otimes\mathcal{L}_2)\oplus(\mathcal{M}_1\otimes\mathcal{L}_2)\otimes\mathcal{P}_3\oplus(\mathcal{L}_2\otimes\mathcal{L}_3)\oplus(\mathcal{M}_1\otimes\mathcal{L}_3)\oplus(\mathcal{M}_1\otimes\mathcal{P}_2\otimes\mathcal{L}_3)$$

and the $\mathcal{O}_X$-module

$$\mathcal{P}_2\oplus\mathcal{P}_3\oplus(\mathcal{P}_2\otimes\mathcal{P}_3)\oplus(\mathcal{M}_1\otimes\mathcal{L}_2)\oplus(\mathcal{M}_1\otimes\mathcal{L}_2\otimes\mathcal{P}_3)\oplus(\mathcal{L}_2\otimes\mathcal{L}_3)\oplus(\mathcal{M}_1\otimes\mathcal{L}_3)\oplus(\mathcal{M}_1\otimes\mathcal{P}_2\otimes\mathcal{L}_3)$$

can be made into an admissible cubic algebra over X.

(E1*) In particular, if $\mathcal{D}_2=\mathcal{D}_3=\mathcal{D}=\mathrm{Cay}(\mathcal{T},\mathcal{P},N)$ with $\mathcal{T}=\mathrm{Cay}(\mathcal{T},\mathcal{L},N_{\mathcal{L}})$ and $\tau_2=\tau_3$, then we have

$$(\mathcal{D},\hat{\ })\otimes(\mathcal{D},\hat{\ })\cong(\mathcal{E}nd_X(\mathcal{D}),\sigma),$$

where σ is the adjoint involution with respect to the involution trace form $T_{(\mathcal{D},\hat{\ })}$. Now

$$(\mathcal{A},\tau)\cong(\mathcal{D}_1,\bar{\ })\otimes(\mathcal{E}nd_X(\mathcal{D}),\sigma)$$

is of type (C*), hence

$$H(\mathcal{A},\tau)\cong\mathrm{Skew}(\mathcal{E}nd_X(\mathcal{D}),\sigma)\otimes\mathcal{M}_1\oplus H(\mathcal{E}nd_X(\mathcal{D}),\sigma)\otimes\mathcal{O}_X$$

with

$$\mathcal{E}nd_X(\mathcal{D})\cong\begin{bmatrix}\mathcal{O}_X & \mathcal{L} & \mathcal{P}\\ \mathcal{L} & \mathcal{O}_X & \mathcal{H}om_X(\mathcal{L},\mathcal{P})\\ \mathcal{P} & \mathcal{H}om_X(\mathcal{P},\mathcal{L}) & \mathcal{E}nd_X(\mathcal{P})\end{bmatrix}$$

and

$$H(\mathcal{E}nd_X(\mathcal{D}),\sigma)\cong\begin{bmatrix}\mathcal{O}_X & 0 & \mathcal{P}\\ 0 & \mathcal{O}_X & 0\\ \mathcal{P} & 0 & \mathcal{E}nd_X(\mathcal{P})\end{bmatrix}.$$

$f=\mathrm{diag}(0,0,1)$ is a non-trivial idempotent in $H(\mathcal{E}nd_X(\mathcal{D}),\sigma)$ of trace 1 and $H(\mathcal{E}nd_X(\mathcal{D}),\sigma)_0\cong\mathcal{O}_X\oplus\mathcal{P}\oplus\mathcal{P}\oplus\mathcal{P}\otimes\mathcal{P}$. Thus

$$\mathcal{J}_0\cong\mathcal{O}_X\oplus(\mathcal{P}\otimes\mathcal{P})\oplus\mathcal{P}\oplus\mathcal{P}\oplus(\mathcal{L}\otimes\mathcal{M}_1)\oplus(\mathcal{L}\otimes\mathcal{M}_1)\oplus(\mathcal{L}\otimes\mathcal{P}\otimes\mathcal{M}_1)\oplus(\mathcal{L}\otimes\mathcal{P}\otimes\mathcal{M}_1)$$

can be made into an Albert algebra and there also exists a pseudo-composition algebra structure on the $\mathcal{O}_X$-module

$$(\mathcal{P}\otimes\mathcal{P})\oplus\mathcal{P}\oplus\mathcal{P}\oplus(\mathcal{L}\otimes\mathcal{M}_1)\oplus(\mathcal{L}\otimes\mathcal{M}_1)\oplus(\mathcal{L}\otimes\mathcal{P}\otimes\mathcal{M}_1)\oplus(\mathcal{L}\otimes\mathcal{P}\otimes\mathcal{M}_1).$$

The Albert algebra $\mathcal{J}_0$ contains $(H(\mathcal{E}nd_X(\mathcal{D}),\sigma))_0$ as a cubic Jordan subalgebra of rank 9.

Question 4. Do we have $(H(\mathcal{E}nd_X(\mathcal{D}),\sigma))_0\cong\mathcal{E}^+$ for some Azumaya algebra $\mathcal{E}$ over X? E.g., for $\mathcal{E}=\mathcal{E}nd_X(\mathcal{O}_X\oplus\mathcal{P})$? Then $\mathcal{J}_0$ would be a first Tits construction and the locally free left $\mathcal{E}$-modules $\mathcal{P}_0$ used would be of the kind $\mathcal{P}_0\cong(\mathcal{O}_X\oplus\mathcal{P})\otimes(\mathcal{L}\otimes\mathcal{M}_3)$, for some selfdual vector bundle $\mathcal{M}_1$ of rank 3 with trivial determinant and a selfdual line bundle $\mathcal{L}$. These have determinant $\mathcal{L}$, the same as $\mathcal{O}_X\oplus\mathcal{P}$, but might not cover all possible cases.

(E2*) Let τ_1 be canonical and τ_2, τ_3 be hat-involutions. Suppose $\mathcal{D}=\mathcal{D}_1=\mathcal{D}_2=\mathrm{Cay}(\mathcal{T},\mathcal{P},n)$ with $\mathcal{T}=\mathrm{Cay}(\mathcal{O}_X,\mathcal{L},N_{\mathcal{N}})$ and that $\mathcal{D}_3=\mathrm{Cay}(\mathcal{T}_3,\mathcal{P}_3,N_3)$ with $\mathcal{T}_3=\mathrm{Cay}(\mathcal{O}_X,\mathcal{L}_3,N_{\mathcal{L}_3})$. We obtain $\mathcal{D}\otimes\mathcal{D}\cong\mathcal{E}nd_X(\mathcal{D})$ with

$$\mathcal{E}nd_X(\mathcal{D})\cong\begin{bmatrix}\mathcal{O}_X & \mathcal{L} & \mathcal{P}\\ \mathcal{L} & \mathcal{O}_X & \mathcal{H}om_X(\mathcal{L},\mathcal{P})\\ \mathcal{P} & \mathcal{H}om_X(\mathcal{P},\mathcal{L}) & \mathcal{E}nd_X(\mathcal{P})\end{bmatrix}.$$

We have $H(\mathcal{A},\tau)\cong\mathrm{Skew}(\mathcal{B},\tau_1)\otimes\mathcal{L}_3\oplus H(\mathcal{B},\tau_1)\otimes(\mathcal{O}_X\oplus\mathcal{P}_3)$ as in (B*) with

$$\mathrm{Skew}(\mathcal{B},\tau_1)\cong\mathcal{L}\oplus\mathcal{L}\oplus\mathcal{P}\oplus(\mathcal{P}\otimes\mathcal{L})\oplus(\mathcal{P}\otimes\mathcal{P})$$

and with

$$H(\mathcal{B},\tau_1)\cong\mathcal{O}_X^2\oplus\mathcal{P}\oplus(\mathcal{P}\otimes\mathcal{L}).$$

Thus

$$H(\mathcal{A},\tau)\cong\mathcal{O}_X\oplus\mathcal{P}\oplus\mathcal{P}_3\oplus(\mathcal{P}\otimes\mathcal{P}_3)\oplus\mathcal{O}_X\oplus(\mathcal{P}\otimes\mathcal{L})\oplus$$
$$\mathcal{P}_3\oplus(\mathcal{P}\otimes\mathcal{L}\otimes\mathcal{P}_3)\oplus(\mathcal{L}\otimes\mathcal{L}_3)\oplus(\mathcal{L}\otimes\mathcal{L}_3)\oplus(\mathcal{P}\otimes\mathcal{L}_3)\oplus(\mathcal{L}\otimes\mathcal{P}\otimes\mathcal{L}_3)\oplus(\mathcal{P}\otimes\mathcal{P}\otimes\mathcal{L}_3).$$

A straightforward calculation shows that $f\otimes 1=\mathrm{diag}(1,0,0)\otimes 1\in\mathcal{E}nd_X(\mathcal{D})\otimes\mathcal{D}_3$ is a non-trivial idempotent in $H^0(X,\mathcal{J})$ of trace $t_{\mathcal{J}}(f\otimes 1)=1$. Thus

$$\mathcal{J}_0\cong\mathcal{O}_X\oplus\mathcal{P}\oplus\mathcal{P}_3\oplus(\mathcal{P}\otimes\mathcal{P}_3)\oplus(\mathcal{P}\otimes\mathcal{L})\oplus\mathcal{P}_3\oplus(\mathcal{P}\otimes\mathcal{L}\otimes\mathcal{P}_3)\oplus(\mathcal{L}\otimes\mathcal{L}_3)\oplus(\mathcal{L}\otimes\mathcal{L}_3)\oplus(\mathcal{P}\otimes\mathcal{L}_3)\oplus(\mathcal{L}\otimes\mathcal{P}\otimes\mathcal{L}_3)\oplus(\mathcal{P}\otimes\mathcal{P}\otimes\mathcal{L}_3)$$

can be made into an Albert algebra. (Note that the choice of the algebra $\mathcal{D}_3$ does not play a role here.) It is not clear a priori if this Albert algebra contains an Azumaya subalgebra of rank 9.

There also exists a pseudo-composition algebra structure on the $\mathcal{O}_X$-module

$$\mathcal{P}\oplus\mathcal{P}_3\oplus(\mathcal{P}\otimes\mathcal{P}_3)\oplus(\mathcal{P}\otimes\mathcal{L})\oplus\mathcal{P}_3\oplus(\mathcal{P}\otimes\mathcal{L}\otimes\mathcal{P}_3)\oplus(\mathcal{L}\otimes\mathcal{L}_3)\oplus(\mathcal{L}\otimes\mathcal{L}_3)\oplus(\mathcal{P}\otimes\mathcal{L}_3)\oplus(\mathcal{L}\otimes\mathcal{P}\otimes\mathcal{L}_3)\oplus(\mathcal{P}\otimes\mathcal{P}\otimes\mathcal{L}_3).$$

(D*) and (E*) cover all possible symplectic involutions on $\mathcal{A}=\mathcal{D}_1\otimes\mathcal{D}_2\otimes\mathcal{D}_3$ which are tensor products of involutions on the $\mathcal{D}_i$'s.

Theorem 8. *Let $\mathcal{J}=H(\mathrm{Mat}_4(\mathcal{D}),*)$ with $\mathcal{D}$ a quaternion algebra over X and with $x^*=\bar{x}^t$. The Albert algebra $\mathcal{J}_0$ as defined in Theorem 5 is reduced and isomorphic to $(H_3(\mathcal{C},*),\star)$ where $\mathcal{C}=\mathrm{Cay}(\mathcal{D},1)$,*

$$x\star y=\widetilde{x}\widetilde{y}-\frac{1}{4}(t(x\cdot y)+t(x)t(y))1$$

with $\widetilde{x}=(\widetilde{x_{ij}})$ for $x=(x_{ij})$, $\widetilde{(d_0,d_1)}=(d_0,-d_1)$.

Proof. Define $\Psi:\mathcal{J}_0\to(H_3(\mathcal{C},*),\star)$ via

$$(d_{ij})\to(d_{ij},\epsilon_{ijk}d_{k4}),$$

where ϵ_{ijk} is the totally anti-symmetric Levi-Civita symbol [E-O, p. 309]. Suppose that $\mathcal{D}$ is a quaternion algebra. By [E-O, p. 311], $\Psi(P)$ is an isomorphism of algebras for all $P\in X$, hence so is Ψ. □

An obvious consequence of this result is the fact that we will not be able to get all the reduced Albert algebras isomorphic to $(H_3(\mathcal{C},*),\star)$ for some octonion algebra $\mathcal{C}$ over X through our construction, when starting from the quartic Jordan algebra $\mathcal{J}=H(\mathrm{Mat}_4(\mathcal{D}),*)$. It only produces those where the octonion algebra $\mathcal{C}$ is a classical Cayley-Dickson doubling.

Example 4. Given a central simple algebra B of degree 4 over a field k, define $A=B\otimes\mathrm{Mat}_2(k)\cong\mathrm{Mat}_2(B)$ with symplectic involution $J_1((b_{ij}))=(\tau_1(b_{ij}))^t$. Then there is a quaternion algebra D over the algebraic closure $\bar{k}$ such that $H(A,J_1)\otimes_k\bar{k}\cong H(\mathrm{Mat}_4(D),*)$ with $x^*=\bar{x}^t$ [J, pp. 208-209].

In our general setup, this is not true: Let X be an elliptic curve over k and let $\mathcal{B}=\mathcal{E}nd(\mathcal{F}_4)\cong\mathcal{O}_X\oplus\mathcal{F}_3\oplus\mathcal{F}_5\oplus F_7$. Then $\mathcal{B}$ carries a symplectic involution induced by a skew-symmetric form on $\mathcal{F}_4$. Let $\mathcal{A}=\mathcal{B}\otimes\mathrm{Mat}_2(\mathcal{O}_X)\cong\mathrm{Mat}_2(\mathcal{B})$ with symplectic involution $J_1((b_{ij}))=(\tau_1(b_{ij}))^t$. Then $H(\mathcal{A},J_1)\cong\mathcal{F}_3\oplus\mathcal{F}_7\oplus\mathcal{O}_X^3\oplus\mathcal{F}_5^3$ (see Section 9). However, since $H(\mathrm{Mat}_4(\mathcal{D}),*)\cong\mathcal{O}_X^{10}\oplus\mathcal{M}^6$ for any quaternion algebra $\mathcal{D}=\mathrm{Quat}(\mathcal{M},N)$ over X, an isomorphism as above is impossible.

7. Azumaya algebras of rank 16 with orthogonal involution over an elliptic curve

For the remainder of the paper, let X be an elliptic curve over a field of characteristic not 2 or 3, see 1.8. In the following, we repeatedly use [AEJ2, 2.2] and [At, Theorem 8, Lemma 21, 22].

Let $\mathcal{A}$ be an Azumaya algebra over X of rank 16, with an orthogonal involution τ. Then $\mathcal{J} = H(\mathcal{A}, \tau)$ is a Jordan algebra of degree 4, which is locally free of rank 10. We apply the results of Section 4.

Theorem 9. *The following $\mathcal{O}_X$-modules can be made into an admissible cubic algebra over X with multiplication as in Theorem 2:*

(1) *The indecomposable $\mathcal{O}_X$-module $tr_{l_1/k}(\mathcal{F}_3 \otimes \mathcal{L}_1)$ (if X has type III);*
(2) $\mathcal{F}_3 \otimes \mathcal{L}_1 \oplus tr_{l_2/k}(\mathcal{F}_3 \otimes \mathcal{L}_2)$ *(if X has type II);*
(3) $(\mathcal{F}_3 \otimes \mathcal{L}_1) \oplus (\mathcal{F}_3 \otimes \mathcal{L}_2) \oplus (\mathcal{F}_3 \otimes \mathcal{L}_3)$ *(if X has type I);*
(4) $\mathcal{F}_3 \oplus \mathcal{F}_3 \oplus \mathcal{F}_3$*;*
(5) $tr_{l_1/k}(\mathcal{L}_1) \oplus tr_{l_1/k}(\mathcal{L}_1) \oplus tr_{l_1/k}(\mathcal{L}_1)$ *(if X has type III);*
(6) $\mathcal{L}_1^3 \oplus [tr_{l_2/k}(\mathcal{L}_2)]^3$ *(if X has type II);*
(7) $\mathcal{L}_1^3 \oplus \mathcal{L}_2^3 \oplus \mathcal{L}_3^3$ *(if X has type I).*

Proof. We look at the quartic Jordan algebra $\mathcal{J} = H(\mathcal{A}, \tau)$, where $(\mathcal{A}, \tau) = (\mathcal{D}_1 \otimes \mathcal{D}_2, {}^- \otimes {}^-)$ is the tensor product of two quaternion algebras $\mathcal{D}_i$ with canonical involutions. The following observations imply the assertion by Theorem 2 and Section 4 (A):

(1) $\mathcal{D}_1 = \mathcal{E}nd_X(\mathcal{F}_2)$, $\mathcal{D}_2 = \mathrm{Quat}(tr_{l_1/k}(\mathcal{L}_1), N)$.
(2) $\mathcal{D}_1 = \mathcal{E}nd_X(\mathcal{F}_2)$, $\mathcal{D}_2 = \mathrm{Cay}(\mathcal{T}, tr_{l_2/k}(\mathcal{L}_2), N_{\mathcal{P}})$ with $\mathcal{T} = \mathrm{Cay}(\mathcal{O}_X, \mathcal{L}_1, N_1)$.
(3) $\mathcal{D}_1 = \mathcal{E}nd_X(\mathcal{F}_2)$, $\mathcal{D}_2 = \mathrm{Cay}(\mathcal{T}, \mathcal{L}_2 \oplus \mathcal{L}_3, N_{\mathcal{P}})$ with $\mathcal{T} = \mathrm{Cay}(\mathcal{O}_X, \mathcal{L}_1, N_1)$.
(4) $\mathcal{D}_1 = \mathcal{E}nd_X(\mathcal{F}_2)$, $\mathcal{D}_2$ is defined over k.
(5) $\mathcal{D}_1$ is defined over k, $\mathcal{D}_2$ is as in (i).
(6) $\mathcal{D}_1$ is defined over k, $\mathcal{D}_2$ is as in (ii).
(7) $\mathcal{D}_1$ is defined over k, $\mathcal{D}_2$ is as in (iii).

□

In cases (1) to (3) of Theorem 9, both $\mathcal{D}_1$ and $\mathcal{D}_2$ only contain trivial idempotents by Proposition 1. From the module structure it is also clear that in all cases $\mathcal{J}$ contains only trivial idempotents. Note that the above list is not exhaustive, we only give some examples of admissible cubic algebras with interesting underlying module structures.

Example 5. The pseudo-composition algebras which arise by Theorem 4 and Section 4 (A1) when looking at the quartic Jordan algebra $\mathcal{J} = H(\mathcal{A}, \tau)$, where $(\mathcal{A}, \tau) = (\mathcal{D} \otimes \mathcal{D}, {}^- \otimes {}^-)$, $\mathcal{D}$ a quaternion algebra with canonical involution, are only a subclass of those which can be obtained out of trace zero elements of some cubic Jordan algebra $\mathcal{E}nd_X(\mathcal{M})^+$. To see this, we look at $\mathcal{E}nd_X(\mathcal{M})^+$, where $\mathcal{M}$ is a vector bundle of rank 3. The following choices

(1) $\mathcal{M} = \mathcal{F}_3$.
(2) $\mathcal{M} = tr_{l_1/k}(\mathcal{L}_1)$ (if X is of type III and l_1/k Galois),
(3) $\mathcal{M} = \mathcal{L}_1 \oplus tr_{l_2/k}(\mathcal{L}_2)$ (if X is of type II),

(4) $\mathcal{M} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$ (if X is of type I),
(5) $\mathcal{M} \in \Omega(3,d)$ absolutely indecomposable, $\gcd(3,d) = 1$, if k has characteristic 0,
(6) $\mathcal{M}$ indecomposable, but not absolutely so, i.e. there is a suitable cubic field extension l of k and a line bundle $\mathcal{N}$ over $Y = X \times_k l$, such that $\mathcal{M} = tr_{l/k}(\mathcal{N})$. Suppose l/k is Galois with $\mathrm{Gal}(l/k) = \{id, \sigma_1, \sigma_2\}$. If X has type III this includes (2).
(7) $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \otimes \mathcal{F}_2$ for some line bundles $\mathcal{M}_i \in \mathrm{Pic}\, X$,
(8) $\mathcal{M} = \mathcal{E} \oplus tr_{l/k}(\mathcal{N})$ is the direct sum of a line bundle $\mathcal{E}$ and an indecomposable vector bundle of rank 2, i.e. l/k is a quadratic field extension with $\mathrm{Gal}(l/k) = \{id, \sigma\}$ and $\mathcal{N}$ a line bundle over $X_l = X \times_k l$, not defined over X; in particular, if X has type II, this includes (3).
(9) $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3$ with $\mathcal{M}_i \in \mathrm{Pic}\, X$,

imply that the following $\mathcal{O}_X$-modules carry the structure of a pseudo-composition algebra over X by Theorem 3:

(1) $\mathcal{F}_3 \oplus \mathcal{F}_5$,
(2) $\mathcal{O}_X^2 \oplus tr_{l_1/k}(\mathcal{L}_1) \oplus tr_{l_1/k}(\mathcal{L}_1)$,
(3) $\mathcal{O}_X^2 \oplus \mathcal{L}_1^2 \oplus [tr_{l_2/k}(\mathcal{L}_2)]^2$ (if X is of type II),
(4) $\mathcal{O}_X^2 \oplus \mathcal{L}_1^2 \oplus \mathcal{L}_2^2 \oplus \mathcal{L}_3^2$ (if X is of type I),
(5) $\mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_m \oplus tr_{l_1/k}(\mathcal{N}_{m+1}) \cdots \oplus tr_{l_j/k}(\mathcal{N}_j)$ if $m < 8$ (m depending on X), where the line bundles $\mathcal{N}_{m+1}$ over $X_{l_1}, \dots, \mathcal{N}_j$ over X_{l_j} are not defined over X, and $\mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_8$ if $m = 8$ (e.g., if k is algebraically closed) [At, Lemma 22],
(6) $\mathcal{O}_X^2 \oplus tr_{l/k}(\mathcal{N} \otimes^{\sigma_1} \mathcal{N}^\vee) \oplus tr_{l/k}(\mathcal{N} \otimes^{\sigma_2} \mathcal{N}^\vee)$,
(7) $\mathcal{O}_X \oplus \mathcal{M}_1 \otimes \mathcal{M}_2^\vee \otimes \mathcal{F}_2 \oplus \mathcal{M}_1^\vee \otimes \mathcal{M}_2 \otimes \mathcal{F}_2 \oplus \mathcal{F}_3$,
(8) $\mathcal{O}_X^2 \oplus tr_{l/k}(\mathcal{N} \otimes {}^\sigma\mathcal{N}^\vee) \oplus (\mathcal{E} \otimes tr_{l/k}(\mathcal{N}^\vee)) \oplus (\mathcal{E}^\vee \otimes tr_{l/k}(\mathcal{N}))$,
(9) $\mathcal{O}_X^2 \oplus \mathcal{M}_1^\vee \otimes \mathcal{M}_2 \oplus \mathcal{M}_1^\vee \otimes \mathcal{M}_3 \oplus \mathcal{M}_2^\vee \otimes \mathcal{M}_1 \oplus \mathcal{M}_2^\vee \otimes \mathcal{M}_3 \oplus \mathcal{M}_3^\vee \otimes \mathcal{M}_1 \oplus \mathcal{M}_3^\vee \otimes \mathcal{M}_2$.

In particular, all the $\mathcal{M}$ in (1) to (4) have trivial determinant and the corresponding pseudo-composition algebras can be obtained by choosing

(1) $\mathcal{D} = \mathcal{E}nd_X(\mathcal{F}_2)$,
(2) $\mathcal{D} = \mathrm{Quat}(tr_{l_1/k}(\mathcal{L}_1), N)$,
(3) $\mathcal{D} = \mathrm{Cay}(\mathcal{T}, tr_{l_2/k}(\mathcal{L}_2), N_{\mathcal{P}})$ with $\mathcal{T} = \mathrm{Cay}(\mathcal{O}_X, \mathcal{L}_1, n_1)$,
(4) $\mathcal{D}_2 = \mathrm{Cay}(\mathcal{T}, \mathcal{L}_2 \oplus \mathcal{L}_3, N_{\mathcal{P}})$ with $\mathcal{T} = \mathrm{Cay}(\mathcal{O}_X, \mathcal{L}_1, N_1)$.

in (A1). Those obtained by choosing $\mathcal{M}$ with non-trivial determinant cannot be obtained this way.

Example 6. Let X be of type II and τ_1 and τ_2 be hat-involutions.
(i) In the setting of Section (B), $\mathcal{J}_0$ cannot always be made into a Jordan algebra: let $\mathcal{D}_1 = \mathrm{Cay}(\mathcal{T}_1, tr_{l_2/k}(\mathcal{L}_2), N_1)$ and $\mathcal{D}_2 = \mathrm{Cay}(T_0 \otimes_k \mathcal{O}_X, \mathcal{L}_1 \oplus \mathcal{L}_1, N_2)$, where $\mathcal{T}_1 = \mathrm{Cay}(\mathcal{O}_X, \mathcal{L}_1, N)$ and T_0 is a quadratic étale defined over k. Then

$$\mathcal{J}_0 \cong \mathcal{L}_1^3 \oplus [tr_{l_2/k}(\mathcal{L}_2)]^3$$

can be made into an admissible cubic algebra over X, but not into a Jordan algebra.
(ii) Let us consider the situation of Section 4 (B1): suppose $\mathcal{D} = \mathrm{Cay}(\mathcal{T}, tr_{l_2/k}(\mathcal{L}_2), N_{\mathcal{P}})$

with $\mathcal{T} = \mathrm{Cay}(\mathcal{O}_X, \mathcal{L}_1, N_1)$ and $\hat{} = \tau_1 = \tau_2$. Then

$$\mathcal{J} = H(\mathcal{A}, \tau) \cong \begin{bmatrix} \mathcal{O}_X & 0 & tr_{l_2/k}(\mathcal{L}_2) \\ 0 & \mathcal{O}_X & 0 \\ tr_{l_2/k}(\mathcal{L}_2) & 0 & \mathcal{E}nd_X(tr_{l_2/k}(\mathcal{L}_2)) \end{bmatrix}$$

and

$$\mathcal{J}_0 \cong \mathcal{O}_X^3 \oplus \mathcal{L}_1^2 \oplus [tr_{l_2/k}(\mathcal{L}_2)]^2$$

can be made into a cubic Jordan algebra and there also exists a pseudo-composition algebra structure on the $\mathcal{O}_X$-module

$$\mathcal{O}_X^2 \oplus \mathcal{L}_1^2 \oplus [tr_{l_2/k}(\mathcal{L}_2)]^2.$$

It is not clear if the algebras obtained this way are isomorphic to $\mathcal{E}nd_X(\mathcal{L}_1 \oplus tr_{l_2/k}(\mathcal{L}_2))$.

8. Cubic Jordan algebras which come from Azumaya algebras of rank 16 over an elliptic curve

Starting with an Azumaya algebra $\mathcal{B}$ of rank 16, we construct examples of admissible cubic algebras and cubic Jordan algebras of rank 15 (whose residue class algebras are central simple) and pseudo-composition algebras of rank 14 over an elliptic curve out of the quartic Jordan algebra $\mathcal{J} = \mathcal{B}^+$. We use the results of Section 5.

Theorem 10. *The following $\mathcal{O}_X$-modules carry the structure of an admissible cubic algebra over X:*

(1)
$$\mathcal{F}_3 \oplus \mathcal{F}_5 \oplus \mathcal{F}_7;$$

(2)
$$\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n \oplus tr_{k_1/k}(\mathcal{H}_{n+1}) \cdots \oplus tr_{k_j/k}(\mathcal{H}_j),$$

if $\mathrm{char}\, k = 0$ and $n < 16$, with the line bundles $\mathcal{H}_1, \ldots, \mathcal{H}_n$ over X of order 4, and the line bundle $\mathcal{H}_i$ of order 4 defined over $X \times_k k_i$ for k_i/k a finite algebraic field extension (and not defined over X), $0 \leq i \leq j$, with the integer j depending on n; in particular we get $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{15}$ if $n = 15$ (e.g., if k is algebraically closed),

(3)
$$\mathcal{O}_X^3 \oplus tr_{l/k}(\mathcal{S} \otimes {}^{\omega_1}\mathcal{S}^\vee) \oplus tr_{l/k}(\mathcal{S} \otimes {}^{\omega_2}\mathcal{S}^\vee) \oplus tr_{l/k}(\mathcal{S} \otimes {}^{\omega_3}\mathcal{S}^\vee)$$

for a suitable quartic Galois field extension l/k with $\mathrm{Gal}(l/k) = \{id, \omega_1, \omega_2, \omega_3\}$ and a line bundle $\mathcal{S}$ over $Y = X \times_k l$;

(4)
$$\mathcal{O}_X \oplus tr_{l/k}(\mathcal{M}') \oplus tr_{l/k}(\mathcal{M} \otimes {}^{\omega}\mathcal{M}^\vee)$$

for a suitable quadratic field extension l of k, $\mathrm{Gal}(l/k) = \{id, \omega\}$ and an absolutely indecomposable vector bundle $\mathcal{M}$ of rank 2 over $Y = X \times_k l$, which is not defined over X. We write $\mathcal{E}nd(\mathcal{M}) \cong \mathcal{O}_X \oplus \mathcal{M}'$. For instance, if $\mathcal{M} \cong \mathcal{L} \otimes \mathcal{F}_2$ for a line bundle $\mathcal{L}$ over Y not defined over X, then we get the $\mathcal{O}_X$-module $\mathcal{O}_X \oplus tr_{l/k}(\mathcal{F}_3) \oplus tr_{l/k}(\mathcal{L} \otimes {}^{\omega}\mathcal{L}^\vee) \oplus tr_{l/k}(\mathcal{L} \otimes {}^{\omega}\mathcal{L}^\vee \otimes \mathcal{F}_3)$.

Proof. Let $\mathcal{B} = \mathcal{E}nd_X(\mathcal{E})$ for a vector bundle $\mathcal{E}$ of rank 4. We use Section 5 (I) and list possible $\mathcal{E}$:

(1) If $\mathcal{E}$ is absolutely indecomposable and $\mathcal{E} = \mathcal{M} \otimes \mathcal{F}_4$, $\mathcal{M} \in \operatorname{Pic} X$ a line bundle, then $\mathcal{B} \cong \mathcal{O}_X \oplus \mathcal{F}_3 \oplus \mathcal{F}_5 \oplus \mathcal{F}_7$ as $\mathcal{O}_X$-module and

$$\mathcal{J}_0 \cong \mathcal{F}_3 \oplus \mathcal{F}_5 \oplus \mathcal{F}_7$$

as $\mathcal{O}_X$-module.

(2) If $\mathcal{E}$ is absolutely indecomposable and $\mathcal{E} \in \Omega(4, d)$, $\gcd(4, d) = 1$, then

$$\overline{\mathcal{B}} \cong \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{15}$$

over $\overline{X}$, thus, if $n < 15$,

$$\mathcal{B} \cong \mathcal{O}_X \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n \oplus tr_{k_1/k}(\mathcal{H}_{n+1}) \cdots \oplus tr_{k_j/k}(\mathcal{H}_j)$$

for a suitable integer n depending on X, $0 \leq n \leq 15$, and

$$\mathcal{J}_0 \cong \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n \oplus tr_{k_1/k}(\mathcal{H}_{n+1}) \cdots \oplus tr_{k_j/k}(\mathcal{H}_j)$$

as $\mathcal{O}_X$-module, with the line bundles $\mathcal{H}_{n+1}, \ldots, \mathcal{H}_j$ of order dividing 4 not defined over X. If $n = 15$ then $\mathcal{J}_0 \cong \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{15}$.

(3) $\mathcal{E} = tr_{l/k}(\mathcal{S})$ and $\mathcal{B} \cong tr_{l/k}(\mathcal{S}) \otimes tr_{l/k}(\mathcal{S}^\vee)$ for a suitable quartic field extension l of k and a line bundle $\mathcal{S}$ over $Y = X \times_k l$. If l/k is Galois, let $\operatorname{Gal}(l/k) = \{id, \omega_1, \omega_2, \omega_3\}$. Then

$$\mathcal{B} \cong \mathcal{O}_X^4 \oplus tr_{l/k}(\mathcal{S} \otimes {}^{\omega_1}\mathcal{S}^\vee) \oplus tr_{l/k}(\mathcal{S} \otimes {}^{\omega_2}\mathcal{S}^\vee) \oplus tr_{l/k}(\mathcal{S} \otimes {}^{\omega_3}\mathcal{S}^\vee)$$

and

$$\mathcal{J}_0 \cong \mathcal{O}_X^3 \oplus tr_{l/k}(\mathcal{S} \otimes^{\omega_1} \mathcal{S}^\vee) \oplus tr_{l/k}(\mathcal{S} \otimes {}^{\omega_2}\mathcal{S}^\vee) \oplus tr_{l/k}(\mathcal{S} \otimes {}^{\omega_3}\mathcal{S}^\vee)$$

as $\mathcal{O}_X$-module.

(4) $\mathcal{E} = tr_{l/k}(\mathcal{M})$ and $\mathcal{A} \cong tr_{l/k}(\mathcal{M}) \otimes tr_{l/k}(\mathcal{M}^\vee)$ for a suitable quadratic field extension l of k and an absolutely indecomposable vector bundle $\mathcal{M}$ of rank 2 over $Y = X \times_k l$, which is not defined over X. Since $\mathcal{E}nd(\mathcal{M}) = \mathcal{O}_X \oplus \mathcal{M}'$,

$$\mathcal{B} \cong tr_{l/k}(\mathcal{O}_X \oplus \mathcal{M}') \oplus tr_{l/k}(\mathcal{M} \otimes {}^{\omega}\mathcal{M}^\vee) \cong \mathcal{O}_X^2 \oplus tr_{l/k}(\mathcal{M}') \oplus tr_{l/k}(\mathcal{M} \otimes {}^{\omega}\mathcal{M}^\vee)$$

and

$$\mathcal{J}_0 \cong \mathcal{O}_X \oplus tr_{l/k}(\mathcal{M}') \oplus tr_{l/k}(\mathcal{M} \otimes^{\omega} \mathcal{M}^\vee)$$

as $\mathcal{O}_X$-module.

□

In cases (1) and (2), $H^0(X, \mathcal{J}_0) = 0$. In none of the above cases, $\mathcal{J}_0$ can be made into a cubic Jordan algebra. There also is no pseudo-composition algebra attached to it.

Theorem 11. *The following $\mathcal{O}_X$-modules can be made into a cubic Jordan algebra over X:*

(1) $\mathcal{O}_X \oplus (\mathcal{M}_1 \otimes \mathcal{M}_2^\vee \otimes \mathcal{F}_3) \oplus (\mathcal{M}_1^\vee \otimes \mathcal{M}_2 \otimes \mathcal{F}_3) \oplus \mathcal{F}_3 \oplus \mathcal{F}_5$ *with* $\mathcal{M}_i \in \operatorname{Pic} X$, $i = 1, 2$, *arbitrary line bundles,*

(2) $\mathcal{O}_X^2 \oplus (\mathcal{M}_1 \otimes \mathcal{M}_2^\vee) \oplus (\mathcal{M}_1 \otimes \mathcal{M}_3^\vee \otimes \mathcal{F}_2) \oplus (\mathcal{M}_2 \otimes \mathcal{M}_1^\vee) \oplus (\mathcal{M}_2 \otimes \mathcal{M}_3^\vee \otimes \mathcal{F}_2) \oplus (\mathcal{M}_1^\vee \otimes \mathcal{M}_3 \otimes \mathcal{F}_2) \oplus (\mathcal{M}_2^\vee \otimes \mathcal{M}_3 \otimes \mathcal{F}_2) \oplus \mathcal{F}_3$, *with* $\mathcal{M}_i \in \operatorname{Pic} X$ $i = 1, 2, 3$, *arbitrary line bundles,*

(3) $\mathcal{O}_X^3 \oplus tr_{l/k}(\mathcal{N} \otimes {}^{\omega_1}\mathcal{N}^\vee) \oplus tr_{l/k}(\mathcal{N} \otimes {}^{\omega_2}\mathcal{N}^\vee) \oplus (\mathcal{M} \otimes tr_{l/k}(\mathcal{N}^\vee)) \oplus (tr_{l/k}(\mathcal{N}) \otimes \mathcal{M}^\vee)$, *with* $\mathcal{M}$ *a line bundle over* X, l/k *a cubic Galois field extension,* $\mathrm{Gal}(l/k) = \{id, \omega_1, \omega_2\}$, *and* $\mathcal{N}$ *a line bundle over* $X_l = X \times_k l$, *which is not defined over* X.

(4) $\mathcal{O}_X^3 \oplus (\mathcal{M}_1 \otimes \mathcal{M}_2^\vee) \oplus (\mathcal{M}_1 \otimes tr_{l/k}(\mathcal{N})^\vee) \oplus (\mathcal{M}_1^\vee \otimes \mathcal{M}_2) \oplus (\mathcal{M}_2 \otimes tr_{l/k}(\mathcal{N})^\vee) \oplus (\mathcal{M}_1^\vee \otimes tr_{l/k}(\mathcal{N})) \oplus (\mathcal{M}_2^\vee \otimes tr_{l/k}(\mathcal{N})) \oplus (tr_{l/k}(\mathcal{N}^\vee \otimes {}^{\omega}\mathcal{N}))$, *for a separable quadratic field extension* l/k *with non-trivial automorphism* ω *and a line bundle* $\mathcal{N}$ *over* $X \times_k l$

(5) $\mathcal{O}_X^3 \oplus (\mathcal{M}_2^\vee \otimes \mathcal{M}_1) \oplus (\mathcal{M}_3^\vee \otimes \mathcal{M}_1) \oplus (\mathcal{M}_4^\vee \otimes \mathcal{M}_1) \oplus (\mathcal{M}_2^\vee \otimes \mathcal{M}_1) \oplus (\mathcal{M}_2^\vee \otimes \mathcal{M}_3) \oplus (\mathcal{M}_2^\vee \otimes \mathcal{M}_4) \oplus (\mathcal{M}_3^\vee \otimes \mathcal{M}_1) \oplus (\mathcal{M}_3^\vee \otimes \mathcal{M}_2) \oplus (\mathcal{M}_3^\vee \otimes \mathcal{M}_4) \oplus (\mathcal{M}_4^\vee \otimes \mathcal{M}_1) \oplus (\mathcal{M}_4^\vee \otimes \mathcal{M}_2) \oplus (\mathcal{M}_4^\vee \otimes \mathcal{M}_3)$.

In particular, these cubic Jordan algebras contain the subalgebra $\mathcal{E}nd_X(\mathcal{E})^+$ *with the following* $\mathcal{E}$*:*

(1) $\mathcal{E} = \mathcal{M}_2 \otimes \mathcal{F}_3$,
(2) $\mathcal{E} = \mathcal{M}_i \oplus (\mathcal{M}_3 \otimes \mathcal{F}_2)$ *with* $i = 1$ *or 2,*
(3) $\mathcal{E} = tr_{l/k}(\mathcal{N})$,
(4) $\mathcal{E} = \mathcal{M}_i \oplus tr_{l/k}(\mathcal{N})$ *with* $i = 1$ *or 2,*
(5) $\mathcal{E} = \mathcal{M}_{i_1} \oplus \mathcal{M}_{i_2} \oplus \mathcal{M}_{i_3}$ *with* $i_1, \dots, i_3 \in \{1, \dots, 4\}$ *pairwise different.*

Proof. We have the situation of Section 5 (I.2):

(1) If $\mathcal{E} = \mathcal{M}_1 \oplus \mathcal{M}_2 \otimes \mathcal{F}_3$, then

$$\mathcal{B} \cong \begin{bmatrix} \mathcal{O}_X & \mathcal{H}om(\mathcal{M}_1, \mathcal{M}_2 \otimes \mathcal{F}_3) \\ \mathcal{H}om(\mathcal{M}_2 \otimes \mathcal{F}_3, \mathcal{M}_1) & \mathcal{E}nd(\mathcal{F}_3) \end{bmatrix}$$

and

$$\mathcal{J}_0 \cong \mathcal{O}_X \oplus (\mathcal{M}_1 \otimes \mathcal{M}_2^\vee \otimes \mathcal{F}_3) \oplus (\mathcal{M}_1^\vee \otimes \mathcal{M}_2 \otimes \mathcal{F}_3) \oplus \mathcal{F}_3 \oplus \mathcal{F}_5.$$

(2) If $\mathcal{E} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus (\mathcal{M}_3 \otimes \mathcal{F}_2)$, then

$$\mathcal{B} \cong \begin{bmatrix} \mathcal{O}_X & \mathcal{M}_1 \otimes \mathcal{M}_2^\vee & \mathcal{H}om(\mathcal{M}_3 \otimes \mathcal{F}_2, \mathcal{M}_1) \\ \mathcal{M}_2 \otimes \mathcal{M}_1^\vee & \mathcal{O}_X & \mathcal{H}om(\mathcal{M}_3 \otimes \mathcal{F}_2, \mathcal{M}_2) \\ \mathcal{H}om(\mathcal{M}_1, \mathcal{M}_3 \otimes \mathcal{F}_2) & \mathcal{H}om(\mathcal{M}_2, \mathcal{M}_3 \otimes \mathcal{F}_2) & \mathcal{E}nd(\mathcal{F}_2) \end{bmatrix}$$

and

$$\mathcal{J}_0 \cong \mathcal{O}_X^2 \oplus (\mathcal{M}_1 \otimes \mathcal{M}_2^\vee) \oplus (\mathcal{M}_1 \otimes \mathcal{M}_3^\vee \otimes \mathcal{F}_2) \oplus (\mathcal{M}_2 \otimes \mathcal{M}_1^\vee) \oplus (\mathcal{M}_2 \otimes \mathcal{M}_3^\vee \otimes \mathcal{F}_2) \oplus (\mathcal{M}_1^\vee \otimes \mathcal{M}_3 \otimes \mathcal{F}_2) \oplus (\mathcal{M}_2^\vee \otimes \mathcal{M}_3 \otimes \mathcal{F}_2) \oplus \mathcal{F}_3.$$

(3) If $\mathcal{E}$ decomposes into the direct sum of a line bundle and an indecomposable (but not absolutely indecomposable) vector bundle of rank 3 then there is a cubic field extension l/k and a line bundle $\mathcal{N}$ over $X_l = X \times_k l$ which is not defined over X such that $\mathcal{E} = \mathcal{M} \oplus tr_{l/k}(\mathcal{N})$ and

$$\mathcal{B} \cong \begin{bmatrix} \mathcal{O}_X & \mathcal{H}om(\mathcal{M}, tr_{l/k}(\mathcal{N})) \\ \mathcal{H}om(tr_{l/k}(\mathcal{N}), \mathcal{M}) & \mathcal{E}nd(tr_{l/k}(\mathcal{N})) \end{bmatrix},$$

so

$$\mathcal{J}_0 \cong (\mathcal{M} \otimes tr_{l/k}(\mathcal{N}^\vee)) \oplus (tr_{l/k}(\mathcal{N}) \otimes \mathcal{M}^\vee) \oplus (tr_{l/k}(\mathcal{N}^\vee) \otimes tr_{l/k}(\mathcal{N})) \cong$$
$$(\mathcal{M} \otimes tr_{l/k}(\mathcal{N}^\vee)) \oplus (tr_{l/k}(\mathcal{N}) \otimes \mathcal{M}^\vee) \oplus \mathcal{O}_X \oplus \mathcal{S}$$

with the vector bundle $\mathcal{S}$ arising from the decomposition $\mathcal{E}nd(tr_{l/k}(\mathcal{N})) \cong \mathcal{O}_X \oplus \mathcal{S}$.

If, in particular, l/k is Galois with $\mathrm{Gal}(l/k) = \{id, \omega_1, \omega_2\}$ then

$$\mathcal{E}nd_X(tr_{l/k}(\mathcal{N})) \cong \mathcal{O}_X^3 \oplus tr_{l/k}(\mathcal{N} \otimes {}^{\omega_1}\mathcal{N}^\vee) \oplus tr_{l/k}(\mathcal{N} \otimes {}^{\omega_2}\mathcal{N}^\vee).$$

In that case, the $\mathcal{O}_X$-module structure is given by

$$\mathcal{J}_0 \cong \mathcal{O}_X^3 \oplus tr_{l/k}(\mathcal{N} \otimes {}^{\omega_1}\mathcal{N}^\vee) \oplus tr_{l/k}(\mathcal{N} \otimes {}^{\omega_2}\mathcal{N}^\vee) \oplus (\mathcal{M} \otimes tr_{l/k}(\mathcal{N}^\vee)) \oplus (tr_{l/k}(\mathcal{N}) \otimes \mathcal{M}^\vee).$$

(4) If $\mathcal{E}$ is the direct sum of two line bundles and an indecomposable bundle of rank 2, which is not absolutely indecomposable, then $\mathcal{E} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus tr_{l/k}(\mathcal{N})$ for a separable quadratic field extension l/k with non-trivial automorphism ω and a line bundle $\mathcal{N}$ over $X \times_k l$. Then

$$\mathcal{B} \cong \begin{bmatrix} \mathcal{O}_X & \mathcal{M}_1 \otimes \mathcal{M}_2^\vee & \mathcal{M}_1 \otimes tr_{l/k}(\mathcal{N})^\vee \\ \mathcal{M}_1^\vee \otimes \mathcal{M}_2 & \mathcal{O}_X & \mathcal{M}_2 \otimes tr_{l/k}(\mathcal{N})^\vee \\ \mathcal{M}_1^\vee \otimes tr_{l/k}(\mathcal{N}) & \mathcal{M}_2^\vee \otimes tr_{l/k}(\mathcal{N}) & \mathcal{O}_X^2 \oplus tr_{l/k}(\mathcal{N}^\vee \otimes {}^{\omega}\mathcal{N}) \end{bmatrix}.$$

Hence

$$\mathcal{J}_0 \cong \mathcal{O}_X^3 \oplus (\mathcal{M}_1 \otimes \mathcal{M}_2^\vee) \oplus (\mathcal{M}_1 \otimes tr_{l/k}(\mathcal{N})^\vee) \oplus (\mathcal{M}_1^\vee \otimes \mathcal{M}_2) \oplus (\mathcal{M}_2 \otimes tr_{l/k}(\mathcal{N})^\vee) \oplus (\mathcal{M}_1^\vee \otimes tr_{l/k}(\mathcal{N}))$$
$$\oplus (\mathcal{M}_2^\vee \otimes tr_{l/k}(\mathcal{N})) \oplus tr_{l/k}(\mathcal{N}^\vee \otimes {}^{\omega}\mathcal{N}).$$

(5) If $\mathcal{E}$ is the direct sum of line bundles $\mathcal{E} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3 \oplus \mathcal{M}_4$ then

$$\mathcal{B} \cong \begin{bmatrix} \mathcal{O}_X & \mathcal{H}om(\mathcal{M}_1, \mathcal{M}_2) & \mathcal{H}om(\mathcal{M}_1, \mathcal{M}_3) & \mathcal{H}om(\mathcal{M}_1, \mathcal{M}_4) \\ \mathcal{H}om(\mathcal{M}_2, \mathcal{M}_1) & \mathcal{O}_X & \mathcal{H}om(\mathcal{M}_2, \mathcal{M}_3) & \mathcal{H}om(\mathcal{M}_2, \mathcal{M}_4) \\ \mathcal{H}om(\mathcal{M}_3, \mathcal{M}_1) & \mathcal{H}om(\mathcal{M}_3, \mathcal{M}_2) & \mathcal{O}_X & \mathcal{H}om(\mathcal{M}_3, \mathcal{M}_4) \\ \mathcal{H}om(\mathcal{M}_4, \mathcal{M}_1) & \mathcal{H}om(\mathcal{M}_4, \mathcal{M}_2) & \mathcal{H}om(\mathcal{M}_4, \mathcal{M}_3) & \mathcal{O}_X \end{bmatrix}$$

and

$$\begin{aligned} \mathcal{J}_0 \cong \mathcal{O}_X^3 &\oplus (\mathcal{M}_2^\vee \otimes \mathcal{M}_1) \oplus (\mathcal{M}_3^\vee \otimes \mathcal{M}_1) \oplus (\mathcal{M}_4^\vee \otimes \mathcal{M}_1) \\ &\oplus (\mathcal{M}_2^\vee \otimes \mathcal{M}_1) \oplus (\mathcal{M}_2^\vee \otimes \mathcal{M}_3) \oplus (\mathcal{M}_2^\vee \otimes \mathcal{M}_4) \\ &\oplus (\mathcal{M}_3^\vee \otimes \mathcal{M}_1) \oplus (\mathcal{M}_3^\vee \otimes \mathcal{M}_2) \oplus (\mathcal{M}_3^\vee \otimes \mathcal{M}_4) \\ &\oplus (\mathcal{M}_4^\vee \otimes \mathcal{M}_1) \oplus (\mathcal{M}_4^\vee \otimes \mathcal{M}_2) \oplus (\mathcal{M}_4^\vee \otimes \mathcal{M}_3). \end{aligned}$$

In all cases (1) to (5), $\mathcal{J}_0$ can be made into a cubic Jordan algebra which contains the claimed cubic Jordan subalgebra by Proposition 7. $\square$

In (1), (2) and (3), for instance, the cubic Jordan algebra $\mathcal{J}_0$ is not isomorphic to a reduced cubic Jordan algebra $H_3(\mathcal{D})$, for some quaternion algebra $\mathcal{D}$, due to its module structure.

Corollary 5. *The following $\mathcal{O}_X$-modules carry the structure of a pseudo-composition algebra of rank 14:*

1. $(\mathcal{M}_1\otimes\mathcal{M}_2^\vee\otimes\mathcal{F}_3)\oplus(\mathcal{M}_1^\vee\otimes\mathcal{M}_2\otimes\mathcal{F}_3)\oplus\mathcal{F}_3\oplus\mathcal{F}_5$ *with* $\mathcal{M}_i\in\operatorname{Pic}X$ $(i=1,2)$ *arbitrary line bundles,*
2. $\mathcal{O}_X\oplus(\mathcal{M}_1\otimes\mathcal{M}_2^\vee)\oplus(\mathcal{M}_1\otimes\mathcal{M}_3^\vee\otimes\mathcal{F}_2)\oplus(\mathcal{M}_2\otimes\mathcal{M}_1^\vee)\oplus(\mathcal{M}_2\otimes\mathcal{M}_3^\vee\otimes\mathcal{F}_2)\oplus(\mathcal{M}_1^\vee\otimes\mathcal{M}_3\otimes\mathcal{F}_2)\oplus(\mathcal{M}_2^\vee\otimes\mathcal{M}_3\otimes\mathcal{F}_2)\oplus\mathcal{F}_3$, *with* $\mathcal{M}_i\in\operatorname{Pic}X$ *arbitrary line bundles,*
3. $\mathcal{O}_X^2\oplus tr_{l/k}(\mathcal{N}\otimes{}^{\omega_1}\mathcal{N}^\vee)\oplus tr_{l/k}(\mathcal{N}\otimes{}^{\omega_2}\mathcal{N}^\vee)\oplus(\mathcal{M}\otimes tr_{l/k}(\mathcal{N}^\vee))\oplus(tr_{l/k}(\mathcal{N})\otimes\mathcal{M}^\vee)$, *with* $\mathcal{M}$ *a line bundle over* X, l/k *a cubic Galois field extension,* $\operatorname{Gal}(l/k)=\{id,\omega_1,\omega_2\}$ *and* $\mathcal{N}$ *a line bundle over* $X_l=X\times_k l$, *which is not defined over* X.
4. $\mathcal{O}_X^2\oplus(\mathcal{M}_1\otimes\mathcal{M}_2^\vee)\oplus(\mathcal{M}_1\otimes tr_{l/k}(\mathcal{N})^\vee)\oplus(\mathcal{M}_1^\vee\otimes\mathcal{M}_2)\oplus(\mathcal{M}_2\otimes tr_{l/k}(\mathcal{N})^\vee)\oplus(\mathcal{M}_1^\vee\otimes tr_{l/k}(\mathcal{N}))\oplus(\mathcal{M}_2^\vee\otimes tr_{l/k}(\mathcal{N}))\oplus(tr_{l/k}(\mathcal{N}^\vee\otimes^\omega\mathcal{N}))$, *for a separable quadratic field extension* l/k *with non-trivial automorphism* ω *and a line bundle* $\mathcal{N}$ *over* $X\times_k l$
5. $\mathcal{O}_X^2\oplus(\mathcal{M}_2^\vee\otimes\mathcal{M}_1)\oplus(\mathcal{M}_3^\vee\otimes\mathcal{M}_1)\oplus(\mathcal{M}_4^\vee\otimes\mathcal{M}_1)\oplus(\mathcal{M}_2^\vee\otimes\mathcal{M}_1)\oplus(\mathcal{M}_2^\vee\otimes\mathcal{M}_3)\oplus(\mathcal{M}_2^\vee\otimes\mathcal{M}_4)\oplus(\mathcal{M}_3^\vee\otimes\mathcal{M}_1)\oplus(\mathcal{M}_3^\vee\otimes\mathcal{M}_2)\oplus(\mathcal{M}_3^\vee\otimes\mathcal{M}_4)\oplus(\mathcal{M}_4^\vee\otimes\mathcal{M}_1)\oplus(\mathcal{M}_4^\vee\otimes\mathcal{M}_2)\oplus(\mathcal{M}_4^\vee\otimes\mathcal{M}_3)$.

By Proposition 7, all of the above contain a pseudo-composition subalgebra of rank 8.

Without exhausting all possible cases, we now look at some interesting examples of the setting described in Section 5 (II):

Theorem 12. *The following $\mathcal{O}_X$-modules carry the structure of an admissible cubic algebra over X:*

1. $tr_{l_1/k}(\mathcal{L}_1)\oplus\mathcal{F}_3\oplus tr_{l_1/k}(\mathcal{L}_1\otimes\mathcal{F}_3)$;
2. $\mathcal{L}_1\oplus tr_{l_2/k}(\mathcal{L}_2)\oplus\mathcal{F}_3\oplus(\mathcal{L}_1\otimes\mathcal{F}_3)\oplus tr_{l_2/k}(\mathcal{L}_2\otimes\mathcal{F}_3)$;
3. $\mathcal{L}_1\oplus\mathcal{L}_2\oplus\mathcal{L}_3\oplus\mathcal{F}_3\oplus(\mathcal{L}_1\otimes\mathcal{F}_3)\oplus(\mathcal{L}_2\otimes\mathcal{F}_3)\oplus(\mathcal{L}_3\otimes\mathcal{F}_3)$.

Proof. Let $\mathcal{J}=\mathcal{D}_1\otimes\mathcal{D}_2$ with $\mathcal{D}_i$ $(i=1,2)$ a quaternion algebra over X as in Section 5 (II). Then $\mathcal{J}_0$ can be provided with a multiplication which makes it an admissible cubic algebra (Theorem 2).

(1) If $\mathcal{D}_1=\mathcal{E}nd_X(\mathcal{F}_2)$ and $\mathcal{D}_2=\operatorname{Quat}(tr_{l_1/k}(\mathcal{L}_1),N)$, then

$$\mathcal{J}\cong\mathcal{O}_X\oplus tr_{l_1/k}(\mathcal{L}_1)\oplus\mathcal{F}_3\oplus tr_{l_1/k}(\mathcal{L}_1\otimes\mathcal{F}_3)$$

and

$$\mathcal{J}_0\cong tr_{l_1/k}(\mathcal{L}_1)\oplus\mathcal{F}_3\oplus tr_{l_1/k}(\mathcal{L}_1\otimes\mathcal{F}_3).$$

(2) If $\mathcal{D}_1=\mathcal{E}nd_X(\mathcal{F}_2)$ and $\mathcal{D}_2=\operatorname{Cay}(\mathcal{T},tr_{l_2/k}(\mathcal{L}_2))$ with $\mathcal{T}=\operatorname{Cay}(\mathcal{O}_X,\mathcal{L}_1,N_{\mathcal{L}_1})$, then

$$\mathcal{J}\cong\mathcal{O}_X\oplus\mathcal{L}_1\oplus tr_{l_2/k}(\mathcal{L}_2)\oplus\mathcal{F}_3\oplus(\mathcal{L}_1\otimes\mathcal{F}_3)\oplus tr_{l_2/k}(\mathcal{L}_2\otimes\mathcal{F}_3)$$

and

$$\mathcal{J}_0\cong\mathcal{L}_1\oplus tr_{l_2/k}(\mathcal{L}_2)\oplus\mathcal{F}_3\oplus(\mathcal{L}_1\otimes\mathcal{F}_3)\oplus tr_{l_2/k}(\mathcal{L}_2\otimes\mathcal{F}_3)$$

(3) If $\mathcal{D}_1 = \mathcal{E}nd_X(\mathcal{F}_2)$ and $\mathcal{D}_2 = \mathrm{Cay}(\mathcal{T}, \mathcal{L}_2 \otimes \mathcal{T}, N_0)$ with $\mathcal{T} = \mathrm{Cay}(\mathcal{O}_X, \mathcal{L}_1, N_{\mathcal{L}_1})$, then

$$\mathcal{J} \cong \mathcal{O}_X \oplus \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \oplus \mathcal{F}_3 \oplus (\mathcal{L}_1 \otimes \mathcal{F}_3) \oplus (\mathcal{L}_2 \otimes \mathcal{F}_3) \oplus (\mathcal{L}_3 \otimes \mathcal{F}_3)$$

and

$$\mathcal{J}_0 \cong \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \oplus \mathcal{F}_3 \oplus (\mathcal{L}_1 \otimes \mathcal{F}_3) \oplus (\mathcal{L}_2 \otimes \mathcal{F}_3) \oplus (\mathcal{L}_3 \otimes \mathcal{F}_3).$$

□

Note that in all of the above cases, $H^0(X, \mathcal{J}_0) = k$, and $\mathcal{J}_0$ cannot be made into a cubic Jordan algebra. There also is no pseudo-composition algebra attached to it.

9. Quartic Jordan algebras of rank 28 over an elliptic curve

Given a quartic Jordan algebra of rank 28 of the type $\mathcal{J} = H(\mathcal{A}, \tau)$ with τ a symplectic involution on an Azumaya algebra $\mathcal{A}$ of rank 64, we construct examples of admissible cubic algebras of rank 27, Albert algebras and pseudo-composition algebras of rank 26 over an elliptic curve. We use the results of Section 6.

Example 7. (a) We start with the set-up described in Section 6 (A*): If a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of constant rank 8 carries a nondegenerate skew-symmetric bilinear form b, the Azumaya algebra $\mathcal{E}nd_X(\mathcal{E})$ has a symplectic involution $\sigma_b(f) = \hat{b}^{-1} \circ f^\vee \circ \hat{b}$. If r is even and $\mathcal{L}$ is a line bundle of order 2, then $\mathcal{L} \otimes \mathcal{F}_r$ carries a nondegenerate skew-symmetric bilinear form [AEJ1, p. 1350]. Let $\mathcal{L}$ be a line bundle of order 2. Then $\mathcal{E}nd_X(\mathcal{L} \otimes \mathcal{F}_8)$ carries a symplectic involution σ_b induced by a skew-symmetric bilinear form on $\mathcal{L} \otimes \mathcal{F}_8$. We have

$$\mathcal{E}nd_X(\mathcal{L} \otimes \mathcal{F}_8) \cong \mathcal{O}_X \oplus \mathcal{F}_3 \oplus \mathcal{F}_5 \oplus \mathcal{F}_7 \oplus \mathcal{F}_9 \oplus \mathcal{F}_{11} \oplus \mathcal{F}_{15}$$

as $\mathcal{O}_X$-module. By counting the ranks of the indecomposable bundles, it follows that $\mathcal{J} = H(\mathcal{E}nd_X(\mathcal{L} \otimes \mathcal{F}_8), \sigma_b)$ is isomorphic to

$$\mathcal{O}_X \oplus \mathcal{F}_3 \oplus \mathcal{F}_9 \oplus \mathcal{F}_{15},$$

$$\mathcal{O}_X \oplus \mathcal{F}_5 \oplus \mathcal{F}_7 \oplus \mathcal{F}_{15},$$

or to

$$\mathcal{O}_X \oplus \mathcal{F}_7 \oplus \mathcal{F}_9 \oplus \mathcal{F}_{11}.$$

Which case appears is not clear. In any case, the Jordan algebra $\mathcal{J}$ can only have trivial idempotents. Hence $\mathcal{J}_0$ cannot be made into an Albert algebra and there also is no pseudo-composition algebra attached to it.

(b) The trivial Azumaya algebra

$$\mathcal{E}nd_X(\mathcal{F}_2 \oplus \mathcal{F}_2 \oplus \mathcal{F}_4) \cong \begin{bmatrix} \mathcal{E}nd_X(\mathcal{F}_2) & \mathcal{E}nd_X(\mathcal{F}_2) & \mathcal{H}om(\mathcal{F}_2, \mathcal{F}_4) \\ \mathcal{E}nd_X(\mathcal{F}_2) & \mathcal{E}nd_X(\mathcal{F}_2) & \mathcal{H}om(\mathcal{F}_2, \mathcal{F}_4) \\ \mathcal{H}om(\mathcal{F}_4, \mathcal{F}_2) & \mathcal{H}om(\mathcal{F}_4, \mathcal{F}_2) & \mathcal{E}nd(\mathcal{F}_4) \end{bmatrix}$$

$\cong \mathcal{O}_X^5 \oplus \mathcal{F}_3^9 \oplus \mathcal{F}_5^5 \oplus \mathcal{F}_7$ carries a symplectic involution σ_b induced by the skew-symmetric bilinear form $b = b_1 \otimes b_2 \otimes b_3$, with b_1, b_2 and b_3 skew-symmetric forms on $\mathcal{F}_2$. Then $(\mathcal{E}nd_X(\mathcal{F}_2), \sigma_{b_i}) \cong (\mathcal{E}nd_X(\mathcal{F}_2), ^-)$ for $i = 1, 2$ and

$$(\mathcal{E}nd_X(\mathcal{F}_2 \oplus \mathcal{F}_2 \oplus \mathcal{F}_4), \sigma_b) \cong (\mathcal{E}nd_X(\mathcal{F}_2), ^-) \otimes (\mathcal{E}nd_X(\mathcal{F}_2), ^-) \otimes (\mathcal{E}nd_X(\mathcal{F}_2), ^-).$$

We obtain

$H(\mathcal{E}nd_X(\mathcal{F}_2 \oplus \mathcal{F}_2 \oplus \mathcal{F}_4), \sigma_b) \cong \mathcal{O}_X \oplus (\mathcal{F}_3 \otimes \mathcal{F}_3) \oplus (\mathcal{F}_3 \otimes \mathcal{F}_3) \oplus (\mathcal{F}_3 \otimes \mathcal{F}_3) \cong \mathcal{O}_X^4 \oplus \mathcal{F}_3^3 \oplus \mathcal{F}_5^3.$

As described in (D1*), we have

$$(\mathcal{D}, ^-) \otimes (\mathcal{D}, ^-) \cong (\mathcal{E}nd_X(\mathcal{D}), \sigma)$$

with $\mathcal{D} = \mathcal{E}nd_X(\mathcal{F}_2)$, and

$$\mathcal{J} = H(\mathcal{E}nd_X(\mathcal{F}_2 \oplus \mathcal{F}_2 \oplus \mathcal{F}_4), \sigma_b) \cong \mathrm{Skew}(\mathcal{E}nd_X(\mathcal{D}), \sigma) \otimes \mathcal{F}_3 \oplus H(\mathcal{E}nd_X(\mathcal{D}), \sigma) \otimes \mathcal{O}_X.$$

Moreover, $f = \mathrm{diag}(0,1)$ is a non-trivial idempotent in $H^0(X, \mathcal{J})$ of trace 1. Therefore the $\mathcal{O}_X$-module

$$\mathcal{J}_0 \cong \mathcal{O}_X^3 \oplus \mathcal{F}_3^3 \oplus \mathcal{F}_5^3$$

can be made into an Albert algebra and

$$\mathcal{O}_X^2 \oplus \mathcal{F}_3^3 \oplus \mathcal{F}_5^3$$

into a pseudo-composition algebra. Indeed, the Albert algebra $\mathcal{J}_0$ is a (classical) first Tits construction starting with $\mathcal{E}nd_X(\mathcal{F}_3) \cong \mathcal{O}_X \oplus \mathcal{F}_3 \oplus F_5$ (Theorem 7). It is not isomorphic to $H_3(\mathcal{C}, *)$ due to its module structure.

Example 8. As in Section 6 (B*), let $\mathcal{D} = \mathrm{Mat}_2(\mathcal{O}_X)$ with the transpose (hence orthogonal) involution t. Let $\mathcal{B} = \mathcal{E}nd_X(\mathcal{F}_4)$, then $\mathcal{B}$ carries a symplectic involution τ_1 induced by the skew-symmetric form on $\mathcal{F}_4$, and

$$H(\mathcal{B}, \tau_1) \cong \mathcal{O}_X \oplus \mathcal{F}_5, \quad \mathrm{Skew}(\mathcal{B}, \tau_1) \cong \mathcal{F}_3 \oplus \mathcal{F}_7$$

(by looking at the module structure this is the only possibility). Let

$$(\mathcal{A}, \tau) = (\mathcal{E}nd_X(\mathcal{F}_4), \tau_1) \otimes (\mathcal{D}, t) \cong (\mathrm{Mat}_2(\mathcal{E}nd_X(\mathcal{F}_4)), J_1)$$

with $J_1(D) = (\tau_1(d_{ij}))^t$, if $D = (d_{ij})$. We have

$$H(\mathcal{A}, J_1) \cong \mathrm{Skew}(\mathcal{B}, \tau_1) \oplus H(\mathcal{B}, \tau_1) \oplus H(\mathcal{B}, \tau_1) \oplus H(\mathcal{B}, \tau_1).$$

Therefore

$$H(\mathcal{A}, J_1) \cong \mathcal{O}_X^3 \oplus \mathcal{F}_3 \oplus \mathcal{F}_7 \oplus \mathcal{F}_5^3.$$

It is not clear if there exist idempotents of trace 1 in $\mathcal{J} = H(\mathcal{A}, \tau) \cong H(\mathcal{A}, J_1)$, the obvious idempotent $1 \otimes \mathrm{diag}(1,0)$ in $\mathcal{B} \otimes \mathrm{Mat}_2(\mathcal{O}_X)$ has trace 2.

Example 9. As in Section 6 (C*), let τ_1 be orthogonal and τ_2 be the canonical involution on $\mathcal{D} = \mathrm{Mat}_2(\mathcal{O}_X)$. Let $\mathcal{B} = \mathcal{E}nd_X(\mathcal{O}_X \oplus \mathcal{F}_3)$ then $\mathcal{B}$ carries an orthogonal involution τ_1 induced by a symmetric bilinear form on $\mathcal{O}_X \oplus \mathcal{F}_3$. We obtain

$$\mathcal{B} \cong \begin{bmatrix} \mathcal{O}_X & \mathcal{H}om(\mathcal{O}_X, \mathcal{F}_3) \\ \mathcal{H}om(\mathcal{F}_3, \mathcal{O}_X) & \mathcal{E}nd(\mathcal{F}_3) \end{bmatrix} \cong \mathcal{O}_X^2 \oplus \mathcal{F}_3^3 \oplus \mathcal{F}_5.$$

and

$$H(\mathcal{A}, \tau) \cong \mathrm{Skew}(\mathcal{B}, \tau_1)^3 \oplus H(\mathcal{B}, \tau_1).$$

Suppose $b = b_1 \otimes b_2$ with b_1, b_2 two skew-symmetric forms on $\mathcal{F}_2$. Then

$$(\mathcal{B}, \tau_1) \cong (\mathcal{E}nd_X(\mathcal{F}_2), ^-) \otimes (\mathcal{E}nd_X(\mathcal{F}_2), ^-).$$

Thus $H(\mathcal{B},\tau_1)\cong \mathcal{O}_X\oplus\mathcal{F}_3\otimes\mathcal{F}_3$ and $\mathrm{Skew}(\mathcal{B},\tau_1)\cong\mathcal{F}_3\oplus\mathcal{F}_3$. Hence

$$\mathcal{J}=H(\mathcal{A},\tau)\cong(\mathcal{F}_3^3\oplus\mathcal{F}_3^3)\oplus(\mathcal{O}_X^2\oplus\mathcal{F}_3\oplus\mathcal{F}_5).$$

As in Example 7, $(\mathcal{B},\tau_1)\cong(\mathcal{E}nd_X(\mathcal{D}),\sigma)$ with $\mathcal{D}=\mathcal{E}nd_X(\mathcal{F}_2)$,

$$H(\mathcal{E}nd_X(\mathcal{D}),\sigma)\cong\begin{bmatrix}\mathcal{O}_X & 0\\ 0 & \mathcal{E}nd_X\mathcal{F}_3\end{bmatrix}\subset\mathcal{J}$$

and $f=\mathrm{diag}(0,1)$ is a non-trivial idempotent in $\mathcal{J}$ of trace 1. Therefore the $\mathcal{O}_X$-module

$$\mathcal{O}_X\oplus\mathcal{F}_3^7\oplus\mathcal{F}_5$$

can be made into an Albert algebra and

$$\mathcal{F}_3^7\oplus\mathcal{F}_5$$

into a pseudo-composition algebra. Again, the Albert algebra $\mathcal{J}_0$ is a first Tits construction starting with $\mathcal{E}nd_X(\mathcal{F}_3)$ (Theorem 7).

(An analogous argument works for $\mathcal{B}=\mathcal{E}nd_X(\mathcal{L}_i\oplus\mathcal{L}_j\otimes\mathcal{F}_3)$.)

In Examples 7 (b) and 9, the Albert algebras $\mathcal{J}_0$ are not isomorphic to an Azumaya algebra of the kind $H_3(\mathcal{C},*)$ due to their module structure.

We conclude with examples arising from the scenario in Section 6 (D*) and (E*). We point out that the algebras covered in case (5) of the following theorem overlap with the ones from Example 9.

Theorem 13. *Let $\mathcal{L}$ be a line bundle over X. The following $\mathcal{O}_X$-modules carry the structure of an Albert algebra over X, which can be obtained by a first Tits construction:*

(1) $\mathcal{O}_X^3\oplus\mathcal{F}_3^3\oplus\mathcal{F}_5^3$, *starting with* $\mathcal{E}nd_X(\mathcal{F}_3)$,
(2) $\mathcal{O}_X\oplus\mathcal{F}_3\oplus\mathcal{F}_5\oplus tr_{l_1/k}(\mathcal{L}_1\otimes\mathcal{F}_3)\oplus tr_{l_1/k}(\mathcal{L}_1\otimes\mathcal{F}_3)$ *starting with* $\mathcal{E}nd_X(\mathcal{F}_3)$ *(if X has type III),*
(3) $\mathcal{O}_X\oplus\mathcal{F}_3\oplus\mathcal{F}_5\oplus[\mathcal{L}_1\otimes\mathcal{F}_3]^2\oplus[tr_{l_2/k}(\mathcal{L}_2\otimes\mathcal{F}_3)]^2$ *starting with* $\mathcal{E}nd_X(\mathcal{F}_3)$ *(if X has type III),*
(4) $\mathcal{O}_X\oplus\mathcal{F}_3\oplus\mathcal{F}_5\oplus[(\mathcal{L}_1\otimes\mathcal{F}_3)\oplus(\mathcal{L}_2\otimes\mathcal{F}_3)\oplus(\mathcal{L}_3\otimes\mathcal{F}_3)]^2$ *starting with* $\mathcal{E}nd_X(\mathcal{F}_3)$ *(if X has type I),*
(5) $\mathcal{O}_X\oplus\mathcal{F}_3^7\oplus\mathcal{F}_5$, *starting with* $\mathcal{E}nd_X(\mathcal{F}_3)$
(6) $\mathcal{O}_X\oplus\mathcal{F}_3^3\oplus\mathcal{F}_5\oplus[(\mathcal{F}_3\otimes\mathcal{L})\oplus(\mathcal{F}_3\otimes\mathcal{L}^\vee)]^2$, *starting with* $\mathcal{E}nd_X(\mathcal{F}_3)$
(7) $\mathcal{O}_X^9\oplus[tr_{l_1/k}(\mathcal{L}_1\otimes^{\sigma_1}\mathcal{L}_1)]^3\oplus[tr_{l_1/k}(\mathcal{L}_1\otimes^{\sigma_2}\mathcal{L}_1)]^3$ *starting with* $\mathcal{E}nd_X(tr_{l_1/k}(\mathcal{L}_1))$ *(if X has type III and l_1/k is Galois),*
(8) $\mathcal{O}_X^3\oplus tr_{l_1/k}(\mathcal{L}_2)\oplus tr_{l_1/k}(\mathcal{L}_3)\oplus[tr_{l_1/k}(\mathcal{L}_1\otimes\mathcal{F}_3)]^2$ *starting with* $\mathcal{E}nd_X(tr_{l_1/k}(\mathcal{L}_1))$ *(if X has type III and l_1/k is Galois),*
(9) $\mathcal{O}_X^3\oplus tr_{l_1/k}(\mathcal{L}_2)\oplus tr_{l_1/k}(\mathcal{L}_3)\oplus[tr_{l_1/k}(\mathcal{L}_1)]^6$, *starting with* $\mathcal{E}nd_X(tr_{l_1/k}(\mathcal{L}_1))$ *(if X has type III and l_1/k is Galois),*
(10) $\mathcal{O}_X^3\oplus tr_{l_1/k}(\mathcal{L}_2)\oplus tr_{l_1/k}(\mathcal{L}_3)\oplus[tr_{l_1/k}(\mathcal{L}_1)\oplus tr_{l_1/k}(\mathcal{L}_1\otimes\mathcal{L})\oplus tr_{l_1/k}(\mathcal{L}_1\otimes\mathcal{L}^\vee)]^2$, *starting with* $\mathcal{E}nd_X(tr_{l_1/k}(\mathcal{L}_1))$ *(if X has type III and l_1/k is Galois), ,*
(11) $\mathcal{O}_X^9\oplus[tr_{l_2/k}(\mathcal{L}_2)]^6\oplus\mathcal{L}_1^6$, *starting with* $\mathcal{E}nd_X(\mathcal{L}_1\oplus tr_{l_2/k}(\mathcal{L}_2))$ *(if X has type II),*
(12) $\mathcal{O}_X^3\oplus[tr_{l_2/k}(\mathcal{L}_2)]^2\oplus\mathcal{L}_1^2\oplus[\mathcal{L}_1\otimes\mathcal{F}_3]^2\oplus[tr_{l_2/k}(\mathcal{L}_2\otimes\mathcal{F}_3)]^2$, *starting with* $\mathcal{E}nd_X(\mathcal{L}_1\oplus tr_{l_2/k}(\mathcal{L}_2))$ *(if X has type II),*

(13) $\mathcal{O}_X^3 \oplus [tr_{l_2/k}(\mathcal{L}_2)]^2 \oplus \mathcal{L}_1^8 \oplus [tr_{l_2/k}(\mathcal{L}_2)]^6$, *starting with* $\mathcal{E}nd_X(\mathcal{L}_1 \oplus tr_{l_2/k}(\mathcal{L}_2))$ *(if X has type II),*
(14) $\mathcal{O}_X^3 \oplus [tr_{l_2/k}(\mathcal{L}_2)]^4 \oplus \mathcal{L}_1^4 \oplus [\mathcal{L}_1 \otimes \mathcal{L}]^2 \oplus [\mathcal{L}_1 \otimes \mathcal{L}^\vee]^2 \oplus [tr_{l_2/k}(\mathcal{L}_2 \otimes \mathcal{L})]^2 \oplus [tr_{l_2/k}(\mathcal{L}_2 \otimes \mathcal{L}^\vee)]^2$, *starting with* $\mathcal{E}nd_X(\mathcal{L}_1 \oplus tr_{l_2/k}(\mathcal{L}_2))$ *(if X has type II),*
(15) $\mathcal{O}_X^9 \oplus \mathcal{L}_1^6 \oplus \mathcal{L}_2^6 \oplus \mathcal{L}_3^6$, *starting with* $\mathcal{E}nd_X(\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3)$ *(if X has type I),*
(16) $\mathcal{O}_X^3 \oplus \mathcal{L}_1^2 \oplus \mathcal{L}_2^2 \oplus \mathcal{L}_3^2 \oplus [(\mathcal{L}_1 \otimes \mathcal{F}_3) \oplus (\mathcal{L}_2 \otimes \mathcal{F}_3) \oplus (\mathcal{L}_3 \otimes \mathcal{F}_3)]^2$, *starting with* $\mathcal{E}nd_X(\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3)$ *(if X has type I),*
(17) $\mathcal{O}_X^3 \oplus \mathcal{L}_1^8 \oplus \mathcal{L}_2^8 \oplus \mathcal{L}_3^8$, *starting with* $\mathcal{E}nd_X(\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3)$ *(if X has type I),*
(18) $\mathcal{O}_X^3 \oplus \mathcal{L}_1^4 \oplus \mathcal{L}_2^4 \oplus \mathcal{L}_3^4 \oplus [\mathcal{L}_1 \otimes \mathcal{L}]^2 \oplus [\mathcal{L}_2 \otimes \mathcal{L}]^2 \oplus [\mathcal{L}_3 \otimes \mathcal{L}]^2 \oplus [\mathcal{L}_1 \otimes \mathcal{L}^\vee]^2 \oplus [\mathcal{L}_2 \otimes \mathcal{L}^\vee]^2 \oplus [\mathcal{L}_3 \otimes \mathcal{L}^\vee]^2$, *starting with* $\mathcal{E}nd_X(\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3)$ *(if X has type I),*
(19) $\mathcal{O}_X^{15} \oplus \mathcal{L}_i^{12}$, *starting with* $\mathcal{D} = \mathcal{D}_3 = \mathrm{Cay}(\mathcal{T}, \mu)$, *where* $\mathcal{T} = \mathrm{Cay}(\mathcal{O}_X, \mathcal{L}_i, n_i)$,
(20) $\mathcal{O}_X^5 \oplus \mathcal{L}_i^4 \oplus \mathcal{F}_3^2 \oplus [\mathcal{L}_i \otimes \mathcal{F}_3]^4$, *starting with* $\mathcal{D} = \mathcal{D}_3 = \mathrm{Cay}(\mathcal{T}, \mu)$, *where* $\mathcal{T} = \mathrm{Cay}(\mathcal{O}_X, \mathcal{L}_i, n_i)$,
(21) $\mathcal{O}_X^{11} \oplus \mathcal{L}_i^{16}$, *starting with* $\mathcal{D} = \mathcal{D}_3 = \mathrm{Cay}(\mathcal{T}, \mu)$, *where* $\mathcal{T} = \mathrm{Cay}(\mathcal{O}_X, \mathcal{L}_i, n_i)$,
(22) $\mathcal{O}_X^9 \oplus \mathcal{L}_1^6 \oplus [tr_{l_2/k}(\mathcal{L}_2)]^6$, *starting with* $\mathcal{D} = \mathcal{D}_3 = \mathrm{Cay}(\mathcal{T}, \mu)$, *where* $\mathcal{T} = \mathrm{Cay}(\mathcal{O}_X, \mathcal{L}_1, n_1)$, *(if X has type II),*
(23) $\mathcal{O}_X^7 \oplus \mathcal{L}_i^8 \oplus [\mathcal{L} \oplus \mathcal{L}^\vee]^2 \oplus [\mathcal{L}_i \otimes \mathcal{L}]^4 \oplus [\mathcal{L}_i \otimes \mathcal{L}^\vee]^4$,
(24) $\mathcal{O}_X^7 \oplus \mathcal{L}_i^8 \oplus [\mathcal{L}_j \oplus \mathcal{L}_j^\vee]^2 \oplus [\mathcal{L}_i \otimes \mathcal{L}_j]^4 \oplus [\mathcal{L}_i \otimes \mathcal{L}_j^\vee]^4$ *for* $i \neq j$ *(if X has type I or II).*
(25) $\mathcal{O}_X^9 \oplus [tr_{l_1/k}(\mathcal{L}_1)]^6$ *starting with some* $\mathcal{D}$ *defined over k (if X has type III),*
(26) $\mathcal{O}_X^9 \oplus \mathcal{L}_1^6 \oplus [tr_{l_2/k}(\mathcal{L}_2)]^6$ *starting with some* $\mathcal{D}$ *defined over k (if X has type II),*
(27) $\mathcal{O}_X^{15} \oplus \mathcal{L}_1^{12}$, *starting with some* $\mathcal{D}$ *defined over k,*
(28) $\mathcal{O}_X^{15} \oplus \mathcal{L}^6 \oplus [\mathcal{L}^\vee]^6$, *starting with some* $\mathcal{D}$ *defined over k, with* $\mathcal{L}$ *any line bundle over X,*
(29) $\mathcal{O}_X^9 \oplus \mathcal{F}_3^6$, *starting with some* $\mathcal{D}$ *defined over k,*
(30) $\mathcal{O}_X^9 \oplus \mathcal{L}_1^6 \oplus \mathcal{L}_2^6 \oplus \mathcal{L}_3^6$, *starting with* $\mathcal{D}$ *as in (15) (if X has type I),*
(31) $\mathcal{O}_X^3 \oplus \mathcal{L}_1^2 \oplus \mathcal{L}_2^2 \oplus \mathcal{L}_3^2 \oplus [(\mathcal{L}_1 \otimes \mathcal{F}_3) \oplus (\mathcal{L}_2 \otimes \mathcal{F}_3) \oplus (\mathcal{L}_3 \otimes \mathcal{F}_3)]^2$, *starting with* $\mathcal{D}$ *as in (15) (if X has type I),*
(32) $\mathcal{O}_X^3 \oplus \mathcal{L}_1^4 \oplus \mathcal{L}_2^4 \oplus \mathcal{L}_3^4 \oplus [(\mathcal{L}_1 \otimes \mathcal{L}_i) \oplus (\mathcal{L}_2 \otimes \mathcal{L}_i) \oplus (\mathcal{L}_3 \otimes \mathcal{L}_i)]^2$, *starting with* $\mathcal{D}$ *as in (15) (if X has type I),*
(33) $\mathcal{O}_X^3 \oplus \mathcal{L}_1^8 \oplus \mathcal{L}_2^8 \oplus \mathcal{L}_3^8$, *starting with* $\mathcal{D}$ *as in (15) (if X has type I),*
(34) $\mathcal{O}_X^3 \oplus \mathcal{L}_1^4 \oplus \mathcal{L}_2^4 \oplus \mathcal{L}_3^4 \oplus [(\mathcal{L}_1 \otimes \mathcal{L}) \oplus (\mathcal{L}_2 \otimes \mathcal{L}) \oplus (\mathcal{L}_3 \otimes \mathcal{L}) \oplus (\mathcal{L}_1 \otimes \mathcal{L}^\vee) \oplus (\mathcal{L}_2 \otimes \mathcal{L}^\vee) \oplus (\mathcal{L}_3 \otimes \mathcal{L}^\vee)]^2$, *starting with* $\mathcal{D}$ *as in (15) (if X has type I),*
(35) $\mathcal{O}_X^{27}$,
(36) $\mathcal{O}_X^9 \oplus \mathcal{L}^6 \oplus \mathcal{L}^{\vee 6} \oplus [\mathcal{L} \otimes \mathcal{L}]^3 \oplus [\mathcal{L}^\vee \otimes \mathcal{L}^\vee]^3$, *for some line bundle* $\mathcal{L}$ *over X,*
(37) $\mathcal{O}_X^3 \oplus \mathcal{L}^2 \oplus \mathcal{L}^{\vee 2} \oplus (\mathcal{L} \otimes \mathcal{L}) \oplus (\mathcal{L}^\vee \otimes \mathcal{L}^\vee) \oplus \mathcal{F}_3^2 \oplus [\mathcal{L} \otimes \mathcal{F}_3]^2 \oplus [\mathcal{L}^\vee \otimes \mathcal{F}_3]^2$ *for some line bundle* $\mathcal{L}$ *over X,*
(38) $\mathcal{O}_X^3 \oplus \mathcal{L}^2 \oplus \mathcal{L}^{\vee 2} \oplus (\mathcal{L} \otimes \mathcal{L}) \oplus (\mathcal{L}^\vee \otimes \mathcal{L}^\vee) \oplus [tr_{l_1/k}(\mathcal{L}_1)]^2 \oplus [tr_{l_1/k}(\mathcal{L} \otimes \mathcal{L}_1)]^2 \oplus [tr_{l_1/k}(\mathcal{L}^\vee \otimes \mathcal{L}_1)]^2$,
(39) $\mathcal{O}_X^3 \oplus \mathcal{L}^2 \oplus \mathcal{L}^{\vee 2} \oplus (\mathcal{L} \otimes \mathcal{L}) \oplus (\mathcal{L}^\vee \otimes \mathcal{L}^\vee) \oplus \mathcal{L}_1^2 \oplus [tr_{l_2/k}(\mathcal{L}_2)]^2 \oplus [\mathcal{L} \otimes \mathcal{L}_1]^2 \oplus [\mathcal{L}^\vee \otimes \mathcal{L}_1]^2 \oplus [tr_{l_2/k}(\mathcal{L} \otimes \mathcal{L}_2)]^2 \oplus [tr_{l_2/k}(\mathcal{L}^\vee \otimes \mathcal{L}_2)]^2$,
(40) $\mathcal{O}_X^5 \oplus \mathcal{L}^4 \oplus \mathcal{L}^{\vee 4} \oplus (\mathcal{L} \otimes \mathcal{L}) \oplus (\mathcal{L}^\vee \otimes \mathcal{L}^\vee) \oplus \mathcal{L}_1^2 \oplus \mathcal{L}_2^2 \oplus \mathcal{L}_3^2 \oplus [\mathcal{L} \otimes \mathcal{L}_1]^2 \oplus [\mathcal{L} \otimes \mathcal{L}_2]^2 \oplus [\mathcal{L} \otimes \mathcal{L}_3]^2 \oplus [\mathcal{L}^\vee \otimes \mathcal{L}_1]^2 \oplus [\mathcal{L}^\vee \otimes \mathcal{L}_2]^2 \oplus [\mathcal{L}^\vee \otimes \mathcal{L}_3]^2$,

(41) $\mathcal{O}_X^{11} \oplus \mathcal{L}^{10} \oplus \mathcal{L}^{\vee 10} \oplus (\mathcal{L} \otimes \mathcal{L}) \oplus (\mathcal{L}^{\vee} \otimes \mathcal{L}^{\vee})$.

Proof. Let $\mathcal{D} = \mathrm{Quat}(\mathcal{M}, N)$ and $\mathcal{D}_3 = \mathrm{Quat}(\mathcal{M}_3, N_3)$ be two quaternion algebras with canonical involution. Let $\mathcal{A} = \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{D}_3$ with involution $\tau = {}^- \otimes {}^- \otimes {}^-$. By Section 6 (D1*), the quartic Jordan algebra $\mathcal{J} = H(\mathcal{A}, \tau)$ contains a non-trivial idempotent of trace 1. Thus $\mathcal{J}_0 \cong \mathcal{E}nd_X(\mathcal{M}) \oplus \mathcal{M} \otimes \mathcal{M}_3 \oplus \mathcal{M} \otimes \mathcal{M}_3$ can be made into an Albert algebra, which is a first Tits construction starting with $\mathcal{E}nd_X(\mathcal{M})$ by Theorem 7. We consider the following cases, which yield the corresponding assertions:

(1) $\mathcal{D} = \mathcal{D}_3 = \mathcal{E}nd_X(\mathcal{F}_2)$.
(2) $\mathcal{D} = \mathcal{E}nd_X(\mathcal{F}_2)$ and $\mathcal{D}_3 = \mathrm{Quat}(tr_{l_1/k}(\mathcal{L}_1), N)$.
(3) $\mathcal{D} = \mathcal{E}nd_X(\mathcal{F}_2)$ and $\mathcal{D}_3 = \mathrm{Cay}(\mathcal{T}, tr_{l_2/k}(\mathcal{L}_2), N_{\mathcal{P}})$ with $\mathcal{T} = \mathrm{Cay}(\mathcal{O}_X, \mathcal{L}_1, n_1)$.
(4) $\mathcal{D} = \mathcal{E}nd_X(\mathcal{F}_2)$ and $\mathcal{D}_3 = \mathrm{Cay}(\mathcal{T}, \mu)$ with $\mathcal{T} = \mathrm{Cay}(\mathcal{O}_X, \mathcal{L}_i, n_i)$.
(5) $\mathcal{D} = \mathcal{E}nd_X(\mathcal{F}_2)$ and $\mathcal{D}_3$ defined over k.
(6) $\mathcal{D} = \mathcal{E}nd_X(\mathcal{F}_2)$ and $\mathcal{D}_3 = \mathcal{E}nd_X(\mathcal{O}_X \oplus \mathcal{L})$ split ($\mathcal{L}$ any line bundle).
(7) $\mathcal{D} = \mathcal{D}_3 = \mathrm{Quat}(tr_{l_1/k}(\mathcal{L}_1), N)$.
(8) $\mathcal{D} = \mathrm{Quat}(tr_{l_1/k}(\mathcal{L}_1), N)$ and $\mathcal{D}_3 = \mathcal{E}nd_X(\mathcal{F}_2)$.
(9) $\mathcal{D} = \mathrm{Quat}(tr_{l_1/k}(\mathcal{L}_1), N)$ and $\mathcal{D}_3$ defined over k.
(10) $\mathcal{D} = \mathrm{Quat}(tr_{l_1/k}(\mathcal{L}_1), N)$ and $\mathcal{D}_3 = \mathcal{E}nd_X(\mathcal{O}_X \oplus \mathcal{L})$ split ($\mathcal{L}$ any line bundle).
(11) $\mathcal{D} = \mathcal{D}_3 = \mathrm{Cay}(\mathcal{T}, tr_{l_2/k}(\mathcal{L}_2), N_{\mathcal{P}})$ with $\mathcal{T}$ as in (3).
(12) $\mathcal{D} = \mathrm{Cay}(\mathcal{T}, tr_{l_2/k}(\mathcal{L}_2), N_{\mathcal{P}})$ as in (11) and $\mathcal{D}_3 = \mathcal{E}nd_X(\mathcal{F}_2)$.
(13) $\mathcal{D} = \mathrm{Cay}(\mathcal{T}, tr_{l_2/k}(\mathcal{L}_2), N_{\mathcal{P}})$ as in (11) and $\mathcal{D}_3$ defined over k.
(14) $\mathcal{D} = \mathrm{Cay}(\mathcal{T}, tr_{l_2/k}(\mathcal{L}_2), N_{\mathcal{P}})$ as in (11) and $\mathcal{D}_3 = \mathcal{E}nd_X(\mathcal{O}_X \oplus \mathcal{L})$ split ($\mathcal{L}$ any line bundle).
(15) $\mathcal{D} = \mathcal{D}_3 = \mathrm{Cay}(\mathcal{T}, \mathcal{P}, N_{\mathcal{P}})$ with $\mathcal{T}$ as in (3) and $\mathcal{P} = \mathcal{L}_2 \otimes \mathcal{T}$, i.e. $\mathcal{P} \cong \mathcal{L}_2 \oplus \mathcal{L}_3$ as $\mathcal{O}_X$-module.
(16) $\mathcal{D} = \mathrm{Cay}(\mathcal{T}, \mathcal{P}, N_{\mathcal{P}})$ as in (15) and $\mathcal{D}_3 = \mathcal{E}nd_X(\mathcal{F}_2)$.
(17) $\mathcal{D} = \mathrm{Cay}(\mathcal{T}, \mathcal{P}, N_{\mathcal{P}})$ as in (15) and $\mathcal{D}_3$ defined over k.
(18) $\mathcal{D} = \mathrm{Cay}(\mathcal{T}, \mathcal{P}, N_{\mathcal{P}})$ as in (15) and $\mathcal{D}_3 = \mathcal{E}nd_X(\mathcal{O}_X \oplus \mathcal{L})$ split ($\mathcal{L}$ any line bundle).
(19) $\mathcal{D} = \mathcal{D}_3 = \mathrm{Cay}(\mathcal{T}, \mu)$ with $\mathcal{T} = \mathrm{Cay}(\mathcal{O}_X, \mathcal{L}_i, n_i)$.
(20) $\mathcal{D} = \mathrm{Cay}(\mathcal{T}, \mu)$ with $\mathcal{T}$ as in (19) and $\mathcal{D}_3 = \mathcal{E}nd_X(\mathcal{F}_2)$.
(21) $\mathcal{D} = \mathrm{Cay}(\mathcal{T}, \mu)$ with $\mathcal{T}$ as in (19) and $\mathcal{D}_3$ defined over k.
(22) $\mathcal{D} = \mathrm{Cay}(\mathcal{T}, \mu)$ with $\mathcal{T}$ as in (19), but with $i = 1$, and $\mathcal{D}_3 = \mathrm{Cay}(\mathcal{T}, tr_{l_2/k}(\mathcal{L}_2), N_{\mathcal{P}})$ as in (3).
(23) $\mathcal{D} = \mathrm{Cay}(\mathcal{T}, \mu)$ with $\mathcal{T}$ as in (19) and $\mathcal{D}_3 = \mathcal{E}nd_X(\mathcal{O}_X \oplus \mathcal{L})$ split ($\mathcal{L}$ any line bundle).
(24) $\mathcal{D} = \mathrm{Cay}(\mathcal{T}, \mu)$ with $\mathcal{T}$ as in (19), and $\mathcal{D} = \mathrm{Cay}(\mathcal{T}', \mu')$ with $\mathcal{T} = \mathrm{Cay}(\mathcal{O}_X, \mathcal{L}_j, n_j)$ with $i \neq j$.
(25) $\mathcal{D}$ defined over k and $\mathcal{D}_3 = \mathrm{Quat}(tr_{l_1/k}(\mathcal{L}_1), N)$.
(26) $\mathcal{D}$ defined over k and $\mathcal{D}_3 = \mathrm{Cay}(\mathcal{T}, tr_{l_2/k}(\mathcal{L}_2), N_{\mathcal{P}})$ with $\mathcal{T} = \mathrm{Cay}(\mathcal{O}_X, \mathcal{L}_1, n_1)$.
(27) $\mathcal{D}$ defined over k and $\mathcal{D}_3 = \mathrm{Cay}(\mathcal{T}, \mu)$ with $\mathcal{T} = \mathrm{Cay}(\mathcal{O}_X, \mathcal{L}_i, n_i)$.
(28) $\mathcal{D}$ defined over k and $\mathcal{D}_3 = \mathcal{E}nd_X(\mathcal{O}_X \oplus \mathcal{L})$ split ($\mathcal{L}$ any line bundle).
(29) $\mathcal{D}$ defined over k and $\mathcal{D}_3 = \mathcal{E}nd_X(\mathcal{F}_3)$.
(30) $\mathcal{D}$ as in (15) and $\mathcal{D} = \mathcal{D}_3$.
(31) $\mathcal{D}$ as in (15) and $\mathcal{D}_3 = \mathcal{E}nd_X(\mathcal{F}_3)$.

(32) $\mathcal{D}$ as in (15) and $\mathcal{D}_3 = \mathrm{Cay}(\mathcal{T}, \mu)$ with $\mathcal{T} = \mathrm{Cay}(\mathcal{O}_X, \mathcal{L}_i, n_i)$.
(33) $\mathcal{D}$ as in (15) and $\mathcal{D}_3$ defined over k.
(34) $\mathcal{D}$ as in (15) and $\mathcal{D}_3 = \mathcal{E}nd_X(\mathcal{O}_X \oplus \mathcal{L})$ split ($\mathcal{L}$ any line bundle).
(35) $\mathcal{D}$ defined over k and $\mathcal{D}_3$ defined over k.
(36) $\mathcal{D} = \mathcal{D}_3 = \mathcal{E}nd_X(\mathcal{O}_X \oplus \mathcal{L})$ split ($\mathcal{L}$ any line bundle).
(37) $\mathcal{D} = \mathcal{E}nd_X(\mathcal{O}_X \oplus \mathcal{L})$ split as in (36) and $\mathcal{D}_3 = \mathcal{E}nd_X(\mathcal{F}_3)$.
(38) $\mathcal{D} = \mathcal{E}nd_X(\mathcal{O}_X \oplus \mathcal{L})$ split as in (36) and $\mathcal{D}_3 = \mathrm{Quat}(tr_{l_1/k}(\mathcal{L}_1), N)$.
(39) $\mathcal{D} = \mathcal{E}nd_X(\mathcal{O}_X \oplus \mathcal{L})$ split as in (36) and $\mathcal{D}_3 = \mathrm{Cay}(\mathcal{T}, tr_{l_2/k}(\mathcal{L}_2), N_{\mathcal{P}})$ with $\mathcal{T} = \mathrm{Cay}(\mathcal{O}_X, \mathcal{L}_1, n_1)$.
(40) $\mathcal{D} = \mathcal{E}nd_X(\mathcal{O}_X \oplus \mathcal{L})$ split as in (36) and $\mathcal{D}_3 = \mathrm{Cay}(\mathcal{T}, \mathcal{P}, N_{\mathcal{P}})$ with $\mathcal{T}$ as in (3) and $\mathcal{P} = \mathcal{L}_2 \otimes \mathcal{T}$, i.e. $\mathcal{P} \cong \mathcal{L}_2 \oplus \mathcal{L}_3$ as $\mathcal{O}_X$-module.
(41) $\mathcal{D} = \mathcal{E}nd_X(\mathcal{O}_X \oplus \mathcal{L})$ split as in (36) and $\mathcal{D}_3$ defined over k.

□

This list covers all possible cases for (D1*). We obtain first Tits constructions starting with an Azumaya algebra $\mathcal{E}nd_X(\mathcal{M})$, where the vector bundle $\mathcal{M}$ (of rank 3) has trivial determinant. However, the locally free right modules used are of the kind $\mathcal{M} \otimes \mathcal{F}$ with $\mathcal{F}$ a *selfdual* vector bundle of rank 3 and trivial determinant. We point out that those cases, where the locally free right modules used in the first Tits construction are of the kind $\mathcal{M} \otimes \mathcal{F}^\vee$ with $\mathcal{F}$ a non-selfdual vector bundle of rank 3 and of trivial, are not covered here.

All the first Tits constructions starting, more generally, with an Azumaya algebra $\mathcal{E}nd_X(\mathcal{M})$ for any vector bundle $\mathcal{M}$ of rank 3, are listed in [Pu4, Section 8].

Corollary 6. *The following $\mathcal{O}_X$-modules can be made into a pseudo-composition algebra of rank 26 over X:*

(1) $\mathcal{O}_X^2 \oplus \mathcal{F}_3^3 \oplus F_5^3$,
(2) $\mathcal{F}_3 \oplus \mathcal{F}_5 \oplus tr_{l_1/k}(\mathcal{L}_1 \otimes \mathcal{F}_3) \oplus tr_{l_1/k}(\mathcal{L}_1 \otimes \mathcal{F}_5)$ *(if X has type III and l_1/k is Galois),*
(3) $\mathcal{F}_3 \oplus \mathcal{F}_5 \oplus [\mathcal{L}_1 \otimes \mathcal{F}_3]^2 \oplus [tr_{l_2/k}(\mathcal{L}_2 \otimes \mathcal{F}_3)]^2$, *(if X has type II),*
(4) $\mathcal{F}_3 \oplus \mathcal{F}_5 \oplus [\mathcal{L}_1 \otimes \mathcal{F}_3 \oplus \mathcal{L}_2 \otimes \mathcal{F}_3 \oplus \mathcal{L}_3 \otimes \mathcal{F}_3]^2$,
(5) $\mathcal{F}_3^7 \oplus \mathcal{F}_5$,
(6) $\mathcal{F}_3^3 \oplus \mathcal{F}_5 \oplus [\mathcal{F}_3 \otimes \mathcal{L} \oplus \mathcal{F}_3 \otimes \mathcal{L}^\vee]^2$,
(7) $\mathcal{O}_X^8 \oplus [tr_{l_1/k}(\mathcal{L}_2)]^3 \oplus [tr_{l_1/k}(\mathcal{L}_3)]^3$ *(if X has type III and l_1/k is Galois),*
(8) $\mathcal{O}_X^2 \oplus tr_{l_1/k}(\mathcal{L}_2) \oplus tr_{l_1/k}(\mathcal{L}_3) \oplus [tr_{l_1/k}(\mathcal{L}_1 \otimes \mathcal{F}_3)]^2$ *(if X has type III and l_1/k is Galois),*
(9) $\mathcal{O}_X^2 \oplus tr_{l_1/k}(\mathcal{L}_2) \oplus tr_{l_1/k}(\mathcal{L}_3) \oplus [tr_{l_1/k}(\mathcal{L}_1)]^6$ *(if X has type III and l_1/k is Galois),*
(10) $\mathcal{O}_X^2 \oplus tr_{l_1/k}(\mathcal{L}_2) \oplus tr_{l_1/k}(\mathcal{L}_3) \oplus [tr_{l_1/k}(\mathcal{L}_1) \oplus tr_{l_1/k}(\mathcal{L}_1 \otimes \mathcal{L}) \oplus tr_{l_1/k}(\mathcal{L}_1 \otimes \mathcal{L}^\vee)]^2$, *(if X has type III and l_1/k is Galois),*
(11) $\mathcal{O}_X^8 \oplus [tr_{l_2/k}(\mathcal{L}_2)]^6 \oplus \mathcal{L}_1^6$ *(if X has type II),*
(12) $\mathcal{O}_X^2 \oplus [tr_{l_2/k}(\mathcal{L}_2)]^2 \oplus \mathcal{L}_1^2 \oplus [\mathcal{L}_1 \otimes \mathcal{F}_3]^2 \oplus [tr_{l_2/k}(\mathcal{L}_2 \otimes \mathcal{F}_3)]^2$ *(if X has type II),*
(13) $\mathcal{O}_X^2 \oplus [tr_{l_2/k}(\mathcal{L}_2)]^2 \oplus \mathcal{L}_1^8 \oplus [tr_{l_2/k}(\mathcal{L}_2)]^6$, *starting with $\mathcal{E}nd_X(\mathcal{L}_1 \oplus tr_{l_2/k}(\mathcal{L}_2))$ (if X has type II),*
(14) $\mathcal{O}_X^2 \oplus [tr_{l_2/k}(\mathcal{L}_2)]^4 \oplus \mathcal{L}_1^4 \oplus [\mathcal{L}_1 \otimes \mathcal{L}]^2 \oplus [\mathcal{L}_1 \otimes \mathcal{L}^\vee]^2 \oplus [tr_{l_2/k}(\mathcal{L}_2 \otimes \mathcal{L})]^2 \oplus [tr_{l_2/k}(\mathcal{L}_2 \otimes \mathcal{L}^\vee)]^2$, *(if X has type II),*
(15) $\mathcal{O}_X^8 \oplus \mathcal{L}_1^6 \oplus \mathcal{L}_2^6 \oplus \mathcal{L}_3^6$, *(if X has type I),*

(16) $\mathcal{O}_X^2 \oplus \mathcal{L}_1^2 \oplus \mathcal{L}_2^2 \oplus \mathcal{L}_3^2 \oplus [\mathcal{L}_1 \otimes \mathcal{F}_3 \oplus \mathcal{L}_2 \otimes \mathcal{F}_3 \oplus \mathcal{L}_3 \otimes \mathcal{F}_3]^2$, *(if X has type I),*
(17) $\mathcal{O}_X^2 \oplus \mathcal{L}_1^8 \oplus \mathcal{L}_2^8 \oplus \mathcal{L}_3^8$, *(if X has type I),*
(18) $\mathcal{O}_X^2 \oplus \mathcal{L}_1^4 \oplus \mathcal{L}_2^4 \oplus \mathcal{L}_3^4 \oplus [\mathcal{L}_1 \otimes \mathcal{L}]^2 \oplus [\mathcal{L}_2 \otimes \mathcal{L}]^2 \oplus [\mathcal{L}_3 \otimes \mathcal{L}]^2 \oplus [\mathcal{L}_1 \otimes \mathcal{L}^\vee]^2 \oplus [\mathcal{L}_2 \otimes \mathcal{L}^\vee]^2 \oplus [\mathcal{L}_3 \otimes \mathcal{L}^\vee]^2$, *(if X has type I),*
(19) $\mathcal{O}_X^{14} \oplus \mathcal{L}_i^{12}$,
(20) $\mathcal{O}_X^4 \oplus \mathcal{L}_i^4 \oplus \mathcal{F}_3^2 \oplus [\mathcal{L}_i \otimes \mathcal{F}_3]^4$,
(21) $\mathcal{O}_X^{10} \oplus \mathcal{L}_i^{16}$,
(22) $\mathcal{O}_X^8 \oplus \mathcal{L}_1^6 \oplus [tr_{l_2/k}(\mathcal{L}_2)]^6$, *(if X has type II),*
(23) $\mathcal{O}_X^6 \oplus \mathcal{L}_i^8 \oplus [\mathcal{L} \oplus \mathcal{L}^\vee]^2 \oplus [\mathcal{L}_i \otimes \mathcal{L}]^4 \oplus [\mathcal{L}_i \otimes \mathcal{L}^\vee]^4$,
(24) $\mathcal{O}_X^6 \oplus \mathcal{L}_i^8 \oplus [\mathcal{L}_j \oplus \mathcal{L}_j^\vee]^2 \oplus [\mathcal{L}_i \otimes \mathcal{L}_j]^4 \oplus [\mathcal{L}_i \otimes \mathcal{L}_j^\vee]^4$ *for $i \neq j$ (if X has type I or II),*
(25) $\mathcal{O}_X^8 \oplus [tr_{l_1/k}(\mathcal{L}_1)]^6$ *starting with some $\mathcal{D}$ defined over k (if X has type III),*
(26) $\mathcal{O}_X^8 \oplus \mathcal{L}_1^6 \oplus [tr_{l_2/k}(\mathcal{L}_2)]^6$ *starting with some $\mathcal{D}$ defined over k (if X has type II),*
(27) $\mathcal{O}_X^{14} \oplus \mathcal{L}_1^{12}$, *starting with some $\mathcal{D}$ defined over k,*
(28) $\mathcal{O}_X^{14} \oplus \mathcal{L}^6 \oplus [\mathcal{L}^\vee]^6$, *starting with some $\mathcal{D}$ defined over k, with $\mathcal{L}$ any line bundle over X,*
(29) $\mathcal{O}_X^8 \oplus \mathcal{F}_3^6$, *starting with some $\mathcal{D}$ defined over k,*
(30) $\mathcal{O}_X^8 \oplus \mathcal{L}_1^6 \oplus \mathcal{L}_2^6 \oplus \mathcal{L}_3^6$, *starting with $\mathcal{D}$ as in (15) (if X has type I),*
(31) $\mathcal{O}_X^2 \oplus \mathcal{L}_1^2 \oplus \mathcal{L}_2^2 \oplus \mathcal{L}_3^2 \oplus [\mathcal{L}_1 \otimes \mathcal{F}_3 \oplus \mathcal{L}_2 \otimes \mathcal{F}_3 \oplus \mathcal{L}_3 \otimes \mathcal{F}_3]^2$, *starting with $\mathcal{D}$ as in (15) (if X has type I),*
(32) $\mathcal{O}_X^2 \oplus \mathcal{L}_1^4 \oplus \mathcal{L}_2^4 \oplus \mathcal{L}_3^4 \oplus [\mathcal{L}_1 \otimes \mathcal{L}_i \oplus \mathcal{L}_2 \otimes \mathcal{L}_i \oplus \mathcal{L}_3 \otimes \mathcal{L}_i]^2$, *starting with $\mathcal{D}$ as in (15) (if X has type I),*
(33) $\mathcal{O}_X^2 \oplus \mathcal{L}_1^8 \oplus \mathcal{L}_2^8 \oplus \mathcal{L}_3^8$, *starting with $\mathcal{D}$ as in (15) (if X has type I),*
(34) $\mathcal{O}_X^2 \oplus \mathcal{L}_1^4 \oplus \mathcal{L}_2^4 \oplus \mathcal{L}_3^4 \oplus [\mathcal{L}_1 \otimes \mathcal{L} \oplus \mathcal{L}_2 \otimes \mathcal{L} \oplus \mathcal{L}_3 \otimes \mathcal{L} \oplus \mathcal{L}_1 \otimes \mathcal{L}^\vee \oplus \mathcal{L}_2 \otimes \mathcal{L}^\vee \oplus \mathcal{L}_3 \otimes \mathcal{L}^\vee]^2$, *starting with $\mathcal{D}$ as in (15) (if X has type I),*
(35) $\mathcal{O}_X^{26}$.
(36) $\mathcal{O}_X^8 \oplus \mathcal{L}^6 \oplus \mathcal{L}^{\vee 6} \oplus [\mathcal{L} \otimes \mathcal{L}]^3 \oplus [\mathcal{L}^\vee \otimes \mathcal{L}^\vee]^3$, *for some line bundle $\mathcal{L}$ over X,*
(37) $\mathcal{O}_X^2 \oplus \mathcal{L}^2 \oplus \mathcal{L}^{\vee 2} \oplus \mathcal{L} \otimes \mathcal{L} \oplus \mathcal{L}^\vee \otimes \mathcal{L}^\vee \oplus \mathcal{F}_3^2 \oplus [\mathcal{L} \otimes \mathcal{F}_3]^2 \oplus [\mathcal{L}^\vee \otimes \mathcal{F}_3]^2$ *for some line bundle $\mathcal{L}$ over X,*
(38) $\mathcal{O}_X^2 \oplus \mathcal{L}^2 \oplus \mathcal{L}^{\vee 2} \oplus \mathcal{L} \otimes \mathcal{L} \oplus \mathcal{L}^\vee \otimes \mathcal{L}^\vee \oplus [tr_{l_1/k}(\mathcal{L}_1)]^2 \oplus [tr_{l_1/k}(\mathcal{L} \otimes \mathcal{L}_1)]^2 \oplus [tr_{l_1/k}(\mathcal{L}^\vee \otimes \mathcal{L}_1)]^2$,
(39) $\mathcal{O}_X^2 \oplus \mathcal{L}^2 \oplus \mathcal{L}^{\vee 2} \oplus \mathcal{L} \otimes \mathcal{L} \oplus \mathcal{L}^\vee \otimes \mathcal{L}^\vee \oplus \mathcal{L}_1^2 \oplus [tr_{l_2/k}(\mathcal{L}_2)]^2 \oplus [\mathcal{L} \otimes \mathcal{L}_1]^2 \oplus [\mathcal{L}^\vee \otimes \mathcal{L}_1]^2 \oplus [tr_{l_2/k}(\mathcal{L} \otimes \mathcal{L}_2)]^2 \oplus [tr_{l_2/k}(\mathcal{L}^\vee \otimes \mathcal{L}_2)]^2$,
(40) $\mathcal{O}_X^4 \oplus \mathcal{L}^4 \oplus \mathcal{L}^{\vee 4} \oplus \mathcal{L} \otimes \mathcal{L} \oplus \mathcal{L}^\vee \otimes \mathcal{L}^\vee \oplus \mathcal{L}_1^2 \oplus \mathcal{L}_2^2 \oplus \mathcal{L}_3^2 \oplus [\mathcal{L} \otimes \mathcal{L}_1]^2 \oplus [\mathcal{L} \otimes \mathcal{L}_2]^2 \oplus [\mathcal{L} \otimes \mathcal{L}_3]^2 \oplus [\mathcal{L}^\vee \otimes \mathcal{L}_1]^2 \oplus [\mathcal{L}^\vee \otimes \mathcal{L}_2]^2 \oplus [\mathcal{L}^\vee \otimes \mathcal{L}_3]^2$,
(41) $\mathcal{O}_X^{10} \oplus \mathcal{L}^{10} \oplus \mathcal{L}^{\vee 10} \oplus \mathcal{L} \otimes \mathcal{L} \oplus \mathcal{L}^\vee \otimes \mathcal{L}^\vee$.

Proof. The cases studied in Theorem 13 yield the assertion. □

Theorem 14. *The following $\mathcal{O}_X$-modules can be made into an admissible cubic algebra over X:*

(1) $tr_{l_1/k}(\mathcal{L}_1 \otimes \mathcal{F}_3) \oplus [tr_{l_1/k}(\mathcal{L}_1)]^3 \oplus \mathcal{F}_3^3$,
(2) $tr_{l_1/k}(\mathcal{L}_1 \otimes \mathcal{F}_3) \oplus tr_{l_1/k}(\mathcal{L}_1) \oplus tr_{l_1/k}(\mathcal{L}_1 \otimes \mathcal{L}) \oplus tr_{l_1/k}(\mathcal{L}_1 \otimes \mathcal{L}^\vee) \oplus \mathcal{F}_3 \oplus (\mathcal{F}_3 \otimes \mathcal{L}) \oplus (\mathcal{F}_3 \otimes \mathcal{L}^\vee)$,
(3) $\mathcal{L}_1 \otimes \mathcal{F}_3 \oplus tr_{l_2/k}(\mathcal{L}_2 \otimes \mathcal{F}_3) \oplus [\mathcal{L}_1 \oplus tr_{l_2/k}(\mathcal{L}_2)]^3 \oplus \mathcal{F}_3^3$,
(4) $(\mathcal{L}_1 \otimes \mathcal{F}_3) \oplus (\mathcal{L}_2 \otimes \mathcal{F}_3) \oplus (\mathcal{L}_3 \otimes \mathcal{F}_3) \oplus [\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3]^3 \oplus \mathcal{F}_3^3$,

(5) $(\mathcal{L}_1 \otimes \mathcal{F}_3) \oplus tr_{l_2/k}(\mathcal{L}_2 \otimes \mathcal{F}_3) \oplus \mathcal{L}_1 \oplus tr_{l_2/k}(\mathcal{L}_2) \oplus (\mathcal{L}_1 \otimes \mathcal{L}) \oplus tr_{l_2/k}(\mathcal{L}_2 \otimes \mathcal{L}) \oplus (\mathcal{L}_1 \otimes \mathcal{L}^\vee) \oplus tr_{l_2/k}(\mathcal{L}_2 \otimes \mathcal{L}^\vee)$.

Proof. Let $\mathcal{D}_i = \mathrm{Quat}(\mathcal{M}_i, N_i)$ be quaternion algebras with canonical involution. Let $\mathcal{A} = \mathcal{D}_1 \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$ with involution $\tau = {}^- \otimes {}^- \otimes {}^-$. The $\mathcal{O}_X$-module

$$\mathcal{J}_0 = \mathcal{M}_1 \otimes \mathcal{M}_2 \oplus \mathcal{M}_2 \otimes \mathcal{M}_3 \oplus \mathcal{M}_1 \otimes \mathcal{M}_3$$

can be made into an admissible cubic algebra over X. We compute $\mathcal{J} = H(\mathcal{A}, \tau)$ using Section 6 (D*) in the following cases:

(1) $\mathcal{D}_1 = \mathcal{E}nd_X(\mathcal{F}_2)$, $\mathcal{D}_2 = \mathrm{Quat}(tr_{l_1/k}(\mathcal{L}_1), N)$ and $\mathcal{D}_3$ defined over X.
(2) $\mathcal{D}_1 = \mathcal{E}nd_X(\mathcal{F}_2)$, $\mathcal{D}_2 = \mathrm{Quat}(tr_{l_1/k}(\mathcal{L}_1), N)$ and $\mathcal{D}_3 = \mathcal{E}nd_X(\mathcal{O}_X \oplus \mathcal{L})$ split ($\mathcal{L}$ any line bundle).
(3) $\mathcal{D}_1 = \mathcal{E}nd_X(\mathcal{F}_2)$, $\mathcal{D}_2 = \mathrm{Cay}(\mathcal{T}, tr_{l_2/k}(\mathcal{L}_2), N_{\mathcal{P}})$ as in Theorem 13 (3) and $\mathcal{D}_3$ defined over X.
(4) $\mathcal{D}_1 = \mathcal{E}nd_X(\mathcal{F}_2)$, $\mathcal{D}_2 = \mathrm{Cay}(\mathcal{T}, \mathcal{P}, N_{\mathcal{P}})$ with $\mathcal{T}$ and $\mathcal{P} = \mathcal{L}_2 \otimes \mathcal{T}$ as in Theorem 11 (15), $\mathcal{D}_3$ defined over X.
(5) $\mathcal{D}_1 = \mathcal{E}nd_X(\mathcal{F}_2)$, $\mathcal{D}_2 = \mathrm{Cay}(\mathcal{T}, tr_{l_2/k}(\mathcal{L}_2), N_{\mathcal{P}})$ as in Theorem 13 (3) and $\mathcal{D}_3 = \mathcal{E}nd_X(\mathcal{O}_X \oplus \mathcal{L})$ split ($\mathcal{L}$ any line bundle).

□

In cases (1) to (4) of Theorem 14, the quartic Jordan algebra $\mathcal{J}$ does not contain idempotents of trace 1 by Corollary 2. If, in case (5), $\mathcal{L} = \mathcal{L}_1$, it is not clear from the module structure if the algebra contains idempotents of trace 1, for all other $\mathcal{L}$ it cannot.

Theorem 15. *The following $\mathcal{O}_X$-modules carry the structure of an Albert algebra over X:*

(1) $\mathcal{O}_X^9 \oplus \mathcal{L}_1^6 \oplus [tr_{l_2/k}(\mathcal{L}_2)]^6$ *(if X has type II),*
(2) $\mathcal{O}_X^3 \oplus \mathcal{L}_1^2 \oplus [tr_{l_2/k}(\mathcal{L}_2)]^2 \oplus [\mathcal{L}_1 \otimes \mathcal{F}_3]^2 \oplus [tr_{l_2/k}(\mathcal{L}_3 \otimes \mathcal{F}_3)]^2$ *(if X has type II),*
(3) $\mathcal{O}_X^3 \oplus \mathcal{L}_1^4 \oplus [tr_{l_2/k}(\mathcal{L}_2)]^4 \oplus [(\mathcal{L}_1 \otimes \mathcal{L}) \oplus (\mathcal{L}_1 \otimes \mathcal{L}^\vee)]^2 \oplus [tr_{l_2/k}(\mathcal{L}_3 \otimes \mathcal{L}) \oplus tr_{l_2/k}(\mathcal{L}_3 \otimes \mathcal{L}^\vee)]^2$ *(if X has type II)*
(4) $\mathcal{O}_X^3 \oplus \mathcal{L}_1^8 \oplus [tr_{l_2/k}(\mathcal{L}_2)]^8$ *(if X has type II),*
(5) $\mathcal{O}_X^7 \oplus \mathcal{L}_1^4 \oplus [tr_{l_2/k}(\mathcal{L}_2)]^8$ *(if X has type II),*
(6) $\mathcal{O}_X^{15} \oplus \mathcal{L}_i^{12}$,
(7) $\mathcal{O}_X^5 \oplus \mathcal{L}_i^4 \oplus [\mathcal{L}_i \otimes \mathcal{F}_3]^4 \oplus \mathcal{F}_3^2$,
(8) $\mathcal{O}_X^7 \oplus \mathcal{L}_i^8 \oplus [\mathcal{L}_i \otimes \mathcal{L}]^4 \oplus [\mathcal{L}_i \otimes \mathcal{L}^\vee]^4 \oplus [\mathcal{L} \oplus \mathcal{L}^\vee]^2$,
(9) $\mathcal{O}_X^{11} \oplus \mathcal{L}_i^{16}$, *starting with* $\mathcal{E}nd_X(\mathcal{O}_X \oplus \mathcal{L}_i^2)$,
(10) $\mathcal{O}_X^9 \oplus \mathcal{L}_1^6 \oplus \mathcal{L}_2^6 \oplus \mathcal{L}_3^6$ *(if X has type I),*
(11) $\mathcal{O}_X^3 \oplus \mathcal{L}_1^2 \oplus \mathcal{L}_2^2 \oplus \mathcal{L}_3^2 \oplus [\mathcal{L}_1 \otimes \mathcal{F}_3]^2 \oplus [\mathcal{L}_2 \otimes \mathcal{F}_3]^2 \oplus [\mathcal{L}_3 \otimes \mathcal{F}_3]^2$ *(if X has type I),*
(12) $\mathcal{O}_X^3 \oplus \mathcal{L}_1^4 \oplus \mathcal{L}_2^4 \oplus \mathcal{L}_3^4 \oplus [\mathcal{L}_1 \otimes \mathcal{L}]^2 \oplus [\mathcal{L}_2 \otimes \mathcal{L}]^2 \oplus [\mathcal{L}_3 \otimes \mathcal{L}]^2 \oplus [\mathcal{L}_1 \otimes \mathcal{L}^\vee]^2 \oplus [\mathcal{L}_2 \otimes \mathcal{L}^\vee]^2 \oplus [\mathcal{L}_3 \otimes \mathcal{L}^\vee]^2$ *(if X has type I),*
(13) $\mathcal{O}_X^3 \oplus (\mathcal{L} \otimes \mathcal{L}) \oplus (\mathcal{L}^\vee \otimes \mathcal{L}^\vee) \oplus [\mathcal{L} \oplus \mathcal{L}^\vee]^2 \oplus \mathcal{F}_3^2 \oplus [(\mathcal{L} \otimes \mathcal{F}_3) \oplus (\mathcal{L}^\vee \otimes \mathcal{F}_3)]^2$,
(14) $\mathcal{O}_X^3 \oplus (\mathcal{L} \otimes \mathcal{L}) \oplus (\mathcal{L}^\vee \otimes \mathcal{L}^\vee) \oplus [\mathcal{L} \oplus \mathcal{L}^\vee]^2 \oplus [tr_{l_1/k}(\mathcal{L}_1)]^2 \oplus [tr_{l_1/k}(\mathcal{L} \otimes \mathcal{L}_1) \oplus tr_{l_1/k}(\mathcal{L}^\vee \otimes \mathcal{L}_1)]^2$, *(if X has type III),*
(15) $\mathcal{O}_X^3 \oplus (\mathcal{L} \otimes \mathcal{L}) \oplus (\mathcal{L}^\vee \otimes \mathcal{L}^\vee) \oplus [\mathcal{L} \oplus \mathcal{L}^\vee]^2 \oplus [\mathcal{L}_1 \oplus tr_{l_2/k}(\mathcal{L}_2)]^2 \oplus [\mathcal{L} \otimes \mathcal{L}_1 \oplus tr_{l_2/k}(\mathcal{L} \otimes \mathcal{L}_2) \oplus \mathcal{L}^\vee \otimes \mathcal{L}_1 \oplus tr_{l_2/k}(\mathcal{L}^\vee \otimes \mathcal{L}_2)]^2$ *(if X has type II).*

Proof. Let $\mathcal{D}_1 = \mathrm{Quat}(\mathcal{M}_1, N_1)$, $\mathcal{D} = \mathrm{Cay}(\mathcal{T}, \mathcal{P}, N)$ with $\mathcal{T} = \mathrm{Cay}(\mathcal{T}, \mathcal{L}, N_{\mathcal{L}})$. By Section 6 (E1*), the quartic Jordan algebra $\mathcal{J} = H(\mathcal{A}, \tau) = (\mathcal{D}_1, ^{-}) \otimes (\mathcal{D}, \hat{\ }) \otimes (\mathcal{D}, \hat{\ })$ contains a non-trivial idempotent of trace 1. Thus $\mathcal{J}_0$ can be made into an Albert algebra. We consider the following cases, which yield the corresponding assertions by Theorem 2 and Section 6 (E1*):

(1) $\mathcal{D} = \mathcal{D}_1 = \mathrm{Cay}(\mathcal{T}, tr_{l_2/k}(\mathcal{L}_2), N_{\mathcal{P}})$ as in Theorem 14 (3).
(2) $\mathcal{D}_1 = \mathcal{E}nd_X(\mathcal{F}_2)$, $\mathcal{D}$ as in (1).
(3) $\mathcal{D}_1 = \mathcal{E}nd_X(\mathcal{O}_X \oplus \mathcal{L})$ for any line bundle $\mathcal{L}$, $\mathcal{D}$ as in (1).
(4) $\mathcal{D}_1$ defined over k, $\mathcal{D}$ as in (1).
(5) $\mathcal{D}_1 = \mathrm{Cay}(\mathcal{T}, \mu)$ with $\mathcal{T} = \mathrm{Cay}(\mathcal{O}_X, \mathcal{L}_1, n_1)$, $\mathcal{D}$ as in (1).
(6) $\mathcal{D} = \mathcal{D}_1 = \mathrm{Cay}(\mathcal{T}, \mu)$ with $\mathcal{T} = \mathrm{Cay}(\mathcal{O}_X, \mathcal{L}_i, n_i)$.
(7) $\mathcal{D}_1 = \mathcal{E}nd_X(\mathcal{F}_2)$, $\mathcal{D}$ as in (6).
(8) $\mathcal{D}_1 = \mathcal{E}nd_X(\mathcal{O}_X \oplus \mathcal{L})$ for any line bundle $\mathcal{L}$, $\mathcal{D}$ as in (6).
(9) $\mathcal{D}_1$ defined over k, $\mathcal{D}$ as in (6).
(10) $\mathcal{D} = \mathcal{D}_1 = \mathrm{Cay}(\mathcal{T}, \mathcal{P}, N_{\mathcal{P}})$ with $\mathcal{T}$ as in (1) and $\mathcal{P} = \mathcal{L}_2 \otimes \mathcal{T}$, i.e. $\mathcal{P} \cong \mathcal{L}_2 \oplus \mathcal{L}_3$ as $\mathcal{O}_X$-module.
(11) $\mathcal{D}_1 = \mathcal{E}nd_X(\mathcal{F}_2)$, $\mathcal{D}$ as in (10).
(12) $\mathcal{D}_1 = \mathcal{E}nd_X(\mathcal{O}_X \oplus \mathcal{L})$ for any line bundle $\mathcal{L}$, $\mathcal{D}$ as in (10).
(13) $\mathcal{D} = \mathcal{E}nd_X(\mathcal{O}_X \oplus \mathcal{L})$ for any line bundle $\mathcal{L}$, $\mathcal{D}_1 = \mathcal{E}nd_X(\mathcal{F}_2)$.
(14) $\mathcal{D}_1 = \mathrm{Quat}(tr_{l_1/k}(\mathcal{L}_1), N)$ and $\mathcal{D}$ as in (13).
(15) $\mathcal{D}_1 = \mathrm{Cay}(\mathcal{T}, tr_{l_2/k}(\mathcal{L}_2), N_{\mathcal{P}})$ as in Theorem 14 (3), $\mathcal{D}$ as in (13).

□

This list is not complete. Note that cases (3) and (5) of Theorem 15 do not appear in Theorem 13. It is not clear, whether the Albert algebras obtained here are first Tits constructions.

Theorem 16. *Let $\mathcal{D}_3$ be a quaternion algebra which is the Cayley Dickson doubling* $\mathrm{Cay}(\mathcal{T}_3, \mathcal{P}, N_{\mathcal{P}})$ *of a quadratic étale algebra $\mathcal{T}_3 \cong \mathcal{O}_X \oplus \mathcal{N}$. The following $\mathcal{O}_X$-modules carry the structure of an Albert algebra over X:*

(1) $\mathcal{O}_X \oplus \mathcal{N}^2 \oplus [\mathcal{L}_1 \otimes \mathcal{N}]^2 \oplus [tr_{l_2/k}(\mathcal{L}_2) \otimes \mathcal{N}]^2 \oplus (\mathcal{P}_3^2 \oplus [tr_{l_2/k}(\mathcal{L}_2) \otimes \mathcal{P}_3]^2) \oplus [tr_{l_2/k}(\mathcal{L}_2)]^2$ *if X has type II,*
(2) $\mathcal{O}_X \oplus (\mathcal{N}^2 \oplus [\mathcal{L}_1 \otimes \mathcal{N}]^4 \oplus [\mathcal{L}_2 \otimes \mathcal{N}]^2 \oplus [\mathcal{L}_3 \otimes \mathcal{N}]^2) \oplus (\mathcal{P}_3^2 \oplus [\mathcal{L}_2 \otimes \mathcal{P}_3]^2 \oplus [\mathcal{L}_3 \otimes \mathcal{P}_3]^2) \oplus [\mathcal{L}_2]^2 \oplus [\mathcal{L}_3]^2$ *if X has type I.*

This list is not exhaustive.

Proof. By Section 6 (E2*), these cases arise with $\mathcal{D}_3 \cong \mathcal{O}_X \oplus \mathcal{N} \oplus \mathcal{P}_3$ $\mathcal{D}$ chosen as follows:

(1) $\mathcal{D} \cong \mathcal{O}_X \oplus \mathcal{L}_1 \oplus tr_{l_2/k}(\mathcal{L}_2)$
(2) $\mathcal{D} \cong \mathcal{O}_X \oplus \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$

□

Our construction method, even when starting with the rather obvious (sometimes trivial) choices for a quartic Jordan algebra as done here and in the previous sections, already yields many examples of how the underlying $\mathcal{O}_X$-module structure of an Albert algebra can look like.

One of the major difficulties is that we do not know enough yet on quartic Jordan algebras over X. In particular, it does not seem to be clear how Azumaya algebras of constant rank 64 with a symplectic involution over an elliptic curve can look like, other than the obvious choices considered here.

Hopefully, more sophisticated choices of quartic Jordan algebras will lead to more interesting examples of cubic Jordan algebras over X.

References

[Ach1] Achhammer, G., *Albert Algebren über lokal geringten Räumen*. PhD Thesis, FernUniversität Hagen, 1995.

[Ach2] Achhammer, G., *The first Tits construction of Albert algebras over locally ringed spaces*. In: Nonassociative algebra and its applications (Oviedo, 1993), Math. Appl. 303 (1994), 8-11. Kluwer Acad. Publ., Dordrecht.

[A-F] Allison, B., Faulkner, J.R., *A Cayley-Dickson process for a class of structurable algebras*. Trans. Amer. Math. Soc. 238 (1) (1984), 185-210.

[AEJ1] Arason, J., Elman, R., Jacob, B., *On indecomposable vector bundles*. Comm. Alg. 20 (1992), 1323-1351.

[AEJ2] Arason, J., Elman, R., Jacob, B., *On the Witt ring of elliptic curves*. Proc. of Symposia in Pure Math. 58.2 (1995), 1-25.

[At] Atiyah, M.F., *Vector bundles over an elliptic curve*. Proc. London Math. Soc. 7 (1957), 414-452.

[E-O] Elduque, A., Okubo, S., *On algebras satisfying $x^2x^2 = N(x)x$*. Math. Z. 235 (2000), 275-314.

[H] Hartshorne, R., "Algebraic geometry". Graduate Texts in Mathematics, vol.52, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[J] Jacobson, N., "Structure and representations of Jordan algebras". AMS Coll. Publ. 39, AMS, Providence, R. I., 1968.

[H-Pe] Hentzel, I. R., Peresi, L. A., *A variety containing Jordan and pseudo-composition algebras*. East-West J. Math. 6 (2004) (1), 67-84.

[Kn] Knus, M.-A., *Quadratic and hermitian forms over rings*. Springer-Verlag, Berlin-Heidelberg-New York, 1991.

[J] Jacobson, N., "Structure and representations of Jordan algebras". AMS Coll. Publ. 39, AMS, Providence, R. I., 1968.

[KMRT] Knus, M.A., Merkurjev, A., Rost, M., Tignol, J.-P., "The Book of Involutions", AMS Coll. Publications, Vol.44 (1998).

[La] Lang, S., "Algebra". Third ed., Addison-Wesley Publishing Company, Reading, Mass., 1997.

[Lo] Loos, O., *Generically algebraic Jordan algebras over commutative rings*. J. Alg. 297 (2006), 474-529.

[McC] McCrimmon, K., " A taste of Jordan algebras." Universitext, Springer Verlag, New York 2004.

[M-O] Meyberg, K., Osborne, J.M., *Pseudo-composition algebras*. Math. Z. 214 (1993), 67-77.

[Pa-S-T1] Parimala, R., Sridharan, R., Thakur, M.L., *Jordan algebras and F_4 bundles over the affine plane*. J. Algebra 198 (1997), 582-607.

[Pa-S-T2] Parimala, R., Sridharan, R., Thakur, M.L., *Tits' constructions of Jordan algebras and F_4 bundles on the plane*. Compositio Mathematica 119 (1999), 13-40.

[P1] Petersson, H.P., *Composition algebras over algebraic curves of genus zero*. Trans. Amer. Math. Soc. 337(1) (1993), 473-491.

[P2] Petersson, H.P., *The Albert algebra of generic matrices*. Comm. Alg. 27(8) (1999), 3703-3717.

[P3] Petersson, H.P., *Idempotent 2-by-2 matrices*. Preprint, 2007.

[P-R1] Petersson, H. P., Racine, M. L., *Jordan algebras of degree 3 and the Tits process.* J. Algebra 98 (1986) (1), 211-243.

[Pu1] Pumplün, S., *Quaternion algebras over elliptic curves.* Comm. Algebra 26 (12), 4357–4373 (1998).

[Pu2] Pumplün, S., *Involutions on composition algebras.* Indag. Math. 14 (2) (2003), 241-248.

[Pu3] Pumplün, S., *Jordan algebras over algebraic varieties.* Submitted, available at http://arXiv:math.RA/0708.4352.

[Pu4] Pumplün, S., *Albert algebras over curves of genus zero and one.* Submitted, available at arXiv:math.RA/0709.2308

[Pu5] Pumplün, S., *Forms permitting composition over algebraic varieties.* Submitted, available at arXiv:math.RA/0705.2522.

[S] Schafer, R.D., "An introduction to nonassociative algebras", Dover Publ., Inc., New York, 1995.

[R-W] Röhrl, H., Walcher, S., *Algebras of complexity one.* Algebras, Groups, Geom. 5 (1988), 61-107.

[W] Walcher, S., *Algebras of rank 3*, Comm. Alg. 27 (7), 3401–3438 (1999).

E-mail address: susanne.pumpluen@nottingham.ac.uk

School of Mathematical Sciences, University of Nottingham, University Park, Nottingham NG7 2RD, United Kingdom