GEOMETRIES, THE PRINCIPLE OF DUALITY, AND ALGEBRAIC GROUPS

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Abstract. J. Tits gave a general recipe for producing an abstract geometry from a semisimple algebraic group. This expository paper describes a uniform method for giving a concrete realization of Tits's geometry and works through several examples. We also give a criterion for recognizing the automorphism of the geometry induced by an automorphism of the group. The E_6 geometry is studied in depth.

Contents

J. Tits's theory of buildings associated with semisimple algebraic groups gives a unified method of extracting a geometry from a group. For example, the group SL_n gives rise to $(n-1)$-dimensional projective space. Tits's geometry however is very abstract. Speaking precisely, one obtains an *incidence geometry*, which consists of an abstract set of objects each with a given type, and a reflexive, symmetric binary relation on the set of objects called incidence. We find it more palatable to think of projective space in

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a concrete way, as the collection of subspaces of some explicit vector space. In Section 2 we give an explicit recipe for concretizing Tits's incidence geometry.

The midsection of this paper consists of explicit descriptions of the concrete realizations of the geometries for groups of type A, D, E_6, E_7, F_4, and G_2. (Readers should have little trouble filling in the missing types B and C. We do not know a good description for the geometry of type E_8, but we make a few comments in 10.3.) Such descriptions may be found in a variety of places in the literature, e.g., [Coh] or [FF]. The main innovation here is that our recipe produces a realization of the geometry by a largely deterministic process beginning from the root system of the group and a fundamental representation, whereas approaches in the literature have the appearance of being ad hoc. Our principal tool is the representation theory of semisimple groups; we only use the most elementary results, but we exploit those ruthlessly. Consequently, throughout this paper, *our base field k is assumed to have characteristic zero.*

Élie Cartan was already aware (see [Ca, p. 362]) that the "outer" automorphism $g \mapsto (g^{-1})^t$ (where t denotes transposition) of SL_3 gives rise to a polarity in the projective plane and so to the principle of duality. In general, an outer automorphism of a semisimple group gives an automorphism ψ of the corresponding geometry, hence also a "principle of duality" (or "triality" or ...). In Sections 11–14 below, we give a criterion for recognizing such an automorphism ψ and apply our criterion to get an explicit description of ψ in all cases.

We hope that readers who are not familiar with, say, exceptional groups will find the presentation here unusually accessible because of the uniform treatment in the common language of representation theory. Experts will note that the treatment of the E_6 geometry in §7 largely avoids any discussion of Jordan algebras. Similarly, we do not need to mention octonions at all in our discussion of triality in §13.

Our hypothetical reader is moderately familiar with the theory of linear algebraic groups as in [Bor], [Hu 81], [Sp], or [St] and the classification of irreducible modules via highest weight vectors from [Hu 80, Ch. VI], [FH, §14], [GW, §5.1], or [Va].

1. Tits's geometry Γ_P

In this section, we describe Tits's recipe for producing an incidence geometry from a certain kind of algebraic group. An *incidence geometry* is a set of objects, each of some type (e.g., point, line), together with a symmetric binary relation known as *incidence*. There is just one further axiom: objects x, y of the same type are incident if and only if they are equal.

Remark. Modern formulations of Tits's recipe take a group and construct a building rather than an incidence geometry. From our perspective, a building is an incidence geometry with extra structure that we do not really need. So we deal only with the much-simpler-to-define incidence geometries as in [T 56]. For a presentation in terms of buildings, see [T 74] or [Br, Ch. V].

We start with a root system Φ in the sense of [Hu 81, §9.2] or [St, §1], a "reduced root system" in the language of [Bou 4–6]. There is a simply connected subgroup G of $GL_n(k)$ (for some n) corresponding to Φ, and it is uniquely determined up to isomorphism. Taking Φ of type A_n, we obtain a group G isomorphic to SL_{n+1}. When k is algebraically closed, every semisimple algebraic group is obtained in this fashion, or is a quotient of a group obtained in this fashion. For example, the group $SO_{2n}(\mathbb{C})$ is a quotient of $\mathrm{Spin}_{2n}(\mathbb{C})$, which is constructed from the root system D_n. For general k, we obtain the "split" semisimple groups.

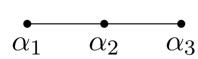
1.1. Parabolic subgroups. A *Borel subgroup* is a maximal closed, connected, solvable subgroup of G. Fix one and call it B; it (combined with a split maximal torus T contained in it) determines a set of simple roots Δ in Φ. We abuse notation by writing Δ also for the associated Dynkin diagram.

A closed subgroup of G is called *parabolic* if it contains a Borel subgroup, and we call a parabolic subgroup *standard* if it contains the Borel B. There is an inclusion-reversing bijection

$$\begin{array}{rcl} \boxed{\text{standard parabolic subgroups of } G} & \leftrightarrow & \boxed{\text{subsets of } \Delta} \\ B & \leftrightarrow & \Delta \\ G & \leftrightarrow & \emptyset \end{array}$$

The maximal proper standard parabolics are in one-to-one correspondence with the elements of Δ. We write P_δ for the standard parabolic corresponding to $\delta \in \Delta$.

1.2. **Example** (Parabolics in SL_4)**.** As an illustration, the Dynkin diagram of SL_4 is



Here we have labeled the vertices as in [Bou 4–6]. The upper triangular matrices are a Borel subgroup, which we take to be B. The maximal proper standard parabolics are

$$P_{\alpha_1} = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}, \quad P_{\alpha_2} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}, \quad P_{\alpha_3} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}. \tag{1.3}$$

1.4. Tits's geometry Γ_P. Tits defines the objects of the incidence geometry to be the maximal proper parabolic subgroups of G. Since the Borel subgroups of G are all conjugate, every parabolic is conjugate to a unique standard parabolic [Bor, 21.12]. Therefore, every maximal proper parabolic subgroup corresponds to a unique element $\delta \in \Delta$; this is the *type* of the parabolic. Two maximal proper parabolics are said to be *incident* if their intersection contains a Borel subgroup. We write Γ_P for this incidence geometry, where the subscript P is meant to remind the reader that the objects are parabolic subgroups of G.

In the case of SL_4, one typically calls the parabolics of type α_1 "points", α_2 "lines", and α_3 "planes". This identifies the geometry Γ_P with 3-dimensional projective space.

2. A concrete geometry Γ_V, part I

We now take Tits's incidence geometry Γ_P—whose objects are certain parabolic subgroups of G—and produce an isomorphic incidence geometry Γ_V whose objects are subspaces of a fixed vector space V. For example, in the case where G is SL_n, Γ_V consists of the nonzero, proper subspaces of k^n.

Fix a representation of G on a finite-dimensional vector space V, i.e., a group homomorphism $G \to GL(V)$ that may be expressed in terms of polynomials in the entries of G. We assume that the representation is nontrivial (i.e., the image of G is not just the identity transformation) and irreducible (i.e., there is no nonzero, proper G-invariant subspace). For each vertex δ of the Dynkin diagram, we choose a nonzero, proper subspace V_δ of V that is invariant under the parabolic P_δ.[1] Every maximal proper parabolic subgroup P' is conjugate to a unique standard parabolic subgroup P_δ, and we define a subspace $V_{P'}$ via

$$V_{P'} := gV_\delta \quad \text{for } g \in G(k) \text{ such that } P' = gP_\delta g^{-1}.$$

Of course, g is not uniquely determined, but if $h \in G(k)$ also satisfies $hP_\delta h^{-1} = P'$, then $h^{-1}g$ normalizes P_δ. Since P_δ is its own normalizer [Bor, 11.16], g equals hp for some $p \in P_\delta(k)$ and $hV_\delta = gV_\delta$. We remark that the stabilizer of $V_{P'}$ in G is precisely P'. Indeed, the stabilizer of $V_{P'}$ is a closed subgroup of G containing P', hence must be P' or G. Since the representation V is irreducible, the stabilizer must be P'.

We define an incidence geometry Γ_V whose objects are the subspaces V_P of V, as P ranges over the maximal proper parabolic subgroups of G. The map $P \mapsto V_P$ is surjective by definition, but it is also injective because P is precisely the stabilizer of V_P. Therefore, the sets of objects in Γ_P and Γ_V are isomorphic. Define the notions of type and incidence in Γ_V by transporting them from Γ_P. Speaking precisely, we say that V_P in Γ_V has type $\delta \in \Delta$ if the parabolic P is of type δ. We define two objects in Γ_V to be incident if and only if the corresponding parabolic subgroups are incident.

In this very simple way, we have obtained a realization of Tits's abstract geometry Γ_P as a collection of subspaces of the concrete vector space V.

[1]For the moment, we assume that such a subspace exists. The doubting reader may wish to glance ahead at Prop. 3.3.

This recipe begs two obvious questions:

(2.1) Are we guaranteed that a nonzero, proper P_δ-invariant subspace of V exists?

(2.2) Is there a way to tell if two subspaces of V in Γ_V are incident without discussing the corresponding parabolic subgroups?

We will address these two questions in the next section and the examples that follow it. But first, here is an example to illustrate the construction.

2.3. **Example** (Γ_V for SL_4)**.** Referring to the description of the parabolic subgroups of SL_4 from Example 1.2, we see that P_{α_i} stabilizes the i-dimensional subspace of k^4 consisting of vectors whose only nonzero entries are in the first i coordinates. We can take this to be V_{α_i}. The objects of Γ_V are all proper, nonzero subspaces of V.

We claim that two elements of Γ_V are incident if and only if one is contained in the other. Indeed, let W, W' be proper, nonzero subspaces, stabilized by maximal proper parabolics P, P' in SL_4. If W and W' are incident, then $P \cap P'$ contains a Borel subgroup. After conjugation we may assume that P and P' are standard, hence appear in (1.3). Then clearly W is contained in W' or vice versa. Conversely, if W is contained in W', there is some $g \in SL_4(k)$ such that gW, gW' are equal to V_{α_i}, $V_{\alpha_{i'}}$ for some $i \leq i'$. Then gPg^{-1}, $gP'g^{-1}$ equal P_{α_i}, $P_{\alpha_{i'}}$, hence are incident.

3. A concrete geometry Γ_V, part II

We will now make the geometry Γ_V from the previous section more concrete by focusing on the case where V is a fundamental irreducible representation of G. We completely answer (2.1) in the affirmative in Prop. 3.3 and we partially answer (2.2) in Prop. 3.4.

We view G as being constructed from the root system Φ by the Chevalley construction as in [St, §6]. That is, it is generated by the images of homomorphisms $x_\alpha : \mathbb{G}_a \to G$ as α ranges over the roots in Φ. Write U_α for the image of x_α. For each $\alpha \in \Phi$, the map $t \mapsto x_\alpha(t)x_\alpha(1)^{-1}$ defines a homomorphism $\mathbb{G}_m \to G$, which we denote by h_α. The images of the h_α's generate a maximal torus T in G. We fix a set of simple roots Δ in Φ and choose our standard Borel subgroup B to be the one generated by T and the U_α for $\alpha \in \Delta$.

Fix a root $\beta \in \Delta$ and let ω be the corresponding fundamental weight. In this section, V denotes the irreducible representation with highest weight ω with respect to our choice of torus T and Borel B. Fix a highest weight vector v in V.

3.1. Write L_β for the image of h_β in G and V_β for the subspace $L_\beta v$ of V. Note that V_β is simply the k-span of the highest weight vector v.

For a simple root $\delta \in \Delta \setminus \{\beta\}$, we define the *$\delta$-component* of Δ to be the connected component of β in $\Delta \setminus \{\delta\}$. We write L_δ for the subgroup of G generated by the root subgroups U_α, $U_{-\alpha}$ for α in the δ-component. The description of G in terms of generators and relations shows that L_δ

is a simple group whose Dynkin diagram is the δ-component. (It is the semisimple part of the Levi subgroup of the parabolic corresponding to the complement of the δ-component.) We set V_δ to be $L_\delta\, v$.

3.2. For each $\delta \in \Delta$, the subspace V_δ is T-invariant because T-normalizes L_δ. Therefore, V_δ is a direct sum of weight spaces in V. It consists of the weight spaces with weights of the form $\omega - \alpha$, where α is the sum of simple roots in the δ-component (possibly with repetition and with the understanding that the "β-component" is the empty set).

Suppose for the moment that $u \in V_\delta$ is a weight vector, with weight $\omega - \alpha$. For $\gamma \in \Delta$ and $t \in k$, we have

$$x_\gamma(t)u = u + (\text{vectors of weight } \omega - \alpha + \gamma,\, \omega - \alpha + 2\gamma, \text{ etc.}).$$

For γ *not* in the δ-component, the weights $\omega - \alpha + \gamma$, $\omega - \alpha + 2\gamma$, etc., are not weights of V, hence U_γ fixes u. That is, V_δ is fixed elementwise by U_γ for every $\gamma \in \Delta$ not in the δ-component.

We also have the following proposition:

3.3. **Proposition.** *For every $\delta \in \Delta$, the subspace V_δ is a nonzero, proper subspace of V stabilized by P_δ.*

Proof. V_δ is clearly nonzero because it contains v. We now show that it is a proper subspace. Let α denote the sum of the simple roots in δ; it is a root since δ is connected [Bou 4–6, VI.1.6, Cor. 3]. In standard root system notation, $\langle \omega, \alpha \rangle$ equals $2/(\alpha, \alpha)$. This is positive (because the inner product is positive definite) and an integer (because ω is a weight). But the weights of V are a saturated set of weights. In particular, V contains a nonzero vector of weight $\omega - \alpha$. But α involves δ, so $\omega - \alpha$ is not a weight of V_δ. This shows that V_δ is proper.

Finally, we show that V_δ is stabilized by P_δ. It suffices to check that the subgroups T, U_γ for $\gamma \in \Delta$, and $U_{-\gamma}$ for $\gamma \in \Delta \setminus \{\delta\}$ stabilize V_δ, since these subgroups generate P_δ by [Bor, 14.18]. For γ in the δ-component, U_γ and $U_{-\gamma}$ are in L_δ, hence they stabilize V_δ. For γ in the other connected components of $\Delta \setminus \{\delta\}$, U_γ and $U_{-\gamma}$ commute with L_δ and fix v because $\langle \omega, \gamma \rangle$ is 0, hence they stabilize V_δ. Finally, T and U_δ fix V_δ elementwise by 3.2. Together, we have seen that V_δ is invariant under P_δ. ◻

The preceding proposition addressed (2.1). Now we address the second question. We call a subspace $X \in \Gamma_V$ of type δ a *δ-space*.

3.4. **Proposition.** *Let X be a δ-space in Γ_V and suppose that the δ-component of Δ is of type A.*

1. *Every 1-dimensional subspace of X is a β-space.*
2. *If the δ-component contains the δ'-component, we have: X is incident to a δ'-space X' if and only if X contains X'.*

Proof. To prove the proposition, we may conjugate X and so assume that X is actually V_δ. Since the δ-component is of type A, the group L_δ is isomorphic to $SL(V_\delta)$.

Every nonzero vector in V_δ is in the same $SL(V_\delta)$-orbit as the highest weight vector. Hence every 1-dimensional subspace of V_δ is L_δ-conjugate to V_β, i.e., is a β-subspace. This proves (1).

Now we prove (2). First suppose that X and X' are incident. Then after conjugation, we may assume that X is V_δ and X' is $V_{\delta'}$. Since L'_δ is contained in L_δ, clearly X' is contained in X.

Conversely, suppose that X' is contained in X. Since L_δ is $SL(V_\delta)$, all subspaces of V_δ with the same dimension are L_δ-equivalent. Hence X' is L_δ-equivalent to $V_{\delta'}$. The parabolics P_δ, $P_{\delta'}$ corresponding to V_δ, $V_{\delta'}$ contain the standard Borel B, hence are incident. □

Before we leave this section, we observe that we know a lot about the spaces V_δ. The case $\delta = \beta$ is trivial, so for the rest of this section we fix a $\delta \in \Delta$ that is not β.

3.5. **Proposition.** *For $\delta \in \Delta \setminus \{\beta\}$, the space V_δ is a fundamental irreducible representation of L_δ.*

Proof. Suppose that $x \in V_\delta$ is fixed by U_α for every α in the δ-component. By 3.2, x is fixed by U_α for all $\alpha \in \Delta$, and since V is an irreducible representation of G, x is in the k-span of v.

The previous paragraph shows that v (and scalar multiples of v) is the only highest weight vector for V_δ relative to the maximal torus $T_\delta = T \cap L_\delta$ of L_δ. Since V_δ is a completely reducible representation of L_δ, it must be irreducible.

The highest weight of V_δ is the restriction of ω to T_δ; we denote it by $\overline{\omega}$. Since ω is a fundamental weight of G, the restriction $\overline{\omega}$ is a fundamental weight of L_δ. □

The dimension of V_δ can be looked up in, e.g., [Bou 7–9, chap. 8, Table 2].

3.6. We also have finer information about the weights of V_δ. Computing relative to L_δ, the weights of V_δ lie between the highest weight $\overline{\omega}$ and the lowest weight $w_0\overline{\omega}$, where w_0 is the longest element in the Weyl group of L_δ. From the tables in [Bou 4–6], one can quickly find the nonnegative integers k_α such that

$$w_0\overline{\omega} = \overline{\omega} - \sum k_\alpha \alpha$$

where α runs over the roots in the δ-component. Considering V_δ as a subspace of the representation V of G, we see that the weights of V_δ are precisely those weights μ of V such that

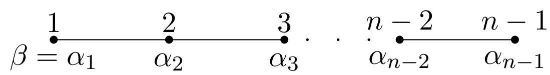
$$\omega - \sum k_\alpha \alpha \leq \mu \leq \omega.$$

4. Example: type A (projective geometry)

In this section, we describe the objects in the geometry Γ_V constructed from $G := SL_n$ as in §3 when the representation is the standard one on

$V := k^n$, corresponding to the simple root $\beta := \alpha_1$. We "discover" that Γ_V is projective $(n-1)$-space. We could do this explicitly in terms of matrices as in Example 2.3, but such an argument would be hard to generalize to other groups. Instead, we give an algebraic-group- and representation-theoretic argument.

We defined V_β, a.k.a. V_{α_1}, to be the 1-dimensional subspace of V spanned by the highest weight vector. For $\alpha_i \in \Delta$ with $i \neq 1$, V_{α_i} is the standard representation of L_{α_i}. Therefore, the dimension of V_{α_i} is precisely i. We summarize this in the Dynkin diagram, where each vertex is labeled with α_i and $\dim V_{\alpha_i}$:



Since SL_n acts transitively on the i-dimensional subspaces of V for all i, we have: the α_i-spaces are the i-dimensional subspaces of V. By Prop. 3.4.2, two subspaces are incident if and only if one contains the other. This is the classical description of $(n-1)$-dimensional projective space as consisting of lines through the origin in k^n.

5. Strategy

In the next few sections, we will fix a split simply connected group G and give an explicit description of the geometry Γ_V. One imagines that the geometry Γ_V we have just constructed will be easiest to visualize if the ambient vector space V is small. With that in mind, we will focus on the case where V is the smallest irreducible representation of G. For G of type A, D_4, or E_6, there are multiple equivalent choices, and we arbitrarily pick one.

(5.1)

type of G	A_n	B_n	C_n	D_n	E_6	E_7	F_4	G_2
β	α_1	α_1	α_1	α_1	α_1	α_7	α_4	α_1
$\dim V$	$n+1$	$2n+1$	$2n$	$2n$	27	56	26	7

We number the elements of Δ as in the tables in [Bou 4–6]. We call a representation V as in the table above a *standard representation* of G. We have omitted type E_8, see 10.3 for comments. (We remind the reader that despite our focus on the standard representation of G, the recipe in §3 gives a concrete realization of Γ_P for every fundamental representation, and one can compute the dimensions of the δ-spaces using Prop. 3.5.)

Roughly speaking, each example consists of three parts: dimensions and properties, transitivity, and incidence.

In "dimensions and properties", for each $\delta \in \Delta$ we compute the dimensions of the δ-spaces and some algebraic properties $\mathcal{P}$ satisfied by the δ-spaces. Here we restrict ourselves to the tools of elementary representation theory, which we exploit mercilessly. This has two advantages. First, no special background is required to understand the exceptional groups versus the more-familiar classical groups. Second, we hope the reader will view our descriptions of the δ-spaces as reasonably canonical and not ad hoc.

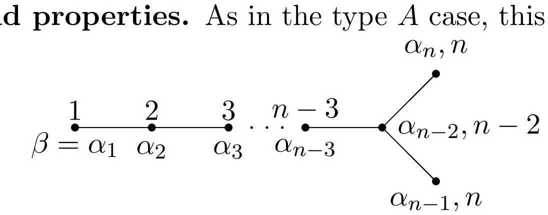
In "transitivity", we prove that the group G acts transitively on the subspaces of V satisfying the properties $\mathcal{P}$. Since the collection of δ-spaces is a G-orbit, this proves that the δ-spaces are precisely the subspaces of V satisfying $\mathcal{P}$. In many cases, we will refer to the literature for a proof. The proofs in the literature use various interpretations of the standard representation as the vector space underlying some algebraic structure. For example, in the type A example in §4, we used the fact that SL_n acts transitively on the subspaces of k^n of a given dimension.

In "incidence", we give a concrete description of how to tell if a δ- and a δ'-subspace are incident. Our final description is purely in terms of subspaces of V, with no mention of the corresponding parabolic subgroups. In most cases, a δ- and δ'-subspace will be incident if and only if one contains the other. When this occurs, we will say that *incidence is the same as inclusion.* In this "incidence" portion, we return to the techniques of representation theory and eschew algebraic interpretations of the representation.

6. Example: type D (orthogonal geometry)

Consider the split simply connected group G of type D_n with $n \geq 4$. This group is sometimes denoted Spin_{2n}. The geometry in this case is more complicated than for type A, apparently because the Dynkin diagram has a fork in it. This case will illustrate the basic principles involved in handling forking diagrams, and we will use them when treating the E-groups later. This geometry will be further investigated in Sections 12 and 13 below.

Dimensions and properties. As in the type A case, this is easy.



The standard representation of G has dimension $2n$. Since $-w_0\omega_1$ is ω_1, where w_0 is the longest element of the Weyl group, there is a nondegenerate G-invariant bilinear form b on V, unique up to multiplication by an element of $k^\times$ [Bou 7–9, 8.7.5, Prop. 12]. Moreover, b is symmetric [Bou 7–9, chap. 8, Table 1]. For v the highest weight vector in V and t an element of the maximal torus T, we have:

$$b(v, v) = b(t \cdot v, t \cdot v) = b(\omega(t)v, \omega(t)v) = \omega(t)^2 b(v, v).$$

Since ω is not the trivial character, $b(v, v)$ is 0. Traditionally, a subspace X is called *isotropic* if $b(X, X)$ is zero. We have just observed that the α_1-spaces are isotropic. By Prop. 3.4.1, the α_i-spaces are isotropic for every i.

Enlarging the geometry. We now add a new type of object to the geometries Γ_P and Γ_V to help us study that α_n- and α_{n-1}-spaces.

Let $\widetilde{\Gamma}_P$ be the geometry whose objects are the objects in Γ_P (the maximal proper parabolics in G) together with the parabolics of type $\{\alpha_n, \alpha_{n-1}\}$. Incidence in $\widetilde{\Gamma}_P$ is as for Γ_P; namely, two parabolics are incident if and only if their intersection contains a Borel subgroup.

6.1. **Lemma.** (Cf. [T 56, pp. 61–64]) *Each parabolic P of type $\{\alpha_n, \alpha_{n-1}\}$ is incident with exactly one parabolic Q_n of type α_n and one parabolic Q_{n-1} of type α_{n-1}. The parabolic P is the intersection $Q_n \cap Q_{n-1}$.*

Proof. First suppose that P is the standard parabolic of type $\{\alpha_n, \alpha_{n-1}\}$. Suppose that P is incident with Q, a parabolic of type α_i for $i = n$ or $n-1$. Then $P \cap Q$ contains a Borel subgroup; let $g \in G(k)$ be such that $g(P \cap Q)g^{-1}$ contains the standard Borel B. In fact, g lies in P by [BT 65, p. 86, 4.4a]. Hence gPg^{-1} is simply P. Since gQg^{-1} contains B, gQg^{-1} is the standard parabolic P_{α_i}. However, P is contained in P_{α_i} and we have

$$Q = g^{-1} P_{\alpha_i} g = P_{\alpha_i}.$$

The second claim—for P the standard parabolic—is well-known.

For P a parabolic of type $\{\alpha_n, \alpha_{n-1}\}$, P is conjugate to the standard parabolic of that type. The lemma now follows from the previous paragraph. □

Similar to what was done in §3, let $L_{\{\alpha_n, \alpha_{n-1}\}}$ be the simple subgroup of G generated by the root subgroups $U_{\alpha_i}, U_{-\alpha_i}$ for $1 \leq i \leq n-2$. Let X be the subspace $L_{\{\alpha_n, \alpha_{n-1}\}} v$ of V. As in Prop. 3.3, we find that X is a nonzero, proper subspace of V stabilized by the standard parabolic P of type $\{\alpha_n, \alpha_{n-1}\}$. As in §2, we find a bijection between parabolics of type $\{\alpha_n, \alpha_{n-1}\}$ and G-conjugates of X. Define the new geometry $\widetilde{\Gamma}_V$ to have the same objects as Γ_V as well as the G-conjugates of X. As in §2, we define objects in $\widetilde{\Gamma}_V$ to be incident precisely when their corresponding parabolics in $\widetilde{\Gamma}_P$ are incident.

6.2. For $\delta = \alpha_n$ or α_{n-1}, the group L_δ is of type A and $L_{\{\alpha_n, \alpha_{n-1}\}} v$ is a subspace of V_δ. The proof of Prop. 3.4.2 shows that an $\{\alpha_n, \alpha_{n-1}\}$-space and a δ-space are incident if and only if one is contained in the other. The lemma gives: each $(n-1)$-dimensional isotropic subspace is the intersection of two uniquely determined and incident n-dimensional subspaces, one of type α_n and one of type α_{n-1} (compare [Ch, III.1.11]).

Transitivity. We claim that G acts transitively on the m-dimensional isotropic subspaces of V for $m < n$. Let X, X' be isotropic of dimension m. They each lie in a direct sum of m hyperbolic planes in V, and one can easily construct an isometry f of b that sends X to X'. Since V is isomorphic to a direct sum of n hyperbolic planes, there is at least one plane where we may choose f as we please. If f has determinant -1, we modify f by a hyperplane reflection in this "extra" hyperbolic plane so that f has determinant 1. Since k is algebraically closed, f is in the image of the map $G(k) \to SO(b)(k)$, which proves the claim. Moreover, we have proved that the α_i-spaces are the i-dimensional isotropic subspaces for $1 \leq i \leq n-2$.

Next let X, X' be δ-spaces for $\delta = \alpha_n$ or α_{n-1}. Fix $(n-1)$-dimensional subspaces (i.e., $\{\alpha_n, \alpha_{n-1}\}$-spaces) U in X and U' in X'. By the previous paragraph, there is some $g \in G(k)$ such that $gU = U'$. Since X, X' have the same type, we must have $gX = X'$ by 6.2. An argument similar to the one in the last paragraph gives that $SO(b)(k)$ has at most two orbits on the n-dimensional subspaces of V. Since the δ-spaces are an orbit for $\delta = \alpha_n$ and α_{n-1}, we have:

$$\{\alpha_n\text{- and } \alpha_{n-1}\text{-spaces}\} = \{\text{isotropic subspaces of dimension } n\}.$$

Incidence. Consider an α_i-space X' and an α_j-space X with $i \leq j$. If (i,j) is not $(n-1,n)$, then Prop. 3.4.2 applies and incidence is the same as inclusion.

Now consider the case $(i,j) = (n-1,n)$. We claim that X, X' are incident if and only if the dimension of $X \cap X'$ is $n-1$, i.e., $X \cap X'$ is an $\{\alpha_n, \alpha_{n-1}\}$-space. If the two spaces are incident, then X, X' are simultaneously G-conjugate to the standard subspaces V_{α_n}, $V_{\alpha_{n-1}}$, and the intersection $V_{\alpha_n} \cap V_{\alpha_{n-1}}$ certainly has dimension $n-1$. Conversely, if the intersection is $(n-1)$-dimensional, then the spaces are incident as observed just after the proof of the lemma.

6.3. **An alternative view.** We now outline the geometry that one obtains from G by considering the fundamental representation with highest weight α_n (a "half-spin" representation) instead of the standard representation. We continue with the same definitions of V, V_{α_i}, etc., as in the rest of this section. We may identify the half-spin representation S with $\wedge^{\text{even}} V_{\alpha_n}$ as described in [Ch, chap. 3].

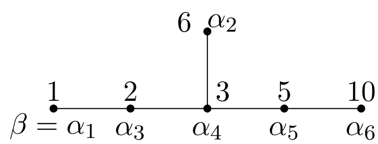
For $i \neq n-1$, the parabolic P_{α_i} stabilizes V_{α_i}, hence also the ideal of $\wedge V_{\alpha_n}$ generated by $\wedge^i V_{\alpha_i}$, hence also its intersection with S. Following the naive algorithm in §2, we define an α_i-space to be an intersection of $(\wedge^i X) \wedge (\wedge V_{\alpha_n})$ with S, where X is an α_i-space relative to the standard representation. (That is, X is isotropic of dimension i, plus an extra condition when $i = n$.) Thinking in terms of exterior powers of vector spaces, it is clear that the α_i-spaces in the half-spin representation have dimension 2^{n-i-1} for $i \leq n-2$; they correspond to the right ideals in the even Clifford algebra constructed in [Ga 99, §1]. The α_n-spaces are 1-dimensional and are the "pure spinors" corresponding to even maximal isotropic subspaces, in the language of [Ch, §3.1]. We do not know how to describe the α_{n-1}-spaces in this geometry.

7. Example: type E_6

The geometry for the split simply connected group G of type E_6 exhibits two complexities. We are prepared for the first—the fork in the diagram—thanks to our work in the previous section on groups of type D. The second complication is new: the root $\delta = \alpha_6$ does not satisfy the hypotheses of our workhorse Prop. 3.4.

This is the last example we will do in detail. In the final section of this paper, we will give an explicit description of the duality in this geometry.

Dimensions and properties. The Dynkin diagram for E_6, labeled with the dimensions of the corresponding spaces is:



In order to find the algebraic properties satisfied by the objects in the geometry, we need to know the weights of the representation V. Figure 7.1 displays the 27 weights of V as a Hasse diagram relative to the usual partial ordering of the weights. The row vectors list the coordinates of the weights with respect to the basis consisting of fundamental weights. An edge joining two weights $\lambda > \mu$ is labeled with i if $\lambda - \alpha_i = \mu$. The lowest weight of V_{α_i} (cf. 3.6) is labeled λ_i.

The automorphism of order 2 of the Dynkin diagram gives an automorphism of the root system, hence an automorphism ϕ of G of order 2. The fixed subgroup is well-known to be simple and split of type F_4, and we denote it simply by F_4. Examining the restrictions of the weights of V to F_4, we find that V decomposes (as a representation of F_4) as a direct sum of a 1-dimensional trivial representation C and the standard representation of F_4, which we denote by V_0.

7.2. **Proposition.** *There is a bilinear form b on V such that*

$$b(\phi(g)x, gy) = b(x, y) \quad \text{for all } g \in G \text{ and } x, y \in V. \tag{7.3}$$

It is unique up to multiplication by a scalar. Moreover, it is symmetric and nondegenerate, and $b|_C$ is not zero.

Proof. First we construct a bilinear form b on V satisfying (7.3). Write $\rho\colon G \to GL(V)$ for the representation of G on V, and write $\rho^*\colon G \to GL(V^*)$ for the dual representation defined by

$$(\rho^*(g)f)(x) := f(\rho(g)^{-1}x) \quad \text{for } g \in G,\ f \in V^*, \text{ and } x \in V.$$

The representations $\rho\phi$ and ρ^* are both irreducible with highest weight ω_6, hence they are isomorphic. Fix an isomorphism $h : V \to V^*$ such that $h\rho\phi(g)h^{-1} = \rho^*(g)$ for all $g \in G$. Define b by setting

$$b(x, y) := h(x)(y).$$

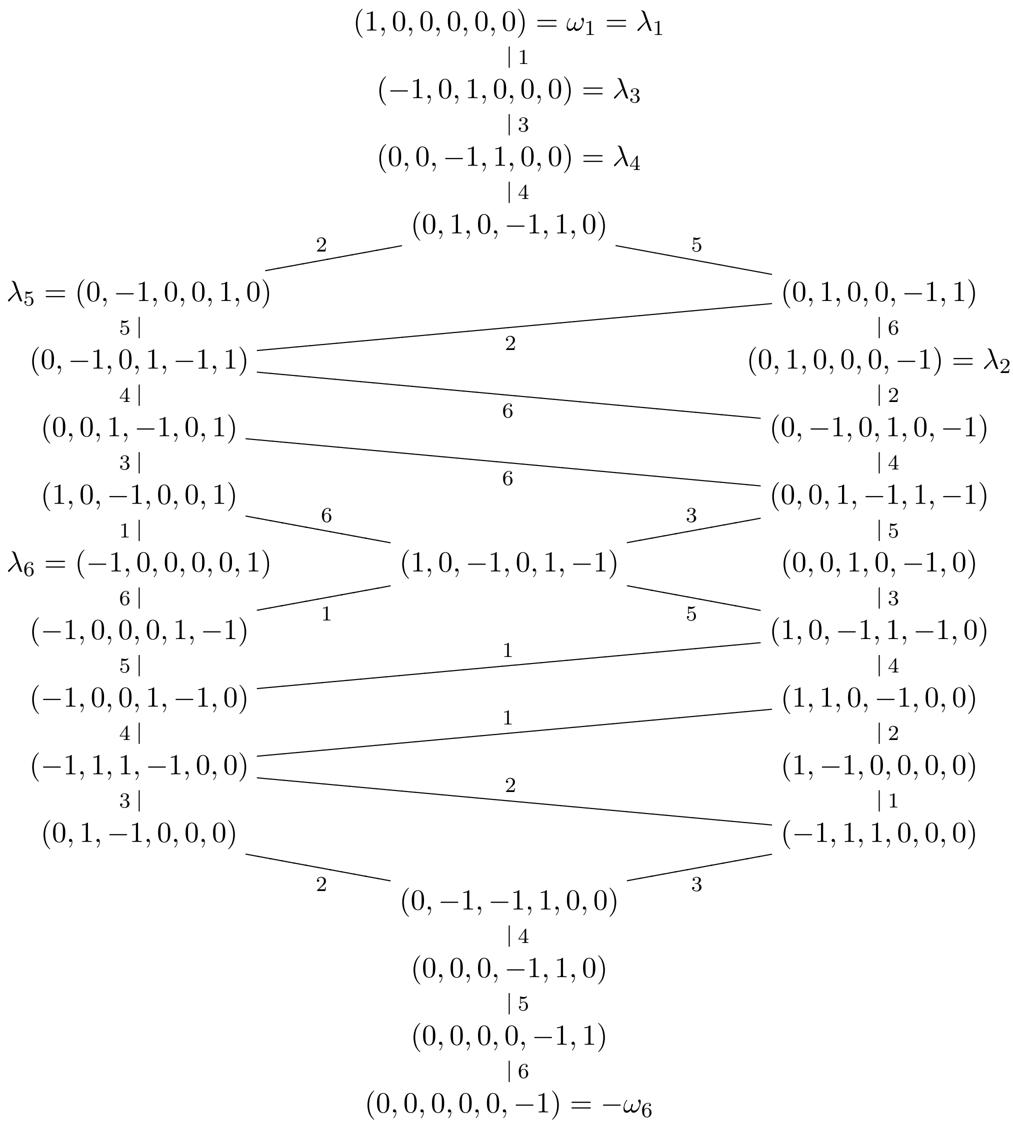


Figure 7.1. Hasse diagram of the weights of V, where $(c_1, c_2, \ldots, c_6)$ denotes the weight $\sum_{i=1}^{6} c_i\omega_i$. The map $-\phi$ reflects the diagram across its horizontal axis of symmetry.

This b is clearly bilinear and

$$b(\phi(g)x, gy) = h(\rho\phi(g)x)(gy) = [\rho^*(g)h(x)]\,(gy) = b(x, y).$$

We now argue that any bilinear form b satisfying (7.3) is symmetric. Set

$$b_\varepsilon(x, y) := b(x, y) + \varepsilon b(y, x).$$

Then b_1 and b_{-1} are bilinear, b_1 is symmetric, b_{-1} is skew-symmetric, and $2b = b_1 + b_{-1}$. We prove that b_{-1} is identically zero. In any case, b_{-1} satisfies (7.3) (using that ϕ is its own inverse), hence b_{-1} is F_4-invariant. But V_0 does not support a nonzero F_4-invariant skew-symmetric form, hence b_{-1} restricts to zero on V_0. Fix $x \in V_0$ a nonzero vector with a nonzero weight λ with respect to F_4, and let c be a nonzero vector in C. Since

$$b_\varepsilon(c, x) = b_\varepsilon(tc, tx) = \lambda(t) b_\varepsilon(c, x)$$

for t in the F_4-torus, $b_\varepsilon(c, x)$ is zero. Since V_0 is an irreducible representation of F_4, $b_\varepsilon(c, V_0)$ is zero. But b_{-1} is skew-symmetric, so $b_{-1}(c, c)$ is also zero, and we have proved the claim.

The previous paragraph also gives more. Continue the assumption that b satisfies (7.3) and suppose that b is not identically zero. Then C and V_0 are orthogonal subspaces. For $x \in V$ and r in the radical of b, we have

$$b(gr, x) = b(r, \phi(g)^{-1} x) = 0,$$

hence the radical is G-invariant. Since V is irreducible and b is not identically zero, the radical is zero, i.e., b is nondegenerate. Since C is 1-dimensional and orthogonal to V_0, we find that b restricts to be nonzero on C.

We now prove uniqueness. Let b, b' be bilinear forms on V satisfying (7.3). The representation V_0 of F_4 supports a unique symmetric bilinear form up to a scalar multiple, so by modifying b' by a factor in $k^\times$, we may assume that b and b' have the same restriction to V_0. Then $b - b'$ is a bilinear form on V satisfying (7.3) that restricts to be zero on V_0. By the previous paragraph, $b - b'$ is identically zero, and we have proved uniqueness. □

We can use representation theory to decompose $(V^*)^{\otimes 3}$ (or, if the reader prefers, $S^3(V^*)$) as a direct sum of irreducible representations; we find just one trivial, 1-dimensional representation. That is, there is a G-invariant cubic form N on V, uniquely determined up to a factor in $k^\times$. We abuse notation and write N also for the trilinear "polarization" of N on V such that $N(x, x, x) = 6N(x)$ for all $x \in V$. Let $\#$ denote the bilinear product defined implicitly by the formula

$$b(x\#y, z) = N(x, y, z) \quad \text{for } x, y, z \in V. \tag{7.4}$$

Using (7.3), we find

$$\phi(g)(x\#y) = (gx)\#(gy) \quad \text{for } g \in G \text{ and } x, y \in V. \tag{7.5}$$

We write simply $x^\#$ for the "half square" $(x\#x)/2$.

The same argument as at the beginning of this section shows that the α_1-spaces are 1-dimensional subspaces consisting of elements $x \in V$ such that $x^\# = 0$. We say a nonzero vector $x \in V$ is *singular* if $x^\# = 0$. (These 1-dimensional subspaces are precisely the singular points for the hypersurface in $\mathbb{P}(V)$ defined by $N = 0$.) We call a subspace of V singular if its nonzero elements are singular. By Prop. 3.4.1, the α_i-spaces are singular for $i \neq 6$.

We will now investigate the restriction of the representation of G on V to the subgroup L_{α_6} of type D_5. This will give us finer information about the product # and lead us to a description of the α_6-spaces. To see how a weight of G restricts to L_{α_6}, one drops the last coordinate and moves the second coordinate to the end of the vector (to allow for the fact that weights of D_5 and E_6 are numbered somewhat incompatibly in [Bou 4–6]).

Let W' be the subgroup of the Weyl group W of G generated by the reflections with respect to the roots α_i for $i \neq 6$. It is the Weyl group of L_{α_6}, and it is the stabilizer of $-\omega_6$ in W [Hu 90, Th. 1.12c].

7.6**. Lemma.** *The orbits of W' in the weights of V are the weights $\geq \lambda_6$, the weights between λ_2 and $(0,0,0,0,-1,1)$, and the weight $-\omega_6$.*

Proof. Since the highest weight ω_1 of V is minuscule, we have $\langle \mu, \alpha \rangle = 1$, 0 or -1 for every weight μ of V and every root α. If μ and $\mu - \delta$ are both weights for some $\delta \in \Delta$, then $\langle \mu, \delta \rangle = 1$, $\langle \mu - \delta, \delta \rangle = -1$, and the reflection s_δ with respect to the root δ interchanges μ and $\mu - \delta$. Consulting Figure 7.1, we see that W' acts transitively on each of the three sets of weights named in the statement of the lemma.

Conversely, ω_1 and λ_2 restrict to the weights $(1,0,0,0,0)$ and $(0,0,0,0,1)$ on L_{α_6}, which are not congruent modulo the D_5 root lattice. Therefore, they lie in different W'-orbits. □

By restricting the weights of V to L_{α_6}, we can decompose V as a direct sum of irreducible representations. The proof of Lemma 7.6 shows that the components of V are

- the standard representation V_{α_6} of V (with highest weight $(1,0,0,0,0)$),
- a half-spin representation (with highest weight $(0,0,0,0,1)$), and
- a 1-dimensional trivial representation (from the lowest weight vector $-\omega_6$).

7.7**. Corollary.** *The Weyl group of type E_6 acts transitively on triples μ_1, μ_2, μ_3 of weights of V such that $\mu_1 + \mu_2 + \mu_3 = 0$.*

Proof. The Weyl group acts transitively on the weights of V, so we may assume that μ_1 is $-\omega_6$. Since $\mu_3 = \omega_6 - \mu_2$ is a weight, μ_2 cannot have last coordinate equal to -1, otherwise μ_3 would have last coordinate -2, which is impossible. In particular, μ_2 cannot be $-\omega_6$ or λ_2. Since the set of triples $-\omega_6, \mu_2, \mu_3$ with sum 0 is stable under the action of W', μ_2 must lie in the W'-orbit with lowest weight λ_6. □

Our preliminary results about the action of the Weyl group can now give us concrete information about the product #.

7.8**. Lemma.** *Let x_1, x_2 be nonzero vectors in V of weight μ_1, μ_2 respectively. The product $x_1 \# x_2$ is nonzero if and only if $\phi(\mu_1 + \mu_2)$ is a weight of V.*

The equations

$$\phi(\lambda_2 + \lambda_5) = \lambda_3 \quad \text{and} \quad \phi(\omega_1 + \lambda_6) = \omega_1 \tag{7.9}$$

furnish specific examples where the product # is not zero.

Proof of Lemma 7.8. If $\phi(\mu_1 + \mu_2)$ is not a weight of V, then the product is zero by (7.5). So suppose that $\phi(\mu_1 + \mu_2)$ is a weight of V. Since $-\phi$ is in the Weyl group, $\mu_3 := -\mu_1 - \mu_2$ is a weight of V; let y be a nonzero vector with that weight.

We claim that $N(x_1, x_2, y)$ is not zero, and hence that $x_1 \# x_2$ is not zero. Indeed, fix a basis $\{b_\lambda\}$ for V consisting of weight vectors and write N in terms of the dual basis $\{x_\lambda\}$. Since N is G-invariant, a monomial $x_{\nu_1} x_{\nu_2} x_{\nu_3}$ has zero coefficient if $\nu_1 + \nu_2 + \nu_3$ is not zero. On the other hand, since N is not identically zero, there exist weights ν_1, ν_2, ν_3 such that the coefficient of $x_{\nu_1} x_{\nu_2} x_{\nu_3}$ is not zero. Since their sum $\nu_1 + \nu_2 + \nu_3$ is zero, there is an element w in the Weyl group such that $w\nu_i = \mu_i$ for each i by Cor. 7.7. A representative of w can be found in G, hence the coefficient of $x_{\mu_1} x_{\mu_2} x_{\mu_3}$ in N is not zero. In particular, $N(x_1, x_2, y)$ is not zero, as claimed. □

Finally, we can give an explicit description of the α_6-space V_{α_6}.

7.10. **Proposition.** *$V_{\alpha_6} = v \# V$, where v is the highest weight vector.*

Proof. ($\supseteq$): Suppose that $x \in V$ has weight μ, which is necessarily at least the minimum weight $-\omega_6$. Since ϕ respects the partial ordering on the weights, $v \# x$ has weight at least $\phi(\omega_1 - \omega_6) = \lambda_6$. But this is the minimal weight of V_{α_6} and every weight space in V is 1-dimensional, so V_{α_6} contains every weight space for weights between λ_6 and the maximal weight ω_1.

($\subseteq$): Let v' be a nonzero vector of the lowest weight $-\omega_6$. Then $v \# v'$ has weight $\lambda_6 \in V_{\alpha_6}$, and $v \# v'$ is not zero by Lemma 7.8. Every weight μ of V_{α_6} is obtained as $w\lambda_6$ for some $w \in W'$ by Lemma 7.6, and $\phi(w)$ fixes ω_1. Hence $w(v \# v') = v \# \phi(w) v'$. This shows that there is a nonzero vector of weight μ in $v \# V$. □

For x singular in V, we call the subspace $x \# V$ a *hyperline*, following Tits'ss terminology from [T 57, p. 25]. Combining the proposition with (7.5), we find that every α_6-space is a hyperline.

7.11. *Remark.* As the standard representation of the group L_{α_6} of type D_5, the space V_{α_6} supports an L_{α_6}-invariant quadratic form, uniquely determined up to a scalar. We claim that it is the form q given implicitly by the equation

$$x^{\#} = q(x) v \quad \text{for } x \in V_{\alpha_6}. \tag{7.12}$$

First, observe that for y, z weight vectors in V_{α_6}, $y \# z$ has weight at least

$$\phi(2\lambda_6) = \omega_1 - (\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6),$$

and the only such weight is ω_1. Therefore, for every $x \in V_{\alpha_6}$, the vector $x^{\#}$ is in the k-span of v and the recipe (7.12) defines a quadratic form on V_{α_6}.

For $g \in L_{\alpha_6}$, we have $q(gx)v = q(x)\phi(g)v$. Since $\phi(g)$ fixes v (by 3.2, if you like), the form q is L_{α_6}-invariant. Moreover, q is not the zero form since

is bilinear and $v\#x$ is not zero for x of weight λ_6, by Lemma 7.8. Combining the two previous sentences, q is the unique invariant quadratic form as claimed, and it is nondegenerate. In particular, the hyperline $v\#V$ cannot contain an isotropic 6-dimensional subspace. In terms of the geometry, an α_6-space cannot contain an α_2-space.

Enlarging the geometry. In a manner completely analogous to the D_n case, we define $\widetilde{\Gamma}_P$ to be the geometry Γ_P with the addition of parabolics of types $\{\alpha_2, \alpha_5\}$ and $\{\alpha_2, \alpha_6\}$. Performing the same enlargement on Γ_V to obtain $\widetilde{\Gamma}_V$, we find that $\{\alpha_2, \alpha_5\}$- and $\{\alpha_2, \alpha_6\}$-spaces are 4- and 5-dimensional respectively. Both types of spaces are singular because they are contained in V_{α_2}. The arguments in §6 show—for example—that for $\delta = \alpha_2$ or α_6, an $\{\alpha_2, \alpha_6\}$-space and a δ-space are incident if and only if one is contained in the other, and furthermore each $\{\alpha_2, \alpha_6\}$-space is contained in precisely one α_2-space and one α_6-space.

Transitivity. As described in §5, we now allow ourselves to use methods from outside of representation theory. Specifically, we view V as the vector space underlying an Albert algebra. The cubic form N is—up to a scalar multiple—the generic norm ("determinant") of the algebra, and b is—again up to a scalar multiple—the symmetric bilinear form induced by the trace.

The group G acts transitively on the i-dimensional singular subspaces for $i = 1, 2, 3$, and 6 by [SV 68, 3.12], [Fa, p. 33], or [A, 6.5(2)]. Thus the α_1-, α_2-, α_3-, and α_4-spaces are as described in Table 7.13 below. Every hyperline is by definition of the form $x\#V$ for a singular $x \in V$. Since G acts transitively on the 1-dimensional singular subspaces, it acts transitively on the hyperlines by (7.5). Therefore, the α_6-spaces are the hyperlines.

Before describing the α_5-spaces, we first treat the $\{\alpha_2, \alpha_6\}$-spaces. Let M denote the 5-dimensional singular subspaces $L_{\{\alpha_2,\alpha_6\}}v$. (Recall that $L_{\{\alpha_2,\alpha_6\}}$ is the subgroup of G generated by $U_{\alpha_i}, U_{-\alpha_i}$ for $i = 1, 3, 4, 5$.) Note that M is not a maximal singular subspace because it is contained in V_{α_6}. Suppose now that X is a 5-dimensional singular subspace of V contained in a 6-dimensional singular subspace of V. Since G acts transitively on the 6-dimensional singular subspaces, we may assume that X is also contained in V_{α_2}. But L_{α_2} is of type A_5, so X is G-equivalent to M. We have just proved that the $\{\alpha_2, \alpha_6\}$-spaces are the 5-dimensional, non-maximal singular subspaces of V.

Consider V_{α_5}; it is a 5-dimensional singular subspace and its stabilizer is P_{α_5}. In contrast, the stabilizer of M is $P_{\{\alpha_2,\alpha_6\}}$, hence V_{α_5} is not in the same G-orbit as M, i.e., V_{α_5} is a 5-dimensional, maximal singular subspace. Since G acts transitively on such subspaces by [SV 68, 3.14], we have proved: the α_5-spaces are the 5-dimensional, maximal singular subspaces of V.

We summarize the descriptions of the α_i-spaces in Table 7.13.

Incidence. Let X' be an α_i-space and let X be an α_j-space with $i \leq j$. If (i, j) is not $(2, 5)$ or $(2, 6)$, Prop. 3.4.2 applies and incidence is the same

__-space	description	name in [SV 68]
α_1	1-dim'l singular	point
α_2	6-dim'l singular	max'l space of 2nd kind
α_3	2-dim'l singular	space of prdim 1
α_4	3-dim'l singular	space of prdim 2
α_5	5-dim'l, maximal singular	max'l space of 1st kind
α_6	hyperline	line

TABLE 7.13. α_i-spaces in the E_6 geometry

as inclusion. As in the D_n case, we quickly find: An α_2- and an α_5-space are incident if and only if their intersection is 4-dimensional. An α_2- and an α_6-space are incident if and only if their intersection is a 5-dimensional (non-maximal) singular subspace.

Bibliographic remarks. We deduced the existence of a G-invariant cubic form on V by decomposing $(V^*)^{\otimes 3}$ as a direct sum of irreducible representations. The program LiE [vLCL] does this purely by computations with characters. But the cubic form was written down explicitly long before these mathematical tools were available, see e.g. [D 01a] and [D 01b]. For a modern derivation of an explicit formula for the cubic form, see [Lu].

Diagrams like Figure 7.1 for different groups and representations can be found, for example, in [PSV].

8. Interlude: more on E_6

We will now take a short break from examples of the geometries to derive some properties of the cubic form N and the product $\#$. These results will only be used in the final section, §14.

Fix a nonzero vector $c \in C$. We claim that $c^\#$ is not zero. For the sake of contradiction, suppose that $c^\#$ is zero, i.e., C is an α_1-space. Then the stabilizer of C is a parabolic of type α_1, which has semisimple part of type D_5 and dimension 45. However, F_4 stabilizes C and has dimension 52, a contradiction.

Equation (7.5) gives that $c^\#$ is fixed by F_4. Therefore $c^\#$ equals λc for some scalar $\lambda \in k^\times$. Combined with Prop. 7.2, we find:

$$N(c) = \frac{1}{6} N(c,c,c) = \frac{1}{6} b(c, c\#c) \neq 0.$$

The forms N and b were only determined up to a factor in $k^\times$, so we are free to choose them so that

$$N(c) = 1 \quad \text{and} \quad b(c,c) = 3. \tag{8.1}$$

Since $\#$ was implicitly defined by equation (7.4), these choices affect $\#$ as well.

8.2. **Proposition.** *With N and b chosen to satisfy* (8.1)*, we have $c^\# = c$ and $x^{\#\#} = N(x)x$ for all $x \in V$.*

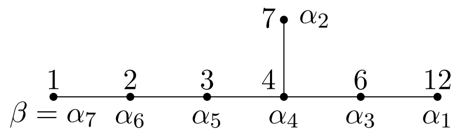
Proof. The first equation follows from the various equations relating N, b, and $\#$:

$$1 = N(c) = \frac{1}{6}b(c\#c, c) = \frac{1}{3}b(\lambda c, c) = \lambda.$$

For the second, we observe that $x \mapsto x^{\#\#}$ and $x \mapsto N(x)x$ are both quartic G-invariant maps $V \to V$. Representation theory gives that there is a unique such map up to a factor in $k^\times$. Since both maps fix c, they agree on all of V. □

9. Example: type E_7

Dimensions and properties. For G of type E_7, the Dynkin diagram looks like



Representation theory shows that there exists a G-invariant symmetric quadrilinear form q (or, if you prefer, a quartic form) and a skew-symmetric bilinear form b on V; both are unique up to a factor in $k^\times$. We define a symmetric trilinear map $t\colon V \times V \times V \to V$ (i.e., a linear map $S^3V \to V$) implicitly via

$$q(x, y, z, w) = b(x, t(y, z, w)) \quad \text{for } x, y, z, w \in V.$$

Since the infinite group G preserves q, one knows by general principles that the hypersurface in $\mathbb{P}(V)$ defined by the equation $q = 0$ is singular, see e.g. [OS, §6]. That is, there are nonzero vectors $x \in V$ such that $t(x, x, x)$ is zero. But in this case, there is more than one kind of singular vector. We will say that an element $x \in V$ *is rank one* if it is nonzero and the image of the linear map $y \mapsto t(x, x, y)$ has dimension ≤ 1.

9.1**. Example.** The highest weight vector $v \in V$ is rank one. Indeed, let $y \in V$ be a weight vector and write its weight as $\omega - \alpha$ where α is a sum of positive roots. If $t(v, v, y)$ is not zero it is a weight vector with weight $3\omega - \alpha$, necessarily equal to $\omega - \alpha'$ for some sum of positive roots α'. However, the lowest weight of V is $w_0\omega$ where w_0 is the longest element of the Weyl group of G, so α is at most $\omega - w_0\omega$. Putting these observations together with the fact that $w_0\omega = -\omega$, we have:

$$2\omega = \alpha - \alpha' \leq \alpha \leq 2\omega.$$

Hence $\alpha = 2\omega$ and $\alpha' = 0$. In particular, $t(v, v, V)$ is contained in the span of v.

By Prop. 3.4.1, the α_i-spaces consist of rank one elements except possibly for the α_1-spaces.

We claim that the δ-spaces are inner ideals for all $\delta \in \Delta$. An *inner ideal* is a subspace X of V such that $t(X, X, V)$ is contained in X. It suffices to check that V_δ is an inner ideal. The lowest weight for the action of G on

V_δ is $\omega_7 - \alpha$, where α is a sum of simple roots in the δ-component, and the lowest weight for V is $-\omega_7$. Therefore, every weight of $t(V_\delta, V_\delta, V)$ is at least $\omega_7 - 2\alpha$. But every weight of V that differs from ω_7 by a sum of simple roots in the δ-component belongs to V_δ. We have proved that V_δ is an inner ideal.

Transitivity. The group G acts transitively on the 1-dimensional subspaces of V spanned by rank one elements by [Fe, 6.2, 7.7]. It acts transitively on the subspaces of a given dimension consisting of rank one elements, and such subspaces necessarily have dimension ≤ 7 [Ga 01b, 6.12]. Consequently, the α_i-spaces are the subspaces consisting of rank one elements for $i \neq 1$. The group also acts transitively on the collection of 12-dimensional inner ideals by [Ga 01b, 6.15], hence the α_1-spaces are the 12-dimensional inner ideals.

For the sake of brevity, we omit the "incidence" portion of this example.

The reader might naturally wonder if the properties of rank one elements are special to our particular representation V of G, or if they generalize to singular vectors for other quartic forms. We close this section with a relevant example.

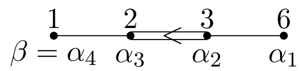
9.2**. Example.** The most familiar example of a quartic form is the determinant of 4-by-4 matrices. Consider SL_4 acting by conjugation on the vector space of 4-by-4 trace zero matrices. This representation is irreducible with highest weight $\omega_1 + \omega_3$. The determinant is clearly SL_4-invariant, as is the nondegenerate symmetric bilinear form defined by $(x, y) \mapsto \mathrm{tr}(xy)$. Let us see how the discussion of rank one elements applies. If we take the usual choices for the maximal torus and the Borel B—the diagonal and the upper-triangular matrices—the highest weight vector v can be taken to be the matrix whose entries are zero except for a 1 in the upper-right corner. Obviously this matrix is rank one in the classical linear algebra sense. Since w_0 acts as -1 on the highest weight, the discussion in Example 9.1 shows that v is rank one in our sense.

We claim that in this case $t(v, v, y)$ is zero for all y. Indeed, for τ an indeterminate, consider the map $y \mapsto \det(\tau v + y)$. Examining the Laplace expansion of the determinant along the top row, we see that the coefficient of τ^2 in $\det(\tau v + y)$ is zero. Therefore, when we polarize det to obtain a symmetric quadrilinear form, we find that the bilinear map $(y, z) \mapsto \det(v, v, y, z)$ is identically zero. Since the bilinear form is nondegenerate, we have $t(v, v, y) = 0$ for all y.

10. Loose ends

To complete our discussion of the concrete realization of the geometries, we address some loose ends. Above, we have skipped the groups of type B and C; the reader should have no trouble filling them in from the previous examples.

10.1. **Example: type F_4.** We now sketch the case where G is of type F_4. We find the following diagram:



Representation theory provides a G-invariant bilinear product on V, which we denote by #. (We have seen the objects G, V, # in §7, where they were known as F_4, V_0, #.) The usual argument shows that the product is identically zero on the α_4-spaces, and it is zero on the α_3- and α_2-spaces by Prop. 3.4.1.

The lowest weight of V_{α_1} is

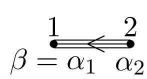
$$\omega_4 - (2\alpha_4 + 2\alpha_3 + \alpha_2) = \alpha_1 + \alpha_2 + \alpha_3$$

by the arguments in 3.6. For x, y weight vectors in V_{α_1}, the product $x\#y$ has weight at least $2\alpha_1 + 2\alpha_2 + 2\alpha_3$. In particular, that character is not a weight of V since the α_1-coordinate of ω_4 is 1. Therefore, the product # is identically zero on the α_1-spaces also. Freudenthal calls α_1-spaces "symplecta", since they are associated with the standard representation of a group of type C_3.

The space V may be interpreted as the trace zero elements in an Albert algebra, i.e., a 27-dimensional simple exceptional Jordan algebra; this identifies G with the group of automorphisms of the algebra. Using this viewpoint, one can prove that the group G acts transitively on the d-dimensional subspaces of V on which the product # is identically zero for $d = 1$ (the α_4-spaces) by [Fr, 28.27], [J, Th. 4], or [A, 8.6], $d = 2, 3$ by [A, 9.5, 9.8], and for $d = 6$ (the α_1-spaces) by [Fr, 28.22]. (We remark that [A] does not use the interpretation in terms of Albert algebras.)

For all of the objects in Γ_V, incidence is the same as inclusion. Aside from α_1-spaces, this is Prop. 3.4.2. For cases involving an α_1-space, one can adapt the proof of 3.4.2, using the fact that a group of type C_3 acts transitively on d-dimensional isotropic subspaces of the standard representation for $d = 1, 2, 3$.

10.2. **Example: type G_2.** The geometry for a group G of type G_2 is similar to that for type F_4, but everything is easier. The dimensions are summarized in the following diagram:



As in the F_4 case, there is a G-invariant bilinear product on V, which we denote by #. The usual argument shows that the multiplication is identically zero on the α_1-spaces, hence also on the α_2-spaces by Prop. 3.4.

The vector space V may be viewed as the trace zero elements in the split octonion algebra; this identifies G with the group of automorphisms of the algebra. Using this viewpoint, it is easy to prove that G acts transitively on the 1-dimensional subspaces of V on which the multiplication is zero using [SV 00, 1.7.3]. The Cayley-Dickson process gives an explicit description of the octonion algebra, which one can use to prove that G acts transitively on the 2-dimensional subspaces of V on which the multiplication # is zero. (This essentially solves Problem 23.54 in [FH], cf. 10.4.)

In summary, the α_i-spaces are the i-dimensional subspaces of V on which the multiplication # is zero. Incidence is the same as inclusion by Prop. 3.4.2.

10.3. Example: type E_8. The recipe in §3 for giving a concrete realization of the geometry associated with a group has been very effective with the examples considered so far. But what of the least familiar case, where G has type E_8? The recipe still works, of course, and for each fundamental representation V, it is easy to write down the dimension of the δ-spaces for each $\delta \in \Delta$. This is already interesting. But a problem occurs when we attempt to describe the algebraic properties that characterize the δ-spaces.

For example, the smallest fundamental representation V of G is the adjoint representation, with highest weight ω_8 and dimension 248. Representation theory shows that V does not support any obvious additional structure, e.g., there is no G-invariant quintic form on V. Therefore, the only description of the δ-spaces that suggests itself would be in terms of the Lie algebra structure.

The next smallest fundamental representation V has highest weight ω_1 and dimension 3875. This representation has G-invariant bilinear and cubic forms, each determined uniquely up to a nonzero scalar multiple. Thus there is also a G-invariant commutative product on V. Unfortunately, one cannot simply translate the analysis in §7 to this case. For example, the proof of Lemma 7.8 does not translate because the weights of V are not all one orbit under the Weyl group and some of the weights occur with multiplicity greater than 1.

10.4. Projective homogeneous varieties. The above examples can all be viewed from the perspective of projective homogeneous varieties, i.e., projective varieties Y such that G acts on Y and the action is transitive on K-points for every algebraically closed extension K of k. We maintain our assumptions that G is split simply connected and V is a fundamental irreducible representation as in §3.

There is a bijection between subsets of Δ and isomorphism classes of projective homogeneous varieties given by sending $S \subseteq \Delta$ to $Y_S := G/P_S$. For example, $Y_\emptyset$ is a point because $P_\emptyset$ is all of G.

For S a singleton, say $\{\delta\}$, the k-points of Y_S are the δ-spaces in V. Indeed, the δ-spaces are defined to be the orbit of V_δ in the appropriate Grassmannian, and V_δ has stabilizer P_δ.

10.5. **Example.** For G of type B or D, the variety Y_{α_1} is a conic. The other Y_δ's are families of linear subspaces of the conic.

For an arbitrary subset $S \subseteq \Delta$, a *flag of type* S is a collection of pairwise incident subspaces $\{X_s \mid s \in S\}$ where X_s is an s-space. The flags of the extreme type Δ are called *chambers*. We call $\{V_s \mid s \in S\}$ the *standard flag of type* S. (What we call the standard chamber is traditionally called the "fundamental chamber".) We need the following consequence of the fact that Γ_V is a building:

10.6. **Proposition.** [T 74, 3.16] *Every flag of type* S *in* Γ_V *is contained in a chamber and is in the* G*-orbit of the standard flag of type* S*.*

In particular, G acts transitively on the collection of flags of type S. The stabilizer of the standard flag is the intersection $\cap_{s\in S} P_s$, which is P_S [Bou 4–6, IV.2.5, Th. 3c]. Hence the k-points of Y_S are the flags of type S in Γ_V.

We view the examples in the preceding sections as giving explicit descriptions of the geometry Γ_V as well as the projective homogeneous varieties under split groups G. When G is not split, the situation is somewhat more complicated. The absolute Galois group of k acts on the Dynkin diagram Δ, and there is a bijection between Galois-invariant subsets S of Δ and projective homogeneous varieties defined over k. The description in the split case can then be altered to give a description in the general case. For groups of type ${}^1A_n$, one finds the generalized Severi-Brauer varieties as in [KMRT, §1]. Examples for groups of type D_4 and E_7 can be found in [Ga 99] and [Ga 01a] respectively.

11. Outer automorphisms

Every automorphism ϕ of G permutes the parabolic subgroups, hence induces an automorphism of Tits's geometry Γ_P. Further, ϕ induces an automorphism of the concrete geometry Γ_V via the isomorphism between Γ_P and Γ_V from §2.

Every $g \in G(k)$ defines an automorphism of G by sending $h \mapsto ghg^{-1}$. Such automorphisms are called *inner*. It is easy to see the effect of such an automorphism on the geometry Γ_V: it sends an object X to gX. In particular, the types of objects in Γ_V are preserved. In classical projective geometry, such an automorphism is called a *collineation*.

On the other hand, some groups have automorphisms that are not of this type; such automorphisms are called *outer*. They have a more interesting action on the geometry Γ_V in that they do not preserve the types of objects. In classical projective geometry, they are called *correlations*. As an example, the map $g \mapsto (g^{-1})^t$ is an automorphism of SL_3, and it is outer because it does not fix the center elementwise. We will see in Example 11.2 below that the induced map ψ is the polarity with respect to a certain conic.

Generally speaking, the existence of an outer automorphism of G implies a principle of duality (for D_4, triality) in the geometry Γ_V. For SL_3—equivalently, $\mathbb{P}^2$—it takes the following form [Cox, 2.3]: "every definition remains significant, and every theorem remains true, when we interchange *point* and *line*, *join* and *intersection*." See [W, p. 155] for an analogous statement of the principle of triality.

Let ϕ be an automorphism of G, and let $\mathrm{SubSp}(V)$ denote the collection of subspaces of V. We want an efficient way to check if a given function $\psi\colon \Gamma_V \to \mathrm{SubSp}(V)$ is the automorphism of the geometry Γ_V induced by ϕ.

11.1. **Theorem.** *If*

(1) $\psi(gX) = \phi(g)\psi(X)$ *for every* $X \in \Gamma_V$ *and* $g \in G$ *and*
(2) *there is a chamber* $\{V_i \mid 1 \le i \le n\}$ *such that* $\{\psi(V_i) \mid 1 \le i \le n\}$ *is also a chamber,*

then ψ *is the automorphism of the geometry* Γ_V *induced by the automorphism* ϕ *of* G.

[The term "chamber" was defined in 10.4.]

Proof. Let X be an object in Γ_V. We first claim that $\psi(X)$ is also in Γ_V. We find a chamber $\{X_i \mid 1 \le i \le n\}$ containing X such that X_i is of type α_i. This chamber is conjugate to the chamber $\{V_i \mid 1 \le i \le n\}$ from (2), i.e., there is some $g \in G$ such that $gV_i = X_i$ for every i. Therefore $\psi(X_i) = \phi(g)\psi(V_i)$, and $\psi(X_i)$ is an object in the geometry for all i.

Let P be the stabilizer of X in G. For $g \in \phi(P)$, we have

$$g\psi(X) = \psi(\phi^{-1}(g)X) = \psi(X) \quad \text{by (1)},$$

hence $\phi(P)$ is contained in the stabilizer of $\psi(X)$. But $\psi(X)$ is an object in Γ_V, hence by definition it is a nonzero, proper subspace of V. In particular, its stabilizer is a proper subgroup of G. Since P is a maximal proper subgroup of G, so is $\phi(P)$, hence $\phi(P)$ is the stabilizer of $\psi(X)$. This proves that the diagram

$$\begin{array}{ccc} \Gamma_P & \xrightarrow{\phi} & \Gamma_P \\ \uparrow & & \uparrow \\ \Gamma_V & \xrightarrow{\psi} & \Gamma_V \end{array}$$

commutes, where the vertical arrows send a subspace of V to its stabilizer in G. Since the vertical arrows are bijections (see §2), ψ is also a bijection. Moreover, ψ respects the notion of incidence in Γ_V, because that relation is the one transported from Γ_P by the vertical isomorphisms. We have proved that ψ is an automorphism of Γ_V, and the commutativity of the diagram shows that it is the one induced by ϕ. □

11.2. **Example** (Type A: projective duality). Let G be SL_n acting on k^n, and let ϕ be the automorphism $g \mapsto (g^{-1})^t$. For a subspace X of k^n, we define $\psi(X)$ to be the orthogonal complement of X with respect to the dot product.

Viewed algebraically, the dot product identifies X with the dual vector space X^*. This identification pairs $\psi(X)$ with the collection of linear forms vanishing on X.

Viewed geometrically, the map ψ is precisely the correspondence between points and hyperplanes giving projective duality in $\mathbb{P}^{n-1}$ described in [Pe] and [Cox, 11.8]. It interchanges a point $[a_1 : a_2 : \ldots : a_n]$ in homogeneous coordinates with the hyperplane consisting of solutions to the equation $\sum a_i x_i = 0$. For $n = 3$, it is the polarity with fundamental conic $x^2 + y^2 + z^2 = 0$, cf. [VY, §98].

We now check that ψ satisfies the conditions of Theorem 11.1. The dot product is compatible with the automorphism ϕ in the sense that

$$x \cdot y = (\phi(g)x) \cdot (gy) \quad \text{for } g \in SL_n \text{ and } x, y \in k^n.$$

Thus, a vector x is in $\psi(gX)$ if and only if $(\phi(g)^{-1}x) \cdot X = 0$, i.e., if and only if x is in $\phi(g)\psi(X)$. Thus ψ satisfies (1).

Consider the collection $\{V_1, V_2, \ldots, V_{n-1}\}$ of subspaces such that V_i consists of the vectors whose bottom $n - i$ coordinates are zero. Clearly, this is a chamber. Applying ψ, we find $\psi(V_i)$ is the set of vectors whose top $n - i$ coordinates is zero. Since these also form a chamber, ψ satisfies (2), hence ψ is the automorphism of Γ_V corresponding to the automorphism ϕ of SL_n.

12. Example: type D (orthogonal duality)

Let G be a group as in §6, constructed from the root system of type D_n for some $n \geq 4$; it is traditionally denoted by Spin_{2n}. Its Dynkin diagram has an automorphism of order 2 given by interchanging the roots α_{n-1} and α_n. Here we will construct the corresponding automorphism ψ of the geometry Γ_V.

12.1. We can draw a Hasse diagram for the weights of V as we did for E_6 in Figure 7.1. The case $n = 4$ is shown in Figure 13.1 below. Figure 12.2 shows the diagram in the general case, rotated counterclockwise by 90° for space considerations.

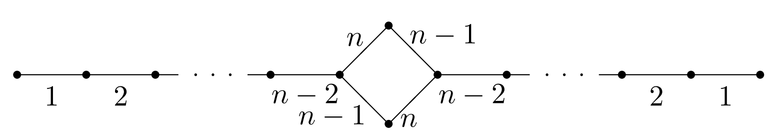


Figure 12.2. Hasse diagram of weights of the standard representation of D_n from [PSV, Fig. 4]. Larger weights are on the left.

Fix nonzero vectors $e_1, e_2, \ldots, e_n$ in V such that e_i has weight

$$\varepsilon_i := \omega_1 - \sum_{j=1}^{i-1} \alpha_j.$$

The longest element of the Weyl group is -1; it is the unique automorphism of the diagram of order 2 that stabilizes none of the weights. The weights ε_1 through ε_{n-1} are those in the string on the left of the diagram and ε_n is the bottom weight in the middle square. The other weights of V are of the form $-\varepsilon_i$ for some i. Let f_i be a nonzero vector of weight $-\varepsilon_i$, so the vectors $e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n$ are a basis of V.

Let b be the G-invariant symmetric bilinear form on V as in §6. Clearly, since ε_i is not $-\varepsilon_j$ for any pair i, j, the subspace of V spanned by the e_i's (respectively, by the f_i's) is isotropic, i.e., $b(e_i, e_j) = b(f_i, f_j) = 0$ for all i, j. Also, $b(e_i, f_j)$ is nonzero if and only if $i = j$. By scaling the f_j's, we may assume that $b(e_i, f_j) = \delta_{ij}$ (Kronecker delta). (We have now obtained the description of G and $SO(b)$ given in [Br, §V.7].) The construction in §6 gives:

$$\begin{aligned} V_{\alpha_i} &:= k\text{-span}\,\{e_1, e_2, \ldots, e_i\} \text{ for } i \leq n-2, \\ V_{\alpha_{n-1}} &= k\text{-span}\,\{e_1, e_2, \ldots, e_n\}, \text{and} \\ V_{\alpha_n} &= k\text{-span}\,\{e_1, e_2, \ldots, e_{n-1}, f_n\}. \end{aligned} \tag{12.3}$$

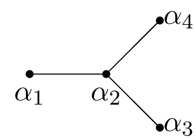
12.4. Since b is G-invariant and G is connected, the representation of G on V is a homomorphism $\rho\colon G \to SO(b)$, where $SO(b)$ denotes the subgroup of $SL(V)$ preserving the bilinear form b. We claim that ρ is a central isogeny. Indeed, every proper, closed normal subgroup of G is central, hence $\ker\rho$ is finite and the image of ρ has the same dimension as G. Root system data gives that the dimension of the Lie algebra of G (equivalently, the dimension of G) is $\binom{2n}{2}$. On the other hand, the Lie algebra of $SO(b)$ is isomorphic over an algebraic closure of k to the space of skew-symmetric $2n$-by-$2n$ matrices, which also has dimension $\binom{2n}{2}$. Since $\operatorname{im}\rho$ and $SO(b)$ are connected and have the same dimension, they are the same. That is, ρ is surjective. The claim now follows because we are in characteristic 0, hence ρ is automatically separable.

Let s denote the matrix in $GL(V)$ that fixes e_i and f_i for $1 \leq i < n$ and interchanges e_n and f_n. It leaves b invariant, and the map $\phi\colon SO(b) \to SO(b)$ defined by $\phi(g) = sgs^{-1}$ is an automorphism of order 2. There is a unique lift of ϕ to an automorphism of G [BT 72, 2.24(i)], which we also denote by ϕ. The description of the root subgroups in $SO(b)$ in [Bor, 23.4] shows that ϕ is, in fact, the automorphism of G induced by the automorphism of the Dynkin diagram that interchanges α_{n-1} and α_n.

For each subspace X of V, put $\psi(X) := sX$. It is a triviality that ψ satisfies condition (1) of 11.1 and that fundamental chamber exhibited in (12.3) is permuted by ψ, hence that ψ satisfies condition (2). That is, ψ is the automorphism of Γ_V induced by the automorphism ϕ of G and $SO(b)$.

13. Example: type D_4 (triality)

Continue the notation of the preceding section, §12, except suppose now that $n = 4$. The Dynkin diagram of G looks like



Let ϕ be the automorphism of order 3 that permutes the arms counterclockwise. We will now describe explicitly the corresponding automorphism ψ of the geometry Γ_V.

Let ρ_0 be the representation of G on V with highest weight ω_1. For $i = 1, 2$, we set $\rho_i := \rho_0 \phi^{-i}$; it is a representation of G on V. The highest weight of ρ_1 is $\phi(\omega_1) = \omega_3$, and the highest weight of ρ_2 is $\phi^2(\omega_1) = \omega_4$.

The weights of V with respect to ρ_0 are listed in Figure 13.1. The weights relative to ρ_1 and ρ_2 are the same except with ϕ or ϕ^2 applied, respectively. Let e_i, f_j be a basis for V as in 12.1. The vector e_i has weight $\phi^j(\varepsilon_i)$ relative to ρ_j. Moreover, the image $\rho_i(G)$ of G in $GL(V)$ is the same for all i, so the symmetric bilinear form b from 12.1 is $\rho_i(G)$-invariant for all i.

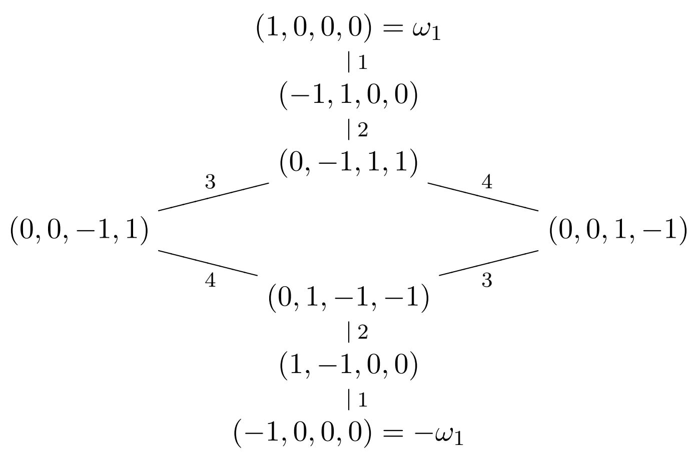


Figure 13.1. Hasse diagram of the weights of V relative to ρ_0.

By representation theory, there is a unique linear map $t \colon V \otimes V \otimes V \to k$ that is G-invariant in the sense that

$$t(\rho_0(g)x_0, \rho_1(g)x_1, \rho_2(g)x_2) = t(x_0, x_1, x_2) \tag{13.2}$$

for all $g \in G$ and $x_0, x_1, x_2 \in V$. We can prove that t is nonzero for certain arguments.

13.3. **Lemma.** *Let $x_i \in V$ be a nonzero vector of weight μ_i relative to ρ_i for $i = 0, 1, 2$. We have: $t(x_0, x_1, x_2)$ is nonzero if and only if $\mu_0 + \mu_1 + \mu_2 = 0$.*

Proof. "Only if" is clear, so we prove "if". Suppose that $\sum \mu_i = 0$. Since $\rho_2(G)$ acts transitively on the weights of V relative to ρ_2, we may assume that μ_2 is ω_4. By the argument in the proof of Lemma 7.6, the subgroup of the Weyl group fixing ω_4 has two orbits on the weights of V relative to ρ_1, with representatives $\pm\omega_3$. Since $-\omega_4 - \omega_3$ is not a weight of V relative to ρ_0, we must have $\mu_1 = -\omega_4 + \omega_3$. We have just proved that the Weyl group acts transitively on the triples μ_0, μ_1, μ_2 such that $\sum \mu_i = 0$. As in the proof of Lemma 7.8, it follows that $t(x_0, x_1, x_2)$ is nonzero. □

Moreover, t is invariant under cyclic permutations.

13.4. **Lemma.** *The value of t is unchanged if its arguments are permuted cyclically.*

Proof. Let $i = 0, 1$, or 2. Consider the linear map $d \colon V \otimes V \otimes V \to k$ defined by

$$d(x_0, x_1, x_2) := t(x_0, x_1, x_2) - t(x_i, x_{i+1}, x_{i+2}),$$

with the subscripts taken modulo 3. This map is G-equivariant because the permutation $x_j \mapsto x_{j+i}$ is cyclic. Indeed, we have:

$$\begin{aligned} t(\rho_i(g)x_i, \rho_{i+1}(g)x_{i+1}, \rho_{i+2}(g)x_{i+2}) &= t(\rho_0(g')x_i, \rho_1(g')x_{i+1}, \rho_2(g')x_{i+2}) \\ &= t(x_i, x_{i+1}, x_{i+2}), \end{aligned}$$

where $g' := \phi^{-i}(g)$.

By the uniqueness of t, the map d must be a scalar multiple of t. The vector $e_4 \in V$ is nonzero of weight $-\omega_3 + \omega_4$ relative to ρ_0. Then $t(e_4, e_4, e_4)$ is not zero by the previous lemma, yet $d(e_4, e_4, e_4)$ is zero. Therefore, d is identically zero. □

We now define products $\cdot_i$ on V for $i = 0, 1, 2$ implicitly via

$$t(x_0, x_1, x_2) = b(x_i, x_{i+1} \cdot_i x_{i+2}).$$

By Lemma 13.4, all three products agree, so we write simply $\cdot$. Because t and b are G-equivariant, so is the product, i.e.,

$$(\rho_i(g)x) \cdot (\rho_{i+1}(g)y) = \rho_{i+2}(g)(x \cdot y). \tag{13.5}$$

This allows us to compute the multiplication, at least up to a scalar factor. Let $x_i \in V$ be nonzero with weight μ_i relative to ρ_i for $i = 1, 2$. It follows from Lemma 13.3 that $x_1 \cdot x_2$ is nonzero if and only if $\mu_1 + \mu_2$ is a weight of V relative to ρ_0, in which case $x_1 \cdot x_2$ has weight $\mu_1 + \mu_2$. We summarize these computations in the table below, where the entry in the row x_1 and column x_2 is "$\cdot$" if $x_1 \cdot x_2$ is zero and, for example, e_3 if $x_1 \cdot x_2$ is a nonzero scalar multiple of e_3. The left column lists the weight of x_1 for the reader's convenience; we omit the weight of x_2 due to space considerations. Since the product is G-equivariant and the weights of ρ_i are preserved under multiplication by -1, one needs only to compute the first four columns of entries;

the remaining four columns can be filled in by symmetry.

(13.6)

			x_2							
			e_1	e_2	e_3	e_4	f_4	f_3	f_2	f_1
	$(0,0,1,0)$	e_1	·	·	·	e_1	·	e_2	e_3	f_4
	$(0,1,-1,0)$	e_2	·	·	e_1	·	e_2	·	e_4	f_3
	$(1,-1,0,1)$	e_3	·	e_1	·	·	e_3	e_4	·	f_2
x_1	$(1,0,0,-1)$	e_4	e_1	·	·	·	f_4	f_3	f_2	·
	$(-1,0,0,1)$	f_4	·	e_2	e_3	e_4	·	·	·	f_1
	$(-1,1,0,-1)$	f_3	e_2	·	f_4	f_3	·	·	f_1	·
	$(0,-1,1,0)$	f_2	e_3	f_4	·	f_2	·	f_1	·	·
	$(0,0,-1,0)$	f_1	e_4	f_3	f_2	·	f_1	·	·	·

(There is a strong resemblance between this multiplication table and the one found by Seligman using the Zorn vector matrices in [Se, p. 287].)

Although the table above is not fine enough to allow us to actually multiply two vectors in V, it is sufficient to describe how the objects in the geometry Γ_V interact with the multiplication. Specifically, we can recover some of the results of [vdBS] without discussing octonion algebras.

13.7. **Proposition.** (Cf. [vdBS, §2])

(1) *If X is an α_1-space (a "point"), then $V \cdot X$ is an α_3-space and $X \cdot V$ is an α_4-space.*
(2) *If X is an α_2-space (a "line"), then $(X \cdot V) \cdot X$ and $X \cdot (V \cdot X)$ are α_2-spaces.*
(3) *If X is an α_3-space (resp., an α_4-space), then there is a unique α_1-space U such that $X = V \cdot U$ (resp., $X = U \cdot V$).*

Proof. By (13.5), it suffices to check (1) for the case where X is the k-span of e_1, i.e., V_{α_1}. In that case, (1) is clear from the multiplication table. A similar argument handles (2).

We now prove (3) for α_3-spaces. As in the previous paragraph, it suffices to check the case where X is the k-span of e_1, e_2, e_3, e_4, i.e., V_{α_3}. The multiplication table shows that X is $V \cdot e_1$, so suppose that $u \in V$ is nonzero and satisfies $V \cdot u = X$. Since V and X are T-invariant, we may assume that u is a weight vector. The multiplication table gives that u is a multiple of e_1. This completes the proof of (3). □

Define ψ via

(13.8) $$\psi(ka) = a \cdot V, \quad \psi(a \cdot V) = V \cdot a, \quad \text{and} \quad \psi(V \cdot a) = ka$$

for a isotropic in V. This is well-defined by the proposition. For X an α_2-space, we define

(13.9) $$\psi(X) = X \cdot (V \cdot X).$$

Applying (13.5), it is easy to check that ψ satisfies condition (1) of Th. 11.1. On the other hand, we checked in the proof of Prop. 13.7 that ψ maps the fundamental chamber to the fundamental chamber, so ψ also satisfies condition (2). Thus ψ is the automorphism of Γ_V corresponding to ϕ.

Remarks. Equation (13.8) defines ψ^2 on the α_1-, α_3-, and α_4-spaces, but (13.9) omits the α_2-spaces. Appealing again to Th. 11.1, we can check that $\psi^2(X)$ is $(X \cdot V) \cdot X$ for X an α_2-space.

We remark that we have recovered a multiplication of the octonions—at least approximately—entirely from first principles of representation theory.

14. Example: type E_6 (duality)

Let G be the split simply connected group of type E_6 with standard representation V as in §7. In this section, we will give an explicit description of the automorphism ψ of the geometry Γ_V corresponding to the automorphism ϕ of the group G.

We define the *brace product* on V following [McC 04, p. 190]:

$$\{x, y, z\} := b(x, y)z + b(z, y)x - (x\#z)\#y.$$

Note that x and z are interchangeable. From (7.5), we find that

$$g\{x, y, z\} = \{gx, \phi(g)y, gz\} \quad \text{for } g \in G \text{ and } x, y, z \in V. \tag{14.1}$$

For each subspace W of V, we set

$$\boxed{\psi(W) := \{x \in V \mid \{W, x, V\} \subseteq W\}}$$

An argument nearly identical to the one in Example 11.2 shows that ψ satisfies hypothesis (1) of Th. 11.1. The rest of this section is spent proving that $\psi(V_\delta) = V_{\phi(\delta)}$ for all $\delta \in \Delta$, i.e., ψ permutes the objects in the fundamental chamber. This will show that ψ satisfies hypothesis (2) of the theorem.

14.2. **Example.** Let U denote the set of weights μ of V such that $\phi(\omega_1 + \mu)$ is *not* a weight of V. Let z be a nonzero vector of weight $\mu \in U$; we claim that z is in $\{v, v', V\}$, where v' is a lowest weight vector of V. First observe that since z does not have weight $-\omega_6$, $b(v, z)$ is zero. Since μ is in U, $v\#z$ is zero and we have: $\{v, v', z\} = b(v, v')z$. But $b(v, v')$ is not zero because b is nondegenerate. This proves the claim.

The subspace of V spanned by weight vectors with weights in U is 17-dimensional, as follows from the proof of Prop. 7.10. Therefore,

$$\dim\{v, v', V\} \geq 17.$$

Connection with Jordan theory. Let c, N, and b be as in §8, so in particular they satisfy (8.1). It is well-known that V is the vector space underlying an exceptional Jordan (a.k.a. Albert) algebra with identity c such that G is the group of isometries of the cubic norm on the algebra. In particular, there exists a G-invariant cubic form N' on V such that $N'(c) = 1$ and a symmetric bilinear form b' on V such that $b'(c, c) = 3$ and b' is compatible with G in the sense of (7.5). By the uniqueness of N and b, the cubic forms N and N' and the bilinear forms b and b' are the same. Therefore we may apply results about Jordan algebras and cubic norm structures.

Specifically, we will use the *5-linear identity* from [McC 04, p. 202]

$$\{x, y, \{z, w, u\}\} = \{\{x, y, z\}, w, u\} - \{z, \{y, x, w\}, u\} + \{z, w, \{x, y, u\}\}.$$

Also, we will use the classification of the inner ideals of V. A subspace X is an *inner ideal* if $\{X, V, X\}$ is contained in X. By [McC 71, §7], the proper inner ideals are the singular subspaces and the hyperlines. The maximal proper inner ideals are the α_2- and α_6-spaces (this follows, for example, from §7).

A straightforward application of the 5-linear identity gives: *If I is an inner ideal in V, then $\psi(I)$ is also an inner ideal.*

14.3. Computation of $\psi(V_{\alpha_1})$. Linearizing the equation $x^{\#\#} = N(x)x$ from Prop. 8.2 as in [McC 69, p. 496], we find the identity (McCrimmon's Equation (12)):

$$(x\#y)\#(x\#z) = b(x^{\#}, y)z + b(x^{\#}, z)y + b(y\#z, x)x - x^{\#}\#(y\#z)$$

Since $v^{\#}$ is zero, we have:

$$(v\#y)\#(v\#z) = b(y\#z, v)v.$$

and

$$\{v, v\#z, y\} = b(v, v\#z)y + b(y, v\#z)v - b(y\#z, v)v.$$

Since $N(-,-,-)$ is symmetric, equation (7.4) shows that the first summand is zero and the second and third summands cancel. Therefore, $\{v, v\#z, y\}$ is zero and $\psi(kv)$ contains $v\#V$.

Since $\psi(kv)$ is an inner ideal containing the hyperline $v\#V$ and it is proper (Example 14.2), the ideal must be the hyperline. Since G acts transitively on the singular 1-dimensional subspaces and ψ satisfies 11.1.1, we obtain the following lemma:

14.4**. Lemma.** *If X is a 1-dimensional singular subspace (= an α_1-space), then $\psi(X)$ is the hyperline $X\#V$.*

14.5. Computation of $\psi(V_{\alpha_2})$. We now show that $\psi(V_{\alpha_2})$ is V_{α_2}. For x, y weight vectors in V_{α_2} and w a weight vector in V, we find that $\{x, y, w\}$ has weight at least

$$\lambda_2 + \phi(\lambda_2) - \omega_6 = \omega_1 - 2(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6).$$

But every weight of V of the form $\omega_1 - (c_1\alpha_1 + c_3\alpha_3 + c_4\alpha_4 + c_5\alpha_5 + c_6\alpha_6)$ with each c_i a nonnegative integer belongs to $L_{\alpha_2}v$, i.e., V_{α_2}. Hence V_{α_2} is contained in $\psi(V_{\alpha_2})$.

Since v' is not in $\psi(V_{\alpha_2})$ by Example 14.2 and $\psi(V_{\alpha_2})$ is an inner ideal, the classification of inner ideals gives that $\psi(V_{\alpha_2})$ is precisely V_{α_2}.

We are left with computing $\psi(V_{\alpha_i})$ for $i \neq 1, 2$. We will use crucially the fact from 3.6 that the weights of V_{α_i} are precisely those lying between the highest weight ω_1 and the weight labeled λ_i in Figure 7.1.

14.6. **Example.** Let y be a nonzero vector of weight λ_2, i.e., a lowest weight vector for L_{α_2} acting on V_{α_2}. We claim that $\{v, y, V\}$ is V_{α_2}. Since y is in $\psi(V_{\alpha_2})$ by 14.5, we need only show that V_{α_2} is contained in $\{v, y, V\}$.

Consulting Figure 7.1, we see that the weights of V_{α_2} are symmetric in the following sense: If $\omega_1 - \alpha$ is a weight of V_{α_2}, then $\lambda_2 + \phi(\alpha)$ is also a weight of V_{α_2}. Consequently, for every weight λ of V_{α_2},

$$f(\lambda) := -\phi(\lambda_2 + \phi(\omega_1 - \lambda)) = -\phi(\lambda_2) + \lambda - \omega_1$$

is a weight of V. For each weight λ of V_{α_2}, fix a nonzero vector z_λ of weight $f(\lambda)$. The vector $\{v, y, z_\lambda\}$ has weight λ, and it suffices to prove that it is not zero for each λ.

For $\lambda = \omega_1$, we note that $\omega_1 + f(\lambda) = \omega_1 - \phi(\lambda_2)$ has 2 as one of its entries, hence $\phi(\omega_1 + f(\lambda))$ is not a weight of V. Therefore, $v \# z_\lambda$ is zero. We have

$$\{v, y, z_\lambda\} = b(z_\lambda, y)v,$$

which is not zero because $f(\lambda) = -\phi(\lambda_2)$.

For the other five weights λ of V_{α_2}, we claim that $v \# z_\lambda$ is not zero. It has weight $\mu := \phi(\omega_1 + f(\lambda)) = \phi(\lambda) - \lambda_2$. For $\lambda = \lambda_3$, Equation (7.9) gives that $\mu = \lambda_5$, a weight of V. For $\lambda = \lambda_4 = \lambda_3 - \alpha_3$, we find that μ is $\lambda_5 - \alpha_5$. Similarly, we find that for each of the the three remaining λ's, the weight μ is a weight of V. That is, $v \# z_\lambda$ is nonzero. The function f was defined so that $(v \# z_\lambda) \# y$ would have weight λ, hence that product is also not zero, i.e.,

$$\{v, y, z_\lambda\} = (v \# z_\lambda) \# y \neq 0.$$

We have proved that $\{v, y, V\}$ is V_{α_2}.

14.7. We can now give a reasonably good description of the space $\{v, w, V\}$ for w a weight vector in V. We say that w and $v \# V$ are *connected* if there is a singular vector $x \in v \# V$ such that x and w are "collinear", i.e., such that x and w span an α_3-space. (In this case, Tits says that w and $v \# V$ are "liés" in [T 57, 3.9].) For example, the vector y from (14.6) and $v \# V$ are connected because y and v are in the α_2-space V_{α_2}. In contrast, the lowest weight vector v' and $v \# V$ are not connected, as it is easily checked that $\phi(-\omega_6 + \mu)$ is a weight of V for every weight μ of $v \# V$. We find

$$\dim\{v, w, V\} \begin{cases} = 0 & \text{if } w \text{ and } v\#V \text{ are incident} \\ = 6 & \text{if } w \text{ and } v\#V \text{ are not incident but are connected} \\ \geq 17 & \text{if } w \text{ and } v\#V \text{ are neither incident nor connected.} \end{cases}$$

We remind the reader that w and $v \# V$ are incident if and only if w is contained in $v \# V$, so the first equality follows from 14.3. The second equality and the inequality are consequences of Lemma 7.6 and Examples 14.2 and 14.6.

14.8. *Remark.* Since G acts "strongly transitively" on the geometry Γ_V, it is sufficient to only consider the case where w is a weight vector. That is, (14.7) holds for *every* pair of singular vectors v, w. We will not use this fact.

14.9**. Lemma.** *Fix a δ in $\Delta \setminus \{\alpha_2\}$. If $\mathcal{X}$ is a basis of V_δ consisting of weight vectors, then*

$$\psi(V_\delta) = \bigcap\nolimits_{x \in \mathcal{X}} x\#V.$$

Proof. Fix a nonzero weight vector $y \in \psi(V_\delta)$. We claim that $\{v, y, V\}$ is the zero subspace. Otherwise, by 14.7 $\{v, y, V\}$ is an α_2-space or has dimension at least 17. If δ is not α_6, then V_δ has dimension at most 5, and we have a contradiction. When δ is α_6, V_δ does not contain an α_2-space by Remark 7.11, and again we find a contradiction. This proves that $\{v, y, V\}$ is zero. Since V_δ and V are T-invariant, so is $\psi(V_\delta)$; that is $\psi(V_\delta)$ is a direct sum of weight spaces. All together, we have that $\{v, \psi(V_\delta), V\}$ is zero, i.e., $\psi(V_\delta)$ is contained in $v\#V$.

For each $x \in \mathcal{X}$, there is some $g \in G$ such that gv is in the span of x and g leaves V_δ invariant. We have

$$\psi(V_\delta) = \phi(g)\psi(V_\delta) \subseteq \phi(g)(v\#V) = x\#V.$$

Conversely, suppose that y is in the intersection of the $x\#V$'s. Then $\{x, y, V\}$ is zero for every x. Since the x's span V_δ, y is in $\psi(V_\delta)$. □

We can now compute $\psi(V_{\alpha_i})$ for $i \neq 1, 2$.

14.10. **Computation of V_{α_3} and V_{α_4}.** The space V_{α_3} is spanned by the highest weight vector v and a vector x of weight λ_3. We wish to compute $\psi(V_{\alpha_3})$, which is $V_{\alpha_6} \cap (x\#V)$ by Lemma 14.9. Each weight τ of V_{α_6} is a weight of $x\#V$ if and only if $\phi(\tau) - \lambda_3$ is a weight of V. The five weights τ of V_{α_6} with a 1 as their last coordinate cannot belong to $x\#V$ because $\phi(\tau) - \lambda_3$ has a 2 as its first coordinate. That is, $\psi(V_{\alpha_3})$ is contained in V_{α_5}.

Since $-\phi(\lambda_2)$ is a weight of V and $\lambda_5 = \phi(\lambda_3 - \lambda(\phi_2))$ by (7.9), the weight λ_5 belongs to $x\#V$. Figure 7.1 shows that $-\phi(\lambda_2) + \alpha_2$ is also a weight of V, hence $\lambda_5 + \alpha_2$ belongs to $x\#V$. Continuing in this manner, we find that V_{α_5} is contained in $x\#V$, hence that $\psi(V_{\alpha_3})$ is V_{α_5}.

The space V_{α_4} is spanned by V_{α_3} and a vector y of weight λ_4. The two weights τ of V_{α_5} that do not belong to V_{α_4} each have a 1 as their 5th coordinate, hence $\phi(\tau) - \lambda_4$ has a 2 as its 3rd coordinate, and such weights τ do not belong to $\psi(V_{\alpha_4})$. The three weights of V_{α_4} are easily checked to be weights of $y\#V$, hence $\psi(V_{\alpha_4})$ is V_{α_4}.

14.11. **Computation of V_{α_5} and V_{α_6}.** By Lemma 14.9 and 14.10, $\psi(V_{\alpha_5})$ is contained in V_{α_4}. Moreover, the 5th coordinate of $\phi(\lambda_4) - \lambda_5$ is -2, hence $\psi(V_{\alpha_5})$ is contained in V_{α_3}.

Equation (7.9) gives that the weight λ_3 belongs to $z\#V$ for z a nonzero vector of weight λ_5. Also,

$$\lambda_1 = \lambda_3 + \alpha_1 = \phi(\lambda_5 + (\lambda_2 + \alpha_6)),$$

so v belongs to $z\#V$. Similar calculations show that V_{α_3} is contained in $x\#V$ for x of weight $\lambda_5 + \alpha_2$, hence V_{α_3} is equal to $\psi(V_{\alpha_5})$.

Lemma 14.9 gives that $\psi(V_{\alpha_6})$ is contained in the 2-dimensional space $\psi(V_{\alpha_5}) = V_{\alpha_3}$. The 6th coordinate of $\phi(\lambda_3) - \lambda_6$ is -2, hence λ_3 does not belong to $u\#V$ for every vector u of weight λ_6, and $\psi(V_{\alpha_6})$ is contained in the k-span of the highest weight vector v.

We now show that v is in $\psi(V_{\alpha_6})$. Linearizing the identity $x^{\#\#} = N(x)x$ as in [McC 69] again (and going through his Equation (19)), we find the identity

$$z\#(y\#(x\#z)) = b(x, z^{\#})y + b(x, y)z^{\#} + b(y, z)(x\#z) - x\#(y\#z^{\#}), \tag{14.12}$$

which holds for every $x, y, z \in V$. Substituting $z \mapsto v$, we find

$$v\#(y\#(x\#v)) = b(y, v)(x\#v).$$

Recalling that $b(v\#x, v)$ is zero, we find that $\{v\#x, v, y\}$ is zero for all $x, y \in V$. That is, v is in $\psi(V_{\alpha_6})$ and $\psi(V_{\alpha_6})$ is V_{α_1}.

We have proved that $\psi(V_{\alpha_i}) = V_{\phi(\alpha_i)}$ for all i. In particular, the image of the fundamental chamber under ψ is just the fundamental chamber. This proves that ψ is the automorphism of Γ_V induced by the automorphism ϕ of G.

In the language of classical projective geometry, ψ is a *hermitian polarity.* Indeed, since ϕ^2 is the identity on G, ψ^2 is the identity on Γ_V, i.e., ψ is a polarity. One says that ψ is hermitian because the "point" V_{α_1} is contained in its "polar" V_{α_6}.

For the sake of a finer description of the map ψ, we record the following corollary to Lemma 14.9.

14.13. **Corollary.** *If $X \in \Gamma_V$ is not of dimension* 6*, then*

$$\psi(X) = \{y \in V \mid \{X, y, V\} = 0\}.$$

Proof. Let $\delta \in \Delta$ be such that X is a δ-space. Fix a $g \in G$ such that gX is V_δ. For every $y \in \psi(X)$, we have

$$g\{x, y, V\} \subseteq \{V_\delta, \psi(V_\delta), V\},$$

which is the zero subspace by the proof of Lemma 14.9. □

Faulkner discussed the geometry Γ_V in terms of the brace product in [Fa], although he focussed on the points (α_1-spaces) and hyperlines. He described the duality on points and hyperlines by the equation displayed in the corollary. However, that definition does not work for our purposes, as the following example shows.

14.14. **Example.** If $X \in \Gamma_V$ has dimension 6, then the set $X^c := \{y \in V \mid \{X, y, V\} = 0\}$ is the zero subspace. To prove this, it suffices to check the case $X = V_{\alpha_2}$. As in the proof of Lemma 14.9 we find that X^c is the intersection of the sets $z\#V$ as z ranges over nonzero vectors of each of the six weights of V between ω_1 and λ_2. Since V_{α_2} contains V_{α_4}, we have $V_{\alpha_2}^c$ is contained in $\psi(V_{\alpha_4}) = V_{\alpha_4}$. Arguing as in 14.10, one quickly sees that the three weights $\lambda_1, \lambda_3, \lambda_4$ of V_{α_4} do not belong to X^c. Hence X^c is zero as claimed.

14.15. **Proposition.** *For $X \in \Gamma_V$, we have*

$$b(X, \psi(X)) = 0 \quad \textit{and} \quad \{\psi(X), X, \psi(X)\} = 0.$$

Proof. By the transitivity of the G-action, we may assume that X is V_{α_i} for some i. Let j be such that $\alpha_j = \phi(\alpha_i)$. Further, let d_i be such that $\omega_1 - d_i = \lambda_i$; it is a sum of positive roots.

We first argue that $b(X, \psi(X))$ is zero. By (7.3), it can only be nonzero if there are vectors $x \in X$ and $x' \in \psi(X)$ of weights λ and λ' such that $\lambda + \phi(\lambda') = 0$. But every weight of X (resp. $\psi(X)$) is at least λ_i (resp. λ_j), and

$$\lambda_i + \phi(\lambda_j) = (\omega_1 + \omega_6) - (d_i + \phi(d_j));$$

we will show that this is > 0 for all i. Consulting the tables in [Bou 4–6], we find:

$$\omega_1 + \omega_6 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6.$$

When $i = 1$, d_1 is zero and d_6 is a sum of positive roots with no root occurring more than twice, as can be seen from Figure 7.1. Therefore, $\lambda_1 + \phi(\lambda_6) > 0$. Applying ϕ to both sides of the equation, we find that $\lambda_6 + \phi(\lambda_1) > 0$. When $i = 2, 3, 4$, or 5, no root appears more than once in d_i. But $2 \leq j \leq 5$, hence the same is also true of d_j. Therefore no root appears in $d_i + \phi(d_j)$ more than twice. We have proved that $\lambda_i + \phi(\lambda_j) > 0$ for all i, hence $b(X, \psi(X))$ is necessarily zero.

We now prove that the second equation holds. The space $\{\psi(X), X, \psi(X)\}$ is a direct sum of its weight spaces, and each weight μ is at least $\phi(\lambda_i) + 2\lambda_j$ by (14.1). That is, each weight μ is of the form $\omega_1 - d$ for

$$0 \leq d \leq \omega_1 - (\phi(\lambda_i) + 2\lambda_j) = (\phi(d_i) + 2d_j) - (\omega_1 + \omega_6). \tag{14.16}$$

For each j, we see from Figure 7.1 that d_j does not include the root α_j, hence neither does $\phi(d_i)$. Therefore, the coefficient of α_j on the right side of (14.16) is negative. In particular, the equation $d \geq 0$ is impossible, and μ cannot be a weight of V. This proves that $\{\psi(X), X, \psi(X)\}$ is zero. □

In [LN, p. 260], Loos and Neher defined

$$\mathrm{Inid}(X) := \{y \in V \mid \{y, X, y\} = 0 \text{ and } \{X, y, V\} \subseteq X\},$$

for X a subspace of V. Clearly, $\psi(X)$ contains $\mathrm{Inid}(X)$, and the preceding proposition shows that the two concepts agree for $X \in \Gamma_V$.

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