On the CR-structure of compact group orbits associated with bounded symmetric domains

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1. Introduction

Let E be a complex vector space of finite dimension and let $\mathsf{K} \subset \mathsf{GL}(E)$ be a compact connected subgroup. Then for fixed $a \in E$ the orbit $K := \mathsf{K}(a)$ is a real-analytic submanifold of E that inherits various structures from E. For instance, choosing a K-invariant positive definite inner product $(x|y)$ on E makes K a Riemannian manifold on which K acts transitively by isometries. On the other hand, K inherits from E a *Cauchy-Riemann structure (CR-structure)*, that is given by the distribution of the maximal complex subspaces $H_xK := T_xK \cap iT_xK$ of the real tangent spaces $T_xK \subset E$, $x \in K$, together with the complex structure on every H_xK (multiplication by i). The subspace H_xK is called the *holomorphic tangent space* to K at x (see [7] and [3] as general references for CR-manifolds).

Of interest for the geometry of the orbit $K = \mathsf{K}(a)$ with respect to its CR-structure is the study of the *CR-functions* (or more generally *CR-mappings*) on K, i.e. of smooth functions $f : K \to \mathbb{C}$ that satisfy the tangential Cauchy-Riemann differential equations in the sense that the restriction of the differential *df* to every holomorphic tangent space is complex linear. For instance, all holomorphic functions defined in an open neighbourhood of $K \subset E$ give by restriction real-analytic CR-functions on K. Actually, we deal with the more general *continuous CR-functions* on K (which satisfy by definition the tangential CR-equations in the distribution sense, or equivalently, which are locally uniform limits of sequences of smooth CR-functions due to the approximation theorem of Baouendi-Treves [4]). In this context it is of interest to determine the space of all points 'to which every continuous CR-function on K can be holomorphically extended', or in a more abstract setting, to determine the spectrum of the Banach algebra of all continuous CR-functions on K. For this also the explicit determination of the corresponding linear, polynomial and rational convex hulls is of help. An important well-known tool and CR-invariant of K is the (vector-valued) *Levi form*, which is a sesqui-linear form defined on the holomorphic tangent space H_xK with values in the complex vector space $(T_xK+iT_xK)/H_xK$ – in a vague sense it can be understood as a 'holomorphic curvature' that measures how far the variation of the subspace $H_xK \subset E$ differs from being CR in $x \in K$. Hence the first step is the understanding of the Levi form. Other natural questions are: *When are two orbits* $\mathsf{K}(a)$ *and* $\mathsf{K}(b)$ *for* $a, b \in E$ *isomorphic as CR-manifolds? When are two CR-isomorphic* K*-orbits in* E *linearly equivalent? When can the orbit* $K = \mathsf{K}(a)$ *be realized as the Shilov boundary of a relatively compact domain in a suitable complex-analytic Stein space?* The last question has been treated (also if K in E is not an orbit) by Harvey-Lawson [17] in case K is of hypersurface type, i.e.

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$\dim(T_aM/H_aM) = 1$ for all $a \in M$. However, if H_aM is of higher codimension in T_aM, the last question, also treated in this paper, is completely open in general.

A well understood case is when $\mathsf{K} \subset \mathsf{GL}(E)$ is a maximal compact subgroup, that is up to isomorphy, $E = \mathbb{C}^n$ with standard inner product $(x|y) = \sum x_j\overline{y}_j$ and $\mathsf{K} = \mathsf{U}(n)$ is the unitary group. Then, choosing a unit vector $a \in E$, the corresponding orbit $K = \mathsf{K}(a)$ is the euclidian unit sphere of E, and for every $x \in K$ the holomorphic tangent space H_xK is the complex orthogonal complement to the vector x in E. As is well known, the holomorphic structure of the open unit ball $\mathcal{D} = \{z \in E : (z|z) < 1\}$ is closely related to the CR-structure of its boundary $K = \partial\mathcal{D}$: Every continuous CR-function on K extends to a holomorphic function on $\mathcal{D}$ which is continuous up to the boundary, and the various convex hulls (e.g. linear, polynomial, rational, holomorphic) of K all coincide with the closed unit ball $\overline{\mathcal{D}}$ (provided E has dimension at least 2, in which case $\mathsf{K}(a) = \mathsf{S}(a)$ holds for $\mathsf{S} = \mathsf{SU}(n)$). Furthermore, the group of all CR-homeomorphisms of K identifies with the group $\mathsf{Aut}(\mathcal{D})$ of all biholomorphic automorphisms of the ball $\mathcal{D}$, which is the group $\mathsf{PSU}(n,1)$ acting transitively by linear fractional transformations on $\mathcal{D}$.

The euclidian unit ball in $\mathbb{C}^n$ is an example of a bounded symmetric domain. Recall that, up to biholomorphic equivalence, the *bounded symmetric domains* are precisely the bounded circular convex domains $\mathcal{D}$ in a complex vector space E of finite dimension such that the group $\mathsf{Aut}(\mathcal{D})$ of all biholomorphic transformations acts transitively on $\mathcal{D}$ (the adjective *symmetric* reflects the fact that then the symmetry $s(z) = -z$ about the origin $0 \in \mathcal{D}$ can be conjugated to a symmetry about any point in $\mathcal{D}$). One of the main invariants is the rank of the bounded symmetric domain $\mathcal{D}$, a certain integer that measures in a way the deviation of $\mathcal{D}$ from being a euclidian ball. In particular, among all bounded symmetric domains the euclidian balls are precisely those of lowest possible rank (namely 1) and also those with smooth boundary.

The next interesting case plays a remarkable role in many seemingly unrelated contexts. It is the bounded symmetric domain of lowest possible dimension whose boundary is not smooth, namely the open unit ball $\mathcal{D}$ with respect to the operator norm in the space E of all complex symmetric 2×2-matrices. Its boundary is the union of two smooth parts: The Shilov boundary of $\mathcal{D}$ (totally real and diffeomorphic to the homogeneous space $\mathsf{U}(2)/O(2)$) and a real hypersurface in E that is the bounded circular realization of the tube over the light cone. This hypersurface is the simplest known example of a real everywhere Levi-degenerate hypersurface that is not locally equivalent to a product of $\mathbb{C}$ with a hypersurface in $\mathbb{C}^2$ (see [12] for these and other related facts on CR-geometry of this hypersurface). In this example the group $\mathsf{K} := \mathsf{GL}(\mathcal{D})$ of all linear transformations $g \in \mathsf{GL}(E)$ with $g(\mathcal{D}) = \mathcal{D}$ consists of all transformations $z \mapsto uzu'$ with $u \in \mathsf{U}(2)$ unitary. The orbits of K and of its commutator subgroup S (isomorphic to $\mathsf{SU}(2)/\{\pm1\}$) in E have been studied in [11] and, in a slightly different formulation, also in [2] and [21]. In particular, it has been shown in [21] that among the S-orbits there are one-parameter families of pairwise CR-inequivalent CR-manifolds, which are all diffeomorphic to the 3-dimensional real projective space. It has been further shown in [2] (see also [34]) that the universal coverings of these CR-manifolds cannot be realized as boundaries of compact complex spaces. Another remarkable feature of this example is the presence of the complex-analytic cone of all singular matrices in E, that realizes the simplest normal singularity and can be seen as the complexification of every orbit $\mathsf{K}(a) = \mathsf{S}(a)$ with a of rank 1. Yet another feature is that the polynomial convex hull of any such orbit $\mathsf{K}(a)$ with $\|a\| = 1$ is the image of the closed unit ball in $\mathbb{C}^2$ under the mapping $(z,w) \mapsto \left(\begin{smallmatrix} zz & zw \\ wz & ww \end{smallmatrix}\right)$, which is the simplest known proper holomorphic mapping between euclidian balls of dimensions > 1 that is not injective.

In this paper we give answers to the above questions for K- and S-orbits in case where the group $\mathsf{K} \subset \mathsf{GL}(E)$ is the connected identity component of the group $\mathsf{GL}(\mathcal{D})$ associated to an arbitrary bounded symmetric domain $\mathcal{D} \subset E$ of rank r and $\mathsf{S} \subset \mathsf{K}$ is the semisimple part of K. Every bounded symmetric domain $\mathcal{D}$ can be written in a unique way as a direct product of *irreducible* ones, i.e. those that cannot be further written as nontrivial direct products. For simplicity we always assume that $\mathcal{D}$ is irreducible, which is equivalent to K acting irreducibly on E or also to $\mathbb{T} := \{z \mapsto tz : |t| = 1\}$ being the center of K. One always has $\mathsf{K} = \mathbb{T}\mathsf{S}$. We shall extensively use the associated Jordan triple product on the ambient space E that allows to carry out computations in an algebraic way. Of special importance are the singular values $\sigma_1(a) \geq \sigma_2(a) \geq \sigma_r(a) \geq 0$ of $a \in E$ that can be defined in a purely Jordan algebraic way and generalize the usual singular values of rectangular matrices.

The irreducible bounded symmetric domains of positive dimension come in 4 classical series and two separate exceptional domains (compare for instance [18] or [29]). These are (without repetitions) precisely the following domains, where n is the dimension and r is the rank. We write $\mathbb{C}^{p\times q}$ for the linear space of all complex $p \times q$-matrices.

$\mathbf{I}_{p,q}$**:** $\mathcal{D} = \{z \in \mathbb{C}^{p\times q} : \mathbb{1}_p - zz^* > 0\}$, where $1 \leq p \leq q$ are arbitrary integers, '> 0' means 'positive-definite', and z^* is the conjugate-transpose of z. Here $n = pq$, $r = p$ and S is the group of all transformations $z \mapsto uzv$ with $u \in \mathsf{SU}(p)$ and $v \in \mathsf{SU}(q)$.

$\mathbf{II}_p$**:** $\mathcal{D} = \{z \in \mathbb{C}^{p\times p} : z' = -z$ and $\mathbb{1}_p - zz^* > 0\}$, where $p \geq 5$ and z' is the transpose of z. Here $n = \binom{p}{2}$, $r = [\frac{p}{2}]$ and S is the group of all transformations $z \mapsto uzu'$ with $u \in \mathsf{SU}(p)$.

$\mathbf{III}_p$**:** $\mathcal{D} = \{z \in \mathbb{C}^{p\times p} : z' = z$ and $\mathbb{1}_p - zz^* > 0\}$, where $p \geq 2$. Here $n = \binom{p+1}{2}$, $r = p$ and S is the group of all transformations $z \mapsto uzu'$ with $u \in \mathsf{SU}(p)$.

$\mathbf{IV}_n$**:** $\mathcal{D} = \{z \in \mathbb{C}^n : (z|z) + \sqrt{(z|z)^2 - |\langle z|z\rangle|^2} < 2\}$ (the Lie ball), where $n \geq 5$, $(z|w) = \sum z_k\overline{w}_k$ and $\langle z|w\rangle = \sum z_k w_k$. Here $r = 2$ and $\mathsf{S} = \mathsf{SO}(n)$ acting in the standard way on $\mathbb{R}^n$ and $\mathbb{C}^n$.

V: An exceptional domain in dimension $n = 16$ with rank $r = 2$ and $\mathsf{S} = \mathsf{SO}(10)$.

VI: An exceptional domain in dimension $n = 27$ with rank $r = 3$. Here S is a compact exceptional group of type E_6.

The types can also be defined for smaller indices. But then there are for instance the coincidences $\mathbf{IV}_3 \approx \mathbf{III}_2$, $\mathbf{IV}_4 \approx \mathbf{I}_{2,2}$, $\mathbf{IV}_6 \approx \mathbf{II}_4$, $\mathbf{II}_3 \approx \mathbf{I}_{1,3}$, and $\mathbf{IV}_2$ is not irreducible.

For every irreducible $\mathcal{D}$ the connected identity component G of the biholomorphic automorphism group $\mathsf{Aut}(\mathcal{D})$ is a simple real Lie group acting transitively on $\mathcal{D}$. In case of the types $\mathbf{I} - \mathbf{IV}$, G is a classical group, whereas for the types **V, IV**, it is exceptional of type E_6 and and E_7 respectively. For this reason, an irreducible bounded symmetric domain is called *classical* if it is of type $\mathbf{I} - \mathbf{IV}$ and *exceptional* otherwise.

An outline of the paper is as follows. In section 2 we illustrate our results in the case of type $\mathbf{I}_{p,q}$ (the space $\mathbb{C}^{p\times q}$ of complex $p \times q$-matrices) containing already a rich class of CR-nonequivalent examples demonstrating main phenomena. In sections 3 – 5 we survey basic facts of the well known Jordan approach to bounded symmetric domains that will be extensively used throughout the paper. In particular in section 4 we describe all formally real Jordan algebras together with their positive cones. These play an important role for the fine structure of the Levi cones later on. In section 6 we generalize the well known Peirce decomposition for tripotents to arbitrary elements $a \in E$. This is motivated by the decomposition of E given by the tangent spaces $H_aK \subset T_aK$ of the orbit $K := \mathsf{K}(a)$ and allows in section 7 to represent the tangent spaces as ranges of certain polynomial functions of operators of low degree. Here the two main structurally different cases become visible: the orbits of invertible and of noninvertible elements (for types $\mathbf{I} - \mathbf{III}$ invertibility here means the usual invertibility of matrices). In the invertible case the K-orbits are nonminimal whereas the S-orbits are minimal submanifolds of codimension 1 (recall that a CR-manifold M is minimal at a point a in the sense of Tumanov [36] if any smooth submanifold through a having the same holomorphic tangent space at every point is necessarily open in M.) In the noninvertible case K- and S-orbits coincide. In both cases we show that the Levi cone of each S-orbit has a nonempty interior. In section 8 we study the orbit of the complexified groups $\mathsf{K}^{\mathbb{C}}$ and $\mathsf{S}^{\mathbb{C}}$ where the K- and S-orbits respectively are embedded as generic submanifolds. We also provide defining equations for both real and complex orbits. We then turn to a more explicit computation of the Levi form and the Levi cone of the orbits in section 9. The Levi cone turns out to be a simplex cone for orbits of elements with pairwise different singular values. On the other hand, if some singular values coincide, the cone becomes more complicated and is not necessarily finitely generated. In each case we give explicit defining equations and inequalities for the Levi cone. We next construct compact subsets of E, naturally associated to the elements of E, having their tangent cones in the direction of the Levi cones that will play an important role in the description of the natural hulls of the orbits. In section 10 we study the interior domains in the above compact sets proving, in particular, that they are Stein. Those domains with automorphism group of maximal dimension will be exactly the corresponding bounded symmetric domains for which we give different characterizations. Section 11 is devoted to the explicit description of convex, polynomial and rational convex hulls of the orbits. Finally, in section 12 we identify the maximal domains of holomorphic extension of CR-functions on the orbits in each case that turn out to be the domains studied in section 10. The extension is obtained by using locally a deformation version of the extension result of Boggess-Polking [8] and constructing one-parameter families of orbits 'moving' everywhere inside the Levi cone. It is shown that such families fill an open dense subset in the domain of consideration whereas the extension to the full domain is obtained by removing real-analytic submanifolds of high codimension. The final continuous extension to the closure is obtained by a linear rescaling argument in the case a is not invertible whereas, if a is invertible, more elaborate arguments involving the fine boundary stratification of the hulls are needed. We conclude by giving applications of our main results to the classification of the orbits with respect to CR-homeomorphisms.

Notation: For every complex vector space E of finite dimension we denote by $\mathcal{L}(E)$ the complex algebra of all linear endomorphisms of E and by $\mathsf{GL}(E)$ the maximal subgroup of invertible operators. $\mathsf{SL}(E)$ is the subgroup of all operators of determinant 1. For every subset $M \subset E$ we denote by $\mathsf{GL}(M) \subset \mathsf{GL}(E)$ the subgroup of all transformations g with $g(M) = M$. Furthermore, $\mathcal{L}_{\mathbb{R}}(E)$ is the complex algebra of all $\mathbb{R}$-linear endomorphisms of E. A sesqui-linear mapping $L : E \times E \to F$ is always understood to be complex linear in the first and conjugate linear in the second argument. For every field $\mathbb{K}$ we denote by $\mathbb{K}^{p\times q}$ the space of all matrices with p rows and q columns and entries from $\mathbb{K}$. By $\mathbb{1}_p$ or simply $\mathbb{1}$ we denote the $p\times p$-identity matrix.
For every real or complex vector space V of finite dimension and every subset $M \subset V$ containing a in its closure, $T_a M$ denotes the (Whitney) tangent cone to M at a, that is the set of all $v \in E$ such that there are sequences (v_j) in M and (t_j) in $\mathbb{R}^+ := \{t \in \mathbb{R} : t > 0\}$ with $\lim_{j\to\infty} v_j = a$ and $\lim_{j\to\infty} t_j(v_j - a) = v$.
If Ω is a topological space, $\mathcal{C}(\Omega)$ denotes the complex algebra of all continuous complex-valued functions on Ω. For every complex manifold (or more generally for every complex space) X we denote by $\mathcal{O}(X)$ the algebra of all holomorphic functions on X and by $\mathsf{Aut}(X)$ the group of all biholomorphic automorphisms of X. For every CR-manifold M we denote by $\mathcal{C}_{\text{CR}}(M) \subset \mathcal{C}(M)$ the complex subalgebra of all continuous CR-functions on M and by $\mathsf{Aut}_{\text{CR}}(M)$ the group of all *CR-homeomorphisms* of M, i.e. homeomorphisms φ such that both φ and φ^{-1} are CR in the distribution sense. The groups $\mathsf{Aut}(X)$ and $\mathsf{Aut}_{\text{CR}}(M)$ are always considered as topological

groups with respect to the compact open topology unless stated otherwise.
For complex vector spaces V, W we simply write $V \otimes W$ instead of $V \otimes_{\mathbb{C}} W$. For subgroups $G \subset \mathsf{GL}(V)$ and $H \subset \mathsf{GL}(W)$ we denote by $G \otimes H \subset \mathsf{GL}(V \otimes W)$ the subgroup of all transformations $g \otimes h$ with $g \in G$ and $h \in H$. Clearly, the canonical surjection $G \times H \to G \otimes H$ is not injective in general.

2. Illustration of the main results

In this section we illustrate our main results in the special situation of matrix spaces. We begin with a general remark: Let E be an arbitrary complex vector space of finite dimension and let $\mathsf{K} \subset \mathsf{GL}(E)$ be a compact connected subgroup. Then $\mathsf{K} = \mathsf{Z}\,\mathsf{S}$, where Z is the connected identity component of the center and S is the (semi-simple) commutator subgroup of K. Every K-orbit is foliated in S-orbits, more precisely, to every $a \in E$ there is a torus subgroup $\mathsf{T} \subset \mathsf{Z}$ such that $\mathsf{T} \times \mathsf{S}(a)$ is a covering space of $\mathsf{K}(a)$ via the mapping $(t, z) \mapsto t(z)$.

2.1 Remark. *For every $a \in E$ the orbit $S := \mathsf{S}(a)$ has finite fundamental group, whereas the fundamental group of $K := \mathsf{K}(a)$ is infinite if $K \neq S$. In case $\mathsf{Z} \subset \mathbb{T} := \{z \mapsto tz : |t| = 1\}$ (for instance if K acts irreducibly on E) the holomorphic tangent spaces satisfy $H_x(K) = H_x(S)$ for every $x \in S$. In particular, K is not minimal as CR-manifold if $\mathsf{Z} = \mathbb{T}$ and $K \neq S$.*
Finiteness of the fundamental group of S follows from the same property for the compact semi-simple group S (compare e.g. [19] p. 144) and the fact that the isotropy subgroup S_a has only finitely many connected components. The equality of the holomorphic tangent spaces follows from $a \notin T_a K = T_a S \oplus \mathbb{R} ia$. Notice that this statement no longer remains true if Z is not contained in $\mathbb{T}$. As a counter-example consider $E := \mathbb{C}^{2\times 2}$ identified with $\mathbb{C}^2 \otimes \mathbb{C}^2$ and set $\mathsf{K} = \mathsf{SU}(2) \otimes \mathsf{SO}(2)$. Then $\mathsf{Z} = \mathbb{1} \otimes \mathsf{SO}(2)$, $\mathsf{S} = \mathsf{SU}(2) \otimes \mathbb{1}$, and for $a := \left(\begin{smallmatrix} t & 0 \\ 0 & 1/t \end{smallmatrix}\right)$, $t > 1$ fixed, the orbit $\mathsf{S}(a)$ is totally real while the orbit $K = \mathsf{K}(a)$ is not. Actually, K is generic in $\mathsf{SL}(2, \mathbb{C})$ and $z \mapsto zz'$ defines for $b := aa'$ a Z-invariant CR-submersion from K onto the minimal CR-submanifold $\{ubu' \in E : u \in \mathsf{SU}(2)\}$ whose differential induces an isomorphism of holomorphic tangent spaces at every point of K. The image is a hypersurface in the affine quadric $\{z \in \mathsf{SL}(2, \mathbb{C}) : z' = z\}$ $\approx \mathsf{SL}(2, \mathbb{C}) / \mathsf{SO}(2, \mathbb{C})$ and is real-analytically equivalent to the real projective space $\mathbb{P}_3(\mathbb{R})$, but its CR-structure does not come from the standard CR-structure on the 3-sphere $S^3 \subset \mathbb{C}^2$ (cf. [21]).

Now fix for the rest of this section integers $p, q \geq 1$ and let $E := \mathbb{C}^{p\times q}$ be the space of all complex $p \times q$-matrices. Denote by $r := \min(p, q)$ the maximal possible rank of matrices in E and call it also the *rank* of E. For simplicity (and without loss of generality) let us assume throughout $p \leq q$, i.e. $r = p$.

On E there is a canonical norm $\|\ \|$, namely the operator norm if every $z \in E$ is considered in the natural way as operator $z : \mathbb{C}^q \to \mathbb{C}^p$ between complex Hilbert spaces (the operator norm $\|z\|$ coincides with the biggest singular value of the matrix $z \in E$, see below for more details). The open unit ball $\mathcal{D} := \{z \in E : \|z\| < 1\}$ is a bounded symmetric domain (the type $\mathbf{I}_{p,q}$, see section 1). The biggest connected subgroup $\mathsf{K} \subset \mathsf{GL}(E)$ leaving the ball $\mathcal{D}$ invariant is the group of all transformations $z \mapsto uzv$ with $u \in \mathsf{U}(p)$ and $v \in \mathsf{U}(q)$, that is, $\mathsf{K} = \mathsf{U}(p) \otimes \mathsf{U}(q)$ if we identify $\mathbb{C}^{p\times q}$ with $\mathbb{C}^p \otimes \mathbb{C}^q$. The subgroup $\mathsf{S} = \mathsf{SU}(p) \otimes \mathsf{SU}(q)$ is semi-simple and of codimension 1. The boundary $\partial\mathcal{D}$ of $\mathcal{D}$ in E is smooth only in case $p = 1$ (and then $\|\ \|$ is a Hilbert norm). In general, there is a stratification $\partial\mathcal{D} = S_1 \cup \ldots \cup S_p$ into (locally-closed) real-analytic submanifolds $S_k \subset E$. Each S_k consists of all those $z \in \partial\mathcal{D}$ for which the hermitian matrix zz^* has the eigenvalue 1 with multiplicity k. The group $G = \mathsf{Aut}(\mathcal{D})$ of all biholomorphic automorphisms of $\mathcal{D}$ acts by analytic continuation over the boundary $\partial\mathcal{D}$ on the closure $\overline{\mathcal{D}}$ and every S_k is a G-orbit.

For every $a \in E$ we are interested in the CR-structure of the orbits $K = \mathsf{K}(a)$ and $S = \mathsf{S}(a)$ in E. It will turn out that the following two cases are structurally different: (1) *a is invertible* (i.e. invertible as operator, which clearly can only happen if $p = q$), and (2) *a is not invertible.* For instance, $K = S$ holds if and only if a is not invertible, and this holds if and only if K is *minimal* as CR-manifold. In any case, we have:

2.2 Remark. *For every $z \in E = \mathbb{C}^{p\times q}$ the orbit $S = \mathsf{S}(z)$ is simply-connected.*

Proof. We may assume $k := \mathrm{rank}(z) > 0$ and write all matrices in $\mathbb{C}^{p\times p}$, $\mathbb{C}^{p\times q}$, $\mathbb{C}^{q\times q}$ as 2×2-block matrices $\binom{ab}{cd}$ with upper left block $a \in \mathbb{C}^{k\times k}$. We may assume furthermore $z = \binom{x0}{00}$ with $x \in \mathsf{GL}(k, \mathbb{C})$. The simply connected group $\mathsf{SU}(p) \times \mathsf{SU}(q)$ acts transitively on S with isotropy subgroup at z given by all pairs $\big(\binom{a0}{0b}, \binom{c0}{0d}\big)$ satisfying $ax = xc$. But this group has the same number of connected components as the group $\{(a, c) \in \mathsf{U}(k)^2 : ax = xc\}$, which is isomorphic to the centralizer $\{a \in \mathsf{U}(k) : ya = ay\}$ of the hermitian matrix $y := xx^*$. This centralizer is isomorphic to a direct product of unitary groups and hence is connected, that is, S is simply connected. □

A rectangular matrix $z = (z_{jk}) \in E$ is called *diagonal* if $z_{jk} = 0$ holds for all $j \neq k$. Identify $\mathbb{C}^p$ in the canonical way with the linear subspace of all diagonal matrices in E. In this sense, the chamber

$$\Delta_p := \{x \in \mathbb{R}^p : x_1 \geq \ldots \geq x_p \geq 0\}$$

is identified with the corresponding set of real diagonal matrices in E. From the singular value decomposition in linear algebra it is known that for every $z \in E$ there is a transformation $g \in \mathsf{K}$ and a unique diagonal matrix $d \in \Delta_p$ with $z = g(d)$. The diagonal entries $\sigma_j(z) := d_{jj}$ for $1 \leq j \leq p$ are called the *singular values* of the matrix z. In particular, $\sigma = (\sigma_1, \ldots, \sigma_p) : E \to \Delta_p$ realizes Δ_p as the orbit space E/K. Another way of saying this is that every K-orbit in E intersects the subset $\Delta_p \subset E$ in a unique point.

The singular values $\sigma_1(z) \geq \sigma_2(z) \geq \ldots \geq \sigma_p(z) \geq 0$ of the matrix $z \in E = \mathbb{C}^{p\times q}$ play a prominent role in our discussion. Notice that $\sigma_j(z)$ also is the j^{th} biggest eigenvalue of the hermitian matrix $\binom{0\ \ z}{z^*\ 0}$ and $\sigma_j(z)^2$ is the j^{th} biggest eigenvalue of the hermitian matrix zz^* (every eigenvalue counted with its multiplicity). One application of our main results states (see section 12 for the proof):

2.3 Theorem. *In case $q > p$ (that is, every $a \in E$ is noninvertible) the following holds:*

(i) *The K-orbits $K, \widetilde{K}$ in E are CR-homeomorphic if and only if $\widetilde{K} = tK$ for some $t > 0$. In particular, the moduli space of all CR-homeomorphy classes of nonzero K-orbits in E can be identified with the space $\{x \in \mathbb{R}^p : 1 = x_1 \geq \ldots \geq x_p \geq 0\}$.*

(ii) *The boundary $\partial\mathcal{D}$ of the bounded symmetric domain $\mathcal{D}$ is the union of pairwise CR-inequivalent K-orbits. Among these is the extremal boundary $\partial_e\mathcal{D}$ of the bounded convex domain $\mathcal{D}$, which satisfies*

$$\partial_e\mathcal{D} \ = \ \{z \in E : \sigma_1(z) = \sigma_p(z) = 1\} \ = \ \{z \in \mathbb{C}^{p\times q} : zz^* = \mathbb{1}_p\}$$

and is also the unique K-orbit in $\partial\mathcal{D}$ with noncompact CR-automorphism group. Furthermore, as is well known, $\partial_e\mathcal{D}$ has dimension $p(2q - p)$ and coincides with the Shilov boundary of $\mathcal{D}$.

(iii) *For $K = \partial_e\mathcal{D}$ the group $\mathsf{Aut}_{\mathrm{CR}}(K)$ coincides with $\mathsf{Aut}(\mathcal{D}) = \mathsf{PSU}(p, q)$. For all other K-orbits $K \subset \partial\mathcal{D}$ the group $\mathsf{Aut}_{\mathrm{CR}}(K)$ coincides with K.*

Theorem 2.3 gives a rich source of pairwise nonequivalent simply-connected homogeneous CR-manifolds. The statements remain only partly true in case $p = q$ for noninvertible K-orbits K

(i.e. orbits, where some and hence every element is noninvertible). For instance, in this situation the moduli space of CR-isomorphy classes of nonzero noninvertible K-orbits in E identifies with $\{x \in \mathbb{R}^p : 1 = x_1 \geq \ldots \geq x_p = 0\}$. On the other hand, the extremal boundary $\partial_e\mathcal{D} = \mathsf{U}(p)$ in case $p = q$ is totally real as CR-manifold and is an invertible K-orbit.

For every $a \in E$ with $q \geq p$ arbitrary, the orbits $\mathsf{K}(a)$ and $\mathsf{S}(a)$ are real-analytic connected submanifolds of E that can be characterized by nice equations: For all $z, w \in E$ and every $1 \leq j \leq r$ denote by $m_j(z, w)$ the sum over all $j \times j$-diagonal-minors of the matrix $zw^* \in \mathbb{C}^{r\times r}$. Then it is clear that $m_j(z, w)$ is holomorphic in z, antiholomorphic in w and homogeneous of bidegree (j, j) in (z, w). For every $z \in E$ the number $m_j(z, z)$ is real, nonnegative and coincides with the j^{th} elementary symmetric function of $\sigma_1(z)^2, \ldots, \sigma_p(z)^2$. Moreover, compare the more general statement (8.11),

$$\mathsf{K}(a) = \{z \in E : m_j(z, z) = m_j(a, a) \ \text{ for all } \ j \leq p\}$$

and, in case $p = q$, $\det(a) = 1$,

$$\mathsf{S}(a) = \{z \in \mathsf{SL}(p, \mathbb{C}) : m_j(z, z) = m_j(a, a) \ \text{ for all } \ j < p\} \, .$$

With respect to the scalar product $(z|w) := \mathsf{tr}(zw^*) = \sum_{j,k} z_{jk}\overline{w}_{j,k}$, which (up to a positive constant) is the unique K-invariant inner product on E, for every $a \in E$ the orbit $K = \mathsf{K}(a)$ induces a unique orthogonal decomposition

$$E = E_1(a) \oplus E_{1/2}(a) \oplus E_0(a)\, ,$$

where $E_1(a) \oplus E_{1/2}(a) = T_aK + iT_aK$ is the $\mathbb{C}$-linear span in E of the tangent space T_aK and $E_{1/2}(a) = T_aK \cap iT_aK$ is the holomorphic tangent space at a to K (also denoted by $\mathbb{C}T_aK$ and H_aK respectively). In addition, there is a unique (real) orthogonal decomposition

$$E_1(a) = A(a) \oplus iA(a)$$

with $iA(a) = T_aK \cap E_1(a)$, that is, $T_aK = iA(a) \oplus E_{1/2}(a)$. It is remarkable that all these linear subspaces get a natural algebraic meaning in terms of the Jordan triple product $\{xyz\} := (xy^*z + zy^*x)/2$ on E, that is associated to the bounded symmetric domain $\mathcal{D} \subset E$. Clearly, $\{xyz\}$ is symmetric complex bilinear in $(x, z) \in E^2$ and conjugate linear in $y \in E$ (see Definition 3.1 in the abstract setting). Of importance are the commuting operators L_a and Q_a on E defined by $L_a(v) = \{aav\}$ and $Q_a(v) = \{ava\}$ for all $a, v \in E$, and derived from these, the operators $\Psi_a := 2(L_a - Q_a)$ and $\Theta_a := 4(L_a^2 - Q_a^2)$. In our special situation of rectangular matrices these operators are given by

$$\Psi_a(v) = aa^*v - 2av^*a + va^*a, \quad \Theta_a(v) = aa^*aa^*v - 2aa^*va^*a + va^*aa^*a\, .$$

The relevance of the operators is due to the fact that the tangent space T_aK is the image of Ψ_a and that the holomorphic tangent space H_aK is the image of Θ_a in E, compare Proposition 7.1 for the general situation. Even more important is the consequence that, for every fixed $v \in E$, the homogeneous (real) polynomial function $z \mapsto X^v_z := \Theta_z(v)$ of degree 4 is a vector field X^v on E with $X^v_z \in H_zK$ for all $z \in K$. This is the key for the explicit calculation of Levi form and Levi cone for the CR-manifold K, compare (9.1) and Proposition 9.12.

To make things even more transparent assume without loss of generality that $a \in \Delta_p$ is a diagonal matrix with diagonal entries $a_j := \sigma_j(a)$. For convenience put $a_j = 0$ for all

$j > p$. Then $E_1(a)$ is the linear subspace of all matrices $z \in E$ such that $z_{jk} \neq 0$ implies $a_j = a_k > 0$, $E_{1/2}(a)$ is the space of all z such that $z_{jk} \neq 0$ implies $a_j \neq a_k$, and $E_0(a)$ is the space of all z such that $z_{jk} \neq 0$ implies $a_j = a_k = 0$. For a visualization of these spaces write $\{a_1, \ldots, a_p, 0\} = \{\lambda_1, \ldots, \lambda_s, 0\}$ with $\lambda_1 > \lambda_2 > \ldots > \lambda_s > 0$ and denote by r_k the multiplicity of λ_k as singular value of a. Then the space $E_1(a)$ consists of all diagonal block matrices in E, where the upper left diagonal block is of size $r_1 \times r_1$, the second block is of size $r_2 \times r_2$ up to the last diagonal block, which is of size $r_s \times r_s$, that is, $E_1(a)$ consists of all matrices in E having zero entries outside the dark-gray area according to Figure 1,

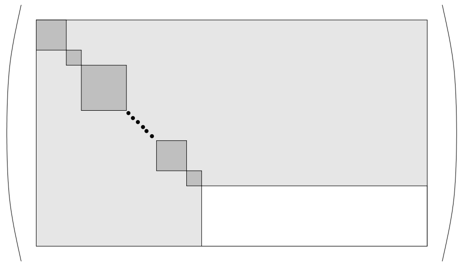


Figure 1

and hence can be identified with $\mathbb{C}^{r_1 \times r_1} \oplus \cdots \oplus \mathbb{C}^{r_s \times r_s}$. The space $E_{1/2}(a)$ consists of all matrices having zero entries outside the semi-gray area and $E_0(a)$ consists of all matrices having zero entries outside the white area. Furthermore, $A(a)$ is the $\mathbb{R}$-linear space of all matrices $z \in E_1(a)$ that are hermitian in the sense of $\overline{z}_{jk} = z_{kj}$ for all $j, k \leq p$, that is, we may identify $A(a)$ with the direct sum $A_1 \oplus \ldots \oplus A_s$, where each A_k is the space of all hermitian matrices in $\mathbb{C}^{r_k \times r_k}$. In every A_k we have the cone $\overline{\Omega}_k$ of all positive semidefinite matrices, which is known to be the closed convex cone generated by all idempotents (= projections) of rank one in A_k. In particular, the cone $\overline{\Omega}(a) := \overline{\Omega}_1 \oplus \ldots \oplus \overline{\Omega}_s$ is a closed convex cone in $A(a)$, the 'semipositive cone' of $A(a)$.

An important invariant of the CR-manifold $K = \mathsf{K}(a)$ is the Levi cone $C(a)$ at the point $a \in K$, which may be considered as a cone in $A(a)$ and for which we obtain an explicit description. In the matrix case $E = \mathbb{C}^{p \times q}$ it is the following: Denote by $X(a) \subset A(a)$ the closed convex cone spanned by all

$$(\lambda_j u_j - \lambda_{j-1} u_{j-1}) \in A(a), \qquad u_j \in A_j,\ u_{j-1} \in A_{j-1} \ \text{ idempotents of rank one}$$

and $j = 2, \ldots, s$. Then $C(a) = X(a)$ holds if a is invertible and $C(a) = X(a) - \overline{\Omega}(a)$ if a is not invertible (see section 9 for the general case). In particular, $-a \in A(a)$ is an interior point of the Levi cone $C(a)$ in case a is not invertible.

Our main results deal with various natural hulls of the orbits $K = \mathsf{K}(a)$, $S = \mathsf{S}(a)$ and with the extension problem for CR-functions on these (compare sections 11 and 12 for the general case). It is not difficult to see that the (linear) convex hull of K is given by

$$\left\{ z \in E : \|z\|_j \leq \|a\|_j \ \text{ for } \ j = 1, \ldots, p \right\},$$

where $\|z\|_j = \sigma_1(z) + \sigma_2(z) + \ldots + \sigma_j(z)$ is the sum of the j largest singular values of the matrix $z \in E$ and actually defines a norm $\|\ \|_j$ on E. As a multiplicative analogue denote for $j = 1, \ldots, p$ by $\mu_j(z) := \sigma_1(z)\sigma_2(z) \cdots \sigma_j(z)$ the product of the j largest singular values of the matrix z. Then, if we define for convenience $\det(z) := 0$ for every nonsquare matrix z, we have (compare the more general case in 11.7 and 12.2):

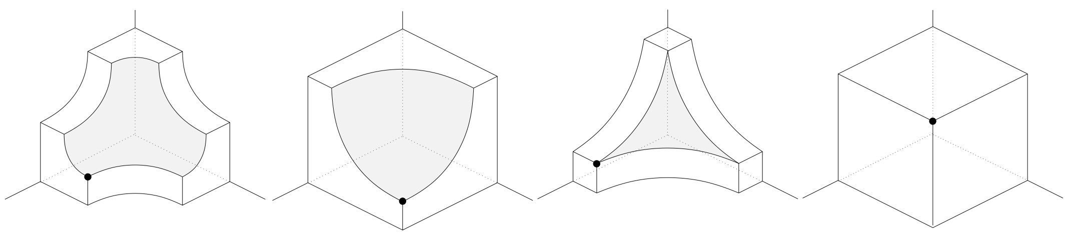
2.4 Theorem. *For every $a \in E$ the polynomial and the rational convex hull of $K = \mathsf{K}(a)$ are*

$$\mathcal{Z}(a) := \{z \in E : \mu_j(z) \le \mu_j(a) \ \text{ for } \ j = 1, \dots, p\} \ \text{ and}$$

$$\mathcal{Y}(a) := \{z \in \mathcal{Z}(a) : |\det(z)| = |\det(a)|\} \ \text{ respectively}\,.$$

For the orbit $S = \mathsf{S}(a)$, both hulls are $\mathcal{X}(a) := \{z \in \mathcal{Z}(a) : \det(z) = \det(a)\}$.

In Figure 2 a visualization of the hulls $\mathcal{Z}(a)$ and $\mathcal{X}(a)$ is given for the special case of 3×3-matrices and a invertible. It is shown the intersection of the hulls with the space of all positive semidefinite real diagonal matrices in $\mathbb{C}^{3\times3}$, identified with the positive octant in $\mathbb{R}^3$. If a is such a diagonal matrix with entries $1 = \lambda_1 \ge \lambda_2 \ge \lambda_3 > 0$, the polynomial convex hull $\mathcal{Z}(a)$ has a 3-dimensional body as section, whereas the rational convex hull $\mathcal{X}(a)$ has the shaded surface as section. Note that the orbit $\mathsf{K}(a)$ is of dimension $15, 13, 13, 9$ and intersects the real octant $6, 3, 3, 1$ times according to the different cases (1),(2),(3),(4) shown in Figure 2. In case (4) the interior of $\mathcal{Z}(a)$ is the bounded symmetric domain $\mathcal{D} \subset \mathbb{C}^{3\times3}$ we started with. In this case $\mathcal{X}(a) = \mathsf{S}(a)$ and the orbits $\mathsf{S}(a) = \mathsf{SU}(3)$, $\mathsf{K}(a) = \mathsf{U}(3)$ are totally real in E. The marked point on every picture corresponds to a.



(1) $\lambda_1{>}\lambda_2{>}\lambda_3$ (2) $\lambda_1{=}\lambda_2{>}\lambda_3$ (3) $\lambda_1{>}\lambda_2{=}\lambda_3$ (4) $\lambda_1{=}\lambda_2{=}\lambda_3$

Figure 2

The pictures in Figure 2 can also be used for $3\times q$-matrices with $q > 3$ (or more generally for all factors of rank 3). But then $\mathsf{K}(a) = \mathsf{S}(a)$ and $\mathcal{Z}(a) = \mathcal{X}(a)$ hold, and the intersection of the Levi cone of $\mathsf{K}(a)$ at a with the subspace $\mathbb{R}^3$ of all real diagonal matrices coincides with the tangent cone to the shown body (see section 9). In case (1) the Levi cone is 3-dimensional, is contained in $\mathbb{R}^3$ and is a simplex cone, i.e. spanned as cone by 3 linearly independent vectors. In cases (2) and (3) the Levi cone is 5-dimensional whereas in case (4) it is 9-dimensional. Furthermore, its intersection with $\mathbb{R}^3$ is generated by 3 linearly independent vectors in cases (1), (2) and (4) and by 4 vectors in case (3). The Levi cone itself is obtained by applying to its intersection with $\mathbb{R}^3$ the isotropy subgroup K_a of K at a.

Now let again $E = \mathbb{C}^{p\times q}$ with $q \ge p \ge 1$ be arbitrary and denote by $k := \operatorname{rank}(a)$ the rank of the matrix a. It is well known that the complex-analytic cone

$$Z := \{z \in E : \operatorname{rank}(z) \le k\}$$

in E has only normal singularities (more generally, see Proposition 8.3). The nonsingular part of Z (the subset of rank-k-matrices in case $k < p$) contains the orbit $K = \mathsf{K}(a)$ as generic CR-submanifold, and the interior of $\mathcal{Z}(a)$ in Z is a bounded balanced domain. Our main result now is (see 12.1, 12.11 and 12.4 for more general statements):

2.5 Theorem. *Every continuous CR-function on* $\mathsf{S}(a)$ *has a unique continuous extension to* $\mathcal{Z}(a)$ *that is holomorphic in its interior with respect to* Z *if the matrix* $a \in E = \mathbb{C}^{p\times q}$ *is not invertible, and has a unique continuous extension to* $\mathcal{X}(a)$ *that is holomorphic in its interior with respect to the complex submanifold* $\{z \in E : \det(z) = \det(a)\}$ *of* E *if the matrix* a *is invertible. The sets* $\mathcal{Z}(a)$ *and* $\mathcal{X}(a)$ *are maximal with respect to these extension properties. If* $\mathsf{K}(a) \neq \mathsf{S}(a)$ *and hence* a *is invertible, every continuous CR-function on* $\mathsf{K}(a)$ *has a unique extension to a continuous function on* $\mathcal{Y}(a)$ *that is CR in its interior in the CR-submanifold* $\{z \in E : |\det(z)| = |\det(a)|\}$ *of* E*.*

In fact we show that, if a is not invertible, $\mathcal{Z}(a)$ identifies via point evaluation with the spectrum of the complex Banach algebra of all continuous CR-functions on $\mathsf{K}(a)$.

3. Jordan-theoretic description

The euclidian unit ball $\mathcal{D} = \{z \in \mathbb{C}^n : 1 - (z|z) > 0\}$ and its boundary, the unit sphere $S = \{z \in \mathbb{C}^n : (z|z) = 1\}$, are well studied objects with respect to their holomorphic and CR-structure. One reason seems to be that many things can be expressed and easily computed in terms of the inner product $(z|w)$ on $\mathbb{C}^n$.

Up to some extent, the same is true for arbitrary bounded symmetric domains if we allow more generally 'operator-valued inner products', more precisely (compare [29] for more details):

3.1 Definition. A finite dimensional complex vector space E together with a sesqui-linear map $L : E^2 \to \mathcal{L}(E)$ is called a *positive hermitian Jordan triple system* (PJT for short) if for all $x, y, z, w \in E$ and $t \in \mathbb{C}$ the following hold:

(i) $\{xyz\} := L(x, y)(z)$ is symmetric bilinear in the outer variables x, z and conjugate linear in the inner variable y.

(ii) $[L(x, y), L(z, w)] = L(\{xyz\}, w) - L(z, \{wxy\})$, where $[\,,\,]$ denotes the commutator of operators.

(iii) $\{xxx\} = tx$ implies $t = |t| > 0$ or $x = 0$.

Condition (ii) is called the *Jordan triple identity*. It implies for instance, that the linear span of all operators $L(x, y)$ is a Lie subalgebra of $\mathcal{L}(E)$. The trace form

$$(x|y) := \mathsf{tr}\big(L(x, y)\big) \tag{3.2}$$

defines a positive-definite (scalar) inner product on E which is invariant under the automorphism group

$$\mathsf{Aut}(E) := \{g \in \mathsf{GL}(E) : g\{xyz\} = \big\{(gx)(gy)(gz)\big\} \text{ for all } x, y, z \in E\big\} \tag{3.3}$$

as a consequence of $L(gx, gy) = gL(x, y)g^{-1}$ for all $g \in \mathsf{Aut}(E)$. In particular, $L(x, y)^* = L(y, x)$ for the corresponding adjoint of $L(x, y)$ – thus justifying the name *hermitian* Jordan triple system. The connection to bounded symmetric domains comes from the fact that the set

$$\mathcal{D} := \{z \in E : \mathsf{id}_E - L(z, z) \;>\; 0\} \tag{3.4}$$

is always a bounded symmetric domain in E, where '> 0' means 'positive-definite' for the hermitian operator $\mathsf{id}_E - L(z, z)$ on E, and also $\mathsf{GL}(\mathcal{D}) = \mathsf{Aut}(E)$. Conversely, every bounded symmetric domain (realized as circular convex domain) occurs this way. For the classical types

I – **IV** (see the end of section 1) the triple product $\{xyz\}$ is given by $(xy^*z + zy^*x)/2$ in case **I** – **III** and by $\big((x|y)z - \langle x|z\rangle\overline{y} + (z|y)x\big)/2$ in case of **IV**, where $z \mapsto \overline{z}$ is the natural conjugation on $\mathbb{C}^n$ and $\langle x|z\rangle$ is the complex product as in section 1. It is known [16] that every $\mathbf{IV}_n$ can be realized as a subtriple $E \subset \mathbb{C}^{p\times p}$ for $p = 2^{n-1}$ in such a way that $z^* \in E$ and $z^2 \in \mathbb{C}\mathbb{1}_p$ for all $z \in E$. On the other hand, every linear subspace $E \subset \mathbb{C}^{p\times p}$ of dimension n satisfying these two conditions is a subtriple isomorphic to $\mathbf{IV}_n$.

Besides the $\mathbb{C}$-linear operator $L(a,b)$ for every $a,b \in E$ we have the conjugate linear operator $Q(a,b)$ on E defined by $z \mapsto \{azb\}$. For every $a \in E$ put

$$L_a := L(a,a) \quad \text{and} \quad Q_a := Q(a,a) \tag{3.5}$$

in the following. The element $a \in E$ is called *invertible* if the operator Q_a is invertible. E is called of *tube type* if it contains invertible elements. This is known to be equivalent to $\mathcal{D}$ being a bounded symmetric domain of tube type. Choose an $\mathsf{Aut}(E)$-invariant inner product $(x|y)$ on E (e.g. the trace form (3.2), a canonical choice will be made later, compare (5.8)). Then, for all x, y, we have to distinguish between (triple) orthogonality (i.e. $L(x,y) = 0$), (complex) orthogonality (i.e. $(x|y) = 0$, this does not depend on the choice of the K-invariant inner product) and (real) orthogonality (i.e. $\mathrm{Re}(x|y) = 0$). Triple orthogonality implies complex orthogonality. Every L_a is self-adjoint with respect to the chosen K-invariant inner product, and $\exp(itL_a) \in \mathsf{K}$ holds for all $t \in \mathbb{R}$.

For every $a \in E$ we can define the complex bilinear *product* $x \circ y := \{xay\}$ (depending on a) on E, which makes E to a commutative (in general not associative) complex algebra that we denote by $E^{(a)}$. Actually, $E^{(a)}$ is a Jordan algebra (see the next section for more details on this type of algebras). Notice also that Lie algebras are in general not associative (but anti-commutative).

4. Some basic facts on Jordan algebras

In this section we recall same basic material on real and complex Jordan algebras that we will use later, see [9], [13] and [31] for further details. By definition, a real vector space A together with a bilinear map

$$A \times A \to A\,, \quad (x,y) \mapsto x \circ y$$

is called a *real Jordan algebra* if for all $x, y \in A$ the following two properties hold

$$x \circ y = y \circ x \quad \text{and} \quad x \circ (x^2 \circ y) = x^2 \circ (x \circ y)\,, \tag{4.1}$$

where $x^2 := x \circ x$. For instance, every associative real algebra V with product $(x,y) \mapsto xy$ becomes a Jordan algebra V^+ with respect to the Jordan product $x \circ y := \frac{1}{2}(xy + yx)$. In both algebras squares are obviously the same.

Every *idempotent* $c \in A$ (that is $c^2 = c$) induces a *Peirce decomposition*

$$A = A_1(c) \oplus A_{1/2}(c) \oplus A_0(c)\,, \tag{4.2}$$

where $A_k(c)$ is the k-eigenspace of $L(c)$, where for every $a \in A$ the multiplication operator $L(a)$ on A is defined by $x \mapsto a \circ x$. The linear subspace $A_1(c)$ is a Jordan subalgebra of A with unit c. The sum $c_1 + c_2$ of orthogonal idempotents in A is again an idempotent, where $x, y \in A$ are called *orthogonal*, if $x \circ y = 0$ holds. The idempotent $c \neq 0$ is called *minimal* if it is not the sum of two orthogonal nonzero idempotents.

We will assume for the rest of the section that the real Jordan algebra $A \neq 0$ has finite dimension and is *formally real*, that is, $x^2 + y^2 = 0$ always implies $x = y = 0$. This is equivalent to A being *euclidian,* i.e. the trace form $(x, y) \mapsto \mathsf{tr}(L(x \circ y))$ being positive definite. As formally real Jordan algebra, A has always a unit e, and for every $x \in A$ the subalgebra $\mathbb{R}[x]$ of A generated by e and x is associative (and commutative by the definition of a Jordan algebra). In particular, all powers x^n, $n \in \mathbb{N}$, are well defined. The element $x \in A$ is called *invertible* if x has an inverse in the associative subalgebra $\mathbb{R}[x] \subset A$ and this inverse then is denoted by x^{-1}. The set A^{-1} of all invertible elements is open and dense in A, furthermore $x \mapsto x^{-1}$ is a rational diffeomorphism of A^{-1} onto itself.

In the formally real Jordan algebra A there exist always nonzero idempotents c, and c is minimal if and only if $A_1(c) = \mathbb{R}c$ holds. Every $x \in A$ has a (not necessarily unique) representation

$$x = \alpha_1 c_1 + \ldots + \alpha_r c_r, \quad c_1 + \ldots + c_r = e \tag{4.3}$$

with pairwise orthogonal minimal idempotents $c_1, \ldots, c_r$ and real coefficients α_j (called the *eigenvalues of* x). The number r in this representation does not depend on the choice of minimal idempotents and also not on the element x, it is called the *rank* of A. The group $\mathsf{Aut}(A)$ of all algebra automorphisms of A is a compact Lie group, and there is a unique $\mathsf{Aut}(A)$-invariant (real) inner product $(x|y)$ on A such that $(c|c) = 1$ for every minimal idempotent $c \in A$. This inner product will be fixed on every formally real Jordan algebra in the following. For x in (4.3) then $(x|x) = \alpha_1^2 + \ldots + \alpha_r^2$ holds.

Although for x the representation (4.3) is not unique in general, for every real-valued function f on $\mathbb{R}$ the element

$$\boldsymbol{f}(x) := f(\alpha_1)c_1 + \ldots + f(\alpha_r)c_r \in A$$

does not depend on (4.3). In particular, for every $x \in A$ and $n \in \mathbb{N}$ the powers $x^n \in A$ correspond to the scalar function $f(t) = t^n$ on $\mathbb{R}$, and x^+ (called the *nonnegative part of* x) is obtained from the function $t \mapsto t^+ := \max(t, 0)$ on $\mathbb{R}$.

In a real vector space V of finite dimension a nonempty subset $C \subset V$ is called a *cone* if $tC \subset C$ holds for every real $t > 0$. With C' we denote the *dual cone* of C, that is the set of all linear forms τ on V with $\tau(C) \geq 0$. It is well known that the bidual cone C'' is the closed convex hull of C in V. An open convex cone C is called *regular* if the interior of C' is not empty, and then this interior is called the *open dual* of the regular cone C. In case that there is given a (positive definite) inner product on V, the dual vector space of V is identified to V in a natural way and then C' can be considered as a cone in V.

In every formally real Jordan algebra A there are two important cones:

$$\Omega = \{x^2 : x \in A^{-1}\} \quad \text{and} \quad \overline{\Omega} = \{x^2 : x \in A\}\,. \tag{4.4}$$

Both cones are convex and contain e in the interior. The first one is open and $\overline{\Omega}$ is the closure of Ω in E. Furthermore

$$A = \overline{\Omega} \ominus \overline{\Omega}\,, \tag{4.5}$$

that is, every $x \in A$ has a unique representation $x = x^+ - x^-$ with orthogonal elements $x^+, x^- \in \overline{\Omega}$. The element x is in $\overline{\Omega}$ if and only if in the representation (4.3) all coefficients

α_j are nonnegative. Ω is also the connected component containing e of the open set A^{-1}. Furthermore, $\exp\colon A \to \Omega$ is a bianalytic diffeomorphism. We call $\overline{\Omega}$ (respectively Ω) the *semipositive* (respectively the *positive*) cone of the formally real Jordan algebra A. They are self-dual in the sense

$$(4.6)\qquad \begin{aligned}\overline{\Omega} =&\{x \in A : (x|y) \geq 0 \text{ for all } y \in \overline{\Omega}\}\\ \Omega =&\{x \in A : (x|y) > 0 \text{ for all } y \in \Omega\}\ .\end{aligned}$$

For all elements $x, y \in A$ we write $x \leq y$ or $y \geq x$ if $y - x \in \overline{\Omega}$ holds, and we write $x < y$ or $y > x$ if $y - x \in \Omega$.

There exists a unique polynomial function $N : A \to \mathbb{R}$ with $N(x) = \alpha_1\alpha_2 \cdots \alpha_r$ for every $x \in A$ given in the form (4.3). N is homogeneous of degree $r = \operatorname{rank}(A)$ and generalizes the determinant function on matrix algebras. Its characteristic property is: $N(x) \neq 0 \Leftrightarrow x \in A^{-1}$ and $N(e) = 1$. The function N is called the *generic norm* of A. In addition, there is a unique A-valued polynomial function $x \mapsto x^{\#}$ on A with $x^{-1} = N(x)^{-1}x^{\#}$ for all $x \in A^{-1}$. Clearly, $x^{\#}$ is homogeneous of degree $r-1$ in x and is called the *adjoint* of x.

We present briefly the classification of all formally real Jordan algebras. From $2x \circ y = (x+y)^2 - x^2 - y^2$ it is clear that the Jordan product is uniquely determined by the square mapping. For every integer $n \geq 1$ let $\mathbb{K}_n$ be the vector space $\mathbb{R}^n$ with the following additional structure: $(x|y) = \sum x_i y_i$ is the usual scalar product and $\overline{x} := (x_1, -x_2, \ldots, -x_n)$ for all $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$. The field $\mathbb{R}$ is identified with $\{x \in \mathbb{K}_n : \overline{x} = x\}$ via $t \mapsto te$, where $e := (1, 0, \ldots, 0)$. In addition, define the product of x and $\overline{x}$ formally as $x\overline{x} := (x|x) \in \mathbb{R} \subset \mathbb{K}_n$. For every integer $r \geq 1$ denote by $\mathcal{H}_r(\mathbb{K}_n) \subset (\mathbb{K}_n)^{r\times r}$ the linear subspace of all hermitian $r\times r$-matrices (x^{ij}) over $\mathbb{K}_n$, that is, $x^{ij} \in \mathbb{K}_n$ and $\overline{x}^{ij} = x^{ji}$ for all $1 \leq i, j \leq r$. Obviously, $\mathcal{H}_r(\mathbb{K}_n)$ has real dimension $r + n\binom{r}{2}$.

Our conventions so far suffice to define all squares x^2 for $x \in \mathcal{H}_2(\mathbb{K}_n)$ (just formally as matrix square). For $r > 2$ we need an additional structure on some $\mathbb{K}_n$: Identify $\mathbb{K}_2$ with the field $\mathbb{C}$, $\mathbb{K}_4$ with the (skew) field $\mathbb{H}$ of quaternions and $\mathbb{K}_8$ with the real division algebra $\mathbb{O}$ of octonions in such a way that $x \mapsto \overline{x}$ is the standard conjugation of these structures. With these identifications also squares are defined in $\mathcal{H}_r(\mathbb{K}_n)$ for all r and $n = 1, 2, 4, 8$ (again in terms of the usual matrix product). Now the complete classification reads as follows:

Every formally real Jordan algebra is a direct sum of simple algebras. The simple formally real Jordan algebras are (without repetition) precisely the following, where r denotes the rank. The Jordan product in any case is derived from the squaring as defined above:
$r = 1 : \mathbb{R}$
$r = 2 : \mathcal{H}_2(\mathbb{K}_n),\ n \geq 1$
$r = 3 : \mathcal{H}_3(\mathbb{R}),\ \mathcal{H}_3(\mathbb{C}),\ \mathcal{H}_3(\mathbb{H}),\ \mathcal{H}_3(\mathbb{O})$
$r > 3 : \mathcal{H}_r(\mathbb{R}),\ \mathcal{H}_r(\mathbb{C}),\ \mathcal{H}_r(\mathbb{H})$.

For $A = \mathcal{H}_r(\mathbb{R})$ or $A = \mathcal{H}_r(\mathbb{C})$ the cone $\overline{\Omega}$ is the set of all positive semidefinite matrices in the usual sense and its interior Ω is the cone of all positive definite matrices. The algebra $A = \mathcal{H}_3(\mathbb{O})$ has dimension 27 and plays a special role. In contrast to the others it does not occur as Jordan subalgebra of V^+ for any associative real algebra V. This algebra is called *exceptional* (as well as every Jordan algebra having $\mathcal{H}_3(\mathbb{O})$ as direct summand).

Now consider an arbitrary formally real Jordan algebra A with unit e. Then by (4.3) for every $x \in A$ there exists an idempotent $c \in A$ with $(x^+|e) = (x|c)$. We will need later the following extremal characterization of $(x^+|e)$ (compare Lemma 9.6).

4.7 Lemma. *Suppose that A is not exceptional. Then*

$$(x^+|e) = \sup_{c^2=c}(x|c) \quad \text{for all} \quad x \in A\,.$$

Proof. Since A is not exceptional there exists an integer r and a realization of A as Jordan subalgebra of $\mathcal{H}_r(\mathbb{C})$ in such a way that $e \in A$ is also the identity in $\mathcal{H}_r(\mathbb{C})$. We may therefore assume without loss of generality that $A = \mathcal{H}_r(\mathbb{C})$ holds. Then $(x|y) = \mathsf{tr}(xy)$ holds for all $x, y \in \mathcal{H}_r(\mathbb{C})$. The claim now is an easy consequence of Theorem 1 in [37]. □

The complex analogs to formally real Jordan algebras are certain Jordan *-algebras. Let us call a complex Jordan algebra U (i.e. the Jordan product is complex bilinear) a Jordan *-algebra if there is fixed a conjugate linear algebra automorphism $z \mapsto z^*$ of period 2 on U. Then the self-adjoint part $A := \{z \in U : z^* = z\}$ is a real Jordan algebra, and the following conditions are equivalent in case U has finite dimension: (1) A is formally real, (2) $z = 0$ for every $z \in U$ with $z \circ z^* = 0$, (3) the trace form $\mathsf{tr}(L(x \circ y^*))$ is positive definite. It is clear that the formally real Jordan algebras are in 1-1-correspondence with Jordan *-algebras that are *positive definite* in the sense of (3). On every such U there also exists a generic norm (a complex homogeneous polynomial $N : U \to \mathbb{C}$ of minimal degree with $N(e) = 1$ and $N(x) \neq 0$ if and only if x is invertible in U). Finally, every positive definite Jordan *-algebra U becomes a PJT by defining the triple left multiplication operators by $L(x, y) := [L(x), L(y^*)] + L(x \circ y^*)$.

5. Joint Peirce decompositions

In the following E is a PJT of dimension n. Then, as already mentioned at the end of section 3, every $a \in E$ makes E into a complex Jordan algebra $E^{(a)}$ with respect to the product $x \circ y = \{xay\}$. In particular, the triple operator $L_a = L(a, a)$ (see (3.5)) coincides with the multiplication operator $L(a)$ in the Jordan algebra $E^{(a)}$. It is clear that a is an idempotent in $E^{(a)}$ if and only if a is a *tripotent* in E, that is, if $\{aaa\} = a$ holds.

As a consequence of (4.2) we have for every tripotent $e \in E$ the *Peirce decomposition*

$$E = E_1(e) \oplus E_{1/2}(e) \oplus E_0(e)\,, \tag{5.1}$$

where $E_k(e)$ is the k-eigenspace of L_e. The operator Q_e vanishes on $E_{1/2}(e) \oplus E_0(e)$ and splits $E_1(e)$ into a direct sum $A(e) \oplus iA(e)$ of +1- and −1-eigenspaces. Actually, $E_1(e)$ is a Jordan subalgebra of $E^{(e)}$ with unit e and $x \mapsto x^* := \{exe\}$ is an algebra involution making $E_1(e)$ a positive definite Jordan *-algebra with self-adjoint part $A(e)$ which is a formally real Jordan algebra with semipositive cone $\overline{\Omega}(e) = \{x^2 : x \in A(e)\}$. The sesqui-linear map

$$F : E_{1/2}(e) \times E_{1/2}(e) \to E_1(e), \qquad F(x, y) := \{xye\} \tag{5.2}$$

satisfies $F(x, x) \in \overline{\Omega}(e)$ for all $x \in E_{1/2}(e)$ and $F(x, x) = 0$ holds if and only if $x = 0$ (compare [29] p. 10.5).

For every pair e, c of orthogonal tripotents in E and every $t \in \mathbb{C}$ with $|t| = 1$ also te and $e + c$ are tripotents. The tripotent $e \neq 0$ is called *minimal* if it cannot be written as a sum $e = e_1 + e_2$ of nonzero orthogonal tripotents, or equivalently, if $A(e) = \mathbb{R}e$ holds. Clearly, minimality for idempotents in $A(e)$ is the same as for tripotents.

Denote by $\mathcal{E}$ the set of all sequences $\boldsymbol{e} = (e_1, \ldots, e_s)$ of nonzero, mutually (triple) orthogonal tripotents $e_j \in E$ and call $l(\boldsymbol{e}) := s$ the *length* of $\boldsymbol{e}$. Then necessarily $l(\boldsymbol{e}) \leq n = \dim E$ and

$r := \max\{l(\boldsymbol{e}) : \boldsymbol{e} \in \mathcal{E}\}$ is called the *rank* of E. Every $\boldsymbol{e} \in \mathcal{E}$ with the maximal possible length $l(\boldsymbol{e}) = r$ is called a *frame* in E. Every tripotent in a frame is minimal.

Every element $a \in E$ has a representation

$$a = \lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_s e_s \tag{5.3}$$

for a suitable sequence $\boldsymbol{e} = (e_1, \ldots, e_s) \in \mathcal{E}$ and real coefficients λ_j. For convenience we put

$$\lambda_0 := 0 \quad \text{and} \quad \lambda_{-j} := -\lambda_j \;\; \text{for} \;\; 1 \le j \le s\,. \tag{5.4}$$

There always exist two extremal choices for the sequence $\boldsymbol{e}$ in (5.3) and the given element $a \in E$.
1. *The maximal length choice:* Here $\boldsymbol{e}$ is a frame, i.e. $s = r$, and we assume in addition that

$$\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_r \ge 0 \tag{5.5}$$

holds. Then the coefficient λ_j in (5.5) is uniquely determined by $a \in E$ and is called the j^{th} *singular value* of a, denoted by $\sigma_j(a)$. In case E is of type $\mathbf{I}_{p,q}$ considered in section 2 these are the usual singular values of matrices which justifies the terminology. For convenience we put $\sigma_j(a) := 0$ for all $j > r$. The integer $\mathrm{rank}(a) := \min\{k \ge 0 : \sigma_{k+1}(a) = 0\}$ is called the *rank* of a (again, in the matrix case one has the usual rank).
2. *The minimal length choice:* Here $\boldsymbol{e}$ is not necessarily a frame, but we require

$$\lambda_1 > \lambda_2 > \ldots > \lambda_s > 0\,. \tag{5.6}$$

Under these assumptions not only the coefficients λ_j but also the tripotents e_j are uniquely determined by the element a. The integer s is called the *reduced rank* of a.

Notice that $\mathrm{rank}(a)$ counts the nonzero singular values of $a \in E$ with multiplicities, whereas the reduced rank ignores multiplicities. Let us call the element $a \in E$ *reduced* if both ranks coincide for a, that is, if and only if all nonzero singular values of a are pairwise different. In case E is a subtriple of a bigger PJT $\widetilde{E}$, the rank of a as element of $\widetilde{E}$ in general is bigger than the one with respect to E. On the other hand, the reduced rank remains the same in both cases. Actually, if we denote by $[a]$ the smallest complex subtriple of E containing a, then the reduced rank of a coincides with the complex dimension of $[a]$.

The functions $\sigma_j : E \to \mathbb{R}$ are K-invariant, continuous and piecewise smooth. Hence also $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_r) : E \to \mathbb{R}^r$ is K-invariant. For every $z \in E$, every $1 \le p < \infty$ and every $k = 1, 2, \ldots, \infty$ put

$$\|z\|_p := \Big(\sum_{j=1}^{r} \sigma_j(z)^p\Big)^{1/p}, \quad \|z\|_\infty := \sigma_1(z) \quad \text{and} \quad |\!|\!|z|\!|\!|_k := \sum_{j=1}^{k} \sigma_j(z)\,. \tag{5.7}$$

As a consequence of [24], Satz 5.2, every $\|\;\|_p$, $1 \le p \le \infty$, and every $|\!|\!|\;|\!|\!|_k$ is a K-invariant norm on E. Clearly, $\|\;\|_\infty = |\!|\!|\;|\!|\!|_1$ and $\|\;\|_1 = |\!|\!|\;|\!|\!|_\infty$.

It should be noted that the bounded symmetric domain $\mathcal{D} \subset E$ given by (3.4) is the open unit ball with respect to the norm $\|\;\|_\infty$. Furthermore, $\|\;\|_2$ is the unique K-invariant Hilbert norm on E such that all minimal tripotents have norm 1. In particular, there is a unique $\mathsf{Aut}(E)$-invariant inner product $(x|y)$ on E with

$$(z|z) = \|z\|_2^2 \quad \text{for all} \quad z \in E\,. \tag{5.8}$$

For the rest of the paper we will always endow E with this inner product. For instance, if E is one of the types $\mathbf{I}_{p,q}$ or $\mathbf{III}_p$, then $\| \|_2$ is the Hilbert-Schmidt norm on E given by the inner product $(x|y) = \mathsf{tr}(xy^*)$. In case E is of type $\mathbf{II}_p$, $(x|y) = \frac{1}{2}\mathsf{tr}(xy^*)$ holds, and $(x|y)$ is the standard inner product on $\mathbb{C}^n$ for the type $\mathbf{IV}_n$.

For every odd function $f : \mathbb{R} \to \mathbb{C}$ and $a \in E$ define

$$\boldsymbol{f}(a) := f(\lambda_1)e_1 + f(\lambda_2)e_2 + \ldots + f(\lambda_s)e_s\,, \tag{5.9}$$

which does not depend on the choice of the representation (5.3) for a. For instance, for the cube function $f(t) = t^3$ on $\mathbb{R}$ we get $\boldsymbol{f}(a) = \{aaa\} =: a^3$. For the signum function on $\mathbb{R}$ defined by $\mathrm{sign}(t) = t/|t|$ for $t \neq 0$ and $\mathrm{sign}(0) = 0$ we get a tripotent $e = \mathrm{sign}(a)$ from a. Finally, the function $t \mapsto t^\dagger$ on $\mathbb{R}$ defined by $t^\dagger = 1/t$ for $t \neq 0$ and $0^\dagger = 0$ gives the *pseudo inverse* $a^\dagger$ of a.

Fix an arbitrary sequence $\boldsymbol{e} = (e_1, e_2, \ldots, e_s) \in \mathcal{E}$ and define for all integers $0 \le j, k \le s$ the linear subspaces

$$E_{j,k} = E_{j,k}(\boldsymbol{e}) = \big\{x \in E : \{e_l e_l x\} = \tfrac{1}{2}(\delta_{jl} + \delta_{lk})x \;\text{ for all }\; 1 \le l \le s\big\} \tag{5.10}$$

which are mutually (complex) orthogonal. Then

$$E = \bigoplus_{0 \le j \le k \le s} E_{j,k} \tag{5.11}$$

holds, and (5.11) is called the *Peirce decomposition* with respect to $\boldsymbol{e}$. The Peirce spaces multiply according to the rules

$$\{E_{j,m} E_{m,n} E_{n,k}\} \subset E_{j,k} \tag{5.12}$$

and all products vanish that cannot be brought into this form (i.e. after writing $E_{s,l}$ as $E_{l,s}$ if necessary).

The Peirce decomposition (5.11) gives the spectral resolution of the operator L_a for $a \in E$ represented in the form (5.3), more precisely, denote by $P_{j,k} \in \mathcal{L}(E)$ the orthogonal projection with range $E_{j,k}$ for each j,k as above. Then by (5.10)

$$L_a = \sum_{0 \le j \le k \le s} \frac{1}{2}(\lambda_j^2 + \lambda_k^2)\, P_{j,k}\,. \tag{5.13}$$

The decomposition must be refined to get a spectral resolution also for the conjugate linear operator Q_a (which commutes with L_a). For this introduce refined (real) Peirce spaces $E^{j,k} \subset E$ in the following way: For all integers j,k with $|j|, |k| \le s$ and $e := e_1 + \ldots + e_s$ put

$$E^{j,k} := \big\{x \in E_{|j|,|k|} : \{exe\} = \mathrm{sign}(jk)\cdot x\big\}\;.$$

Then every $E^{j,k}$ is an $\mathbb{R}$-linear subspace of E with $E^{-j,k} = iE^{j,k} = E^{j,-k}$, and

$$E = \bigoplus_{|j| \le k \le s} E^{j,k} \tag{5.14}$$

is a direct sum of pairwise (real) orthogonal summands, called the *refined Peirce decomposition* with respect to $\boldsymbol{e}$. Notice that $E_{j,k} = E^{j,k} \oplus E^{-j,k}$ and $E^{0,k} = E_{0,k}$ holds for all $j, k > 0$. If we denote by $P^{j,k} \in \mathcal{L}_{\mathbb{R}}(E)$ the (real) orthogonal projection with range $E^{j,k}$, we get in addition to (5.13) the spectral resolutions

$$L_a = \sum_{|j| \le k \le s} \frac{1}{2}(\lambda_j^2 + \lambda_k^2)\, P^{j,k}, \qquad Q_a = \sum_{|j| \le k \le s} \lambda_j \lambda_k\, P^{j,k}\,, \tag{5.15}$$

where our convention (5.4) is in force.

A PJT E is called *reducible* if there exists a decomposition $E = E_1 \oplus E_2$ into positive dimensional linear subspaces satisfying $L(E_1, E_2) = 0$, otherwise *irreducible*. E is irreducible if and only if the corresponding bounded symmetric domain (3.4) is irreducible, i.e. is not biholomorphically equivalent to a direct product of bounded symmetric domains of lower dimensions. If $\boldsymbol{e} = (e_1, \ldots, e_r)$ is a frame in E, then E is irreducible if and only if $E_{j,k} \neq 0$ holds for all $j, k > 0$. In this case the integers $\alpha := \dim E_{j,k}$ and $\beta := \dim E_{0,k}$ do not depend on the indices $j > k > 0$ (in case $r = 1$ we put $\alpha = 2$ for convenience) whereas $\dim E_{j,j} = 1$. They even do not depend on the chosen frame $\boldsymbol{e}$ and hence are invariants of the Jordan triple structure on E. Clearly $n = (1+\beta)r + \binom{r}{2}\alpha$ is the dimension of E. It is known that the invariants r, α, β determine E up to isomorphism. Furthermore, E is of tube type (i.e. containing invertible elements) if and only if $\beta = 0$. For the 6 different types we have:

$\mathbf{I}_{p,q}$: $\alpha = 2$, $\beta = q - p$	$\mathbf{II}_p$: $\alpha = 4$, $\beta = 0$ if p is even and $\beta = 2$ otherwise
$\mathbf{III}_p$: $\alpha = 1$, $\beta = 0$	$\mathbf{IV}_n$: $\alpha = n - 2$, $\beta = 0$
$\mathbf{V}$: $\alpha = 6$, $\beta = 4$	$\mathbf{VI}$: $\alpha = 8$, $\beta = 0$.

Instead of *'irreducible PJT'* we shall simply say *'factor'* in the following. The factor E is called *classical* if it is one of types $\mathbf{I} - \mathbf{IV}$ and is called *exceptional* if it is one of the types **V, VI**. All factors of type **IV** are also called *spin factors*.

6. Yet another Peirce decomposition

We use the Peirce decompositions (5.11) and (5.14) to generalize the decomposition (5.1) from tripotents to arbitrary elements of E. For this let the fixed element $a \in E$ be given in the form (5.3) satisfying (5.6) and put for $E_{j,k} = E_{j,k}(e_1, \ldots, e_s)$

$$\begin{gathered} E_1(a) := \bigoplus_{1 \le j \le s} E_{j,j}, \quad E_{1/2}(a) := \bigoplus_{0 \le j < k \le s} E_{j,k}, \quad E_0(a) := E_{0,0} \quad \text{and} \\ A(a) := \bigoplus_{1 \le j \le s} E^{j,j}\,. \end{gathered} \tag{6.1}$$

Then

$$E = E_1(a) \oplus E_{1/2}(a) \oplus E_0(a) \quad \text{and} \quad E_1(a) = A(a) \oplus iA(a)\,. \tag{6.2}$$

$A(a)$ is the 1-eigenspace of the conjugate linear operator $Q(a, a^\dagger)$ and $E_1(a)$ is the 1-eigenspace of the complex linear operator $Q(a, a^\dagger)^2$, where $a^\dagger = \lambda^{-1}e_1 + \ldots + \lambda^{-1}e_s$ is the pseudo inverse of

a as defined in section 5. In general, $E_{1/2}(a)$ is not a subtriple of E, whereas $E_1(a)\oplus E_{1/2}(a) = E_1(e)\oplus E_{1/2}(e)$ and $E_0(a)=E_0(e)$ for the tripotent $e:=\operatorname{sign}(a)$.

$$A(a) = A(e_1)\oplus\cdots\oplus A(e_s) \tag{6.3}$$

is a Jordan subalgebra of $A(e)$ and hence a formally real Jordan algebra with semipositive cone

$$\overline{\Omega}(a) = \overline{\Omega}(e_1)\oplus\cdots\oplus\overline{\Omega}(e_s) \;=\; A(a)\cap\overline{\Omega}(e)\,. \tag{6.4}$$

Notice that for the representation (5.3) without the assumption (5.6) the Peirce spaces with respect to a become

(6.5)

$$E_1(a) = \bigoplus_{\substack{0\le j\le k\le s\\ \lambda_j^2=\lambda_k^2>0}} E_{j,k}, \quad E_{1/2}(a) = \bigoplus_{\substack{0\le j\le k\le s\\ \lambda_j^2\ne\lambda_k^2}} E_{j,k}, \quad E_0(a) = \bigoplus_{\substack{0\le j\le k\le s\\ \lambda_j=\lambda_k=0}} E_{j,k}, \quad A(a) = \bigoplus_{\substack{|j|\le|k|\le s\\ \lambda_j=\lambda_k>0}} E^{j,k}\,.$$

This makes it more transparent how the Peirce spaces depend on the coefficients λ_j. For instance, some summands of $A(a)$ get multiplied by the imaginary unit i if λ_j passes through $\lambda_0=0$.

The decomposition (6.2) will play an important role in the study of the orbits $K=\mathsf{K}(a)$ and $S=\mathsf{S}(a)$. For this we also need a characterization of the Peirce spaces $E_{1/2}(a)$ and $A(a)\oplus E_{1/2}(a)$ in terms of our basic operators L_a and Q_a. First of all, it is clear that $E_1(a)\oplus E_{1/2}(a)$ is the range and that $E_0(a)$ is the kernel of L_a. Now put

$$\Phi_a := 2(L_a+Q_a),\ \Psi_a := 2(L_a-Q_a) \ \text{ and } \ \Theta_a := \Phi_a\Psi_a = 4(L_a^2-Q_a^2)\,,$$

where the last operator is complex linear in contrast to the other two. Obviously,

$$\Phi_a = \sum_{|j|\le k\le s}(\lambda_j+\lambda_k)^2\,P^{j,k}\,, \quad \Psi_a = \sum_{|j|\le k\le s}(\lambda_j-\lambda_k)^2\,P^{j,k}$$

$$\Theta_a = \sum_{0\le j\le k\le s}(\lambda_j^2-\lambda_k^2)^2\,P_{j,k}\,. \tag{6.6}$$

Every Peirce projection $P_{j,k}$ is a real polynomial in the operators L_a and Q_a^2 (but in general not a polynomial in L_a alone). Therefore the same holds for the orthogonal projection of E with range $E_1(a)$, that we denote by $\Pi_a\in\mathcal{L}(E)$. The following statement is easily verified:

6.7 Lemma. *The operators Φ_a, Ψ_a, Θ_a satisfy*

$$\Phi_a(E) \;=\; E_{1/2}(a)\,\oplus A(a) \;=\; i\Psi_a(E)\,, \tag{6.8}$$

and

$$\Theta_a(E) \;= E_{1/2}(a) \;=\; \Phi_a(E)\cap\Psi_a(E)$$

is the maximal complex linear subspace of $\Phi_a(E)$ as well as of $\Psi_a(E)$.

7. Tangent spaces to orbits

For the rest of the paper E is always a factor, that is, an irreducible PJT. This is not an essential restriction since the reducible case is obtained by taking direct products of irreducible objects. As before, $\mathsf{K} = \mathbb{T}\,\mathsf{S}$ is the connected identity component of $\mathsf{GL}(\mathcal{D}) = \mathsf{Aut}(E)$, the circular group $\mathbb{T} = \{z \mapsto tz : |t| = 1\}$ is the center of K and S is the commutator subgroup of K. Clearly, S is also the connected identity component of the group $\mathsf{K} \cap \mathsf{SL}(E)$. It is known that the Lie algebra $\mathfrak{k} \subset \mathcal{L}(E)$ of the Lie group K is the $\mathbb{R}$-linear span of all operators iL_x with $x \in E$, which coincides with the $\mathbb{R}$-linear span of all operators $L(x,y) - L(y,x)$ with $x, y \in E$. Consequently, the Lie algebra $\mathfrak{s}$ of S is the $\mathbb{R}$-linear span of all commutators $[L_x, L_y]$ with $x, y \in E$. The following proposition gives a characterization of the tangent spaces to the orbits S and K in terms of the generalized Peirce decomposition defined in section 6.

7.1 Proposition. *For every $a \in E$ the tangent spaces to the orbits $S = \mathsf{S}(a)$ and $K = \mathsf{K}(a)$ at a satisfy $T_aS \subset T_aK = iA(a) \oplus E_{1/2}(a)$ and $H_aS = H_aK = E_{1/2}(a)$.*

Proof. T_aK is the $\mathbb{R}$-linear span of all vectors $\{xya\} - \{yxa\}$ with $x, y \in E$. This implies (for $y = a$) that the image of Ψ_a is in T_aK and hence that $iA(a) \oplus E_{1/2}(a)$ is contained in T_aK by Lemma 6.7. For the proof of the opposite inclusion assume that a is given in the form (5.3) satisfying (5.6) and fix an arbitrary $z = (z_{j,k}) \in iT_aK$, where $z_{j,k} \in E_{j,k}$ are the Peirce components of z. Because of Lemma 6.7 and (6.1) it is enough to show $z_{j,j} \in \Phi_a(E)$ for all $j \geq 0$. Without loss of generality we may assume $z = L_x(a) = \{xxa\}$ for some $x = (x_{j,k}) \in E$. By the multiplication rules of Peirce spaces (5.12) we get

$$z_{j,j} = \sum_{l \geq 0} \lambda_j \{x_{j,l} x_{l,j} e_j\}\ .$$

We may therefore assume $j > 0$ and $z = \{uuc\}$ for $c = e_j$ and $u \in E_{1/2}(c)$. But then by 3.1.ii

$$z = \{uu\{ccc\}\} = 2\{\{uuc\}cc\} - \{c\{uuc\}c\} = 2z - \{czc\}$$

implies $z = \{czc\} \in A(c) \subset A(a)$ and hence $z \in \Phi_a(E)$, that is, $T_aK \subset \Psi_a(E)$ and hence $T_aK = \Psi_a(E) = iA(a) \oplus E_{1/2}(a)$.

For every $x \in E$ the vector $[L_x, L_a](a) = \{xx\{aaa\}\} - \{aa\{xxa\}\}$ is contained in T_aS. Polarization implies $\{va\{aaa\}\} + \{av\{aaa\}\} - \{aa\{vaa\}\} - \{aa\{ava\}\} \in T_aS$ for all $v \in E$. Applying the Jordan triple identity 3.1.ii to the first two terms and using that L_a, Q_a commute yields $\{\{vaa\}aa\} - \{a\{ava\}a\} \in T_aS$, i.e. $E_{1/2}(a) = \Theta_a(E) \subset T_aS \subset T_aK$. □

7.2 Corollary. *The minimal codimension of a K-orbit in E is the rank of E and is attained precisely for all orbits $\mathsf{K}(a)$ where all singular values of a are nonzero and pairwise distinct.*

Proposition 7.1 can be used together with (6.5) and the table at the end of section 5 to compute the *CR-dimension* and *CR-codimension* of the orbit $K = \mathsf{K}(a)$ at a, which are by definition the complex dimension of the holomorphic tangent space H_aK and the real codimension of this space in the full tangent space T_aK, respectively. These dimensions depend on the multiplicities $r_1, \ldots, r_s$ of the nonzero singular values of a, which can also be characterized in the following way: Represent a uniquely in the form (5.3) satisfying (5.6). Then r_j is the rank of the tripotent e_j for $j = 1, \ldots, s$. For instance, the multiplicity sequences are $1,1,1$ and $2,1$ and $1,2$ and 3 according to the 4 different cases in `Figure 2`. Our computations above show that

$$\dim_{\mathrm{CR}} K = \dim E_{1/2}(a) = \alpha \sum_{i<j} r_i r_j + \beta \sum_j r_j,$$

$$\text{codim}_{\text{CR}} K = \dim E_1(a) = \sum_j r_j + \alpha \sum_j \frac{r_j(r_j-1)}{2},$$

where the numbers α and β are chosen as at the end of section 5 and depend only on E. Note that both dimensions above as well as the diffeomorphism type of K do not depend on the order of the multiplicities. In contrast to this, the geometric form of the various hulls of K depends essentially on this order (see e.g. `Figure 2`).

Proposition 7.1 does not determine the tangent space T_aS. Since the subgroup $S \subset K$ has codimension 1, the codimension of T_aS in T_aK is at most 1. Since Proposition 7.1 implies $T_aS = H_a(S) \oplus \Pi_a(T_aS)$, it will be enough to determine the real subspace $\Pi_a(T_aS) \subset iA(a)$, where Π_a is the orthogonal projection with range $E_1(a)$.

We will consider mappings $\xi : E \to E$ also as vector fields on E and write ξ_z for the value at $z \in E$. Then a smooth vector field ξ on E is called a real CR vector field on S if $\xi_z \in H_zS$ holds for all $z \in S$. Denote by

$$H_a^2 S \subset T_aS \quad \text{the } \mathbb{R}\text{-linear span of all vectors } \Pi_a([\xi,\eta]_a)\,,$$

where ξ, η run over all real CR vector fields on S. As a consequence of Proposition 7.1 and Lemma 6.7, for every $v \in E$ the real-analytic vector field ξ^v on E defined by $\xi^v_z := \Theta_z(v)$ is CR on S (and also on K). On the other hand, every vector in H_aS can be written as ξ^v_a for a unique $v \in H_aS$ since the restriction of Θ_a to H_aS is an invertible operator on H_aS.

For every $z, w \in E$ put

$$\Theta(z,w) := 2L(z,w)L_z + 2L_zL(z,w) - 4Q_zQ(z,w)\,.$$

Then $\Theta_z = \Theta(z,z)$ holds and a simple calculation gives for all $v \in E$ and $u := \Theta_a(v) = \xi^v_a$

$$[\xi^v, \xi^{iv}]_a \;=\; 16i\,\Theta(a,u)(v) \;\in\; T_aS\,. \tag{7.3}$$

Recall that $a \in E$ is called invertible if the operator Q_a is invertible on E.

7.4 Proposition. *Suppose that $a \in E$ is not invertible. Then the orbits $S = \mathsf{S}(a)$ and $K = \mathsf{K}(a)$ coincide and $H_a^2S = iA(a)$ as well as $T_aS \;=\; H_aS \oplus H_a^2S \;=\; E_{1/2}(a) \oplus iA(a)$ hold. In particular, $S = K$ is minimal as CR-manifold (in fact of finite type 2).*

Proof. We may assume (5.6) for a in the decomposition (5.3). This implies $A(a) = \sum_{j=1}^s A(e_j)$ and it is enough to show for $1 \le j \le s$ that $A(e_j)$ is the linear span of all vectors $\Theta(a,u)(v)$ with $v \in E_{j,0}$ and $u := \Theta_a(v)$. The assumption on E and Q_a implies $E_{j,0} \neq 0$ for all $j > 0$. Every subtriple in E of the form $E_{j,j} \oplus E_{j,0} \oplus E_{0,0}$ is irreducible, we may therefore assume without loss of generality that $s = 1$ and that $a = e_1$ is a tripotent. But then $u = v$ and $\Theta(a,u)(v) = -\{avv\}$. But it is known that the convex hull of all vectors $\{avv\}$, $v \in E_{1/2}(a)$, is the cone $\overline{\Omega}(a)$ (compare [26], Proposition 8.15). Since $\overline{\Omega}(a)$ has nonempty interior, the statement follows. □

7.5 Corollary. *Suppose that E is not of tube type. Then there does not exist an invertible element in E and hence the conclusion in Proposition 7.4 holds for every $a \in E$ in this case.*

Let us now come to the case left out in Proposition 7.4, that is, where a is invertible. Then E is necessarily of tube type and becomes a complex Jordan $E^{(e)}$ algebra with unit $e := \text{sign}(a)$ in the product $z \circ w = \{zew\}$. Denote by $N : E^{(e)} \to \mathbb{C}$ the generic norm of the complex Jordan algebra $E^{(e)}$, which is a complex homogeneous polynomial of degree $r := \text{rank}(E)$ (compare e.g. [9], [31] and section 4). For every frame $(e_1, \ldots, e_r)$ in E with $e = e_1 + \cdots + e_r$ and every

$z = z_1e_1 + \cdots + z_re_r$ with $z_1, .., z_r \in \mathbb{C}$ then $N(z) = z_1z_2\cdots z_r$ holds. Also, there exists a character $\chi : \mathsf{K} \to \mathsf{U}(1)$ with $N(gz) = \chi(g)N(z)$ for all $g \in \mathsf{K}$ and $z \in E$. More generally, let us call a (complex) homogeneous polynomial $N : E \to \mathbb{C}$ of degree $r := \mathrm{rank}(E)$ a *generic norm* on E if

(i) $N(e) = 1$ for some tripotent $e \in E$ and

(ii) $z \in E$ is invertible if and only if $N(z) \neq 0$ for every $z \in E$.

From the above it is clear that the factor E has a generic norm if and only if it is of tube type and then any two generic norms on E differ by a complex factor of absolute value 1. For instance, in case E is of type $\mathbf{I}_{p,p}$ or of type $\mathbf{III}_p$, then the usual determinant function is a generic norm on E. In case E is of type $\mathbf{II}_p$ with p even, then the *Pfaffian determinant* (i.e. the square root of the usual determinant) is a generic norm on E.

Now fix an invertible element a in E and let N be a generic norm on E. Then $N(ga) = \chi(g)N(a) = N(a) \neq 0$ for all $g \in \mathsf{S}$, since S is semisimple. On the other hand, $N(ta) = t^rN(a) \neq N(a)$ for some $t \in \mathsf{U}(1)$, that is, $S = \mathsf{S}(a)$ is a submanifold of $K = \mathsf{K}(a)$ of lower dimension having everywhere the same holomorphic tangent space, in particular, K is not a minimal CR-manifold. Since T_aS has codimension 1 in T_aK, in order to describe it, it is sufficient to find a nontrivial linear form on T_aK that vanishes on T_aS. Since the generic norm N is constant on S but not on K such a form is easily found: Let $R := dN_a : E \to \mathbb{C}$ be the derivative of N at a. Then $R(T_aS) = 0$ and $R(a) = rN(a)$, in particular, $R(ia) \neq 0$ for the tangent vector $ia \in T_aK$. For computational purposes this can be made more specific in the following way: Assume for the decomposition (5.3) of a that $\boldsymbol{e} = (e_1, \ldots, e_s)$ is a frame (that is, $s = r$). Then $\lambda_j \neq 0$ for all j by the invertibility of a. For every x and every $1 \leq j \leq r$ define $x_j \in \mathbb{C}$ by $P_{j,j}x = x_je_j$. Then the pseudo inverse $a^\dagger$ of a (see section 5) satisfies $(x|a^\dagger) = \sum_{j=1}^r x_j/\lambda_j$ and we have

7.6 Proposition. *Suppose that $a \in E$ is invertible. Then*

$$T_aS \;=\; \big\{x \in T_aK : (x|a^\dagger) = 0\big\}\,. \tag{7.7}$$

In particular, K is not minimal and $S \subset E$ is not generic as CR-manifold.

Every odd function $f : \mathbb{R} \to \mathbb{C}$ induces by (5.9) an odd K-equivariant mapping $\boldsymbol{f} : E \to E$. With f also $\boldsymbol{f}$ is of class $\mathcal{C}^1$ and the derivative of $\boldsymbol{f}$ at a is (compare [1])

$$d\boldsymbol{f}_a = \sum_{|j|\leq k\leq s} m_f(\lambda_j, \lambda_k)P^{j,k} \;\subset\; \mathcal{L}_{\mathbb{R}}(E)\,, \tag{7.8}$$

where the divided difference $m_f : \mathbb{R}^2 \to \mathbb{C}$ is given by

$$m_f(x,y) = \frac{f(x) - f(y)}{x - y} \;\text{ if }\; x \neq y \quad\text{ and }\quad = f'(x) \text{ otherwise.}$$

The restriction $\varphi := \boldsymbol{f}|_K$ to the orbit $K = \mathsf{K}(a)$ realizes K as fiber bundle over the orbit $\widetilde{K} := \mathsf{K}(\widetilde{a})$ of $\widetilde{a} := \boldsymbol{f}(a)$. The differential $d\varphi_a : T_aK \to T_{\widetilde{a}}\widetilde{K}$ is the operator

$$d\varphi_a = \sum_{0\leq\lambda_j\neq\lambda_k} \frac{f(\lambda_j) - f(\lambda_k)}{\lambda_j - \lambda_k}\, P^{j,k} \quad\text{restricted to}\quad T_aK = \bigoplus_{0\leq\lambda_j\neq\lambda_k} E^{j,k}\,, \tag{7.9}$$

where the indices run over $|j| \leq k \leq s$. This implies by a simple computation that $\varphi : K \to \widetilde{K}$ is a CR-map if and only if $f(\lambda_j) = c\,\lambda_j$ for all j and some $c \in \mathbb{C}$ not depending on j. Under the assumption that (5.6) holds for the representation (5.3), the fiber $F := \boldsymbol{f}^{-1}(a)$ has tangent space

$$T_aF \;=\; \bigoplus_{\substack{1\le j\le s\\ f(\lambda_j)=0}} iA(e_j) \;\oplus\; \bigoplus_{\substack{|j|<k\le s\\ f(\lambda_j)=f(\lambda_k)\neq 0}} E^{j,k} \;\oplus\; H_aF \quad \text{with} \quad H_aF = \bigoplus_{\substack{1\le j<k\le s\\ f(\lambda_j)=f(\lambda_k)=0}} E_{j,k}$$

the holomorphic tangent space to F at a.

Denote as before by $[a]$ the smallest (complex) subtriple of E containing a. It is clear that $[a] = \mathbb{C}e_1 \oplus \ldots \oplus \mathbb{C}e_s$ holds if a is given in the form (5.3) satisfying (5.6). For every subgroup $H \subset \mathsf{GL}(E)$ denote by $\mathsf{Fix}(H) := \big\{x \in E : H(x) = \{x\}\big\}$ the fixed point set of H. Also let $\mathsf{K}_a = \{g \in \mathsf{K} : g(a) = a\}$ be the *isotropy subgroup* at $a \in E$.

7.10 Lemma. $\mathsf{Fix}(\mathsf{K}_a) = [a]$.

Proof. Choose a representation (5.3) for a satisfying $s = r$ and (5.5). For every real $t > 0$ consider the tripotent $c_t := \sum_{\lambda_j=t} e_j$ (empty sums are 0 by definition). Since every c_t is of the form $f(a)$ for some odd polynomial $f \in \mathbb{R}[t]$ (see (5.9)) we conclude $c_t \in \mathsf{Fix}(\mathsf{K}_a)$ for every $t > 0$. On the other hand, $[a]$ is the linear span of all c_t, i.e. $[a] \subset \mathsf{Fix}(\mathsf{K}_a)$. Suppose on the contrary that $x \in \mathsf{Fix}(\mathsf{K}_a)$ is an arbitrary element. For every $0 \le j \le k \le r$ let $x_{j,k} = P_{j,k}(x)$ be the corresponding Peirce component. Then $x_{0,0} = 0$ since $E_{0,0} = 0$, and for every $k > 0$ the transformation $g := \exp(2\pi i L(e_k, e_k)) \in \mathsf{K}_a$ satisfies $g(x_{j,k}) = -x_{j,k}$ for all $j < k$, that is, $x = \sum_{j=1}^r \alpha_j e_j$ for certain complex coefficients α_j. Consider $j,k > 0$ with $\lambda_j = \lambda_k$. By the irreducibility of E there exists $g \in \mathsf{K}$ with $g^{\pm1}(e_j) = e_k$ and $g(e_l) = e_l$ for all $l \neq j,k$. Then $g \in \mathsf{K}_a$ and $g(x) = x$ implies $\alpha_j = \alpha_k$, that is, x is a linear combination of the tripotents c_t and hence is in $[a]$. □

The group K acts transitively on frames in E (since we assumed E to be irreducible). Therefore, the orbit space $E/\,\mathsf{K}$ is homeomorphic to

$$\Delta_r := \big\{\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^r : \lambda_1 \ge \ldots \ge \lambda_r \ge 0\big\}\,. \tag{7.11}$$

The canonical homeomorphism is induced by the singular value map $\sigma : E \to \Delta_r$. In the same way, if $E \neq 0$ is of tube type, the orbit space $E/\,\mathsf{S}$ is homeomorphic to

$$\big\{\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^{r-1} \times \mathbb{C} : \lambda_1 \ge \ldots \ge \lambda_{r-1} \ge |\lambda_r|\big\}\,. \tag{7.12}$$

A (non canonical) homeomorphism is obtained as follows: Choose a frame $(e_1, \ldots, e_r)$ in E and associate to every λ from the set (7.12) the orbit $\mathsf{S}(\lambda_1 e_1 + \ldots + \lambda_r e_r)$.

By definition, two orbits $\mathsf{K}(a)$ and $\mathsf{K}(b)$ in E are isomorphic as K-spaces if the isotropy subgroups K_a and K_b are conjugate in K, or equivalently, if there exists a K-equivariant diffeomorphism $\varphi : \mathsf{K}(a) \to \mathsf{K}(b)$. It can be seen that in our situation this is the case if and only if there is an odd bijection $f : \mathbb{R} \to \mathbb{R}$ with $\sigma_j(b) = f(\sigma_j(a))$ for all j, and then the mapping $\boldsymbol{f} : \mathsf{K}(a) \to \mathsf{K}(b)$ induced by the odd functional calculus gives the corresponding K-equivariant diffeomorphism. As a consequence, the set of all K-isomorphism classes of nonzero K-orbits in E can be identified with the set of all finite ordered sequences $r_1 \ge r_2 \ge \ldots \ge r_s$ of positive integers r_j satisfying $r_1 + r_2 + \ldots + r_s \le r = \mathrm{rank}(E)$. In general, the diffeomorphism $\boldsymbol{f} : \mathsf{K}(a) \to \mathsf{K}(b)$ is not CR.

7.13 Proposition. *Let $a,b\in E$ be arbitrary elements. Then there exists a* K*-equivariant CR-map* $\mathsf{K}(a)\to\mathsf{K}(b)$ *if and only if* $\mathsf{K}(b)=t\,\mathsf{K}(a)$ *for some real $t\ge 0$, that is, if and only if $\sigma_j(b)=t\sigma_j(a)$ for some real $t\ge0$ and all j.*

Proof. Represent a in the form (5.3) for a suitable frame $(e_1,\dots,e_r)$ and fix a K-equivariant CR-map $\varphi:\mathsf{K}(a)\to\mathsf{K}(b)$. Without loss of generality we may assume $b=\varphi(a)$. Then $K_a\subset K_b$ and Lemma 7.10 implies $[b]\subset[a]$. Therefore b has a representation $b=\beta_1e_1+\ldots+\beta_re_r$ with complex coefficients such that always $\lambda_j=\lambda_k$ implies $\beta_j=\beta_k$. As a consequence, there exists an odd function $f:\mathbb{R}\to\mathbb{C}$ with $f(\lambda_j)=\beta_j$ for all j, and φ is the restriction of $\boldsymbol{f}$ to $\mathsf{K}(a)$. Since φ is CR there is a constant $c\in\mathbb{C}$ by (7.9) with $\boldsymbol{f}=c\cdot\mathsf{id}_E$ and hence $\mathsf{K}(b)=t\,\mathsf{K}(a)$ for $t:=|c|$. □

8. Complex orbits

The CR-manifold $K=\mathsf{K}(a)$ is not always generic in E. As follows from Proposition 7.1, K is generic in E if and only if $E_0(a)=\ker(L_a)=0$ holds, that is, if and only if a has maximal rank in E. We therefore will consider complex orbits of a in which K is always generic.

Denote by $\mathsf{K}^{\mathbb{C}}$, $\mathsf{S}^{\mathbb{C}}$ and $\mathsf{T}^{\mathbb{C}}$ the smallest complex Lie subgroups of $\mathsf{GL}(E)$ containing K, S and $\mathbb{T}$ respectively. All these groups are connected, $\mathsf{K}^{\mathbb{C}}=\mathsf{T}^{\mathbb{C}}\mathsf{S}^{\mathbb{C}}$ is reductive, $\mathsf{T}^{\mathbb{C}}$ is the center and $\mathsf{S}^{\mathbb{C}}$ is the semisimple part of $\mathsf{K}^{\mathbb{C}}$. Clearly, $\mathsf{T}^{\mathbb{C}}=\{z\mapsto tz: t\in\mathbb{C}^*\}$ and $\mathsf{S}^{\mathbb{C}}$ is the connected identity component of $\mathsf{K}^{\mathbb{C}}\cap\mathsf{SL}(E)$. The group $\mathsf{K}^{\mathbb{C}}$ can be identified with the connected identity component of the *structure group*

$$\mathsf{Str}(E):=\big\{g\in\mathsf{GL}(E): g\{xyz\}=\{(gx)(\widetilde{g}y)(gz)\}\ \text{ for some }\ \widetilde{g}\in\mathsf{GL}(E)\ \text{ and all }\ x,y,z\big\}\,.$$

For every $g\in\mathsf{Str}(E)$ the operator $\widetilde{g}$ is uniquely determined by g, lies again in $\mathsf{Str}(E)$ and $g\mapsto\widetilde{g}$ defines an anti-holomorphic group automorphism of $\mathsf{Str}(E)$ with fixed point set $\mathsf{Aut}(E)\subset\mathsf{Str}(E)$.

The Lie algebra of $\mathsf{K}^{\mathbb{C}}$ is $\mathfrak{k}^{\mathbb{C}}:=\mathfrak{k}\oplus i\mathfrak{k}\subset\mathcal{L}(E)$, where $\mathfrak{k}$ is the Lie algebra of the linear group K. The complex Lie algebra $\mathfrak{k}^{\mathbb{C}}$ is the $\mathbb{R}$-linear span of all operators $L(x,y)$, $x,y\in E$. In particular, if $\boldsymbol{e}=(e_1,\dots,e_r)$ is a frame in E with joint Peirce decomposition (5.11), all operators

$$\sum_{0\le j\le k\le r} c_jc_kP_{j,k}\,,\qquad c_0=1\ \text{ and }\ c_1,\dots,c_r\in\mathbb{C}^* \tag{8.1}$$

are in $\mathsf{K}^{\mathbb{C}}$, where as before $P_{j,k}\in\mathcal{L}(E)$ denotes the orthogonal projection with range $E_{j,k}$.

From section 5 recall the notion of rank of an element $a\in E$, which by definition is the maximal index k with $\sigma_k(a)\ne0$ (and also coincides with the rank of the Peirce space $E_1(a)$). For every $0\le\rho\le r$ put furthermore

$$E_{[\rho]}:=\{a\in E:\operatorname{rank}(a)=\rho\}\,. \tag{8.2}$$

For the factors of type **I** and **III** this notion of rank coincides with the usual matrix rank. In case of type **II** for every skew symmetric matrix the usual matrix rank is even and is twice the triple rank defined above.

8.3 Proposition. *The closure $\overline{E}_{[\rho]}$ is an irreducible complex-analytic cone in E and $E_{[\rho]}$ is a connected open dense complex submanifold of $\overline{E}_{[\rho]}$ on which the complex linear group* $\mathsf{K}^{\mathbb{C}}$ *acts transitively. In case $\rho>0$, the analytic set $\overline{E}_{[\rho-1]}$ has complex codimension ≥2 in $\overline{E}_{[\rho]}$ unless $E_{[\rho]}$ contains invertible elements of E. Furthermore, in case E is classical or $\rho\ne1$ holds, the complex space $\overline{E}_{[\rho]}$ is normal. In particular, the complex space $E_{[\rho]}\backslash\{0\}$ is always normal.*

Proof. E can be realized as a subtriple $E \subset F$ with F a factor of tube type in such a way that every minimal tripotent of E is also minimal in F. Let $m := \mathrm{rank}(F)$ and fix a generic norm N on F (see section 7). Fix furthermore a tripotent $e \in F$ with $N(e) = 1$. Expanding

$$N(te - z) = \sum_{j+k=m} (-1)^j N_j(z) t^k \quad \text{implies} \tag{8.4}$$

$$\overline{E}_{[\rho]} = \{z \in E : N_j(z) = 0 \ \text{ for all } \ \rho < j \leq r\}\,, \tag{8.5}$$

where every N_j is a complex homogeneous polynomial of degree j on E, that is, $\overline{E}_{[\rho]}$ is a complex-analytic cone in E. Since K acts transitively on frames in E and since all transformations (8.1) are in $\mathsf{K}^{\mathbb{C}}$ we derive that $\mathsf{K}^{\mathbb{C}}$ acts transitively on $E_{[\rho]}$. In particular, $E_{[\rho]}$ is a connected complex submanifold of $\overline{E}_{[\rho]} \backslash \overline{E}_{[\rho-1]}$, that is, $\overline{E}_{[\rho]}$ is irreducible.
For every $a \in E_{[\rho]}$ the codimension of $E_{[\rho]}$ in E is $\dim(E_0(a))$. Therefore $\dim(E_0(b)/E_0(a))$ is the codimension of $E_{[\rho-1]}$ in $E_{[\rho]}$ for every $b \in E_{[\rho-1]}$. But this number is $1+\beta+(r-\rho)\alpha$, where r is the rank of E and α, β are the invariants of E as defined at the end of section 5.
The normality statement is well known in the matrix case $E = \mathbb{C}^{p\times q}$, see e.g. [10] Theorem 6.3, where the proof proceeds as follows: The group $\mathsf{H} := \mathsf{GL}(\rho, \mathbb{C})$ acts on $U := \mathbb{C}^{p\times\rho} \oplus \mathbb{C}^{\rho\times q}$ by $(x, y) \mapsto (xg^{-1}, gy)$, and the H-invariant map $\varphi : U \to E$ defined by $\varphi(x, y) = xy$ has image $\overline{E}_{[\rho]}$. Actually, the function ring of $\overline{E}_{[\rho]}$ identifies via φ with the ring of H-invariant functions on U, implying normality of the space $\overline{E}_{[\rho]}$. The other two matrix types follow in a similar way (compare also [27] and [28]): In case $E = \mathbf{III}_p$ is the space of symmetric complex $p\times p$-matrices, set $U := \mathbb{C}^{p\times\rho}$ on which the complex orthogonal group $\mathsf{H} := \{g \in \mathsf{GL}(\rho, \mathbb{C}) : gg' = 1\!\!1\}$ acts from the right. Then $\varphi : U \to E$ defined by $x \mapsto xx'$ yields the claim. In case $E = \mathbf{II}_p$ is the space of skew-symmetric $p\times p$-matrices, set $U := \mathbb{C}^{p\times 2\rho}$ and fix a skew-symmetric matrix $j \in \mathsf{GL}(2\rho, \mathbb{C})$. Put $\mathsf{H} := \{g \in \mathsf{GL}(2\rho, \mathbb{C}) : gjg' = j\}$ and define $\varphi : U \to E$ by $x \mapsto xjx'$.
For the remaining cases we may assume that E is of tube type and that $\rho = \mathrm{rank}(E)-1$ holds. Then $\overline{E}_{[\rho]} = N^{-1}(0)$, where N is a generic norm on E. The analytic set $\{z \in \overline{E}_{[\rho]} : dN_z = 0\}$ is contained in $\overline{E}_{[\rho-1]}$ and hence has codimension ≥ 3 in E, proving normality also in this situation. □

8.6 Corollary. *For $\rho := \mathrm{rank}(a)$, the tangent space to $\mathsf{K}^{\mathbb{C}}(a) = E_{[\rho]}$ at a is $E_1(a) \oplus E_{1/2}(a)$. In case a is invertible and N is a generic norm on E we have $\rho = r$ and*

$$\mathsf{S}^{\mathbb{C}}(a) = \big\{z \in E : N(z) = N(a)\big\} \ \subset \ E_{[r]}\,. \tag{8.7}$$

It is clear that for every E and every $a \in E$ the orbits $\mathsf{K}(a)$ and $\mathsf{S}(a)$ are generic CR-submanifolds of the complex manifolds $\mathsf{K}^{\mathbb{C}}(a)$ and $\mathsf{S}^{\mathbb{C}}(a)$ respectively. Having determined the complex orbits (and their closures) fairly explicitly by holomorphic equations we want to do the same with the real orbits (by real-analytic equations). Denote by

$$m(t, z, w) = \sum_{j+k=r} (-1)^j m_j(z, w)\, t^k \tag{8.8}$$

the *generic minimal polynomial* of E, which is monic of degree $r := \mathrm{rank}(E)$ in the indeterminate t with complex coefficients depending holomorphically on z and anti-holomorphically on $w \in E$, compare [29] p. 4.13. Furthermore, every coefficient $m_j(z, w)$ is homogeneous of bidegree

(j,j), and $m(t,gz,gw) = m(t,z,w)$ holds for every $g \in \mathsf{Aut}(E)$. Clearly, the generic minimal polynomial of E is uniquely determined by all $m(t,z,z)$, $z \in E$. On the other hand, for every $z \in E$ the roots of $m(t,z,z)$ as polynomial in t are the squares of the singular values of z, more precisely,

$$m(t,z,z) = \prod_{k=1}^{r} \big(t - \sigma_k^2(z)\big) \quad \text{and} \quad m_j(z,z) = \sum_{k_1<\cdots<k_j} \sigma_{k_1}^2(z)\,\sigma_{k_2}^2(z)\cdots\sigma_{k_j}^2(z)\,. \tag{8.9}$$

In particular, $m_1(z,w) = (z|w)$ is the inner product as defined in (5.8) and $m_r(z,z) = \mu_r(z)^2$ as defined in (9.14). Also, for every $1 \le j < r$ and all $z \in E$ the inequalities

$$m_j(z,z)^2 \;\ge\; \tfrac{j+1}{j}\,\tfrac{r-j+1}{r-j}\, m_{j-1}(z,z)\, m_{j+1}(z,z) \tag{8.10}$$

are well known, see e.g. [32] p. 95. For all types **I - VI**, explicit expressions for $m(t,z,w)$ can be found in [29]. We recall only the first four of them: In the cases $\mathbf{I}_{p,q}$ with $p \le q$ and $\mathbf{III}_p$ the generic minimal polynomial is given by $m(t,z,w) = \det(t\mathbb{1}_p - zw^*)$ and $m_j(z,w)$ is the sum of all diagonal $j\times j$-minors of zw^*, whereas $t^\varepsilon m(t,z,w)^2 = \det(t\mathbb{1}_p - zw^*)$ in case $\mathbf{II}_p$ holds with $\varepsilon = 0$ if p is even and $\varepsilon = 1$ otherwise. Finally, for $\mathbf{IV}_n$ the generic minimal polynomial is $t^2 - (z|w)t + (z|\overline{z})(\overline{w}|w)/4$.

Since the group K acts transitively on frames in E we get

$$\begin{aligned} \mathsf{K}(a) &= \{z \in E : \sigma(z) = \sigma(a)\} \;=\; \{z \in E : \|z\|_k = \|a\|_k \ \text{ for } \ 1 \le k \le r\} \\ &= \{z \in E : \mu_k(z) = \mu_k(a) \ \text{ for } \ 1 \le k \le r\} \\ &= \{z \in E : m_k(z,z) = m_k(a,a) \ \text{ for } \ 1 \le k \le r\}\,. \end{aligned} \tag{8.11}$$

From $m_j(z,z) = 0$ for all $j > \mathrm{rank}(z)$ we derive in addition

$$\mathsf{K}(a) = \{z \in E_{[\rho]} : m_j(z,z) = m_j(a,a) \ \text{ for } \ 1 \le j \le \rho\} \quad \text{if} \quad \rho = \mathrm{rank}(a)\,. \tag{8.12}$$

In case a is invertible and N is a generic norm on E we have as a consequence of (8.7)

$$\begin{aligned} \mathsf{S}(a) &= \big\{z \in \mathsf{K}(a) : N(z) = N(a)\big\} \\ &= \{z \in E : N(z) = N(a) \ \text{ and } \ m_j(z,z) = m_j(a,a) \ \text{ for } \ 1 \le j < r\} \end{aligned} \tag{8.13}$$

since $m_r(z,z) = |N(z)|^2$.

Equation (8.11) describes K-orbits by polynomial equations of degree $\le 2r$. For equations in a small neighbourhood of $\mathsf{K}(a)$ we can do often with lower degrees, more precisely: Suppose $a \in E$ satisfies (5.3) with (5.4) and (5.6). Then

$$M_f := \{z \in E : \boldsymbol{f}(z) = 0\} \quad \text{for} \quad f(t) := \prod_{|j|\le s} (t - \lambda_j) \;\in\; \mathbb{R}[t] \tag{8.14}$$

is a real-analytic submanifold of E consisting of a finite number of K-orbits. In particular, $K = \mathsf{K}(a)$ is a connected component of M_f. The odd polynomial $f(t)$ has degree $2s+1$, therefore in a small neighbourhood U of K the orbit K is given by the system of scalar equations $\{z \in U : \tau(\boldsymbol{f}(z)) = 0 \ \text{ for all } \ \tau \in E'\}$, where E' is the dual of E, and all equations are of polynomial degree $\le 2s+1$. But $f(\lambda_j) = 0 \ne f'(\lambda_j)$ for all j, that is, $d\boldsymbol{f}_a = \sum_{j=0}^{s} f'(\lambda_j)P^{j,j}$ has the tangent space T_aK as kernel, see Proposition 7.1 and equation (7.8). In this sense, (8.14) is a local defining equation for the orbit $K = \mathsf{K}(a)$.

8.15 Example. Suppose E is irreducible with rank r and $a \ne 0$ is a tripotent. Then $s = 1$, $\lambda_1 = 1$ and $f(t) = t^3 - t$, that is, $\boldsymbol{f}(z) = \{zzz\} - z$. The manifold M_f has exactly $r+1$ connected components, in each $E_{[k]}$ exactly one. Obviously, $d\boldsymbol{f}_a = 2L_a + Q_a - \mathsf{id}_E$ holds, which coincides with $2P^{1,1} - P^{0,0}$ in view of (7.8).

9. Levi forms of orbits

With the notation of the sections before let $a \in E$ be fixed and $K = \mathsf{K}(a)$. If $\widetilde{E}$ is another factor with given point $\widetilde{a} \in \widetilde{E}$ we may ask when the orbits $K = \mathsf{K}(a)$ and $\widetilde{K} = \tilde{\mathsf{K}}(\widetilde{a})$ are isomorphic (or locally isomorphic) as CR-manifolds. Besides the obvious CR-invariant $\dim(H_aK)$ the Levi form is an invariant of the CR-structure that contains important information. Denote by $\mathbb{C}T_aK := T_aK + iT_aK \subset E$ the $\mathbb{C}$-linear span of T_aK. Recall that the *Levi form* at a

$$\Lambda_a : H_aK \times H_aK \to \mathbb{C}T_aK/H_aK$$

is given by

$$\Lambda_a(x, y) \;\equiv\; \big([\xi, \eta]_a + i[i\xi, \eta]_a\big) \quad \mathrm{mod} \quad H_aK\,,$$

where ξ, η are any real CR vector vector fields on K with $x = \xi_a$ and $y = \eta_a$. Because of $\mathbb{C}T_aK = E_1(a) \oplus H_aK$ we may, and will henceforth, identify $\mathbb{C}T_aK/H_aK$ in the canonical way with the Peirce space $E_1(a)$. Then Λ_a is sesqui-linear and hermitian in the sense $\Lambda_a(y, x) = \Lambda_a(x, y)^*$, where $z \mapsto z^* := \{eze\}$ with $e := \mathrm{sign}(a)$ is the Jordan algebra involution of $E_1(a)$. In particular, $\Lambda_a(u, u) \in A(a)$ holds for all $u \in H_aK$. Denote by $C_a \subset A(a)$ the convex hull of all such vectors and call it the *Levi cone* of K at a. The same can be done with the orbit $S = \mathsf{S}(a)$. But, because of $H_aS = H_aK$ we get the same Levi form Λ_a for S.

From (7.3) we derive for all $v \in E_{1/2}(a)$ and $u = \Theta_a(v)$ (recall that $H_aS = H_aK = E_{1/2}(a)$ by Proposition 7.1):

$$\Lambda_a(u, u) \;\equiv\; i[\xi^{iv}, \xi^{v}]_a \;=\; 16\,\Theta(a, u)(v) \;=\; 16\,\Theta(a, u)(\Theta_a^{-1}u) \quad \mathrm{mod} \quad E_{1/2}(a)\,. \tag{9.1}$$

Here the inverse Θ_a^{-1} is taken for the restriction of Θ_a to $E_{1/2}(a)$. For a we choose the representation (5.3) and assume throughout this section that (5.6) holds. Consider arbitrary elements $u, v \in E_{1/2}(a)$ and put $u_{j,k} := P_{j,k}u$ as well as $v_{j,k} := P_{j,k}v$ for all $0 \le j < k \le s$. Then the multiplication rules for Peirce spaces (5.12) yield

$$\Lambda_a(u, v) = \sum_{0 \le j < k \le s} \Lambda_a(u_{j,k}, v_{j,k}) \;\in\; A(a)\,. \tag{9.2}$$

For every $j < k$ denote by $Z_{j,k}(a) \subset A(a)$ the cone spanned by all $\Lambda_a(u, u)$ with $u \in E_{j,k}$. From (9.2) it is clear that the Levi cone C_a is the sum of all cones $Z_{j,k}(a)$, $j < k$. Now fix integers $0 \le j < k \le s$ and put for convenience $e_0 := 0 \in E$. Then $A(e_0) = \overline{\Omega}(e_0) = 0$ and $Z_{j,k}(a) \subset A(e_j) \oplus A(e_k)$. For every $u \in E_{j,k}$ put $u_j := 2\{uue_j\}$ and $u_k := 2\{uue_k\}$. Then $u_j \in \overline{\Omega}(e_j)$, $u_k \in \overline{\Omega}(e_k)$ by [29] p. 10.5 and furthermore

$$\Lambda_a(u, u) = c_{j,k}(\lambda_j u_j - \lambda_k u_k) \quad \text{for} \quad c_{j,k} := 8(\lambda_k^2 - \lambda_j^2)^{-1} < 0\,. \tag{9.3}$$

9.4 Corollary. *$\Lambda_a(x, y) = 0$ for all (triple) orthogonal $x, y \in E_{j,k}$ with $j < k$. In particular, the cone $Z_{j,k}(a)$ is spanned by all $\Lambda_a(u, u)$ with $u \in E^{j,k}$ a minimal tripotent.*

Proof. If x, y are orthogonal, $\Lambda_a(u, u) = \Lambda_a(x, x) + \Lambda_a(y, y)$ holds for $u := x + y$ as a consequence of (9.3), i.e. $\Lambda_a(x, y) + \Lambda_a(y, x) = 0$ and hence $\Lambda_a(x, y) = 0$. Since every $u \in E_{j,k}$ is a linear combination of orthogonal minimal tripotents the cone $Z_{j,k}(a)$ is spanned by all $\Lambda_a(u, u)$ with $u \in E_{j,k}$ a minimal tripotent. But then $\Lambda_a(u, u) = \Lambda_a(tu, tu)$ for some complex number t with $tu \in E^{j,k}$. □

In order to describe the Levi cones explicitely denote again by $a^\dagger = \sum_{j=1}^s \lambda_j^{-1} e_j \in A(a)$ the pseudo inverse of a and define the following two closed cones in $A(a)$

$$Z(a) := \{x \in A(a) : (x_j^+|a^\dagger) + \sum_{k<j} (x_k|a^\dagger) \;\le\; 0 \quad \text{for} \quad j = 1, \ldots, s\} \tag{9.5}$$
$$X(a) := \{x \in Z(a) : (x|a^\dagger) = 0\} \;\subset\; Z(a)\,.$$

Here $x_k \in A(e_k)$ is the component of x with respect to the direct sum decomposition $A(e) = A(e_1) \oplus \cdots \oplus A(e_s)$, and $x_j^+ \in \overline{\Omega}(e_j)$ is the nonnegative part of x_j, compare (4.5).

9.6 Lemma. *The cones $Z(a)$, $X(a)$ are convex and*

$$Z(a) = \big\{x \in A(a) : (x|w) \le 0 \;\text{ for all }\; w \in W(a)\big\}\,, \tag{9.7}$$

where $W(a) \subset \overline{\Omega}(a)$ is the convex cone generated by all vectors $w \in \overline{\Omega}(a)$ satisfying for some $1 \le j \le s$ the conditions

$$\lambda_k w_k = \begin{cases} e_k & k < j \\ 0 & k > j \end{cases} \qquad \text{and} \qquad \lambda_j w_j \textit{ is a minimal idempotent in } A(e_j)\,. \tag{9.8}$$

Proof. By our general assumption E is a factor and hence all Jordan algebras $A(e_j)$, $1 \le j \le s$, are simple. In case $s = 1$ we have $Z(a) = -\overline{\Omega}(a)$ and the claim is obvious. Therefore we may assume $s > 1$. But then every Jordan algebra $A(e_j)$ is not exceptional (since it is a proper subalgebra of an irreducible formally real Jordan algebra) and by Lemma 4.7 we have

$$(x_j^+|a^\dagger) \;=\; \lambda_j^{-1} \sup_{c^2=c} (x_j|c)$$

for every $x_j \in A(e_j)$. This implies the convexity of the cones as well as the identity (9.7). □

Consequently, $Z(a)$ is the dual of the cone $-W(a)$ and $X(a)$ is the dual of the cone $\mathbb{R}a^\dagger - W(a)$ in $A(a)$. The cones $Z(a)$, $X(a)$ and $W(a)$ are invariant under the isotropy subgroup K_a. Actually, K_a leaves every idempotent e_j fixed and acts transitively on the set of all idempotents of fixed rank in $A(e_j)$. Therefore, by (9.8), there are finitely many vectors $w^0, \ldots, w^\rho$ for $\rho := \operatorname{rank}(a)$ and $w^0 := 0$ such that $W(a)$ is the convex cone spanned by the union of all orbits $\mathsf{K}_a(w^i)$, $0 \le i \le \rho$.

9.9 Lemma. $Z(a) = X(a) - \overline{\Omega}(a)$.

Proof. We proceed by induction on s. For $s = 1$ the claim is obvious, so assume $s > 1$ and fix $z \in Z(a)$. For $\widetilde{a} := \lambda_2 e_2 + \ldots + \lambda_s e_s$ we have $Z(\widetilde{a}) = X(\widetilde{a}) - \overline{\Omega}(\widetilde{a})$ by induction hypothesis. Write $z = z_1 + \ldots + z_s$ with $z_j \in A(e_j)$. Then $\widetilde{z} := z_2 + \ldots + z_s$ has a representation $\widetilde{z} = \widetilde{x} - \widetilde{y}$ with $\widetilde{x} \in X(\widetilde{a}) \subset X(a)$ and $\widetilde{y} \in \overline{\Omega}(\widetilde{a}) \subset \overline{\Omega}(a)$. The inequality in the first line of (9.5) for $j = 1$ implies $z_1^+ = 0$, that is, $y := \widetilde{y} - z_1 \in \overline{\Omega}(a)$ and $z = \widetilde{x} - y \in X(a) - \overline{\Omega}(a)$. □

9.10 Lemma. *In case a is not invertible, $Z_{0,k}(a) = -\overline{\Omega}(e_k)$ is the seminegative cone of $A(e_k)$, otherwise $Z_{0,k} = 0$ holds for all k.*

Proof. In case a is invertible, $E_{0,k} = 0$ holds for all k. So assume that a is not invertible and fix an integer $k > 0$ with $k \le s$. Then $E_{0,k} \ne 0$ and it is well known that the set of all vectors $\{uue_k\}$ with $u \in E_{0,k}$ spans the cone $\overline{\Omega}(e_k)$, compare e.g. Proposition 8.15 in [26]. □

9.11 Lemma. *For every $1 \le j < k \le s$ the cone $Z_{j,k}(a)$ is spanned by all vectors $\lambda_k v - \lambda_j w$, where $v \in \overline{\Omega}(e_j)$ and $w \in \overline{\Omega}(e_k)$ are minimal tripotents.*

Proof. For every minimal tripotent $u \in E^{j,k}$ the elements $u_j = 2\{uue_j\}$ and $u_k = 2\{uue_k\}$ are minimal tripotents. On the other hand, by the irreducibility assumption, every pair of minimal tripotents $v \in \overline{\Omega}(e_j)$, $w \in \overline{\Omega}(e_k)$ occurs this way. The claim now follows with (9.3) and Lemma 9.10. □

9.12 Proposition. *The Levi cone C_a at a is $X(a)$ if a is invertible and is $Z(a)$ if a is not invertible. In particular, C_a is always a closed cone.*

Proof. From 9.10 and 9.11 we derive $C_a \subset Z(a)$ and also that $C_a \subset X(a)$ holds in case a is invertible. For the proof of the opposite inclusions it is enough to show $X(a) \subset C_a$ because of 9.9 and 9.10. We show by induction on s that $X(a)$ is in the sum of all $Z_{j,k}(a)$ with $1 \le j < k \le s$. For $s = 1$ the claim is obvious, so assume $s > 1$. Fix $x = x_1 + \ldots + x_s \in X(a)$ with $x_j \in A(e_j)$. Then (9.5) implies

$$(x_s^+ | a^\dagger) \quad \le \quad \sum_{k<s} (x_k^- | a^\dagger)\,,$$

and after subtracting from x a suitable element of $Z_{1,s} + \cdots + Z_{s-1,s} \subset X(a)$ we may therefore assume without loss of generality that $x_s^+ = 0$. But then $(x|a^\dagger) = 0$, $(x_s|a^\dagger) \le 0$ and $\sum_{k<s}(x_k|a^\dagger) \le 0$ imply the equalities and hence $x_s = 0$. By the induction hypothesis, x is in the sum of all $Z_{j,k}(a)$ with $1 \le j \le s-1$. □

For applications it is important to know that the Levi cone C_a is big. Clearly $Z(a)$ always contains inner points. On the other hand, $X(a) = 0$ holds in case $s \le 1$, that is, if a is proportional to a tripotent.

9.13 Lemma. *In case $s > 1$ the cone $X(a)$ has inner points with respect to the hyperplane $\{x \in A(a) : (x|a^\dagger) = 0\}$.*

Proof. For every j denote by $r_j \ge 1$ the rank of the Jordan algebra $A(e_j)$ (which is the maximal length of a sequence of orthogonal minimal idempotents in $A(e_j)$). Define inductively positive real numbers $\alpha_2, \ldots, \alpha_s$ with $\alpha_{j+1} r_{j+1} > \alpha_j r_{j-1}$ for all $1 < j < s$. Then

$$\begin{aligned} v :=& \sum_{j=2}^{s} \alpha_j (r_{j-1}\lambda_j e_j - r_j \lambda_{j-1} e_{j-1}) \\ =& -\alpha_2 r_2 \lambda_1 e_1 + \Big(\sum_{j=2}^{s-1} (\alpha_j r_{j-1} - \alpha_{j+1} r_{j+1}) \lambda_j e_j\Big) + \alpha_s r_{s-1} \lambda_s e_s \end{aligned}$$

is in $X(a)$ and the components $v_j < 0$ in $A(e_j)$ for $1 \le j < s$, that is, v is an inner point of $X(a)$ with respect to the hyperplane $(x|a^\dagger) = 0$. □

We want to give a more geometrical meaning to the cones $X(a)$, $Z(a)$ and hence to the Levi cone C_a. For this define the multiplicative analogue to $\| \; \|_k$ in (5.7) by

$$\mu_k := \prod_{j=1}^{k} \sigma_j \quad \text{for all} \quad k \ge 1\,, \tag{9.14}$$

where $\sigma_1, \ldots, \sigma_r$ are the singular values defined in section 5. Then every μ_k is a continuous, piecewise smooth K-invariant function on E with $\mu_k(tz) = |t|^k \mu_k(z)$ for all $t \in \mathbb{C}$ and all $z \in E$. Consider for $\rho := \operatorname{rank}(a)$ the K-invariant compact sets

$$\begin{aligned} \mathcal{Z}(a) :=& \big\{ z \in E : \mu_k(z) \le \mu_k(a) \text{ for } 1 \le k \le r \big\} \\ \mathcal{Y}(a) :=& \big\{ z \in \mathcal{Z}(a) : \mu_\rho(z) = \mu_\rho(a) \big\}\,. \end{aligned} \tag{9.15}$$

It is clear that $\mathcal{Z}(a)$ is a compact subset of $\overline{E}_{[\rho]}$. Recall that the orbit $K = \mathsf{K}(a)$ is generic in the complex manifold $E_{[\rho]}$ and that the tangent space to $E_{[\rho]}$ at a is $E_1(a) \oplus E_{1/2}(a) = A(a) \oplus T_a K$.

In case a is invertible in E and N is a generic norm on E we also consider the compact S-invariant set

$$\mathcal{X}(a) := \{z \in \mathcal{Z}(a) : N(z) = N(a)\} . \tag{9.16}$$

It is clear that $\mathcal{Y}(a) = \mathbb{T}(\mathcal{X}(a))$ is isomorphic to $(\mathbb{T}/3.3_\rho) \times \mathcal{X}(a)$, where $\mathbb{T} \subset \mathsf{GL}(E)$ is the circle group and 3.3_ρ is identified with the subgroup $\{t \in T : t\mathcal{X}(a) = \mathcal{X}(a)\}$. The following proposition describes a relation between the (Whitney) tangent cone to $\mathcal{Z}(a)$ and the Levi cone of the orbit $\mathsf{K}(a)$.

9.17 Proposition. *The tangent cone at a to $\mathcal{Z}(a)$ satisfies $T_a\mathcal{Z}(a) = Z(a) \oplus T_aK$ for $K = \mathsf{K}(a)$.*

Proof. In the above notation let r_j be the rank of the Jordan algebra $A(e_j)$ and choose a representation $e_j = e_j^1 + \ldots + e_j^{r_j}$ with orthogonal minimal idempotents as summands for $1 \le j \le s$. This choice is unique up to a transformation from the isotropy group K_a. Denote by F the $\mathbb{R}$-linear span of all e_j^k and identify F with $\mathbb{R}^\rho$ for $\rho = r_1 + \ldots + r_s = \text{rank}(a)$ by fixing (e_j^k) as a basis. In this sense, the point $a \in F$ corresponds to $(a_j^k) \in \mathbb{R}^\rho$ with $a_j^k = \lambda_j$ for all j, k. Now put $J := \{(j,k) \in \mathbb{N}^2 : 1 \le j \le s,\ 1 \le k \le r_j\}$ and denote by $\mathcal{F}$ the set of all functions $f = f_I$ on F of the form $f(x) = \prod_{(j,k)\in I} x_j^k$, where $I \subset J$ is any nonempty subset satisfying the following property: $(j,k) \in I$, $(n,m) \in J$ *and* $n < j$ *imply* $(n,m) \in I$. For a suitable compact neighbourhood $U \subset F$ of a we have

$$U \cap \mathcal{Z}(a) = \{x \in U : f(x) \le f(a) \ \ \forall f \in \mathcal{F}\}$$

and

$$T_a(F \cap \mathcal{Z}(a)) = \{x \in F : df_a(x) \le 0 \ \ \forall f \in \mathcal{F}\} . \tag{9.18}$$

But $df_a(x) = f(a) \sum_I \lambda_j^{-1} x_j^k$ for all $f = f_I \in \mathcal{F}$, that is,

$$T_a(F \cap \mathcal{Z}(a)) = F \cap Z(a)$$

as a consequence of (9.7). Since $Z(a) \subset A(a) = \mathsf{K}_a(F)$, it follows that

$$A(a) \cap T_a\mathcal{Z}(a) \supset Z(a). \tag{9.19}$$

On the other hand, since K_a is compact and both $Z(a)$ and $\mathcal{Z}(a)$ are K_a-invariant, (9.18) implies also the opposite inclusion in (9.19) and hence the required statement. □

In a similar way it can be shown:

9.20 Proposition. *$T_a\mathcal{Y}(a) = X(a) \oplus T_aK$ for $K = \mathsf{K}(a)$. In case $a \in E$ is invertible, also $T_a\mathcal{X}(a) = X(a) \oplus T_aS$ holds for the orbit $S = \mathsf{S}(a)$.*

As a consequence of Propositions 9.12, 9.17 and 9.20 we obtain:

9.21 Corollary. *If a is not invertible, the tangent cone of $\mathcal{Z}(a)$ at a coincides with the sum of the Levi cone of $K = S$ and $T_aK = T_aS$. If a is invertible, the tangent cone of $\mathcal{Y}(a)$ (resp. $\mathcal{X}(a)$) at a coincides with the sum of the Levi cone of K and T_aK (resp. of S and T_aS).*

10. The domains $\mathcal{D}(a)$ and $\mathcal{B}(a)$

In the following let E be a factor and $a \in E$ an arbitrary element. Then

$$(10.1) \qquad \mathcal{D}(a) := \big\{z \in \overline{E}_{[\rho]} : \mu_k(z) < \mu_k(a) \text{ for } 1 \le k \le \rho\big\}, \qquad \rho := \operatorname{rank}(a),$$

is a bounded balanced domain in the complex-analytic cone $\overline{E}_{[\rho]}$, that is, $t\mathcal{D}(a) \subset \mathcal{D}(a)$ for every complex number t with $|t| \le 1$. In case $a \ne 0$ the domain $\mathcal{D}(a)$ is nonempty, its closure is the compact set $\mathcal{Z}(a)$ defined in (9.15), and its boundary $\partial\mathcal{D}(a)$ consists of all $z \in \mathcal{Z}(a)$ with $\mu_k(z) = \mu_k(a)$ for some $k \le \rho$. The orbit $K = \mathsf{K}(a)$ coincides with the subset $\{z \in \mathcal{Z}(a) : \mu_k(z) = \mu_k(a) \text{ for all } k \le \rho\} \subset \partial\mathcal{D}(a)$.

We are interested in the holomorphic structure as well as the boundary structure of $\mathcal{D}(a)$. We start with a technical lemma in the special case when all nonzero singular values of a have multiplicity 1, that is, when a is reduced in the sense of section 5.

10.2 Lemma. *Assume that $a \in E$ is reduced and has rank $\rho > 0$. Then for every $1 \le k \le \rho$ and every $c \in \mathcal{Z}(a)$ with $\mu_k(c) = \mu_k(a)$ the function μ_k is real-analytic in a neighbourhood of $c \in E$ and its differential $d\mu_k$ does not vanish at c.*

Proof. Suppose that $\mu_k(c) = \mu_k(a)$, i.e. $\sigma_1(c)\cdots\sigma_k(c) = \lambda_1\cdots\lambda_k$, where $\lambda_j := \sigma_j(a)$ for all j. Together with $\mu_{k-1}(c) \le \mu_{k-1}(a)$ and $\mu_{k+1}(c) \le \mu_{k+1}(a)$ this implies

$$(10.3) \qquad \sigma_k(c) \ge \lambda_k > \lambda_{k+1} \ge \sigma_{k+1}(c)\,.$$

Denote by $\mathcal{I}$ the set of all subsets of $\{1,\dots,r\}$ having cardinality k. Then (10.3) implies $\mu_k(c) > \sigma_I(c) := \prod_{i\in I}\sigma_i(c)$ for all $I \in \mathcal{I}$ that are different from $\{1,\dots,k\}$. Let $\mathcal{C} := \mathcal{C}(E,\mathbb{R})$ be the algebra of all real valued continuous functions on E and define $p \in \mathcal{C}[t]$ by

$$(10.4) \qquad p(z,t) := \prod_{I\in\mathcal{I}}\big(t - \sigma_I(z)^2\big)\,.$$

The coefficients of p are symmetric polynomials in $\sigma_1(z)^2,\dots,\sigma_r(z)^2$ and hence are polynomials in the corresponding elementary symmetric functions $m_j(z,z)$, compare (8.9). This implies $p \in \mathcal{A}[t]$, where $\mathcal{A} \subset \mathcal{C}$ is the subalgebra of all real polynomials. Moreover, $\mu_k(c)^2$ is a simple root of $p(c,t) \in \mathbb{R}[t]$, hence by the implicit function theorem μ_k^2 is real-analytic in a neighbourhood of $c \in E$. Because of $\mu_k(c) \ne 0$, also μ_k is real-analytic near c.
Write $c = \lambda_1c_1 + \ldots + \lambda_rc_r$ for a suitable frame $(c_1,\dots,c_r)$ in E and put $F := \mathbb{R}c_1 \oplus\cdots\oplus \mathbb{R}c_r$. Then $\mu_k(x) = x_1\cdots x_k$ for all $x = x_1c_1 + \ldots + x_rc_r \in F$ near c as a consequence of (10.3). This together with $\mu_k(c) \ne 0$ implies that the differential of the restriction $\mu_k|_F$ does not vanish at c. □

10.5 Proposition. *For every $a \in E$ the complex space $\mathcal{D}(a)$ is Stein.*

Proof. Let us first assume that a has the maximal rank r and is reduced. For every $1 \le k \le r$ put

$$\mathcal{D}_k := \{z \in E : \mu_k(z) < \mu_k(a)\}\,.$$

Every $\mathcal{D}_k$ is a K-invariant domain in E and $\mathcal{D}(a)$ is the intersection of all $\mathcal{D}_k$. In particular, any boundary point $c \in \partial\mathcal{D}(a)$ is contained in the boundary $\partial\mathcal{D}_k$ for some k. Consider an arbitrary $c \in \partial\mathcal{D}(a) \cap \partial\mathcal{D}_k$. By Lemma 10.2 the boundary $M := \partial\mathcal{D}_k$ is smooth in a neighbourhood of $c \in E$. On the other hand, M is fibered in K-orbits, which implies $H_cK \subset H_cM$ for the holomorphic tangent spaces at c.
We next compute the Levi form of M at c. By Proposition 9.17, $\Lambda_c(u,u) \in Z(c) = \pi(T_c\mathcal{Z}(c))$

for any $u \in H_cK$, where Λ_c denotes the Levi form of K at c and $\pi\colon E \to A(c)$ is the orthogonal projection. Since $\mathcal{Z}(c) \subset \mathcal{D}_k$, it follows that the restriction of the Levi form Λ_c^M of M to H_cK points inside $\mathcal{D}_k$, i.e. is positive semidefinite. The full holomorphic tangent space H_cM is the direct sum of $H_cK = E_{1/2}(c)$ and $E_1(c) \cap H_cM$. We next claim that these spaces are orthogonal with respect to the Levi form Λ_c^M. Indeed, using the action of K on M and the local slice $A(c) \cap M$, we can choose real-analytic coordinates and real CR vector fields ξ and η on M with $\xi_w \in E_1(w) \cap H_wM$, $\eta_w \in E_{1/2}(w)$ for w near c such that ξ has constant coefficients and the coefficients of η are constant along $E_1(c) \cap M$. Taking Lie brackets of ξ and η at c verifies the claim.
Finally, the intersection $\partial\mathcal{D}_k \cap E_1(c)$ is Levi flat since it is locally given by $|w_1 \cdots w_k| = \mu_k(a)$. Summing up, we obtain that the Levi form of $\partial\mathcal{D}_k$ is positive semidefinite and therefore $\mathcal{D}_k$ is locally Stein near c, i.e. every point $b \in \partial\mathcal{D}_k$ sufficiently close to c has an open neighbourhood W in E such that $\mathcal{D}_k \cap W$ is Stein. Since $\mathcal{D}$ is locally an intersection of $\mathcal{D}_k$'s for $1 \le k \le r$, it is also locally Stein. Since $\mathcal{D}(a)$ is a bounded domain in E, it is Stein (see e.g. [20], Theorem 2.6.10).
Let now a be of maximal rank r but not necessarily reduced. Then it is easy to construct a sequence (a_m) in $A(a) \cap \mathcal{D}(a)$ of reduced elements converging to a in such a way that $\mathcal{D}(a)$ is the increasing union of $\mathcal{D}(a_m)$ and $\mathcal{D}(a_m)$ is relatively compact in $\mathcal{D}(a_{m+1})$ for all m. Then $\mathcal{D}(a)$ is Stein by a theorem of Behnke-Stein (see e.g. [15]).
Suppose finally that a is of rank $\rho < r$ and choose a representation (5.3) satisfying (5.5). Then $\lambda_1 \ge \cdots \ge \lambda_\rho > \lambda_{\rho+1} = 0$ and the element $b := r_1e_1 + \cdots + r_re_r$, where $r_j = \lambda_j$ for $1 \le j \le \rho$ and $r_{\rho+1} = \cdots = r_r = \lambda_\rho$, is of rank r. Since, $\mathcal{D}(b)$ is Stein by the above arguments, the intersection $\mathcal{D}(a) = \mathcal{D}(b) \cap \overline{E_{[\rho]}}$ is also Stein as required. □

Next let $G := \mathsf{Aut}(\mathcal{D}(a))$ be the group of all biholomorphic automorphisms of the complex space $\mathcal{D}(a)$, $a \ne 0$, endowed with the compact open topology.

10.6 Proposition. *For every $a \ne 0$ the following conditions are equivalent:*
(i) *$\mathcal{D}(a)$ is a bounded symmetric domain in E.*
(ii) *$g(0) \ne 0$ for some $g \in G = \mathsf{Aut}(\mathcal{D}(a))$.*
(iii) *G is not compact.*
(iv) *$\mathcal{D}(a)$ is convex.*
(v) *$\sigma_1 = \sigma_r$ for $r = \mathrm{rank}(E)$.*
(vi) *a is proportional to a tripotent e of rank r.*

Proof. (i) $\Longrightarrow$ (ii) follows from the fact that every bounded symmetric domain is homogeneous.
(ii) $\Longrightarrow$ (iii) By [23] the group G is a real Lie group acting properly on $\mathcal{D}(a)$. In particular, the orbit $M := G(0)$ is a closed real submanifold of $\mathcal{D}(a)$. The tangent space $W := T_aM \subset E$ is K-invariant and hence a complex linear subspace of E. Since the action of K is irreducible on E only $W = 0$ or $W = E$ are possible. Suppose, $b := g(0) \ne 0$ for some $g \in G$. Since $\mathsf{K} \subset G$, the orbit $G(b) = G(0)$ has positive dimension, i.e. $W = E$, $\mathcal{D}(a)$ is open in E and G acts transitively on $\mathcal{D}(a)$. Therefore G cannot be compact.
(iii) $\Longrightarrow$ (i) In case G is not compact it acts transitively on $\mathcal{D}(a)$ by the above argument and hence $\mathcal{D}(a)$ is symmetric.
The remaining implications are easy and left to the reader. □

For every relatively compact domain D in a Stein complex space X one has the notion of *minimal boundary.* This is a minimal subset $M \subset \overline{D}$ such that every $f \in \mathcal{C}(\overline{D}) \cap \mathcal{O}(D)$ attains its maximum modulus in M. Existence and uniqueness of the minimal boundary for D is well known (even in a much more general context, compare [6]), and its closure is called the *Shilov boundary* of D. Clearly, both boundaries are contained in the topological boundary ∂D of D

by the maximum principle. In particular, for the domain $\mathcal{D}(a) \subset E_{[\rho]}$, $a \in E$ arbitrary of rank $\rho > 0$, both boundaries coincide with the orbit $\mathsf{K}(a) \subset \partial\mathcal{D}(a)$. This is verified in the following way: Let $e_1, \ldots, e_\rho$ be pairwise orthogonal minimal tripotents in E and denote by F its complex linear span. Then

$$F \cap \mathcal{D}(a) = \Big\{ \sum_{j=1}^{\rho} z_j e_j := |z_1 z_2 \cdots z_k| < \mu_k(a) \ \text{ for } \ k = 1, \ldots, \rho \Big\}$$

easily gives the claim.

Recall that for every $a \in E$ the orbit $\mathsf{K}(a)$ is a generic CR-submanifold of the complex manifold $\mathsf{K}^{\mathbb{C}}(a)$ and that $\mathsf{K}(a)$ is minimal only if a is not invertible, in which case also $\mathsf{K}^{\mathbb{C}}(a) = \mathsf{S}^{\mathbb{C}}(a)$ holds. In case a is invertible, the orbit $\mathsf{S}(a)$ is a generic and minimal CR-submanifold of the complex manifold $\mathsf{S}^{\mathbb{C}}(a) = \{z \in E : N(z) = N(a)\}$, where N is a generic norm on E. For such a we put

$$(10.7), \qquad \mathcal{B}(a) := \{z \in \mathsf{S}^{\mathbb{C}}(a) : \mu_k(z) < \mu_k(a) \ \text{ for all } \ k < r\},$$

if $r := \operatorname{rank}(a) = \operatorname{rank}(E) > 1$, and $\mathcal{B}(a) := \emptyset$ otherwise. It can be seen as above that $\mathsf{S}(a)$ coincides with the Shilov boundary of $\mathcal{B}(a)$ in $\mathsf{S}^{\mathbb{C}}(a)$.

For shorter notation let us write $X := \mathsf{S}^{\mathbb{C}}(a)$ for the rest of this section, where $a \in E$ is a fixed invertible element. It is easy to see that $\mathcal{B}(a)$ is a relatively compact domain in X and that $\mathcal{B}(a)$ is empty if and only if the orbit $\mathsf{S}(a)$ is totally real as CR-manifold. This is also equivalent to $\sigma_1(a) = \sigma_r(a)$ and to E being of tube type. Clearly, $\mathcal{B}(a)$ is invariant under the group S, and in case $\mathcal{B}(a) \neq \emptyset$ there is a unique S-orbit in $\mathcal{B}(a)$ that plays the role of the origin 0 in $\mathcal{D}(a)$: This is the unique $\mathsf{S}(b) \subset \mathcal{B}(a)$ with $\mathcal{B}(b) = \emptyset$. Then $\mathsf{S}(b)$ is also the set of all points $z \in X$ where the function $\dim \mathsf{S}(z)$ attains its minimum. In analogy to 10.5 we have:

10.8 Proposition. *$\mathcal{B}(a)$ is Stein.*

Proof. We may assume $\mathcal{B}(a) \neq \emptyset$ and hence $\lambda_1 > \lambda_1$. Write a in the form (5.3) satisfying (5.6), where $(e_1, \ldots, e_r)$ is a frame in E. Then $\lambda_r > 0$ and in a first case we assume $\lambda_{r-1} > \lambda_r$. Then for every $t > 0$ with $\lambda_{r-1} > \lambda_r + t$ and $b := a + te_r$ we have $\mathcal{B}(a) = \mathcal{D}(b) \cap X$. But $\mathcal{D}(b)$ is Stein by Proposition 10.5, therefore also $\mathcal{B}(a)$ is Stein.
Now suppose $\lambda_{r-1} = \lambda_r$ and consider $k := \min\{j \leq r : \lambda_j = \lambda_r\} > 1$. For every integer $n \geq 1$ and $\varepsilon_n := n/(n+1)$ define $\delta_n > 1$ (uniquely) in such a way that

$$a_n := \sum_{j=1}^{k-1} \varepsilon_n \lambda_j e_j \ + \ \sum_{j=k}^{r-1} \delta_n \lambda_j e_j \ + \ \sqrt{\delta_n}\lambda_r e_r \ \in X .$$

There exists $n_0 \in \mathbb{N}$ with $\varepsilon_n \lambda_{k-1} > \delta_n \lambda_k$ for all $n > n_0$. This implies that $\mathcal{B}(a_n)$ is a Stein domain by case 1 and is contained relatively compact in $\mathcal{B}(a_{n+1})$ for every $n > n_0$. The claim now follows from

$$\mathcal{B}(a) = \bigcup_{n > n_0} \mathcal{B}(a_n) . \qquad \square$$

In case $r > 1$ the compact set $\mathcal{X}(a) = \{z \in X : \mu_k(z) \leq \mu_k(a), \ k < r\}$ defined in (9.16) has $\mathcal{B}(a)$ as interior in X. As a consequence of Propositions 10.5 and 10.8 we obtain:

10.9 Lemma. *For every a with* $\operatorname{rank}(a) = \rho$, *the compact set* $\mathcal{Z}(a) \subset \overline{E}_{[\rho]}$ *has a Stein neighbourhood basis in* $\overline{E}_{[\rho]}$. *If a is invertible, also* $\mathcal{X}(a) \subset X$ *has a Stein neighbourhood basis in X.*

Proof. Write again a in the form (5.3) satisfying (5.5). For every $t > 1$ consider $a_t := t\lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_\rho e_\rho \in \overline{E}_{[\rho]}$ and $b_t := t\lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_{r-1}e_{r-1} + t^{-1}\lambda_r e_r \in X$ in case a is invertible. Then the sets $\mathcal{D}(a_t)$ and $\mathcal{B}(b_t)$ form the required bases. □

Let us consider in more detail the special case where $K = \mathsf{K}(a)$ and $S = \mathsf{S}(a)$ bound the domains $\mathcal{D}(a)$ and $\mathcal{B}(a)$ respectively, that is, when these orbits are of hypersurface type. To avoid trivialities assume for the rest of the section that the factor E has dimension > 1.

case 1: *K is of hypersurface type.* This is precisely the case when $a \in E$ has rank 1, and then $K = S$. Therefore, up to multiplication with a real factor $t > 0$, there exists a unique K-orbit of this type in E, and we may assume that a is a minimal tripotent in E. In particular, $K = \{z \in E_{[1]} : \|z\|_2 = 1\}$ and $\mathcal{D}(a) = \{z \in \overline{E_{[1]}} : \|z\|_2 < 1\}$, where the K-invariant euclidian norm $\|\,\|_2$ on E is defined in (5.8). Then every continuous CR-function on K has a unique continuous extension to $\mathcal{Z}(a) = \mathcal{D}(a) \cup K$ (cf. Theorem 12.1 below), which is holomorphic on $\mathcal{D}(a)\backslash\{0\}$. Except for the types $\mathbf{I}_{1,q}$ of rank 1 the origin in the domain $\mathcal{D}(a)$ is (the only) singular point. In case E is of type $\mathbf{I}$, the orbit K is simply connected, compare 2.2, and the same is true if E is of type $\mathbf{II}$. In case E is of type $\mathbf{III}_p$, the orbit K is isomorphic to the real projective space $\mathbb{P}_{2p-1}(\mathbb{R})$. The universal covering of K then is the euclidian sphere $M := \{z \in \mathbb{C}^{1\times p} : zz^* = 1\}$ with its induced CR-structure. The covering map $M \to K$ is CR and is given by $z \mapsto z'z$, where z' is the transpose of z.

case 2: *S is of hypersurface type and $S \neq K$.* This case can only happen if E has rank 2 and is of tube type, i.e. if E is of type $\mathbf{IV}_n$ for some $n \geq 3$ (recall the list of coincidences for low dimensions at the end of section 1). Then $\mathsf{S} = \mathsf{SO}(n)$, a generic norm on $E = \mathbb{C}^n$ is given by $N(z) = 2^{-1}\sum_j z_j^2$ and (e_1, e_2) is a frame in E for

$$e_1 := \big(1/\sqrt{2}, i/\sqrt{2}, 0, \ldots, 0\big) \quad \text{and} \quad e_2 := \big(1/\sqrt{2}, -i/\sqrt{2}, 0, \ldots, 0\big) = \overline{e_1}\,.$$

For every $0 \leq t \leq 1$ denote by S_t the S-orbit of the point $e_1 + te_2$ in E. Then it is easily seen that every nonzero S-orbit in E is of the form $c \cdot S_t$ for some complex factor c and and some $0 \leq t \leq 1$. Every S_t is contained in the complex affine quadric $Q_t := \{z \in E : N(z) = t\}$ and is a real hypersurface there if $t < 1$, in which case S_t is strictly pseudo-convex and isomorphic as $\mathsf{SO}(n)$-space to the Stiefel manifold $\mathsf{SO}(n)/\,\mathsf{SO}(n-2)$, which is simply connected except for $n = 3$. The orbit S_1 is totally real and isomorphic to the $(n-1)$-sphere and S_0 is the rank-1-orbit from case 1. None of the S-orbits S_t, $0 < t < 1$, is a K-orbit in this case.

Let us now specialize to the case $n = 3$, for which the orbits S_t appear e.g. in [21] (p. 162–165) in a different realization. Then E is isomorphic to the factor $F := \{z \in \mathbb{C}^{2\times 2} : z' = z\}$ of type $\mathbf{III}_2$, on which the group $\mathsf{SU}(2)/\{\pm\mathbb{1}\} = \mathsf{SO}(3)$ acts by $z \mapsto uzu'$. Via this identification every CR-manifold S_t can be identified with the $\mathsf{SU}(2)/\{\pm\mathbb{1}\}$-orbit of the diagonal matrix $\left(\begin{smallmatrix}1 & 0\\ 0 & t\end{smallmatrix}\right)$ in F. For $t < 1$ the orbit S_t is diffeomorphic to the real projective space $\mathbb{P}_3(\mathbb{R})$ and thus its universal covering $\widetilde{S}_t$ gives a CR-structure on the 3-sphere S^3. By [2] and [21] every Levi-nondegenerate $\mathsf{SU}(2)$-invariant CR-structure on S^3 occurs precisely once in the family $(\widetilde{S}_t)_{0\leq t<1}$.

11. Hulls of orbits

In the following let E be a factor of rank r. As defined in section 5 let $\sigma_1, \sigma_2, \ldots, \sigma_r$ be the singular value functions on E and let $\|\,\|_k = \sum_{j=1}^k \sigma_j$ be the corresponding K-invariant norms. Notice that the triangle inequality for $\|\,\|_k$ in the matrix case has already been established in [14], compare also [37].

Now fix $a \in E$ and choose a frame $(e_1, \ldots, e_r)$ in E such that (5.3) and (5.5) hold. Then $\lambda_k = \sigma_k(a)$ for $1 \le k \le r$. Throughout this section let $F := \sum_{j=1}^{r} E_{j,j}$ be the $\mathbb{C}$-linear span of $e_1, \ldots, e_r$ and identify $\mathbb{C}^r$ with F via $z \mapsto z_1 e_1 + \ldots + z_r e_r$. In this sense, the element $a \in F$ is identified with the vector $\sigma(a) \in \mathbb{R}^r$. Also, the subgroup $\{g|_F : g \in \mathsf{K}, g(F) = F\} \subset \mathsf{GL}(F)$ is identified with the subgroup $\Sigma_r \subset \mathsf{GL}(r, \mathbb{C})$ consisting of all transformations $(z_1, \ldots, z_r) \mapsto (t_1 z_{\pi(1)}, \ldots, t_r z_{\pi(r)})$, where $\pi \in \mathcal{S}_r$ is a permutation and $t_1, \ldots, t_r$ are complex numbers of absolute value 1.

11.1 Proposition. *The (linear) convex hull of the orbit $K = \mathsf{K}(a)$ is given by*

$$\mathsf{ch}(K) = \big\{ z \in E : \|z\|_k \le \|a\|_k \ \text{ for } \ k = 1, \ldots, r \big\}. \tag{11.2}$$

Proof. It is clear that the right hand side of (11.2) is convex, contains K and hence also $\mathsf{ch}(K)$. For the proof of the opposite inclusion consider the intersection $S := \mathbb{R}^r \cap K$, where as above $\mathbb{C}^r$ is identified with the subspace $F \subset E$. Then S consists of all vectors in $\mathbb{R}^r$ obtained from $\sigma(a)$ by applying all transformations $(x_1, \ldots, x_r) \mapsto (\varepsilon_1 x_{\pi(1)}, \ldots, \varepsilon_r x_{\pi(r)})$ where $\pi \in \mathcal{S}_r$ is a permutation and $\varepsilon_j = \pm 1$ for all j. By [30], Theorem 1.2, the convex hull of S consists precisely of all $x \in \mathbb{R}^r$ satisfying $\sum_{j \le k} |x_j| \le \sum_{j \le k} \sigma_j(a)$ for all k. Since $\mathsf{ch}(K)$ is K-invariant we get the claimed inclusion. □

Next consider the polynomial convex hull

$$\mathsf{pch}(K) = \big\{ z \in E : |f(z)| \le \sup |f(K)| \quad \forall f \in \mathbb{C}[z_1, \ldots, z_r] \big\}$$

of the orbit $K = \mathsf{K}(a)$. It is clear that $\mathsf{pch}(K)$ is invariant under the group K and that it is contained in the complex-analytic cone $\overline{E}_{[\rho]}$, $\rho := \operatorname{rank}(a)$, since all N_j in (8.5) are polynomials. For the compact set $\mathcal{Z}(a)$ defined in (9.15) we have

11.3 Proposition. *The polynomial convex hull $\mathsf{pch}(K)$ contains the set $\mathcal{Z}(a)$.*

Proof. Fix $b \in \mathcal{Z}(a)$. Then $\operatorname{rank}(b) \le \rho = \operatorname{rank}(a)$. Identify $\mathbb{C}^\rho$ with the linear subspace $\{z \in F : z_j = 0 \text{ for } j > \rho\}$ of E and put $c_j := \log(\sigma_j(a))$ for $1 \le j \le \rho$. After applying a suitable transformation from K to b we may assume $b \in \mathbb{R}^\rho \subset F$. We have to show $b \in B := \mathbb{R}^\rho \cap \mathsf{pch}(K)$. Since the polydisk $\{z \in \mathbb{C}^\rho : |z_j| \le \exp(c_j)\}$ clearly is contained in $\mathsf{pch}(K)$ we have $\exp(C) \subset B$ for $C := \{x \in \mathbb{R}^\rho : x \le c\}$, where $\exp : \mathbb{R}^\rho \to \mathbb{R}^\rho$ is defined coordinate-wise and $x \le c = (c_1, \ldots, c_\rho)$ means $x_j \le c_j$ for all $1 \le j \le \rho$. The symmetric group $\mathcal{S}_\rho$ acts on $\mathbb{R}^\rho$ by permuting coordinates and hence also $\exp(\pi(C)) \subset B$ for every $\pi \in \mathcal{S}_\rho$. But then also $\exp(\widehat{C}) \subset B$, where $\widehat{C}$ is the convex hull of the subset $\mathcal{S}_k(C) \subset \mathbb{R}^\rho$. But $x \in \mathbb{R}^\rho$ is in $\widehat{C}$ if and only if $x \le r$ for some $r \in R$, where we denote by R the convex hull of the finite set $\mathcal{S}_\rho(c) \subset \mathbb{R}^\rho$. But by Theorem 1.1 in [30]

$$\begin{aligned} R \ &= \mathcal{S}_\rho\Big(\big\{ x \in P : \sum_{j=1}^{h} x_j \le \sum_{j=1}^{h} c_j, \ h = 1, \ldots, \rho \big\} \Big) \text{ for} \\ P &:= \big\{ x \in \mathbb{R}^\rho : x_1 \ge x_2 \ge \ldots \ge x_\rho \big\} \,. \end{aligned}$$

This implies $b \in \exp(\widehat{C}) \subset B$ in case $b > 0$. But then also $b \in B$ without any further assumption on b since B is compact. □

For the proof of the opposite inclusion in Proposition 11.3 it is necessary to find suitable *peak functions:* For every tripotent $c \in E$ denote by $P_c : E \to E_1(c)$ the canonical projection with respect to the Peirce decomposition (5.1) and by N_c the generic norm of the complex Jordan algebra $E_1(c)$. Then $f_c := N_c \circ P_c$ is a complex polynomial of degree rank(c) on E.

11.4 Lemma. *In case E is a classical factor, i.e. one of the types* $\mathbf{I} - \mathbf{IV}$, $|f_c(z)| \le \mu_k(z)$ *holds for every $z \in E$ and every tripotent c of rank k.*

Proof. We begin with the special case where $E = \mathbb{C}^{p\times p}$ is the type $\mathbf{I}_{p,p}$. We realize c as diagonal matrix with diagonal entries $c_{jj} = \sigma_j(c)$. Then we have to show for every $z \in E$ that the absolute value of its k-principal minor is bounded by $\mu_k(z)$, more precisely, if we write z as block matrix $\binom{xy}{uv}$ with $x \in \mathbb{C}^{k\times k}$, $y \in \mathbb{C}^{k\times l}$, $u \in \mathbb{C}^{l\times k}$ and $v \in \mathbb{C}^{l\times l}$ for $l := p - k$, then $|\det(x)| \le \mu_k(z)$ must be shown, or equivalently, $\det(xx^*) = |\det(x)|^2 \le \mu_k(z)^2$.
Denote for every hermitian matrix h (of any size) by $\lambda_1(h) \ge \lambda_2(h) \ge \cdots$ its eigenvalues (with multiplicities counted) in decreasing order. Then $\mu_k(z)^2 = \lambda_1(zz^*)\cdots\lambda_k(zz^*)$. On the other hand $\lambda_j(xx^* + yy^*) \le \lambda_j(zz^*)$ holds for every $j \le k$ since $xx^* + yy^*$ is the upper $k \times k$-diagonal block of zz^*, compare e.g. [37] p. 107. From Theorem 2 in [37] we get $\lambda_j(xx^*) \le \lambda_j(xx^* + yy^*) + \lambda_1(-yy^*)$ and hence $\lambda_j(xx^*) \le \lambda_j(xx^* + yy^*)$ since $\lambda_1(-yy^*) \le 0$. Then $\det(xx^*) = \lambda_1(xx^*)\cdots\lambda_k(xx^*)$ proves the claim in the special case of type $\mathbf{I}_{p,p}$.
Now let E be arbitrary (classical). Then E can be realized as a subtriple $E \subset \widetilde{E} := \mathbb{C}^{p\times p}$ for some $p \ge 1$. Denote by $\psi : E \hookrightarrow \widetilde{E}$ the canonical embedding. Since every $g \in \mathsf{K}$ can be extended to a triple automorphism of $\widetilde{E}$, there is an integer $d \ge 1$ with rank$(\psi e) = d$ for every minimal tripotent $e \in E$. In particular, $\sigma_j(z) = \sigma_{d\cdot j-l}(\psi z)$ for all $z \in E$, $j \ge 1$ and $0 \le l < d$. Then, by the special case above, $|f_c(z)|^d = |f_{\psi c}(\psi z)| \le \mu_{d\cdot k}(\psi z) = \mu_k(z)^d$, proving the Lemma. □

Notice that in the above proof $d = 1$ holds if E is of type **I** or **III** and that $d = 2$ if E is of type **II**. In the latter case $f_c(z)$ is the Pfaffian determinant and $f_{\psi c}(\psi z)$ is the usual determinant of the upper $2k\times 2k$-diagonal block of z. The claim for the type **IV** also is a consequence of the following Lemma.

11.5 Lemma. *Suppose that E is arbitrary and that the tripotent $c \in E$ is minimal or invertible in E. Then $|f_c(z)| \le \mu_k(z)$ holds for $k :=$ rank(c) and every $z \in E$.*

Proof. In case $k = 1$ the claim follows from the fact that then $\mu_1 = \sigma_1$ is a norm (in the sense of a Banach space) on E and that the Peirce projection $P_c : E \to E_1(c) = \mathbb{C}c$ is a contraction with respect to this norm. In case c is invertible, the claim follows from $|f_c| = \mu_k(z)$ for all z since then f_c is a generic norm on E. □

As a consequence we get in case E is classical

11.6 Corollary. *Let $a, b \in E$ be points with $\mu_k(b) = \mu_k(a)$ for some k with $1 \le k \le r$. Then there is a tripotent $c \in E$ of rank k with $|f_c(z)| \le |f_c(b)| = \mu_k(a)$ for all $z \in \mathcal{Z}(a)$.*

Proof. Write b in the form $b = \beta_1 c_1 + \ldots + \beta_r c_r$ with $\beta_j := \sigma_j(b)$ and $(c_1, \ldots, c_r)$ a frame in E. For the tripotent $c := c_1 + \ldots + c_k$ then $f_c(b) = \beta_1\cdots\beta_k = \mu_k(b) = \mu_k(a)$ holds. By Lemma 11.4 therefore $|f_c(z)| \le \mu_k(a)$ holds for all $z \in K$. But then this inequality holds for all $z \in \mathcal{Z}(a)$ as a consequence of Proposition 11.3. □

Putting together Proposition 11.3 and Corollary 11.6 now gives immediately

11.7 Theorem. *For every classical factor E and every $a \in E$ the compact set $\mathcal{Z}(a)$ is the polynomial convex hull of the orbit $K = \mathsf{K}(a)$.*

We expect that Theorem 11.7 also holds in the exceptional case. For a proof it would be necessary to find a substitute in the nonassociative case for the elaborate estimates of eigenvalues of hermitian matrices in [37].

An easy consequence of Theorem 11.7 is the inclusion $\mathsf{GL}(\mathsf{K}(a)) \subset \mathsf{GL}(\mathcal{D}(a))$. Since $\mathsf{K}(a)$ is the Shilov boundary of $\mathcal{D}(a)$, we actually have the equality $\mathsf{GL}(\mathsf{K}(a)) = \mathsf{GL}(\mathcal{D}(a))$. We now use a characterization of the structure group of E from [25] to show that all groups $\mathsf{GL}(\mathsf{K}(a))$, $a \neq 0 \in E$, are the same.

11.8 Proposition. *Let E be a classical factor and let $\mathcal{D} = \{z \in E : \sigma_1(z) < 1\}$ be the corresponding bounded symmetric domain. Then $\mathsf{GL}(S) \subset \mathsf{GL}(K) = \mathsf{GL}(\mathcal{D}) = \mathsf{GL}(\mathcal{D}(a))$ for every $a \neq 0$ and $S := \mathsf{S}(a)$, $K := \mathsf{K}(a)$. In particular, K is the connected identity component of the compact group $\mathsf{GL}(K)$. In case $a \in E$ is invertible, S is the connected identity component of $\mathsf{GL}(S)$.*

Proof. By definition, K is the connected identity component of the compact group $\mathsf{GL}(\mathcal{D})$. Since the action of $\mathsf{GL}(\mathcal{D})$ does not change singular values, $\mathsf{GL}(\mathcal{D}) \subset \mathsf{GL}(K)$. For $\rho := \mathrm{rank}(a)$ every $g \in \mathsf{GL}(K)$ leaves invariant the cone $E_{[\rho]} \subset E$ and hence is in the structure group of E, compare [25] Proposition 5.3 for details. But, $\mathsf{GL}(\mathcal{D})$ is a maximal compact subgroup of the structure group, i.e. $\mathsf{GL}(K) = \mathsf{GL}(\mathcal{D})$. □

Next we consider the rational convex hull

$$\mathsf{rch}(K) := \{z \in E : |f(z| \le \sup |f(K)| \;\; f \in \mathcal{R}\} ,$$

where $\mathcal{R}$ denotes the space of all rational functions f on E that are holomorphic in a suitable neighbourhood (depending on f) of K and where $|f(z)| \le t$ for some real $t > 0$ in particular includes that f is holomorphic in z. Clearly, $\mathsf{rch}(K) \subset \mathsf{pch}(K)$ always holds. As an application of our main results in the next section we will see that, if E is classical, the equality $\mathsf{rch}(K) = \mathsf{pch}(K)$ holds if and only if $a \in E$ is not invertible (see Corollary 12.2).

12. Global extension of CR-functions on orbits and applications

In this section we discuss for every factor E (i.e. an irreducible PJT) the following problem: Given $a \in E$ and a continuous CR-function f on the orbit $K = \mathsf{K}(a)$ (resp. $S = \mathsf{S}(a)$), to which subsets $\mathcal{H} \subset E$ containing K (resp. S) in its closure can f be uniquely 'holomorphically' extended in a reasonable sense. We have seen in section 8 that K and S are contained in the closed complex-analytic subsets

$$Z := \overline{\mathsf{K}^{\mathbb{C}}(a)} \subset E \quad \text{and} \quad X := \overline{\mathsf{S}^{\mathbb{C}}(a)} \subset E$$

respectively, where $Z = \overline{E}_{[\rho]}$ if a is not invertible and $\rho := \mathrm{rank}(a)$ and the orbit $\mathsf{S}^{\mathbb{C}}(a)$ is closed in E if a is invertible. Hence any subset $\mathcal{H} \subset E$ as above must be contained in the corresponding complex-analytic subsets. It is also clear that such $\mathcal{H}$ must be contained in the polynomial convex hull. Hence it is suggested by Theorem 11.7 that the best choice for $\mathcal{H}$ is $\mathcal{H} = \mathcal{Z}(a)$ in case a is not invertible and $\mathcal{Y}(a)$ (resp. $\mathcal{X}(a)$) otherwise. Our main result generalizing Theorem 2.5 shows that this choice is indeed possible and is the best, i.e. $\mathcal{H}$ cannot be chosen any larger. We write $\widehat{Z}$ for the normalization of Z that is homeomorphic to Z and biholomorphic outside 0 and $\widehat{\mathcal{Z}}(a) \subset \widehat{Z}$ for the preimage of $\mathcal{Z}(a)$ under the normalization map. In case E is classical, $\widehat{Z} = Z$ and $\widehat{\mathcal{Z}}(a) = \mathcal{Z}(a)$ by Proposition 8.3.

12.1 Theorem. *If a is not invertible, every continuous CR-function on $K = S$ has a unique continuous extension to $\widehat{\mathcal{Z}}(a)$ that is holomorphic in its interior with respect to $\widehat{Z}$. If a is invertible, every continuous CR-function on K (resp. S) has a unique continuous extension to $\mathcal{Y}(a)$ (resp. $\mathcal{X}(a)$) that is holomorphic (resp. CR) in its interior with respect to Z (resp. X). Furthermore, the sets $\mathcal{Z}(a)$, $\mathcal{Y}(a)$ and $\mathcal{X}(a)$ are maximal in the following sense. If $\mathcal{H}$ is any* *domain in $\widehat{Z}$ (resp. Z or X) containing the interior of $\widehat{\mathcal{Z}}(a)$ (resp. $\mathcal{Y}(a)$ or $\mathcal{X}(a)$) with the above extension property, then necessarily $\mathcal{H} \subset \widehat{\mathcal{Z}}(a)$ (resp. $\mathcal{H} \subset \mathcal{Y}(a)$ or $\mathcal{H} \subset \mathcal{X}(a)$).*

As an immediate application of Theorems 11.7 and 12.1, we obtain:

12.2 Corollary. *Let E be classical. If $a \in E$ is not invertible, the rational convex hull of $K = S$ is $\mathcal{Z}(a)$. If $a \in E$ is invertible, the rational convex hulls of S and K are $\mathcal{X}(a)$ and $\mathcal{Y}(a)$ respectively. In the last case also the polynomial convex hull of S is $\mathcal{X}(a)$.*

We shall obtain Theorem 12.1 as a consequence of the following two statements.

12.3 Proposition. *Suppose that E is a factor and $a \in E$ is not invertible. Then every continuous CR-function f on $K = \mathsf{K}(a)$ has a unique continuous extension to $\mathcal{Z}(a)$ that is holomorphic on $\mathcal{D}(a)\backslash\{0\}$.*

12.4 Proposition. *Suppose that E is a factor and $a \in E$ is invertible. Then every continuous CR-function f on $S = \mathsf{S}(a)$ (resp. on $K = \mathsf{K}(a)$) has a unique continuous extension to $\mathcal{X}(a)$ (resp. to $\mathcal{Y}(a) = \mathbb{T}(\mathcal{X}(a))$) that is holomorphic on $\mathcal{B}(a)$ (resp. CR on $\mathbb{T}(\mathcal{B}(a))$).*

If E is classical, the complex space $\mathcal{D}(a)$ is normal by Proposition 8.3. Hence, in this case, the conclusion of Proposition 12.3 can be slightly strengthened:

12.5 Corollary. *If E is classical, the extension to $\mathcal{Z}(a)$ given by Proposition 12.3 is in fact holomorphic on $\mathcal{D}(a)$.*

For the proof of 12.3 and 12.4 we need some general extension results. Let X be a complex manifold and $M \subset X$ be a smooth real submanifold. Suppose there is given a smooth submersion $\varphi : M \to \mathbb{R}$ such that $M_x := \varphi^{-1}(\varphi(x))$ is a generic submanifold of X for every $x \in M$ (i.e. $T_xM_x + iT_xM_x = T_xX$). Assume furthermore that there is fixed a hermitian metric on X and denote by $N_x \subset T_xX$ for every $x \in M$ the (real) orthogonal complement to T_xM_x in T_xX. Then the Levi cone C_x of M_x at x can be considered in a natural way as a convex cone in the normal space N_x. For every cone $B \subset N_x$ we write $B \ll C_x$ if the intersection of the closure $\overline{B}$ with the unit sphere in N_x is contained in the interior of the cone C_x. Furthermore, for every $\varepsilon > 0$ denote by B_ε the intersection of B with the ball with center $0 \in N_x$ and radius ε.

The following local extension result is a deformation version of Theorem 1.1 in [8] whose proof can be obtained by a direct adaptation of the proof given there (a simpler proof can be obtained with a method of [5]) and of the proof of the approximation theorem in [4].

12.6 Lemma. *Let $X = \mathbb{C}^n$ with hermitian metric given by the standard scalar product on and let $M \subset X$ be a smooth submanifold. Let $\varphi : M \to \mathbb{R}$ be a smooth submersion such that the submanifold $M_x := \varphi^{-1}(\varphi(x))$ is generic in X for every $x \in M$. Suppose furthermore that $B \ll C_y$ is an open cone in N_y for some $y \in M$, where $N_y \subset T_yX$ is the normal space and $C_y \subset N_y$ is the Levi cone of M_y at y. Then there exists an open neighbourhood U of y in M and an $\varepsilon > 0$ such that for every $x \in U$*

(i) *$W_x := (U \cap M_x) + B_\varepsilon$ is open in X and*
(ii) *every continuous CR-function on M_x extends to a function in $\mathcal{C}(M_x \cup W_x) \cap \mathcal{O}(W_x)$.*

We use Lemma 12.6 in the proof of the following global extension result that will play an important role for the proof of Propositions 12.3 and 12.4.

12.7 Proposition. *Let X be a complex manifold, $M \subset X$ a smooth connected submanifold and $\varphi : M \to \mathbb{R}$ a smooth function such that $M_x := \varphi^{-1}(\varphi(x))$ is a connected compact generic submanifold of X for every $x \in M$. Assume that there exists on M a smooth vector field ξ with $d\varphi(\xi) > 0$ such that $\pi_x(\xi_x)$ is in the interior of the Levi cone C_x of M_x for every $x \in M$, where π_x is the canonical projection $T_xX \to T_xX/T_xM_x$. Then for every $a \in M$ and*

$W := \{x \in M : \varphi(x) > \varphi(a)\}$ there exists an open neighbourhood U of W in X such that every continuous CR-function on M_a extends to a function in $\mathcal{C}(U \cup M_a) \cap \mathcal{O}(U)$.

Proof. Fix a point $a \in M$ and denote by I the set of all real numbers t with the following property: *There exists an open neighbourhood $V_t \subset X$ of $\{x \in M : \varphi(a) < \varphi(x) < t\}$ such that every continuous CR-function on M_a has a continuous extension to $V_t \cup M_a$ that is holomorphic on V_t.* It is enough to show $\tau = +\infty$ for $\tau := \sup(I) \geq \varphi(a)$.
Assume to the contrary $\tau < +\infty$. Then there exists an element $y \in \varphi^{-1}(\tau)$, and $d\varphi(\xi_y) > 0$ implies the existence of an open cone $B \ll C_y \subset T_yX/T_yM_y$ with $\pi_y(\xi_y) \in B$, where π_y is the canonical projection mod T_yM_y. Since M_y is compact we derive from Lemma 12.6 the existence of a φ-saturated (i.e. containing with each point x the set $\varphi^{-1}(\varphi(x))$) neighbourhood $Q \subset M$ of M_y such that the following is true: *For every $x \in Q$ there is an open neighbourhood $P_x \subset X$ of $\{q \in Q : \varphi(q) > \varphi(x)\}$ such that every continuous CR-function on M_x has a continuous extension to $P_x \cup M_x$ which is holomorphic on P_x.* Since τ is an inner point of $\varphi(Q)$ we conclude that $\tau = \varphi(a)$ cannot be true. But also $\tau > \varphi(a)$ leads to a contradiction. Indeed, fix an $x \in Q$ with $\varphi(a) < \varphi(x) < \tau$. Then every continuous CR-function on M_a extends to V_τ and hence in particular to a continuous CR-function on M_x. Putting together the two extensions to V_τ and P_x leads to an extension to the union (after making both open sets smaller if necessary). This gives a number $t \in I$ with $t > \tau$ contrary to the definition of τ. □

We will also need the following elementary removability result that can be easily proved by a Hartogs type argument:

12.8 Lemma. *Let Y be a complex manifold and let $A \subset Y$ be a closed real-analytic submanifold of real codimension ≥ 2. Then every bounded holomorphic function on $Y \backslash A$ has a holomorphic extension to Y.*

Proof of Proposition 12.3. Fix a continuous CR-function f on $K = \mathsf{K}(a)$. Without loss of generality we may assume $a \neq 0$, i.e. $\rho := \mathrm{rank}(a) > 0$. For every $k \leq \rho$ denote by $\mathcal{D}_k \subset \mathcal{D}(a)$ the open subset of all elements $x \in \mathcal{D}(a)$ of rank ρ that have at least k pairwise different singular values $\neq 0$. Then $\mathcal{D}_\rho \subset \mathcal{D}_{\rho-1} \subset \ldots \subset \mathcal{D}_1 = \mathcal{D}(a) \cap E_{[\rho]}$ and $\mathcal{D}_\rho$ is the set of all reduced elements of rank ρ in $\mathcal{D}(a)$. Furthermore, for every $k \leq \rho$ the complement $A_k := \mathcal{D}_{k-1} \backslash \mathcal{D}_k$ is a (not necessarily connected) real-analytic submanifold of $\mathcal{D}_{k-1}$. All K-orbits in $\mathcal{D}_\rho$ have the same dimension and all other K-orbits in $\mathcal{D}(a)$ have lower dimensions. In particular, A_k has codimension ≥ 2 in $\mathcal{D}_{k-1}$ for all $k \leq \rho$. We first prove that f extends holomorphically to $\mathcal{D}(a) \backslash \{0\}$.
case 1: a is reduced, that is,

$$a = \sum_{j=1}^{\rho} \lambda_j e_j$$

for real coefficients $\lambda_1 > \ldots > \lambda_\rho > 0$ and suitable orthogonal minimal tripotents $e_1, \ldots, e_\rho$ in E. Denote by $\mathcal{S}$ the set of all elements

$$x = \sum_{j=1}^{\rho} x_j e_j \ \in \ \mathcal{D}(a)$$

with $x_1 > \ldots > x_\rho > 0$. Then clearly $\mathcal{D}_\rho = \mathsf{K}(\mathcal{S})$ holds. For every $x \in \mathcal{S}$ define $\gamma : \mathbb{R} \to E$ by

$$\gamma(t) := \sum_{j=1}^{\rho} \lambda_j^{1-t} x_j^t e_j \,. \qquad (12.9)$$

Then $\gamma(0) = a$, $\gamma(1) = x$ and there is an open interval $I \subset \mathbb{R}$ with $0, 1 \in I$ and $\gamma(t) \in \mathcal{D}_\rho$ for every $t \in I$. In particular, the orbits $\mathsf{K}(\gamma(t))$ all have the same dimension for $t \in I$ and there is a unique K-invariant map $\varphi : M \to I$ with $\varphi \circ \gamma|_I = \mathsf{id}_I$ for $M := \mathsf{K}(\varphi(I))$. Also, there is a unique K-invariant smooth vector field ξ on the smooth submanifold $M \subset E_{[\rho]}$ with $\xi_{\gamma(t)} = \gamma'(t)$ for all $t \in I$. For $X := E_{[\rho]}$ it follows from Propositions 9.12 and 9.17 that the assumptions of Proposition 12.7 are satisfied. We conclude that there is an open subset W in X with $\varphi^{-1}(t) \subset W$ for all $0 < t \leq 1$ such that $f \in \mathcal{C}(W \cup K) \cap \mathcal{O}(W)$, where we use the same letter for the extension of f. Since $x \in \mathcal{S}$ was arbitrary we get a holomorphic extension of f to $\mathcal{D}_\rho$, also denoted by f, that is continuous up to K in the *nontangential sense*, i.e. in any wedge $W = (U \cap K) + B_\varepsilon$, where U is a sufficiently small neighbourhood of a point $b \in K$, $B \ll C_b$ (where C_b is the Levi cone of K at b) and $\varepsilon > 0$ also sufficiently small. Since for every $c \in \mathbb{C}\backslash f(K)$ the CR-function $(f - c)^{-1}$ on K also has a holomorphic extension to $\mathcal{D}_\rho$ we have $f(\mathcal{D}_\rho) \subset f(K)$. In particular, f is bounded on $\mathcal{D}_\rho$. Now suppose that for $k \leq \rho$ the function f has a holomorphic extension to $\mathcal{D}_k$. Then for $Y := \mathcal{D}_k$ and $A := \mathcal{D}_k \backslash \mathcal{D}_{k-1}$ Lemma 12.8 can be applied and f has a further holomorphic extension to $\mathcal{D}_{k-1}$. Using induction down from $k = \rho$ we conclude that f extends holomorphically to $\mathcal{D}_1 = \mathcal{D}(a) \cap E_\rho$. Since $\mathcal{D}(a)\backslash\{0\}$ is normal by Proposition 8.3 and since $\mathcal{D}_1$ has a complex-analytic complement in $\mathcal{D}(a)\backslash\{0\}$ we get that f has a holomorphic extension to $\mathcal{D}(a)\backslash\{0\}$.
case 2: a is not reduced. As a consequence of Proposition 9.12 we can choose a cone $B \ll C_a$ and a sequence (a_n) of reduced points in $(a + B) \cap \mathcal{D}(a)$ with $\lim a_n = a$ and $\mathcal{D}(a_n) \subset \mathcal{D}(a_{n+1})$ for all n. Then it follows from Lemma 12.6 that there exists $n_0 \in \mathbb{N}$ and a K-invariant open subset V of $\mathcal{D}(a)$ with $a_n \in V$ for all $n \geq n_0$ such that f has a holomorphic extension to V. But then by case 1 the function f extends holomorphically to $V \cup \big(\mathcal{D}(a_n)\backslash\{0\}\big)$ for all $n \geq n_0$ and hence to $\mathcal{D}(a)\backslash\{0\} = \bigcup_{n \geq n_0} \mathcal{D}(a_n)\backslash\{0\}$. In any case, the normalization of $\mathcal{D}(a)$ is homeomorphic to $\mathcal{D}(a)$ and as a consequence we get a holomorphic extension $f \in \mathcal{O}\big(\mathcal{D}(a)\backslash\{0\}\big)$ that is continuous up to K in the nontangential sense. For every $0 < t < 1$ the function f_t defined by $f_t(z) = f(tz)$ is holomorphic on a neighbourhood of $\mathcal{Z}(a)\backslash\{0\}$ in $\overline{E}_\rho$. Since $f = \lim_{t \nearrow \infty} f_t$ is a uniform limit on K it is also uniform on $\mathcal{D}(a)$, that is, f extends from $\mathcal{D}(a) \cup K$ to a continuous function on $\mathcal{Z}(a) = \overline{\mathcal{D}(a)}$. □

For the proof of Proposition 12.4 we have to extend the known tool of analytic discs to *analytic annuli* in $\mathbb{C}^n$. We use these here in the following sense: An analytic annulus in $\mathbb{C}^n$ is a complex submanifold $R \subset \mathbb{C}^n$ such that there is an annulus $A := \{\zeta \in \mathbb{C} : s < |\zeta| < t\}$ for $0 < s < t$ suitable and a biholomorphic mapping $A \to R$ that extends to a homeomorphism $\overline{A} \to \overline{R}$ of the closures. A special difficulty with analytic annuli (in contrast to the case of analytic discs) is that different annuli may not be biholomorphically equivalent. Nevertheless, one still has the following elementary property.

12.10 Lemma. *Let (R_n) be a sequence of analytic annuli in E converging to an analytic annulus $R \subset E$ in the following sense: There is a sequence (φ_n) of homeomorphisms $\varphi_n : \overline{R} \to \overline{R_n}$ converging uniformly to the identity transformation on $\overline{R}$. Suppose that (f_n) is a sequence of functions $f_n \in \mathcal{C}(\overline{R_n}) \cap \mathcal{O}(R_n)$ such that the sequence $(f_n \circ \varphi_n)$ converges uniformly on the boundary $\partial R := \overline{R}\backslash R$. Then $(f_n \circ \varphi_n)$ converges uniformly on $\overline{R}$ to a function $f \in \mathcal{C}(\overline{R}) \cap \mathcal{O}(R)$.*

It is easy to see that the proof of 12.10 can be reduced to the special case where $R = \{\zeta \in \mathbb{C} : 1 < |\zeta| < t\}$, $R_n = \{\zeta \in \mathbb{C} : 1 < |\zeta| < t_n\}$ and φ_n is given by $\varphi_n(z) = (z/|z|)\theta_n(|z|)$, where θ_n is the unique affine transformation of $\mathbb{R}$ satisfying $\theta_n(1) = 1$ and $\theta_n(t) = t_n$. Then convergence of annuli means $t = \lim_n t_n$, and the claim is obtained by writing every f_n as a sum $f_n^+ + f_n^-$, where f_n^+ is holomorphic on the disc $\{|\zeta| < t_n\}$, f_n^- is holomorphic on the disc $\{|\zeta| > 1\} \cup \{\infty\}$ and both functions f_n^+ and f_n^- extend continuously to the corresponding boundary circles.

Proof of Proposition 12.4. Without loss of generality, assume $\mathcal{B}(a) \neq \emptyset$ which happens precisely when the orbit S is not totally real (i.e. $H_aS \neq 0$). For the extension to $\mathcal{B}(a)$, the main steps of the proof are similar to those of 12.3. Fix a continuous CR-function f on S and put $r := \operatorname{rank}(a) = \operatorname{rank}(E)$ as well as $X := \mathsf{S}^{\mathbb{C}}(a)$. For every $k \leq r$ denote by $\mathcal{B}_k \subset \mathcal{B}(a)$ the open subset of all elements $x \in \mathcal{B}(a)$ that have at least k pairwise different singular values. Again, in a first case suppose that a is reduced and define $\mathcal{S}$ with $\mathcal{B}_r = \mathsf{S}(\mathcal{S})$ as in the proof of 12.3. Also, for every $x \in \mathcal{S}$ define $\gamma : \mathbb{R} \to E$ by formula (12.9). Then $\gamma(t) \in \mathcal{S}$ for $0 \leq t < 1$ and as in the proof of 12.3 we conclude that f has a holomorphic extension to $\mathcal{B}(a)$ that is continuous up to S in the nontangential sense (see the proof of 12.3). The same extension property follows in case a is not reduced (by using a suitable sequence (a_n) of reduced points in $\mathcal{B}(a)$ converging to a, compare the proof of 12.3).

The proof that the extension of f to $\mathcal{B}(a) \cup S$ can be further extended continuously to the closure $\overline{\mathcal{B}(a)} = \mathcal{X}(a)$ requires some more care. Here the final step of the proof of 12.3 cannot be carried out here since the existence of a suitable family (f_t) of holomorphic functions on $\mathcal{B}(a)$ converging uniformly to f is not clear. The proof given here uses the fine stratification structure of the set $\mathcal{X}(a)$ and a construction of analytic annuli connecting different strata.

Put $N := \{1, 2, \ldots, r-1\}$. Then $\mathcal{X}(a)$ is the disjoint union of the 'strata'

$$\mathcal{X}_I := \mathcal{X}_I(a) := \big\{ z \in \mathcal{X}(a) : \mu_k(z) = \mu_k(a) \iff k \in I \big\},$$

where I runs over all subsets $I \subset N$ with $\mathcal{X}_I \neq \emptyset$. For every integer $j \geq 1$ denote by $\mathcal{K}_j$ the union of all $\mathcal{X}_I$ with I of cardinality $\geq j$. Clearly $\mathcal{K}_{r-1} = S$ and $\mathcal{K}_1 = \partial\mathcal{D}(a)$. Denote by m the smallest integer $m \geq 1$ such that the holomorphic extension $f \in \mathcal{O}(\mathcal{B}(a))$ is continuous up to $\mathcal{K}_m$ in the following nontangential sense: For every $b \in \mathcal{K}_m$ the limit $\lim_n f(z_n)$ exists for every sequence (z_n) in $\mathcal{B}(a)$ converging to b and satisfying for some $\varepsilon > 0$ and all n the inequality $\operatorname{dist}(z_n, \partial\mathcal{B}(a)) \geq \varepsilon \operatorname{dist}(z_n, \mathcal{K}_m)$, where 'dist' stands for the distance from a point to a subset with respect to a fixed norm on the vector space E. Since we have nontangential continuous extension of f to $S = \mathcal{K}_{r-1}$ by the first part of the proof we have $m \leq r-1$.

We wish to show that $m = 1$. Assume on the contrary that $m > 1$ holds and fix a point $b \in \mathcal{K}_{m-1} \subset \partial\mathcal{B}(a)$ in the following. Fix $q \in N$ in such a way that $\sigma_q(a) > \sigma_{q+1}(a)$ in case $b \in \mathcal{K}_m$ (that is possible since $\mathcal{B}(a) \neq \emptyset$) and that $q \notin I$ in case $b \notin \mathcal{K}_m$, where $I \subset N$ is determined by $b \in \mathcal{X}_I$. For every $z \in \mathcal{B}(a)$ we construct an analytic annulus $R(z) \subset \mathcal{B}(A)$ with $z \in R(z)$ in the following way: Write z as linear combination $z = \sum_j \sigma_j(z) e_j$ for some frame $\boldsymbol{e} = (e_1, \ldots, e_r)$, compare section 5, and define $R(z)$ to be the set of all complex linear combinations $\sum_j z_j e_j$ in $\mathcal{B}(a)$ with coefficients satisfying $z_j = \sigma_j(z)$ for all $j \neq q, q+1$. Clearly, $R(z)$ depends on the choice of the frame $\boldsymbol{e}$ for z. Consider a sequence (z_n) in $\mathcal{B}(a)$ converging to b and satisfying $\operatorname{dist}(z_n, \partial\mathcal{B}(a)) \geq \varepsilon \operatorname{dist}(z_n, \mathcal{K}_{m-1})$ for some $\varepsilon > 0$ and all n. For every n there is a decomposition $z_n = \sum_j \sigma_j(z_n) e^n_j$ for some frame $\boldsymbol{e}^n = (e^n_1, \ldots, e^n_r)$ in E. Define with respect to this frame the analytic annulus $R(z_n) \subset \mathcal{B}(a)$ as above. The space of all frames in E is compact, therefore the sequence $(\boldsymbol{e}^n)$ has a frame $\boldsymbol{e}$ as point of accumulation. Let us assume for a while that $\boldsymbol{e}$ actually is a limit. Then the sequence $(R(z_n))$ converges to an analytic annulus R in the sense of 12.10. For this annulus $b \in \overline{R}$ as well as $\partial R \subset \mathcal{K}_m$ holds by the choice of the index q. By choosing smaller annuli $R_n \Subset R(z_n)$ with $z_n \in R_n$ we can achieve that the sequence (R_n) also converges to the annulus R and that in addition the boundaries ∂R_n converge to ∂R in the nontangential sense with respect to $\mathcal{K}_m$. But then Lemma 12.10 guarantees the existence of $\lim_n f(z_n)$, which so far may depend on the limit annulus R and hence on the frame $\boldsymbol{e}$. Suppose that $(\widetilde{z}_n)$ is another sequence converging to b as above such that the corresponding sequence of frames $\widetilde{\boldsymbol{e}}^n$ converges to a frame $\widetilde{\boldsymbol{e}}$ and hence gives a limit to the sequence $(f(\widetilde{z}_n))$. We claim that the two limits coincide. Indeed, choose a sequence (w_n) in $\mathcal{B}(a)$ converging nontangentially to b

with respect to $\mathcal{K}_{m-1}$, where every w_n has the form $w_n = \sum_j w_{n,j}e_j$ for suitable coefficients $w_{n,j}$ satisfying $\sigma_j(b) = \lim_n w_{n,j}$ and $w_{n,j} = w_{n,k}$ if $\sigma_j(b) = \sigma_k(b)$. Then $\lim f(z_n) = \lim f(w_n)$ is clear since both limit frames agree. Now $b = \sum_j \sigma_j(b)e_j = \sum_j \sigma_j(b)\widetilde{e}_j$ implies that every w_n also has the representation $w_n = \sum_j w_{n,j}\widetilde{e}_j$, which implies $\lim f(\widetilde{z}_n) = \lim f(w_n)$. As a consequence, $f(b) := \lim f(z_n)$ does not depend on the sequence (z_n). Since $b \in \mathcal{K}_{m-1}$ was arbitrarily chosen, f has a continuous extension to $\mathcal{K}_{m-1}$ in the nontangential sense and therefore m is not minimal with respect to the property used for its definition, that is, $m = 1$, or equivalently, for every $b \in \partial\mathcal{B}(a)$ and every sequence (z_n) in $\mathcal{B}(a)$ converging to b the sequence $(f(z_n))$ converges. Since every convergent sequence in $\partial\mathcal{B}(a)$ can be approximated by a sequence in $\mathcal{B}(a)$ we derive that f has a continuous extension to $\overline{\mathcal{B}(a)} = \mathcal{X}(a)$, completing the proof for the extension to $\mathcal{X}(a)$ of CR-functions on the orbit S.
Since the orbit $K = \mathsf{K}(a) = \mathbb{T}(S)$ is foliated by S-orbits, every continuous CR-function on K extends to a function on $\mathcal{Y}(a)$ that is continuous on each subset $t(\mathcal{X}(a)) \subset \mathcal{Y}(a)$ with $t \in \mathbb{T}$ and holomorphic on its interior. Since the norm of the extension equals the norm of the function itself and $\mathbb{T}$ acts transitively on S-orbits in $\mathcal{Y}(a)$, the extension is continuous on $\mathcal{Y}(a)$. Moreover, since the holomorphic tangent spaces of $\mathbb{T}(\mathcal{B}(a))$ coincide with those of $t(\mathcal{B}(a))$, $t \in \mathbb{T}$, the extension is also CR on $\mathbb{T}(\mathcal{B}(a))$. The proof of 12.4 is complete. □

Proof of Theorem 12.1. The theorem is a consequence of Propositions 12.3 and 12.4. The fact that the sets $\mathcal{X}(a)$, $\mathcal{Y}(a)$ and $\mathcal{Z}(a)$ are maximal follows the existence of Stein neighbourhood bases provided by Lemma 10.9. □

We now give a more precise meaning to the property that $\mathcal{Z}(a)$ is the maximal possible set of extension of CR-functions on K by identifying it with the spectrum of the algebra of these CR-functions. Recall that the spectrum $\mathsf{Spec}\big(\mathcal{C}_{\text{CR}}(K)\big)$ of the sup-normed Banach algebra $\mathcal{C}_{\text{CR}}(K)$ of all continuous CR-functions on K is the space of all nonzero continuous multiplicative linear functionals on $\mathcal{C}_{\text{CR}}(K)$ endowed with the w^*-topology (also called the weak* topology) from the dual Banach space of $\mathcal{C}_{\text{CR}}(K)$. By Proposition 12.3, if a is not invertible, every $f \in \mathcal{C}_{\text{CR}}(K)$ has a holomorphic extension to the normalization of $\mathcal{Z}(a)$ and hence the point evaluation $f \mapsto f(z)$ is well-defined for $z \in \mathcal{Z}(a)$ and yields an element in $\mathsf{Spec}\big(\mathcal{C}_{\text{CR}}(K)\big)$. Conversely, we obtain as an application of Propositions 10.5 and 12.3 that every element in $\mathsf{Spec}\big(\mathcal{C}_{\text{CR}}(K)\big)$ is the evaluation at some point in $\mathcal{Z}(a)$:

12.11 Proposition. *Suppose that $a \in E$ is not invertible. Then every element in $\mathsf{Spec}\big(\mathcal{C}_{\text{CR}}(K)\big)$ is a point evaluation at some $z \in \mathcal{Z}(a)$. As a consequence, the point evaluation defines a homeomorphism between $\mathcal{Z}(a)$ and $\mathsf{Spec}\big(\mathcal{C}_{\text{CR}}(K)\big)$.*

Proof. Let $\varphi \in \mathsf{Spec}\big(\mathcal{C}_{\text{CR}}(K)\big)$ be a nonzero continuous multiplicative linear functional. For every $s > 1$ the domain $s\mathcal{D}(a)$ in $\overline{E}_\rho$ contains $K = \mathsf{K}(a)$ and is a complex Stein space by Proposition 10.5. Hence also the normalization $\mathcal{N}_s(a)$ of $s\mathcal{D}(a)$ is Stein, where the normalization map $\mathcal{N}_s(a) \to s\mathcal{D}(a)$ is a homeomorphism and is biholomorphic outside 0 (see Proposition 8.3). It is well known that the restriction of φ to the subalgebra $\mathcal{O}(\mathcal{N}_s(a)) \subset \mathcal{C}_{\text{CR}}(K)$ is a point evaluation for some $z \in s\mathcal{D}(a)$, compare e.g. [22] Proposition 57.1. Since $s > 1$ is arbitrary, $z \in \bigcap_{s>1} s\mathcal{D}(a) = \mathcal{Z}(a)$ and $\varphi(f) = f(z)$ holds for every holomorphic function f in a neighbourhood of $\mathcal{Z}(a)$. If now $f \in \mathcal{C}_{\text{CR}}(K)$, it extends continuously to a holomorphic function on the normalization of $\mathcal{D}(a)$ by Proposition 12.3. Hence f is uniformly approximated in $\mathcal{C}_{\text{CR}}(K)$ by the functions $f(tz)$, $t < 1$, that are holomorphic in a neighbourhood of the normalization of $\mathcal{Z}(a)$ (in the normalization of the cone $\overline{E}_{[\rho]}$). Since φ is continuous, it coincides with the point evaluation at z for all f. □

12.12 Proposition. *Let E, $\widetilde{E}$ be classical factors and let $a \in E$, $\widetilde{a} \in \widetilde{E}$ be elements with a not invertible. Then the orbits $\mathsf{K}(a)$ and $\widetilde{\mathsf{K}}(\widetilde{a})$ are CR-homeomorphic if and only if they are linearly* *equivalent, i.e. if there is a (complex) linear bijection* $\lambda : E \to \widetilde{E}$ *with* $\lambda(\mathsf{K}(a)) = \widetilde{\mathsf{K}}(\widetilde{a})$.

Proof. Suppose that $\varphi\colon \mathsf{K}(a) \to \widetilde{\mathsf{K}}(\widetilde{a})$ is a CR-homeomorphism with $\widetilde{a} = \varphi(a)$. By Corollary 12.5, φ extends to a continuous map $\widehat{\varphi}\colon \mathcal{Z}(a) \to E$ that is holomorphic on $\mathcal{D}(a)$. We claim that $\widetilde{(a)}$ is also not invertible. Indeed, otherwise $\widetilde{\mathsf{K}}(\widetilde{a}) \subset Y := \overline{\mathsf{K}^{\mathbb{C}}(\widetilde{a})}$ and hence $\widehat{\varphi}(\mathcal{Z}(a)) \subset Y$ by the maximum principle. Since Y is given by $|N(z)| = |N(\widetilde{a})|$, where N is a generic norm on $\widetilde{E}$, the holomorphic function $N \circ \widehat{\varphi}$ must be constant on $\mathcal{Z}(a)$. On the other hand, since N is not constant on $\widetilde{\mathsf{K}}(\widetilde{a})$, $N \circ \varphi$ cannot be constant and we reach a contradiction proving the claim.
By applying Corollary 12.5 to φ^{-1}, we conclude that φ extends to a biholomorphic map between $\mathcal{D}(a)$ and $\mathcal{D}(\widetilde{a})$. The linear span of the tangent cone $T_0\mathcal{D}(a)$ is invariant under the linear group K acting irreducibly on E and thus coincides with E, where we assume $a \neq 0$ without loss of generality. Denote by $G = \mathsf{Aut}(\mathcal{D}(a))$ the group of all biholomorphic automorphisms of $\mathcal{D}(a)$. Then by Proposition 10.6 only the following two cases are possible: Either $G(0) = \{0\}$ or $G(0) = \mathcal{D}(a)$. In the first case we have that $0 \in \mathcal{D}(a)$ is the unique fixed point of the group G. Since the same holds for the point $\widetilde{a}$ we get $\varphi(0) = 0 \in \widetilde{E}$ in this case. In the second case the group G acts transitively on $\mathcal{D}(a)$ and induces real-analytic CR-diffeomorphisms on $\mathsf{K}(a)$. Without loss of generality we may therefore assume $\varphi(0) = 0$ in any of both cases. Then φ has a further holomorphic extension to an open neighbourhood of 0 in E – denote by $\lambda : E \to \widetilde{E}$ the derivative at the origin of this extension. Then it is clear that λ is a linear bijection. Furthermore, a variation of Cartan's Uniqueness Theorem shows that $\varphi(tz) = t\varphi(z)$ holds for all $z \in \mathcal{D}(a)$ and all unimodular $t \in \mathbb{C}$. This implies $\varphi(z) = \lambda(z)$ for all $z \in \mathcal{D}(a)$ and hence $\lambda(\mathsf{K}(a)) = \widetilde{\mathsf{K}}(\widetilde{a})$ as stated. □

12.13 Proposition. *Let E be a classical factor and $\mathcal{D} \subset E$ the corresponding bounded symmetric domain. Then for every noninvertible $a \in \partial\mathcal{D}$ the orbit $K := \mathsf{K}(a)$ satisfies one of the following two properties.*

(i) $\mathcal{D}(a) = \mathcal{D}$ *and* $\mathsf{Aut}_{\text{CR}}(K) = \mathsf{Aut}(\mathcal{D})$ *is a simple real Lie group (noncompact).*

(ii) $\mathcal{D}(a) \neq \mathcal{D}$ *and* $\mathsf{Aut}_{\text{CR}}(K) = \mathsf{GL}(\mathcal{D})$ *is a real-algebraic linear subgroup of* $\mathsf{GL}(E)$ *(compact).*

In any case we have $\mathsf{Aut}_{\text{CR}}(K) = \mathsf{Aut}(\mathcal{D}(a))$.

Proof. Corollary 12.5 implies $\mathsf{Aut}_{\text{CR}}(K) \subset \mathsf{Aut}(\mathcal{D}(a))$. In case $\mathcal{D}(a) = \mathcal{D}$ the orbit K is the Shilov boundary of $\mathcal{D}$ and the opposite inclusion is well known. Now suppose $\mathcal{D}(a) \neq \mathcal{D}$. Then Proposition 10.6 and Cartan's Uniqueness Theorem imply $\mathsf{Aut}_{\text{CR}}(K) = \mathsf{GL}(K)$. But $\mathsf{GL}(K) = \mathsf{GL}(\mathcal{D})$ by Proposition 11.8 and $\mathsf{GL}(\mathcal{D}) = \mathsf{Aut}(E)$ (compare (3.3)) is a real-algebraic group. □

Notice that in Proposition 12.13 the equality $\mathsf{Aut}_{\text{CR}}(K) = \mathsf{Aut}(\mathcal{D}(a))$ even holds topologically if $\mathsf{Aut}(\mathcal{D}(a))$ is endowed with the topology of uniform convergence on $\mathcal{D}(a)$. But it happens that on $\mathsf{Aut}(\mathcal{D}(a))$ the topology of global uniform convergence and the compact open topology coincide. For (ii) this is evident from the compactness of the groups and for the bounded symmetric domain $\mathcal{D}$ in (i) this is well known. In case $a \in E$ is invertible the group $\mathsf{Aut}_{\text{CR}}(K)$ always has infinite dimension and hence never is a Lie group. But by the extension property the group $\mathsf{Aut}_{\text{CR}}(S)$ for $S := \mathsf{S}(a)$ with $\mathcal{B}(a) \neq \emptyset$ can be realized as a closed subgroup of $\mathsf{Aut}(\mathcal{B}(a))$ provided the latter group is endowed with the topology of global uniform convergence on $\mathcal{B}(a)$. On the other hand, the group of all real-analytic CR-automorphisms of S (in a suitable jet topology) has the structure of a Lie group, compare the proof of Corollary 1.3 in [39].

Proof of Theorem 2.3. (i) The case $p = 1$ is trivial, so we may assume $p > 1$ in the following. Suppose that $K, \widetilde{K}$ are nonzero CR-isomorphic K-orbits in E. By Proposition 11.8 and $\mathsf{GL}(\mathcal{D}) = K$ we have $\mathsf{GL}(K) = \mathsf{GL}(\widetilde{K}) = \mathsf{K}$. By Proposition 12.12 there exists $\lambda \in \mathsf{GL}(E)$ with $\widetilde{K} = \lambda(K)$. Then $g \mapsto \lambda g \lambda^{-1}$ induces a group automorphism φ of the commutator subgroup S of K, which lifts to an automorphism of the universal covering group $\mathsf{SU}(p) \otimes \mathsf{SU}(q)$. Since $q > p$ and $\mathsf{SU}(q)$,

$\mathsf{SU}(q)$ are simple, the automorphism φ comes from automorphisms of $\mathsf{SU}(p)$ and $\mathsf{SU}(q)$. But every automorphism ψ of $\mathsf{SU}(n)$ is of the form $g \mapsto hgh^{-1}$ or of the form $g \mapsto h\overline{g}h^{-1}$, where $h \in \mathsf{SU}(n)$ is a fixed element and $g \mapsto \overline{g}$ is complex conjugation in $\mathbb{C}^{n\times n}$, compare [33] p. 48. In particular, $\psi(g) = hgh^{-1}$ for every g in the orthogonal group $\mathsf{SO}(p) = \{g \in \mathsf{SU}(p) : \overline{g} = g\}$. This implies that there is an element $s \in \mathsf{S}$ with $\varphi(g) = sgs^{-1}$ for all $g \in \mathsf{SO}(p) \otimes \mathsf{SO}(q)$. Replacing λ by $s^{-1}\lambda : K \to \widetilde{K}$ we may assume $s = \mathsf{id}$ in the following, i.e. $\lambda g = g\lambda$ for all $g \in \mathsf{SO}(p) \otimes \mathsf{SO}(q)$. In case $p > 2$ the group $\mathsf{SO}(p)$ acts irreducibly on $\mathbb{C}^p$. Therefore $\mathsf{SO}(p) \otimes \mathsf{SO}(q) \subset \mathsf{S}$ acts irreducibly on E, compare [35] p. 34(f), and by Schur's Lemma λ is a multiple of the identity. Therefore we only have to consider the case $p = 2$. In this situation we use that every automorphism of the group $\mathsf{SU}(2)$ is inner. Since the group $\mathsf{SU}(2) \times \mathsf{SO}(q)$ again acts irreducibly on E we get the result by the same argument as in the case $p > 2$.
(ii) and (iii) are easy consequences of Proposition 12.13. □

References

1. Arazy, J., Kaup, W.: On continuous Peirce decompositions, Schur multipliers and the perturbation of triple functional calculus. Math. Ann. **320**, 431–461 (2001).
2. Azad, H., Huckleberry, A., Richthofer, W.: Homogeneous CR-manifolds. Reine Angew. Math. **358**, 125–154 (1985).
3. Baouendi, M.S.; Ebenfelt, P.; Rothschild, L.P.: *Real Submanifolds in Complex Spaces and Their Mappings.* Princeton Math. Series, Princeton Univ. Press, 1998.
4. Baouendi, M.S., Treves, F.: A property of the functions and distributions annihilated by a locally integrable system of complex vector fields. Ann. Math. (2) **114**, 387–421, (1981).
5. Baracco, L., Zampieri, G.: The Boggess-Polking extension theorem for CR functions on manifolds with corners. Israel J. Math., to appear.
6. Bishop, E.: A minimal boundary for function algebras. Pacific J. Math. **9**, 629–642 (1959).
7. Boggess, A.: *CR Manifolds and the Tangential Cauchy-Riemann Complex.* Studies in Advanced Mathematics. CRC Press. Boca Raton Ann Arbor Boston London 1991.
8. Boggess, A., Polking, J.C.: Holomorphic extension of CR functions. Duke Math. J. **49**, 757–784 (1982).
9. Braun, H., Koecher, M.: *Jordan-Algebren.* Berlin-Heidelberg-New York: Springer 1966.
10. Bruns, W., Vetter, U.: Determinantal Rings. Lecture Notes in Mathematics Vol. 1327. Berlin-Heidelberg-New York: Springer 1988.
11. Debiard, A., Gaveau, B.: Équations de Cauchy-Riemann sur SU(2) et leurs enveloppes d'holomorphie. Can. J. Math. **38**, 1009–1024 (1986).
12. Ebenfelt, P.: Uniformly Levi degenerate CR manifolds: the 5-dimensional case. Duke Math. J. **110**, no. 1, 37–80 (2001).
13. Faraut, J., Korányi, A.: *Analysis on Symmetric Cones.* Clarendon Press, Oxford 1994.
14. Fan, K.: On a theorem of Weyl concerning eigenvalues of linear transformations. Proc. Nat. Acad. Sci. USA **37**, 760–766 (1951).
15. Gunning, R.C.: *Introduction to holomorphic functions of several variables. Vol. I. Function theory.* The Wadsworth & Brooks/Cole Mathematics Series. Pacific Grove, CA, 1990.
16. Harris, L.A.: Bounded symmetric homogeneous domains in infinite dimensional spaces, In: Lecture Notes in Mathematics Vol. 364. Berlin-Heidelberg-New York: Springer 1973.
17. Harvey, F.R., Lawson, H.B.: On boundaries of complex analytic varieties I. Ann. of Math. **102**, 223–290 (1975).
18. Helgason, S.: Differential Geometry, Lie groups and Symmetric Spaces, Academic Press 1978.
19. Hochschild, G.:*The structure of Lie groups.* Holden. San Francisco London Amsterdam 1965.

20. Hörmander, L.: *An Introduction to Complex Analysis in Several Variables*. Third edition. North-Holland Publishing Co. Amsterdam - New York, 1990.
21. Huckleberry, A., Richthofer, W.: *Recent Developments in homogneous CR-hypersurfaces.* Contributions to several complex variables, 149–177, Aspects Math. E9, Vieweg 1986.
22. Kaup, L., Kaup, B.: *Holomorphic Functions of Several Variables.* de Gruyter. Berlin New York 1983.
23. Kaup, W.: Reelle Transformationsgruppen und invariante Metriken auf komplexen Räumen. Inventiones math. **3**, 43-70 (1967).
24. Kaup, W.: Über die Klassifikation der symmetrischen Hermiteschen Mannigfaltigkeiten unendlicher Dimension I, II. Math. Ann. **257**, 463–483 (1981); **262**, 503–529 (1983).
25. Kaup, W.: On Grassmannians associated with JB*-triples. Math. Z. **236**, 567–584 (2001).
26. Kaup, W., Zaitsev, D.: On Symmetric Cauchy-Riemann Manifolds. Adv. in Math. **149**, 145–181 (2000).
27. Kleppe, H., Laksov, D.: The algebraic structure and deformation of Pfaffian schemes. J. Algebra **64**, 167–189 (1980).
28. Kutz, R.E.: Cohen-Macaulay rings and ideal theory of invariants of algebraic groups. Transactions Amer. Math. Soc. **194**, 115–129 (1974).
29. Loos, O.: *Bounded symmetric domains and Jordan pairs.* Mathematical Lectures. Irvine: University of California at Irvine 1977.
30. Markus, A.S.: The eigen- and singular values of the sum and product of linear operators. Russ. Math. Surv. **19**, 91–120 (1964).
31. McCrimmon, K.: The generic norm of an isotope of a Jordan algebra. Scripta Math. **29**, 229–236 (1973).
32. Mitrinović, D.S.: *Analytic inequalities.* Berlin-Heidelberg-New York: Springer 1970.
33. Onishchik, A.L.: *Topology of transitive transformation groups.* Barth, Leipzig Berlin Heidelberg 1994.
34. Rossi, H.: Attaching analytic spaces to an analytic space along a pseudoconcave boundary. 1965. Proc. Conf. Complex Analysis (Minneapolis, 1964) pp. 242–256. Springer, Berlin.
35. Serre, J.P.: *Linear Representation of Finite Groups* Berlin-Heidelberg-New York: Springer 1977.
36. Tumanov, A.E.: Extension of CR-functions into a wedge from a manifold of finite type. Mat. Sb. (N.S.) **136 (178)**, 128–139 (1988); translation in *Math. USSR-Sb.* **64**, 129–140 (1989).
37. Wielandt, H.: An extremum property of sums of eigenvalues. Proc. Amer. Math. Soc. **6**, 106–110 (1955).
38. Wolf, A.J.: *Fine Structure of Hermitian Symmetric Spaces.* Symmetric spaces (Short Courses, Washington Univ., St. Louis, Mo., 1969–1970), pp. 271–357. Pure and App. Math., Vol. 8, Dekker, New York, 1972.
39. Zaitsev, D.: Germs of local automorphisms of real analytic CR structures and analytic dependence on the k-jets. *Math. Res. Lett.* **4**, 1–20 (1997).