SYMPLECTIC DUALITY OF SYMMETRIC SPACES

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Abstract

Let $(M, 0) \subset \mathbb{C}^n$ be a complex n-dimensional Hermitian symmetric space endowed with the hyperbolic form ω_B. Denote by (M^*, ω_B^*) the compact dual of (M, ω_B), where ω_B^* is the Fubini–Study form on M^*. Our first result is Theorem 1.1 where, with the aid of the theory of Jordan triple systems, we construct an explicit diffeomorphism $\Psi_M : M \to \mathbb{R}^{2n} = \mathbb{C}^n \subset M^*$ satisfying $\Psi_M^*(\omega_0) = \omega_B$ and $\Psi_M^*(\omega_B^*) = \omega_0$. Amongst other properties of the map Ψ_M, we also show that it takes (complete) complex and totally geodesic submanifolds of M through the origin to complex linear subspaces of $\mathbb{C}^n$. As a byproduct of the proof of Theorem 1.1 we get an interesting characterization of the Bergman form on a Hermitian space in terms of its restriction to classical Hermitian symmetric spaces of noncompact type (see Theorem 5.4 below).

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1. Introduction

Dusa McDuff [15] proved a global version of Darboux theorem for a n-dimensional Kähler manifold with a pole p such that its radial curvature is nonpositive, by showing that there exists a diffeomorphism $\psi_p : M \to \mathbb{R}^{2n} = \mathbb{C}^n$ (depending on p) such that $\psi_p(p) = 0$ $\psi_p^*(\omega_0) = \omega$, where $\omega_0 = \sum_{j=1}^n dx_j \wedge dy_j$ is the standard symplectic form on $\mathbb{R}^{2n}$. The interest for these kind of questions comes, for example, after Gromov's discovery [8] of the existence of exotic symplectic structures on $\mathbb{R}^{2n}$. Eleonora Ciriza [3] (see also [2] and [4]) proves that the image $\psi_p(T)$ of any (complete) complex and totally geodesic submanifold T of M which contains the pole is a complex linear subspace of $\mathbb{C}^n$. It is worth pointing out that McDuff's proof deals only with the existence problem and the expression of the symplectomorphism ψ_p is, in general, very hard to find (see [5] for the construction of explicit symplectic coordinates on some complex domains in $\mathbb{C}^n$).

In this paper we deal with the symplectic geometry of Hermitian symmetric spaces of noncompact type. We are going to regard such spaces as bounded symmetric domains $(M, 0) \subset \mathcal{M}$ centered in the origin of their associated Hermitian positive Jordan triple system $\mathcal{M}$. Furthermore M will be equipped with the hyperbolic form. Let (M^*, ω_B^*) be the compact dual of (M, ω_B) equipped with the Fubini–Study form ω_B^* (see the next section for the definition of hyperbolic and Fubini–Study form). We denote with the same symbol the Kähler form ω_B^* on $\mathcal{M}$ obtained by the restriction of ω_B^* via the Borel embedding $\mathcal{M} \subset M^*$. Finally, we denote by $HSSNT$ the space of all Hermitian symmetric spaces of noncompact type $(M,0)$ and by $\mathcal{P}$ the set of all diffeomorphisms $\psi : M \to \mathcal{M}$, $M \in HSSNT$, such that $\psi(0)=0$.

Our main result is the following theorem which establishes a bridge among the symplectic geometry of HSSNT, their duals and the theory of Jordan triple systems.

Theorem 1.1. *Under the above assumptions, the map*

$$\Psi_M : M \to \mathcal{M},\ z \mapsto B(z,z)^{-\frac{1}{4}} z, \tag{1}$$

has the following properties:

(D) Ψ_M *is a (real analytic)* diffeomorphism *and its inverse* Ψ_M^{-1} *is given by:*

$$\Psi_M^{-1} : \mathcal{M} \to M,\ z \mapsto B(z,-z)^{-\frac{1}{4}} z\,;$$

(H) *The map* $\Psi : HSSNT \to \mathcal{P}$ *which takes an* $M \in HSSNT$ *into the diffeomorphism* Ψ_M *is* hereditary *in the following sense: for any* $(T,0) \overset{i}{\hookrightarrow} (M,0)$ *complex and totally geodesic embedded submanifold* $(T,0)$ *through the origin* 0, *i.e.* $i(0)=0$ *one has:*

$$\Psi_{M|_T} = \Psi_T.$$

Moreover

$$\Psi_M(T) = \mathcal{T} \subset \mathcal{M}, \tag{2}$$

where $\mathcal{T}$ *is the Hermitian positive Jordan triple system associated to* T;

(I) Ψ_M *is a (non-linear)* interwining *map w.r.t the action of the isotropy group* $K \subset Iso(M)$ *at the origin, where* $Iso(M)$ *is the group of isometries of* M, *i.e. for every* $\tau \in K$

$$\Psi_M \circ \tau = \tau \circ \Psi_M;$$

(S) Ψ_M *is a* symplectic duality, *i.e. the following holds*

$$\Psi_M^*(\omega_B^*) = \omega_0; \tag{3}$$

$$\Psi_M^*(\omega_0) = \omega_B, \tag{4}$$

where ω_0 *is the flat Kähler form on* $\mathcal{M}$ *(see formula (14) below for its definition).*

From the point of view of inducing geometric structures as in Gromov's programme [8] the importance of property (S) relies on the existence of a smooth map (i.e. Ψ_M) which is a simultaneous symplectomorphism with respect to different symplectic structures, i.e. satisfying (3) and (4). (We refer the reader to [6] and [7] and the reference therein for the case of induction of different pairs like symplectic forms and Riemannian metrics or connections and Riemannian metrics). Notice also that property (H) is exactly the above mentioned property observed by Ciriza for the McDuff map, namely the image of a totally geodesic submanifold (through the origin) via the map Ψ_M is sent to a complex linear subspace of $\mathcal{M}$.

The map $\Psi_M : M \to \mathcal{M}$ above was defined, independently from the authors, by Guy Roos in [18] (see Definition VII.4.1 at p. 533). There (see Theorem VII.4.3) he proved the analogous of (S) for volumes, namely $\Psi_M^*(\omega_0^n) = \omega_B^n$ and $\Psi_M^*((\omega_B^*)^n) = \omega_0^n$ (n is the complex dimension of M) which is, of course a corollary of (S).
The case where M is the first Cartan domain $D_I[n]$ (namely the dual of $\mathrm{Grass}_n(\mathbb{C}^{2n})$) the map Ψ_M was already considered by John Rawnsley in a unpublished work of 1989, where

he proved property (3) for this case (see Section 3 below for details). Actually the proof of (S) for classical HSSNT (i.e. those Hermitian spaces which do not contain exceptional factors in theirs De–Rham decomposition) follows from the property (S) for $D_I[n]$ (see 4.1 below). Regarding the proof of (S) in the general case we present here two proofs. The first one, presented in Section 5, is actually a "partial proof" since it is obtained by assuming that one already knows that the symplectic form $\Psi_M^*(\omega_0)$ and $(\Psi_M^{-1})^*(\omega_B^*)$ are of type $(1,1)$. The second (and complete) proof, more algebraic in nature, is due to Guy Roos. His proof is, as it often occurs, more or less an adaptation of the proof for matrices into the language of Jordan triples and their operators.
There are two reasons of having included our (partial) proof in this paper. First, it is of geometric nature and second because, the techniques employed, heuristically, have suggested us how to attack and prove Theorem 5.4 below which gives an interesting (and to the authors' knowledge unknown) characterization of the Bergman metric in terms of its restriction to classical HSSNT.

The paper is organized as follows. In the next section we collect some basic material about Hermitian positive Jordan triple systems, Hermitian symmetric spaces and their dual. Section 3 is dedicated to the proof of (D) and (S) of Theorem 1.1 for the first Cartan domain. The result of these sections are used in Section 4 to prove (H) and (I) of Theorem 1.1 in the general case and Theorem 1.1 in the classical case. In Section 5, after recalling some basic facts on Jordan algebras we prove (D) and (S) of Theorem 1.1 by reduction to the classical case (property (S) is proved only in the hypothesis mentioned above). Moreover at the end of this section we state Theorem 5.4 whose proof can be easily obtained by the same method used in the proof of (S). Finally Section 6 contains Roos's proof of (S). The paper ends with an appendix containing two technical results on Hermitian positive Jordan triple systems.

We would like to thank Guy Roos for giving us the opportunity of including his proof in this paper and for useful discussions about his work on Jordan triple systems and for his interest in ours.

2. Jordan triple systems, Hermitian spaces of noncompact type and their compact dual

2.1. **Jordan triple systems and Hermitian spaces.** We briefly recall some standard material about Hermitian symmetric spaces of noncompact type and Hermitian positive Jordan triple systems. We refer to [18] for details, notations and further results.

An Hermitian Jordan triple system is a pair $(\mathcal{M}, \{,\ ,\})$, where $\mathcal{M}$ is a complex vector space and $\{,\ ,\}$ is a $\mathbb{R}$-trilinear map

$$\{,\ ,\}: \mathcal{M}\times\mathcal{M}\times\mathcal{M}\to\mathcal{M}, (u,v,w)\mapsto\{u,v,w\}$$

which is $\mathbb{C}$-bilinear and symmetric in u and w, $\mathbb{C}$-antilinear in v and such that the following *Jordan identity* holds:

$$\{x,y,\{u,v,w\}\}-\{u,v,\{x,y,w\}\}=\{\{x,y,u\},v,w\}-\{u,\{v,x,y\},w\}. \tag{5}$$

For $u,v\in\mathcal{M}$, denote by $D(u,v)$ the operator of $\mathcal{M}$

$$D(u,v)(w)=\{u,v,w\}.$$

An Hermitian Jordan triple system is called *positive* if the Hermitian form $(u, v) \mapsto \operatorname{tr} D(u, v)$ is positive definite. In the sequel we will write $HPJTS$ to denote an Hermitian positive Jordan triple system. We also denote (with a slight abuse of notation) by $HPJTS$ the set of all Hermitian positive Jordan triple systems on a fixed complex vector space $\mathcal{M}$. An $HPJTS$ is always *semi-simple*, that is a finite family of simple subsystems with component-wise triple product. A $HPJTS$ is called *simple* if it is not the product of two non-trivial Hermitian positive Jordan triple subsystems. The *quadratic representation* $Q : \mathcal{M} \to End_{\mathbb{R}}(\mathcal{M})$ is defined by

$$2Q(u)(v) = \{u, v, u\}, \ u, v \in \mathcal{M}.$$

The *Bergman operator* is defined by

$$B(u, v) = \operatorname{id} - D(u, v) + Q(u)Q(v), \tag{6}$$

where $\operatorname{id} : \mathcal{M} \to \mathcal{M}$ denotes the identity map of $\mathcal{M}$.

An element $c \in \mathcal{M}$ is called *tripotent* if $\{c, c, c\} = 2c$. Two tripotents c_1 and c_2 are called *orthogonal* if $D(c_1, c_2) = 0$. A non zero tripotent c is called *primitive* if it is not the sum of non-zero orthogonal tripotents. Due to the positivity of the Jordan triple system $\mathcal{M}$, each element $x \in \mathcal{M}$ has a unique *spectral decomposition*

$$x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_p c_p, \tag{7}$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_p > 0$ and $(c_1, \ldots, c_p)$ is a system of mutually orthogonal tripotents. Moreover, each $x \in \mathcal{M}$ may also be written

$$x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_r c_r, \tag{8}$$

with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$ and $(c_1, \ldots, c_r)$ is a *frame* (that is a maximal system of mutually orthogonal tripotents). The decomposition (23) is also called *spectral decomposition*; it is unique only for elements x of maximal rank r, which form a Zariski dense open subset of $\mathcal{M}$.
There exist polynomials $m_1, \ldots, m_r$ on $\mathcal{M} \times \overline{\mathcal{M}}$, homogeneous of respective bidegrees $(1, 1), \ldots, (r, r)$, such that for $x \in \mathcal{M}$, the polynomial

$$m(T, x, y) = T^r - m_1(x, y)T^{r-1} + \cdots + (-1)^r m_r(x, y)$$

satisfies

$$m(T, x, x) = \prod_{i=1}^{r} (T - \lambda_i^2),$$

where x is the spectral decomposition of $x = \sum \lambda_j c_j$.
The inohomogeneous polynomial

$$N(x, y) = m(1, x, y)$$

is called the generic norm.
Denote by $\mathcal{N}$ and $\mathcal{N}_*$ the associated functions

$$\mathcal{N}(z) = N(z, z) = 1 - m_1(x, x) + \cdots + (-1)^k m_k(x, x) + \cdots + (-1)^r m_r(x, x), \tag{9}$$

and

$$\mathcal{N}_*(z) = N(z, -z) = 1 + m_1(x, x) + \cdots + m_k(x, x) + \cdots + m_r(x, x). \tag{10}$$

2.2. **HSSNT associated to HPJTS.** M. Koecher ([11], [12]) discovered that to every HPJTS $(\mathcal{M}, \{\ ,\ ,\})$ one can associate an Hermitian symmetric space of noncompact type, i.e. a bounded symmetric domain $(M, 0)$ centered at the origin $0 \in \mathcal{M}$. The domain $(M, 0)$ is defined as the connected component containig the origin of the set of all $u \in \mathcal{M}$ such that $B(u, u)$ is positive definite with respect to the Hermitian form $(u, v) \mapsto \operatorname{tr} D(u, v)$. The Bergman form ω_{Berg} on M is given by

$$\omega_{Berg} = -\frac{i}{2\pi}\partial\bar{\partial}\log\det B.$$

The *hyperbolic metric* ω_B (which appears in the statement of our Theorem 1.1) is given by:

$$\omega_B = -\frac{i}{2\pi}\partial\bar{\partial}\log\mathcal{N}, \tag{11}$$

where $\mathcal{N}(z)$ is given by (9). If M is irreducible, or equivalently $\mathcal{M}$ is simple, then $\det B = \mathcal{N}^g$, where g is the genus of M, and hence, in this case, $\omega_B = \frac{\omega_{Berg}}{g}$. Observe that in the rank one case, that is when M is the complex Hermitian ball, the form ω_B is the standard hyperbolic metric (cfr. formula (17) in the next section).
The HPJTS $(\mathcal{M}, \{\ ,\ ,\})$ can be recovered by its associated Hermitian symmetric space of noncompact type $(M, 0)$ by defining $\mathcal{M} = T_0M$ (the tangent space to the origin of M) and

$$\{u, v, w\} = -\frac{1}{2}\left(R_0(u, v)w + J_0R_0(u, J_0v)w\right), \tag{12}$$

where R_0 (resp. J_0) is the curvature tensor of the Bergman metric (resp. the complex structure) of M evaluated at the origin. For more informations on the correspondence between $HPJTS$ and $HSSNT$ we refer to p. 85 in Satake's book [20] (see also [14]). We refer also to [1] for some deep implications of formula (12).

2.3. **Totally geodesic submanifolds of HSSNT.** In the proof of our theorems we need the following result whose proof follows by equality (12) and the well-known correspondence between totally geodesic submanifolds and Lie triple systems (see Theorem 4.3 p. 237 in [10]).

Proposition 2.1. *Let $(M, 0)$ be a HSSNT with origin $0 \in M$ and let M be its associated $HPJTS$. Then there exists a one to one correspondence between (complete) complex totally geodesic submanifolds and sub-$HPJTS$ of $\mathcal{M}$. This correspondence sends $(T, 0) \subset (M, 0)$ to $\mathcal{T} \subset \mathcal{M}$, where $\mathcal{T}$ denotes the HPJTS associated to T.*

2.4. **The compact dual of an $HSSNT$.** Let (M^*, ω_B^*) be the compact dual of (M, ω_B). It is well known (see e.g. [21]) that (M^*, ω_B^*) is a compact homogeneous simply-connected Kähler manifold and M^* admits a holomorphic embedding $\mathrm{BW} : M^* \to \mathbb{C}P^N$ (the *Borel–Weil embedding*) into a N-dimensional complex projective space satisfying $\mathrm{BW}^*(\omega_{FS}) = \omega_B^*$, where ω_{FS} denotes the Fubini–Study form on $\mathbb{C}P^N$, namely the Kähler form which, in the homogeneous coordinates $[z_0, \dots z_N]$ on $\mathbb{C}P^N$, is given by $\omega_{FS} = \frac{i}{2\pi}\partial\bar{\partial}\log(|z_0|^2 + \cdots + |z_N|^2)$. In our Theorem 1.1 and in the sequel we will call ω_B^* the *Fubini–Study form* on M^*. In order to write its local expression, let $p \in M^*$ and assume, without loss of generality, that $\mathrm{BW}(p) = [1, 0, \dots, 0] \in \mathbb{C}P^N$. Let $H_p \subset \mathbb{C}P^N$ be the hyperplane at infinity corresponding to the point $BW(p)$ and set $Y_p = \mathrm{BW}^{-1}(H_p)$. One can prove (see [22]) that $M^* \setminus Y_p$ is biholomorphic to $\mathcal{M} = T_0M$ (the $HPJTS$ associated to M). Moreover, under the previous biholomorphism, p can be made to correspond to the origin

$0 \in M$. Hence we have the following inclusions $M \subset \mathcal{M} \subset M^*$ and one can prove that the restriction to $\mathcal{M}$ of the Kähler form ω_B^* is given by:

$$\omega_B^* = \frac{i}{2\pi}\partial\bar{\partial}\log\det\mathcal{N}_*, \tag{13}$$

where $\mathcal{N}_*(z)$ is given by (10) (see also [13] for the relations between the two Kähler forms ω_B and ω_B^*).

2.5. **The flat form on $\mathcal{M}$.** The *flat Kähler form* on $\mathcal{M}$ is defined by

$$\omega_0 = \frac{i}{2\pi}\partial\bar{\partial}m_1(x,x), \tag{14}$$

where $m_1(x,x)$ is the polynomial appearing in (9). Observe that if $\mathcal{M}$ is simple then $\operatorname{tr} D(x,y) = g m_1(x,y)$ and hence $\omega_0 = \frac{i}{2g\pi}\partial\bar{\partial}D(x,x)$. Notice also that in the rank-one case ω_0 is (up to the factor 2π) the standard Euclidean form on $\mathcal{M} = \mathbb{C}^n$ (cfr. formula (20) below).

3. The proof of (D) and (S) of Theorem 1.1 for the first Cartan's domain

Let $M = D_1[n]$ be the complex noncompact dual of $M^* = G_n(\mathbb{C}^{2n})$, where $G_n(\mathbb{C}^{2n})$ is the complex Grassmannian of complex n subspaces of $\mathbb{C}^{2n}$. In its realization as a bounded domain, $D_1[n]$ is given by

$$D_1[n] = \{Z \in M_n(\mathbb{C}) |\ I_n - ZZ^* >> 0\}, \tag{15}$$

The triple product on $\mathbb{C}^{n^2}$ making it an HPJTS is given by

$$\{U,V,W\} = UV^*W + WV^*U,\ U,V,W \in M_n(\mathbb{C}). \tag{16}$$

Hence the Bergman operator is given by

$$B(U,V)W = (I_n - UV^*)W(I_n - V^*U)$$

A simple computation shows that the hyperbolic form and the duality map $\Psi_M : D_I[n] \to M_n(\mathbb{C}) = \mathbb{C}^{n^2}$ are given by:

$$\omega_B = -\frac{i}{2\pi}\partial\bar{\partial}\log\det(I_n - ZZ^*). \tag{17}$$

$$\Psi_M(Z) = (I_n - ZZ^*)^{-\frac{1}{2}}Z, \tag{18}$$

respectively. In this case the Borel–Weil embedding is given by the Plücker embedding $G_n(\mathbb{C}^{2n}) \hookrightarrow \mathbb{C}P^N, N = \left(\begin{array}{c}2n\\ n\end{array}\right) - 1$ and the local expression (13) of ω_B^* on $\mathcal{M} = \mathbb{C}^{2n}$ reads as

$$\omega_B^* = \frac{i}{2\pi}\partial\bar{\partial}\log\det(I_n + XX^*), \tag{19}$$

with $X \in M_n(\mathbb{C})$. Moreover the flat Kähler form (14) is given by

$$\omega_0 = \frac{i}{2\pi}\partial\bar{\partial}\log\operatorname{tr}(ZZ^*). \tag{20}$$

By using the equality

$$XX^*(I_n + XX^*)^{\frac{1}{2}} = (I_n + XX^*)^{\frac{1}{2}}XX^*$$

it is easy to verify that the map

$$\Phi_M : \mathbb{C}^{n^2} \to D_1[n], X \mapsto (I_n + XX^*)^{-\frac{1}{2}}X \tag{21}$$

is the inverse of Ψ_M.
We are now ready to prove (S), namely the equalities

$$\Psi_M^*(\omega_0) = \omega_B \tag{22}$$

$$\Psi_M^*(\omega_B^*) = \omega_0. \tag{23}$$

As we already pointed out, the proof of the equation (22), is due to J. Rawnsley (unpublished). Here we present his proof. First of all observe that we can write

$$\begin{aligned}
\omega_B &= -\frac{i}{2\pi}\partial\bar\partial \log\det(I_n - ZZ^*) = \frac{i}{2\pi}\,\mathrm{d}\,\partial \log\det(I_n - ZZ^*)\\
&= \frac{i}{2\pi}\,\mathrm{d}\,\partial\,\mathrm{tr}\log(I_n - ZZ^*) = \frac{i}{2\pi}\,\mathrm{d}\,\mathrm{tr}\,\partial\log(I_n - ZZ^*)\\
&= -\frac{i}{2\imath}\,\mathrm{d}\,\mathrm{tr}[Z^*(I_n - ZZ^*)^{-1}dZ],
\end{aligned}$$

where we use the decomposition $\mathrm{d} = \partial + \bar\partial$ and the identity $\log\det A = \mathrm{tr}\log A$. By substituting $X = (I_n - ZZ^*)^{-\frac{1}{2}}Z$ in the last expression one gets:

$$-\frac{i}{2\pi}\,\mathrm{d}\,\mathrm{tr}[Z^*(I_n - ZZ^*)^{-1}dZ] = -\frac{i}{2\pi}\,\mathrm{d}\,\mathrm{tr}(X^*dX) + \frac{i}{2\pi}\,\mathrm{d}\,\mathrm{tr}\{X^*d[(I_n - ZZ^*)^{-\frac{1}{2}}]Z\}.$$

Observe now that $-\frac{i}{2\pi}\,\mathrm{d}\,\mathrm{tr}(X^*dX) = \omega_0$ and the 1-form $\mathrm{tr}[X^*\,\mathrm{d}(I_n - ZZ^*)^{-\frac{1}{2}}Z]$ on $\mathbb{C}^{n^2}$ is exact being equal to $\mathrm{d}\,\mathrm{tr}(\frac{C^2}{2} - \log C)$, where $C = (I_n - ZZ^*)^{-\frac{1}{2}}$. Therefore ω_B in the X-coordinates equals ω_0 and this concludes the proof of equality (22).
The proof of (23) follows the same line. Indeed, by (19) we get

$$\begin{aligned}
\omega_B^* &= \frac{i}{2\pi}\partial\bar\partial\log\det(I_n + XX^*) = -\frac{i}{2\pi}\,\mathrm{d}\,\mathrm{tr}\,\partial\log(I_n + XX^*)\\
&= -\frac{i}{2\pi}\,\mathrm{d}\,\mathrm{tr}[X^*(I_n + XX^*)^{-1}dX]
\end{aligned}$$

By substituting $Z = (I_n + XX^*)^{-\frac{1}{2}}X$ in the last expression one gets:

$$\begin{aligned}
-\frac{i}{2\pi}\,\mathrm{d}\,\mathrm{tr}[X^*(I_n - XX^*)^{-1}dX] &= -\frac{i}{2\pi}\,\mathrm{d}\,\mathrm{tr}(Z^*dZ) + \frac{i}{2\pi}\,\mathrm{d}\,\mathrm{tr}\{Z^*d[(I_n + XX^*)^{-\frac{1}{2}}]X\}\\
&= \omega_0 + \frac{i}{2\pi}\mathrm{d}^2\,\mathrm{tr}(\log D - \mathrm{tr}\,\frac{D^2}{2}) = \omega_0,
\end{aligned}$$

where $D = (I_n + XX^*)^{-\frac{1}{2}}$ and this concludes the proof of (S) for $D_I[n]$.

4. Proof of (H) and (I) and the proof of Theorem 1.1 for classical domains

Let $(M, 0)$ be any HSSNT. Since the map Ψ_M depends only on the triple product $\{,\ ,\}$ properties (H) is a straightforward consequence of Proposition 2.1 above. Let $\mathcal{M}$ be the HPJTS associated to M. As usual, let us write $M = G/K$, where G is the isometry group of M and $K \subset G$ is the (compact) isotropy subgroup of the origin $0 \in M$. Due to a theorem of E. Cartan (see [16], p. 63) the group K consists entirely of linear transformations, i.e. $K \subset \mathrm{GL}(\mathcal{M})$. In order to prove (I) of Theorem 1.1, observe that the Bergman operator associated to $\mathcal{M}$ is invariant by the group of isometry of M, namely for every isometry $\tau \in K$

$$B(\tau(u), \tau(v))(\tau(w)) = \tau\left(B(u, v)(w)\right), \forall u, v, w \in \mathcal{M},$$

which implies that

$$B(\tau(z)\,,\,\tau(z))^{-1/4}(\cdot) = \tau(B(z, z)^{-1/4}(\tau^{-1}(\cdot))),\ \forall z \in M.$$

Hence

$$\Psi_M \circ \tau = \tau \circ \Psi_M$$

for all $\tau \in K$ and we are done. □

4.1. **Proof of Theorem 1.1 for classical HSSNT.** Observe that since now we have proved properties (H), (I) for any HSSNT and property (D) and (S) for $D_I[n]$. Let $(M, 0)$ be a classical HSSNT and let $\mathcal{M}$ be its associated $HPJTS$. It is well-known that $(M, 0)$ can be complex and totally geodesic embedded into $D_I[n]$, for n sufficiently large. We can assume that this embedding takes the origin $0 \in M$ to the origin $0 \in D_I[n]$. Therefore, by Proposition 2.1, the HPJTS $\mathcal{M}$ is a sub-HPJTS of $(\mathbb{C}^{n^2}, \{,,\})$. Hence properties (D), (S) for M are consequences of property (H) and the fact (proved in Section 3) that these properties hold true for $D_I[n]$.

5. Proof of Theorem 1.1 by reduction to the classical case

Let $(M, 0)$ be an exceptional $HSSNT$. Properties (H) and (I) were proved in the previous section. In this section we prove (D) and (S) In order to prove (D) and (S), we pause to prove Lemmata 5.1, 5.2 and its Corollary 5.3 below which will be the bridges between the exceptional case and the classical one. Before stating and proving these lemmata let us briefly recall the concept of *Jordan algebras* (see e.g. [18] for details). A Jordan algebra (over $\mathbb{R}$ or $\mathbb{C}$) is a (real or complex) vector space $\mathcal{A}$ endowed with a commutative bilinear product

$$\circ : \mathcal{A} \times \mathcal{A} \to \mathcal{A}, (a, b) \mapsto a \circ b$$

satisfying the following identity:

$$a \circ (a^2 \circ b) = a^2 \circ (a \circ b), \forall a, b \in \mathcal{A},$$

where $a^2 = a \circ a$. Given a Jordan algebra $\mathcal{A}$ over $\mathbb{C}$ the triple product given by

$$\{x, y, z\} = 2\left((x \circ \bar{y}) \circ z + (z \circ \bar{y}) \circ x - (x \circ z) \circ \bar{y}\right)$$

defines a structure of Jordan triple system on $\mathcal{A}$ (cfr. Proposition II.3.1 at page 459 and formula (6.18) at page 514 in [18]). Not all HPJTS $\mathcal{M}$ arises form a Jordan algebra. If this happens the HSSNT associated to $\mathcal{M}$ is called of *tube type*. Nevertheless we have the following result

Lemma 5.1. *Let $(M, 0)$ be a HSSNT and let $\mathcal{M}$ be its associated HPJTS. Then, there exists a HSSNT $(\widetilde{M}, 0)$ such that:*

(i) $(M, 0) \hookrightarrow (\widetilde{M}, 0)$ *complex and totally geodesic embedded,*
(ii) *The HPJTS $\widetilde{\mathcal{M}}$ associated to $(\widetilde{M}, 0)$ arises from a Jordan Algebra.*

Proof: Assume first that M is irreducible. If M is of classical type take a suitable n and a complex and totally geodesic embedding $(M, 0) \hookrightarrow D_I[n]$. The HPJTS associated to $D_I[n])$ comes from a Jordan algebra and so, by Proposition 2.1, the lemma is proved for classical HSSNT. If M is of exceptional type, it is known (see Appendix in [19]) that the HPJTS associated to E_6 (the exceptional HSSNT of dimension 16) is a sub-HPJTS of the HPJTS $(H_3(\mathcal{O}_{\mathbb{C}}), \{,,\})$ of dimension 27 associated to the exceptional HSSNT E_7. Since $(H_3(\mathcal{O}_{\mathbb{C}}), \{,,\})$ arises from a Jordan algebra (i.e. E_7 is of tube type, see Appendix in [19]), the prove of the lemma follows again by Proposition 2.1.

For a reducible HSSNT one simply takes the product of the Jordan algebras associated to each factor. □

Lemma 5.2. *Let M be a HSSNT. Let p be a point of M, $a, b \in T_pM = \mathcal{M}$ be two non-zero vectors and $\pi \subset T_pM$ be the real plane generated by these vectors. Then there exists a classical HSSNT $C \hookrightarrow M$ complex and totally geodesically emdedded in M passing trough p such that $\pi \subset T_pC$.*

Proof: Without loss of generality we can assume that p is equal the origin 0 of M. Consider the Jordan subalgebra $\mathcal{C}_{ab} \subset \widetilde{\mathcal{M}}$ generated by a and b, where $\widetilde{\mathcal{M}}$ is the Jordan algebra given by the previous lemma. A deep result due to Jacobson-Shirsov [9] asserts that this Jordan algebra is special, namely its associated HSSNT is of classical type. Therefore, by (i) of the previous lemma, the HSSNT $C \hookrightarrow M \hookrightarrow \widetilde{M}$ associated to the HPJTS $\mathcal{C}_{ab} \cap \mathcal{M} \subset \mathcal{M}$ satisfies the desired properties. □

Corollary 5.3. *Let M be a HSSNT. Let $p \in M$ be a point of M and let γ be a geodesic of M passing trough p. Then, for any complex line $\mathcal{L} \subset T_pM$ there exists a classical HSSNT $C \hookrightarrow M$ complex and totally geodesically emdedded in M passing trough p such that $\gamma \subset C$ and $\mathcal{L} \subset T_pC$.*

Proof: Let $a \in T_pM$ be a non zero vector tangent to γ and let $b \in T_pM$ any nonzero vector of $\mathcal{L}$. The desired C is then obtained by applying Lemma 5.2 to these vectors. □

5.1. **Proof of (D).** Let $p \in (M, 0)$ be any point and let $v_p \in T_pM$ any vector tangent to M. Then, by Lemma 5.2 we can construct a classical (complete) complex totally geodesic $(C, 0) \hookrightarrow (M, 0)$ such that $p \in N$ and $v_p \in T_pC$. By property (H) we know that $\Psi_M|_C$ is a diffeomorphism and therefore we get that $(d\Psi_M)_p$ is bijective and therefore by the inverse function theorem Ψ_M is a local diffeomorphism. In order to prove the injectivity of the ψ_M let $p, q \in M$ and let $C \hookrightarrow M$ be a classical HSSNT containing $0, p, q \in C$, whose existence is guaranteed by Lemma 5.2. Then, property (H) implies that Ψ_M is $1-1$.
In order to prove the surjectivity of Ψ_M, let $q \in \mathcal{M}$ be an arbitrary point. We can assume that $q \neq 0$, since $\Psi_M(0) = 0$. We have to show that there exist $p \in M$ such that $\Psi_M(p) = q$. Let now γ be the 1-dimensional real subspace of $\mathcal{M}$ generated by q. Observe that regarding $M \subset \mathcal{M}$ as a bounded domain, it follows that $\widetilde{\gamma} = \gamma \cap M$ is (as a subset) a geodesic of M. Let C be a classical totally geodesic complex submanifold of M containing $\widetilde{\gamma}$ given by Lemma 5.2. Notice that by Proposition 2.1 the point q belongs to $\mathcal{C}$ (the HPJTS associated to C). Since C is classical, we know (by Section 4) that there exists $p \in C \subset M$ such that $q = \Psi_C(p) = \Psi_M(p)$, (last identity follows again by property (H)), and we are done. Finally, note that the expression for the inverse of Ψ_M also follows from property (H) by restriction to classical (complete) complex totally geodesic submanifolds. □

5.2. **Proof of (S) under the assumptions that $\Psi_M^*(\omega_0)$ and $(\Psi_M^{-1})^*(\omega_B^*)$ are of type $(1,1)$.** We only give a proof of (3) since (4) is obtained in a similar manner by applying the following argument to the map Ψ_M^{-1}.
First of all notice that if we set $\omega_{\Psi_M} = \Psi_M^*(\omega_0)$ equality (3) is equivalent to the validity of the following two equations

$$(\omega_{\Psi_M})_p(u, Ju) = (\omega_B)_p(u, Ju), \tag{24}$$

$$(\omega_{\Psi_M})_p(Ju, Jv) = (\omega_B)_p(Ju, Jv), \tag{25}$$

for all $p \in M$, $u, v \in T_pM$, where J denotes the almost complex structure of M evaluated at the point p. The second equation, namely (25), is precisely our assumption that $\Psi_M^*(\omega_0)$ is of type $(1,1)$.
Thus it remains to prove (24). Fix $p \in M$ and $u \in T_pM$. Consider the complex line $\mathcal{L} = \mathrm{span}_{\mathbb{C}}(u) \subset T_pM$ and a classical complex and totally geodesic submanifold $(C, 0) \hookrightarrow (M, 0)$ such that $\mathcal{L} \subset T_pC$ (whose existence is guaranteed by Corollary 5.3). If we denote by $\omega_{B,C}$ and $\omega_{0,\mathcal{C}}$ the hyperbolic form on C and the flat Kähler form on $\mathcal{C}$ (the HPJTS associated to C) we get:

$$(\omega_{\Psi_M})_p(u, Ju) = (\Psi_C^*(\omega_{0,\mathcal{C}}))_p(u, Ju) = (\omega_{B,C})_p(u, Ju) = (\omega_B)_p(u, Ju),$$

where the first and third equalities follow by the hereditary property (H) and the fact that the embedding $(C, 0) \hookrightarrow (M, 0)$ is a Kähler embedding while the second equality is true since C is of classical type (and hence $\Psi_C^*(\omega_{0,\mathcal{C}}) = \omega_{B,C}$ by the results of Section 4). □

As a byproduct of the previous proof one immediately gets the following theorem.

Theorem 5.4. *Let $(M, 0)$ be a HSSNT equipped with its Bergman form $\omega_{Berg,M}$. Let ω be a two form of type $(1,1)$ on M. Assume that the restriction of ω to all classical HSSNT $(C, 0)$ passing through the origin equals the Bergman form of C. Then $\omega = \omega_{Berg,M}$.*

6. Roos' proof of properties (D) and (S)

Let $\mathcal{M}$ be a HPJTS of rank r. In this section we denote by $\Psi = \Psi_M : M \to \mathcal{M}, z \mapsto B(z,z)^{-1/4}z$ the duality map. Let

$$z = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_r c_r$$

be a spectral decomposition of $z \in M$. As (see [18], Proposition V.4.2, (5.8))

$$\begin{aligned} B(z,z)c_j &= \left(1 - \lambda_j^2\right)^2 c_j, \\ D(z,z)c_j &= 2\lambda_j^2 c_j, \end{aligned}$$

we have

$$\Psi(z) = \sum_{j=1}^{r} \frac{\lambda_j}{\left(1 - \lambda_j^2\right)^{1/2}} c_j \tag{26}$$

and

$$\left(\mathrm{id} - \frac{1}{2} D(z,z)\right) c_j = \left(1 - \lambda_j^2\right) c_j. \tag{27}$$

Thus

$$\Psi(z) = \left(\mathrm{id} - \frac{1}{2} D(z,z)\right)^{-1/2} z = \left(\mathrm{id} - z \square z\right)^{-1/2} z, \tag{28}$$

where we use the operator

$$z \square z = \frac{1}{2} D(z,z). \tag{29}$$

From the previous equation, it is easily seen that Ψ is bijective and that the inverse map $\Psi^{-1} : \mathcal{M} \to M$ is given by

$$\Psi^{-1}(u) = \sum_{j=1}^{r} \frac{\mu_j}{\left(1 + \mu_j^2\right)^{1/2}} c_j, \tag{30}$$

if $u = \sum_{j=1}^{r} \mu_j c_j$ is the spectral decomposition of $u \in V$. The relation (30) is equivalent to

$$\Psi^{-1}(u) = B(u, -u)^{-1/4} u \qquad (u \in V), \tag{31}$$

so that Ψ is a diffeomorphism. Therefore (D) in Theorem 1.1 is proved.

In order to prove (S) of Theorem 1.1, set $p_1(z) = m_1(z, z)$. We then have $\overline{\partial} p_1 = m_1(z, \mathrm{d}\, z)$ and

$$\begin{aligned}\Psi^*\left(\overline{\partial} p_1\right) &= m_1\left(\Psi(z), \mathrm{d}\,\Psi(z)\right)\\ &= m_1\left(\Psi(z), \left(\mathrm{d}\left(\mathrm{id} - z\Box z\right)^{-1/2}\right) z\right)\\ &+ m_1\left(\Psi(z), \left(\left(\mathrm{id} - z\Box z\right)^{-1/2}\right) \mathrm{d}\, z\right),\end{aligned} \tag{32}$$

where we have used the identity

$$\mathrm{d}\,\Psi(z) = \left(\mathrm{d}\left(\mathrm{id} - z\Box z\right)^{-1/2}\right) z + \left(\mathrm{id} - z\Box z\right)^{-1/2} \mathrm{d}\, z.$$

As $z\Box z$ is self-adjoint w.r. to the Hermitian metric m_1, we have

$$\begin{aligned}m_1\left(\Psi(z), \left(\left(\mathrm{id} - z\Box z\right)^{-1/2}\right) \mathrm{d}\, z\right) &= m_1\left(\left(\mathrm{id} - z\Box z\right)^{-1/2} z, \left(\left(\mathrm{id} - z\Box z\right)^{-1/2}\right) \mathrm{d}\, z\right)\\ &= m_1\left(\left(\mathrm{id} - z\Box z\right)^{-1} z, \mathrm{d}\, z\right).\end{aligned}$$

If $z = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_r c_r$ is a spectral decomposition of $z \in \mathcal{M}$, we have

$$\left(\mathrm{id} - z\Box z\right)^{-1} z = \sum_{j=1}^{r} \frac{\lambda_j}{1 - \lambda_j^2} c_j = z^z, \tag{33}$$

where z^z denotes the *quasi-inverse* in the Jordan triple system $\mathcal{M}$.
Using (37), (33) and Lemma 7.1 in the Appendix, we get the last term in (32), namely

$$m_1\left(\Psi(z), \left(\left(\mathrm{id} - z\Box z\right)^{-1/2}\right) \mathrm{d}\, z\right) = -\frac{\overline{\partial}\mathcal{N}}{\mathcal{N}}. \tag{34}$$

Applying Lemma 7.2 in the Appendix, we get

$$\begin{aligned}m_1\left(\Psi(z), \left(\mathrm{d}\left(\mathrm{id} - z\Box z\right)^{-1/2}\right) z\right) &= m_1\left(\Psi(z), \frac{1}{2}\left(\mathrm{id} - z\Box z\right)^{-3/2}\left(\mathrm{d}\left(z\Box z\right)\right) z\right)\\ &= \frac{1}{2} m_1\left(\left(\mathrm{id} - z\Box z\right)^{-2} z, \left(\mathrm{d}\left(z\Box z\right)\right) z\right).\end{aligned}$$

We finally obtain , using this last result and (34):

$$\Psi^*\left(\overline{\partial} p_1\right) = -\frac{\overline{\partial}\mathcal{N}}{\mathcal{N}} + \frac{1}{2} m_1\left(\left(\mathrm{id} - z\Box z\right)^{-2} z, \left(\mathrm{d}\left(z\Box z\right)\right) z\right). \tag{35}$$

Along the same lines, one proves

$$\left(\Psi^{-1}\right)^*\left(\overline{\partial} p_1\right) = \frac{\overline{\partial}\mathcal{N}_*}{\mathcal{N}_*} - \frac{1}{2} m_1\left(\left(\mathrm{id} + z\Box z\right)^{-2} z, \left(\mathrm{d}\left(z\Box z\right)\right) z\right). \tag{36}$$

To prove (S) in Theorem 1.1, it is now enough to check that the forms

$$\begin{aligned}\beta(z) &= m_1\left((\operatorname{id} - z\Box z)^{-2} z, (\mathrm{d}\,(z\Box z))\, z\right),\\ \beta_*(z) &= m_1\left((\operatorname{id} + z\Box z)^{-2} z, (\mathrm{d}\,(z\Box z))\, z\right)\end{aligned}$$

are d-closed (or d-exact, as M and $\mathcal{M}$ are simply connected). We verify it for $\beta(z)$ in the following proposition (the proof for $\beta_*(z)$ is similar).

Proposition 6.1. *Let G be the analytic function defined on $]-1,+1[$ by*

$$G(t) = \frac{1}{t}\int_0^t \frac{u}{(1-u)^2}\,\mathrm{d}\,u$$

and $\gamma : M \to \mathbb{R}$ the function defined by

$$\gamma(z) = m_1\left(G\,(z\Box z)\, z, z\right).$$

Then $\beta(z) = \mathrm{d}\,\gamma(z)$.

Proof:
By using Lemma 7.2, in the appendix one has

$$\mathrm{d}\,\gamma(z) = m_1\left(G'\,(z\Box z)\,(\mathrm{d}\,(z\Box z))\, z, z\right) + m_1\left(G\,(z\Box z)\,\mathrm{d}\,z, z\right) + m_1\left(G\,(z\Box z)\, z, \mathrm{d}\,z\right)$$

Using the identity $G(t) + tG'(t) = \frac{t}{(1-t)^2}$, we get

$$\begin{aligned}\mathrm{d}\,\gamma(z) &= m_1\left(G'\,(z\Box z)\,(\mathrm{d}\,(z\Box z))\, z, z\right)\\ &- m_1\left(G'\,(z\Box z)\,(z\Box z)\,\mathrm{d}\,z, z\right) - m_1\left(G'\,(z\Box z)\,(z\Box z)\, z, \mathrm{d}\,z\right)\\ &+ m_1\left((\operatorname{id} - z\Box z)^{-2}\,(z\Box z)\,\mathrm{d}\,z, z\right) + m_1\left((\operatorname{id} - z\Box z)^{-2}\,(z\Box z)\, z, \mathrm{d}\,z\right).\end{aligned}$$

Now

$$\begin{aligned}m_1\left(G'\,(z\Box z)\,(\mathrm{d}\,(z\Box z))\, z, z\right) &= m_1\left(G'\,(z\Box z)\,(\mathrm{d}\,z\Box z)\, z, z\right) + m_1\left(G'\,(z\Box z)\,(z\Box\,\mathrm{d}\,z)\, z, z\right)\\ &= m_1\left(G'\,(z\Box z)\,(z\Box z)\,\mathrm{d}\,z, z\right) + m_1\left(G'\,(z\Box z)\,(z\Box\,\mathrm{d}\,z)\, z, z\right)\end{aligned}$$

and using the commutativity between $z\Box z$ and $Q(z)$ and the identity (4.55) in [18], p.495) one gets

$$\begin{aligned}m_1\left(G'\,(z\Box z)\,(z\Box\,\mathrm{d}\,z)\, z, z\right) &= m_1\left(Q(z)\,\mathrm{d}\,z, G'\,(z\Box z)\, z\right)\\ &= \overline{m_1\left(\mathrm{d}\,z, Q(z)G'\,(z\Box z)\, z\right)}\\ &= \overline{m_1\left(\mathrm{d}\,z, G'\,(z\Box z)\, Q(z) z\right)}\\ &= m_1\left(G'\,(z\Box z)\,(z\Box z)\, z, \mathrm{d}\,z\right).\end{aligned}$$

So we get

$$\mathrm{d}\,\gamma(z) = m_1\left((\operatorname{id} - z\Box z)^{-2}\,(z\Box z)\,\mathrm{d}\,z, z\right) + m_1\left((\operatorname{id} - z\Box z)^{-2}\,(z\Box z)\, z, \mathrm{d}\,z\right).$$

By the same argument as before, with G' replaced by $F'(t) = \frac{t}{(1-t)^2}$, we have

$$\begin{aligned}\beta(z) &= m_1\left((\operatorname{id} - z\Box z)^{-2} z, (\mathrm{d}\,(z\Box z))\, z\right) = m_1\left((\operatorname{id} - z\Box z)^{-2}\,(z\Box z)\,\mathrm{d}\,z, z\right)\\ &+ m_1\left((\operatorname{id} - z\Box z)^{-2}\,(z\Box z)\, z, \mathrm{d}\,z\right) = \mathrm{d}\,\gamma(z).\end{aligned}$$

□

7. Appendix: some technical results on HPJTS

The following general result holds in Jordan triple systems (see Lemma 2 in [19]):

Lemma 7.1. *Let $\mathcal{M}$ be an Hermitian positive Jordan triple system with generic trace m_1 and generic norm N. Let $\mathcal{N}(z) = N(z,z)$ and $\mathcal{N}_*(z) = N(z,-z)$. Then*

$$\frac{\overline{\partial}\mathcal{N}}{\mathcal{N}} = -m_1\left(z^z, \mathrm{d}\, z\right), \tag{37}$$

$$\frac{\overline{\partial}\mathcal{N}_*}{\mathcal{N}_*} = m_1\left(z^{-z}, \mathrm{d}\, z\right), \tag{38}$$

where z^z denotes the quasi inverse in the Jordan triple system $\mathcal{M}$.

Lemma 7.2. *Let $f :]-1,1[\to \mathbb{R}$ and $F :]-1,1[\to \mathbb{R}$ be real-analytic functions. Then*

$$m_1\left(f\left(z\Box z\right)z, \left(\mathrm{d}\, F\left(z\Box z\right)\right)z\right) = m_1\left(f\left(z\Box z\right)z, F'\left(z\Box z\right)\mathrm{d}\left(z\Box z\right)z\right). \tag{39}$$

Proof. It suffices to prove (39) for $f = t^p$ and $F = t^k$. For $k > 0$, we have

$$\left(\mathrm{d}\left(\left(z\Box z\right)^k\right)\right)z = \sum_{j=0}^{k-1}\left(z\Box z\right)^{k-1-j}\mathrm{d}\left(z\Box z\right)\left(z\Box z\right)^j z.$$

Recall that the *odd powers* $z^{(2j+1)}$ in a Hermitian Jordan triple system are defined recursively by

$$z^{(1)} = z, \quad z^{(2j+1)} = Q(z)z^{(2j-1)} \tag{40}$$

and that they satisfy the identity

$$z^{(2j+1)} = \left(z\Box z\right)^j z. \tag{41}$$

Using the commutativity between $z\Box z$ and $Q(z)$ we then have

$$\begin{aligned}
\mathrm{d}\left(z\Box z\right)Q(z) &= \left(\mathrm{d}\, z\Box z + z\Box\,\mathrm{d}\, z\right)Q(z) = Q(z)\,\mathrm{d}\left(z\Box z\right), \\
\mathrm{d}\left(z\Box z\right)\left(z\Box z\right)^j z &= \mathrm{d}\left(z\Box z\right)Q(z)^j z = Q(z)^j\,\mathrm{d}\left(z\Box z\right)z, \\
\left(\mathrm{d}\left(\left(z\Box z\right)^k\right)\right)z &= \sum_{j=0}^{k-1}\left(z\Box z\right)^{k-1-j}Q(z)^j\,\mathrm{d}\left(z\Box z\right)z.
\end{aligned}$$

As $z\Box z$ is self-adjoint w.r. to m_1, we have

$$m_1\left(z^{(2p+1)}, \left(\mathrm{d}\left(z\Box z\right)^k\right)z\right) = \sum_{j=0}^{k-1} m_1\left(\left(z\Box z\right)^{p+k-1-j}z, Q(z)^j\,\mathrm{d}\left(z\Box z\right)z\right).$$

Using the identity (4.55) in [18], p.495, we obtain (denoting by τ the conjugation of complex numbers)

$$\begin{aligned}
m_1\left(z^{(2p+1)}, \left(\mathrm{d}\left(z\Box z\right)^k\right)z\right) &= \sum_{j=0}^{k-1}\tau^j m_1\left(Q(z)^j\left(z\Box z\right)^{p+k-1-j}z, \mathrm{d}\left(z\Box z\right)z\right) \\
&= \sum_{j=0}^{k-1}\tau^j m_1\left(\left(z\Box z\right)^{p+k-1}z, \left(z\Box z\right)\mathrm{d}\, z + Q(z)\,\mathrm{d}\, z\right).
\end{aligned}$$

But

$$\begin{aligned} m_1\left((z\Box z)^{p+k-1}\, z, \mathrm{d}\,(z\Box z)\, z\right) &= m_1\left((z\Box z)^{p+k-1}\, z, (z\Box z)\,\mathrm{d}\, z + Q(z)\,\mathrm{d}\, z\right) \\ &= m_1\left(z^{(2p+2k+1)}, \mathrm{d}\, z\right) + \tau m_1\left(z^{(2p+2k+1)}, \mathrm{d}\, z\right) \end{aligned}$$

is real, so that

$$\begin{aligned} m_1\left(z^{(2p+1)}, \left(\mathrm{d}\,(z\Box z)^k\right) z\right) &= k m_1\left((z\Box z)^{p+k-1}\, z, \mathrm{d}\,(z\Box z)\, z\right) \\ &= m_1\left((z\Box z)^p\, z, k\,(z\Box z)^{k-1}\,\mathrm{d}\,(z\Box z)\, z\right), \end{aligned}$$

which is precisely (39) for $f = t^p$, $F = t^k$. $\Box$

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