SYMMETRIC COMPOSITION ALGEBRAS OVER ALGEBRAIC VARIETIES

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ABSTRACT. Let R be a ring such that $2, 3 \in R$. Let $(X, \mathcal{O}_X)$ be a locally ringed space such that $2, 3 \in H^0(X, \mathcal{O}_X^\times)$. We investigate the structure of symmetric composition algebras over X and thus in particular over R. Symmetric composition algebras are constructed on the trace 0 elements of cubic alternative algebras $\mathcal{A}$ over X, if $H^0(X, \mathcal{O}_X)$ contains a primitive third root of unity, generalizing a method first presented by J. R. Faulkner. If $\mathcal{A}$ is an Azumaya algebra of rank 9 satisfying certain additional conditions, an isotope of such a symmetric composition algebra is a Hurwitz algebra. We find examples of Okubo algebras over elliptic curves which cannot be made into octonion algebras.

INTRODUCTION

Unital composition algebras (also called Hurwitz algebras) were first studied over locally ringed spaces by Petersson [P1] and were classified over curves of genus zero [P1, 4.4]. Quaternion and octonion algebras over curves of genus one were investigated in [Pu1, 2].

Petersson [P2], Okubo [O] and later, in a more general setting, Faulkner [F], provided the elements of trace 0 of a central simple associative algebra A of degree 3 over a field k with the structure of an eight-dimensional non-unital symmetric composition algebra, under the assumption that the base field k has characteristic not 2 or 3 and contains the cube roots of unity. This algebra (called an *Okubo algebra*), in turn, determined a unique octonion algebra structure on the elements of trace 0 in A. Both the Okubo algebra obtained from this construction and the octonion algebra have the same norm. Expanding these earlier approaches, Elduque and Myung [E-M] set up a categorical equivalence between finite dimensional flexible composition algebras and finite dimensional separable alternative algebras of degree 3, the latter with or without involutions of the second kind, depending on whether the base field contains the third roots of unity or not. Given a primitive sixth root of unity in the base ring R, Loos [L] constructed functors between the category of generalized symmetric compositions and the category of unital algebras with multiplicative cubic forms, which are equivalences if $3 \in R^\times$.

In this paper we initiate the study of symmetric composition algebras over locally ringed spaces. We restrict our investigations to algebras with nondegenerate norms and construct symmetric composition algebras over a locally ringed space X using Faulkner's idea of taking the trace zero elements of a separable cubic alternative algebra over X and supplying them with a suitable multiplication. We only consider the case where $(X, \mathcal{O}_X)$ is a locally ringed

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space such that $2, 3 \in H^0(X, \mathcal{O}_X^\times)$ and such that $H^0(X, \mathcal{O}_X)$ contains a primitive third root of unity ω. The case where $H^0(X, \mathcal{O}_X)$ does not contain a primitive third root of unity will be treated in another paper, to keep the length of this one within reason.

In the process we find algebras $\mathcal{A}$ over X whose residue class algebras $\mathcal{A}(P) = \mathcal{A}_P \otimes k(P)$ are isometric to the Okubo algebra $P_8(k(P))$ (Theorem 2, 7.1), and thus can be viewed as canonical generalizations of Okubo algebras over fields. These algebras have the additional property that, if we take the attached algebra $\mathcal{A}^-$ with multiplication $[x, y] = xy - yx$, we obtain examples of Lie algebras over X, whose residue class algebras are central simple Lie algebras of type A_2 which arise from central simple associative algebras of degree 3 [E-M, p. 2499].

Special emphasis is put on symmetric composition algebras over algebraic curves of genus zero and on elliptic curves. It is interesting to see which (selfdual) vector bundles carry the structure of a symmetric composition algebra, and if there are any indecomposable such bundles or bundles, which are direct sums of vector bundles of large ranks. The results on these algebras over curves also serve as examples on what underlying module structures can appear when studying symmetric composition algebras over rings.

The vector bundles over curves of genus zero are of relatively simple type (there is no indecomposable vector bundle of rank greater than 2, and the only absolutely indecomposable ones are the line bundles). Thus it is worth going one step further and looking at algebras over curves of genus one. The vector bundles over elliptic curves and their behaviour are well-known [At, AEJ1, 2, 3]. There exist absolutely indecomposable selfdual vector bundles of arbitrary rank, and indecomposable ones of different types. Over elliptic curves (or more generally, over curves of genus one, since, indeed, our arguments generally would work also for curves without rational points), a classification of symmetric composition algebras seems to be still out of reach. One of the problems is that the Azumaya algebras of constant rank 9 over an elliptic curve are not sufficiently well understood, at least to the author's knowledge, which would be important in order to at least get an idea on all the constructions which are possible when using such an algebra.

We give examples of symmetric composition algebras of constant rank 2 over locally ringed spaces (and over rings) which cannot be turned into a quadratic étale algebra (Examples 2, 3), and of Okubo algebras over elliptic curves, for which there exists no multiplication which makes them into a Hurwitz algebra (Proposition 11). For composition algebras over fields this is not possible: every symmetric composition algebra over a field has an isotope which is a unital composition algebra.

The hope is that by using this construction it should be possible to find octonion algebras over X which cannot be constructed by one of the construction we know so far, like the generalized Cayley-Dickson doubling process [P1], or the Thakur's method [T], which yields all octonion algebras containing a quadratic étale subalgebra. A better understanding of the Azumaya algebras of constant rank 9 over a curve would be important in order to get an idea on all the underlying module structures which are possible.

The contents of the paper are as follows: After defining (not necessary unital) composition algebras over rings and locally ringed spaces as well as recalling some other well-known facts on algebras and cubic forms in Section 1, we extend some results from the theory of composition algebras over fields to the setting of composition algebras over rings and locally ringed spaces in Section 2. We do not strive for completeness but rather choose a selection of results needed later, in order to keep the paper within a reasonable length. Some observations on symmetric algebras and Petersson algebras over rings are collected in Section 3. We then move on to describe how to define a flexible symmetric composition algebra on the trace zero elements of a cubic alternative algebra over X in Section 5. Before that, the results needed for alternative algebras over X are briefly summarized in Section 4. The converse of Theorem 2, the main result of Section 5, is treated in Section 6. In Section 7, we introduce Petersson algebras over X and in Section 8, we look at flexible symmetric composition algebras of rank 2. Sections 9 and 10 deal with symmetric composition algebras over curves of genus zero and elliptic curves, respectively.

We use the standard terminology from algebraic geometry, see Hartshorne's book [H]. In the following, let $(X, \mathcal{O}_X)$ be a locally ringed space and R a unital commutative associative ring.

1. Preliminaries

1.1. Let M be an R-module. M has *full support* if $\operatorname{Supp} M = \operatorname{Spec} R$. The *rank* of M is defined to be $\sup\{\operatorname{rank}_{R_P} M_P \mid P \in \operatorname{Spec} R\}$. In the following, the term R-algebra always refers to non-associative algebras over R which are finitely generated projective as R-modules. Let C be an R-algebra with full support. A quadratic form $N\colon C \to R$ on C is *multiplicative* (one also says that it *permits composition*) if $N(uv) = N(u)N(v)$ for all $u, v \in C$.

Following [KMRT, p. 454 ff.], C is called a *composition algebra* over R if it admits a multiplicative quadratic form $N\colon C \to R$ which is *nondegenerate*; i.e., its induced symmetric bilinear form $N(u, v) = N(u+v) - N(u) - N(v)$ determines a module isomorphism $C \xrightarrow{\sim} C^\vee = Hom_R(C, R)$. A unital composition algebra is called a *Hurwitz algebra*. For a Hurwitz algebra C, the nondegenerate multiplicative form N on C is uniquely determined up to isometry and called the *norm* of C. A Hurwitz algebra is quadratic alternative and its norm N satisfies $N(1) = 1$. Hurwitz algebras exist only in ranks 1, 2, 4 or 8. Those of constant rank 2 (resp., 4, 8) are called *quadratic étale* (resp., *quaternion algebra*, *octonion algebra*). Every Hurwitz algebra possesses a *canonical involution* $^- : C \to C$, $\bar{x} = N(x, 1)1 - x$ (see for instance [P1, 1.6]). If C is a Hurwitz algebra over R, then C_P is locally free as an R_P-module for all $P \in \operatorname{Spec} R$, so the canonical morphism $R \to C$, $r \to r1_C$ is injective, since it is injective locally. Thus C is a faithful R-module; i.e., $\operatorname{Ann}_R(C) = 0$ (so $rC = 0$ implies $r = 0$ for all $r \in R$). However, a non-unital composition algebra need not be defined on a faithful R-module C.

1.2. Let $(X, \mathcal{O}_X)$ be a locally ringed space. For $P \in X$ let $\mathcal{O}_{P,X}$ be the local ring of $\mathcal{O}_X$ at P and m_P the maximal ideal of $\mathcal{O}_{P,X}$. The corresponding residue class field is denoted by $k(P) = \mathcal{O}_{P,X}/m_P$. For an $\mathcal{O}_X$-module $\mathcal{F}$ the stalk of $\mathcal{F}$ at P is denoted by $\mathcal{F}_P$. $\mathcal{F}$ is said to have *full support* if $\operatorname{Supp} \mathcal{F} = X$; i.e., if $\mathcal{F}_P \neq 0$ for all $P \in X$. We call $\mathcal{F}$ *locally free of finite rank* if for each $P \in X$ there is an open neighborhood $U \subset X$ of P such that $\mathcal{F}|_U = \mathcal{O}_U^r$ for some integer $r \geq 0$. The *rank* of $\mathcal{F}$ is defined to be $\sup\{\mathrm{rank}_{\mathcal{O}_{P,X}} \mathcal{F}_P \,|\, P \in X\}$. The term "$\mathcal{O}_X$-algebra" (or "algebra over X") always refers to a non-associative $\mathcal{O}_X$-algebra which is locally free of finite rank as $\mathcal{O}_X$-module. An $\mathcal{O}_X$-algebra $\mathcal{A}$ is called *alternative* if it is unital and if $x^2y = x(xy)$ and $yx^2 = (yx)x$ for all sections x, y of $\mathcal{A}$ over the same open subset of X. A unital algebra $\mathcal{A}$ over $\mathcal{O}_X$ is called *separable* if $\mathcal{A}(P)$ is a separable $k(P)$-algebra for all $P \in X$. A unital associative $\mathcal{O}_X$-algebra $\mathcal{A}$ is called an *Azumaya algebra* if $\mathcal{A}_P \otimes_{\mathcal{O}_{P,X}} k(P)$ is a central simple algebra over $k(P)$ for all $P \in X$ [K].

1.3. Let $\mathcal{C}$ be an $\mathcal{O}_X$-algebra with full support. A quadratic form $N \colon \mathcal{C} \to \mathcal{O}_X$ on $\mathcal{C}$ is *multiplicative* (or *permits composition*), if $N(uv) = N(u)N(v)$ for all sections u, v of $\mathcal{C}$ over the same open subset of X. Following [P1], $\mathcal{C}$ is called a *composition algebra* over X if it admits a multiplicative quadratic form $N \colon \mathcal{C} \to \mathcal{O}_X$ such that the induced symmetric bilinear form $N(u, v) = N(u + v) - N(u) - N(v)$ is *nondegenerate*; i.e., it determines a module isomorphism $\mathcal{C} \xrightarrow{\sim} \mathcal{C}^\vee = \mathcal{H}om(\mathcal{C}, \mathcal{O}_X)$. A unital composition algebra is called a *Hurwitz algebra*. If $\mathcal{C}$ is a Hurwitz algebra over X with nondegenerate multiplicative form N then N is uniquely determined and called the *norm* of $\mathcal{C}$. It is often denoted by $N_{\mathcal{C}}$. Hurwitz algebras over X exist only in ranks 1, 2, 4 or 8. A Hurwitz algebra of constant rank 2 (resp. 4 or 8) is called a *quadratic étale* algebra (resp. *quaternion algebra* or *octonion algebra*). A Hurwitz algebra over X of constant rank is called *split*, if it contains a Hurwitz subalgebra isomorphic to $\mathcal{O}_X \oplus \mathcal{O}_X$.

If X is an R-scheme with structure morphism $\tau \colon X \to \operatorname{Spec} R$, then a composition algebra $\mathcal{C}$ over X is *defined over* R if there exists a composition algebra C over R such that $\mathcal{C} \cong \tau^* C = C \otimes_R \mathcal{O}_X$.

1.4. **Construction methods for Hurwitz algebras.** There exists a Cayley-Dickson doubling for Hurwitz algebras of constant rank over locally ringed spaces [P1]: let $\mathcal{D}$ be a Hurwitz algebra of constant rank ≤ 4 over X. If a locally free right $\mathcal{D}$-module $\mathcal{P}$ of rank one has *norm one* as defined in [P1], there exists a nondegenerate quadratic form $N \colon \mathcal{P} \to \mathcal{O}_X$ satisfying $N(w \cdot u) = N(w)N_{\mathcal{D}}(u)$ for all sections w in $\mathcal{P}$, u in $\mathcal{D}$, where $\cdot$ denotes the right $\mathcal{D}$-module structure of $\mathcal{P}$. N is uniquely determined up to an invertible factor in $H^0(X, \mathcal{O}_X)$ and called a *norm* on $\mathcal{P}$. N determines a unique $\mathcal{O}_X$-bilinear map $\mathcal{P} \times \mathcal{P} \to \mathcal{D}$, written multiplicatively and satisfying $(w \cdot u)(w \cdot v) = N(w)v^*u$ for u, v in $\mathcal{D}$, w in $\mathcal{P}$. The $\mathcal{O}_X$-module

$$\mathrm{Cay}(\mathcal{D}, \mathcal{P}, N) = \mathcal{D} \oplus \mathcal{P}$$

becomes a Hurwitz algebra under the multiplication

$$(u, w)(u', w') = (uu' + ww', w' \cdot u + w \cdot u'^*),$$

with norm $N_{\mathrm{Cay}(\mathcal{D},\mathcal{P},N)} = N_{\mathcal{D}} \oplus (-N)$.

If $\mathcal{C}$ is a Hurwitz algebra of constant rank r containing a Hurwitz subalgebra of constant rank $r/2$, then there are $\mathcal{P}$, N as above such that $\mathcal{C} \cong \mathrm{Cay}(\mathcal{D}, \mathcal{P}, N)$. This construction is called the *(generalized) Cayley-Dickson doubling* of $\mathcal{D}$. The globally free right $\mathcal{D}$-module $\mathcal{D}$ itself has norm one and $\mu N_{\mathcal{D}}$, for any invertible $\mu \in H^0(X, \mathcal{O}_X)$, is a norm on $\mathcal{D}$. This case yields the *classical* doubling $\mathrm{Cay}(\mathcal{D}, \mu) = \mathrm{Cay}(\mathcal{D}, \mathcal{D}, \mu N_{\mathcal{D}})$.

Moreover, it is possible to construct octonion algebras which do not necessarily arise from a Cayley-Dickson doubling but contain a quadratic étale subalgebra (cf. Petersson and Racine [P-R, 3.8] or Thakur [T]).

1.5. **Cubic forms over X.** Let $(X,\mathcal{O}_X)$ be a locally ringed space such that $2,3\in H^0(X,\mathcal{O}_X^\times)$. Let $\mathcal{M}$, $\mathcal{N}$ be two $\mathcal{O}_X$-modules which are locally free of finite rank. A map $N:\mathcal{M}\to\mathcal{N}$ is called a *cubic map*, if $N(ax)=a^3N(x)$ for all sections a in $\mathcal{O}_X$, x in $\mathcal{M}$ over the same open subset of X, where the map $\theta:\mathcal{M}\times\mathcal{M}\times\mathcal{M}\to\mathcal{N}$ defined by

$$\theta(x,y,z)=\frac{1}{6}(N(x+y+z)-N(x+y)-N(x+z)-N(y+z)+N(x)+N(y)+N(z))$$

for x,y,z sections in $\mathcal{M}$ over the same open subset of X, is a trilinear form over $\mathcal{O}_X$. We have $N(x)=\theta(x,x,x)$ for all sections x of $\mathcal{M}$ over the same open subset of X.

A trilinear map $\theta:\mathcal{M}\times\mathcal{M}\times\mathcal{M}\to\mathcal{N}$ is called *symmetric* if $\theta(x,y,z)$ is invariant under all permutations of its variables. We canonically identify symmetric trilinear maps and cubic maps.

If $\mathcal{N}=\mathcal{O}_X$, then a cubic map $N:\mathcal{M}\to\mathcal{O}_X$ is called a *cubic form* and $\mathcal{M}$ together with a symmetric trilinear map $\theta:\mathcal{M}\times\mathcal{M}\times\mathcal{M}\to\mathcal{O}_X$ a *trilinear space*. A cubic form $N:\mathcal{M}\to\mathcal{O}_X$ on a locally free $\mathcal{O}_X$-module $\mathcal{M}$ of finite rank with full support (or, respectively, the associated trilinear form θ) is called *nondegenerate* if, for all $P\in X$, the induced maps $N(P):\mathcal{M}(P)\to k(P)$ are nondegenerate in the sense that the residue maps $\theta'\otimes k(P)$ of the maps $\theta':\mathcal{M}\to\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}\otimes\mathcal{M},\mathcal{O}_X)$ defined by

$$x\to\theta_x(y\otimes z)=\theta(x,y,z)$$

are injective. This notion of nondegeneracy is invariant under base change.

Two trilinear spaces $(\mathcal{M}_i,\theta_i)$ $(i=1,2)$ are called *isomorphic* if there exists an $\mathcal{O}_X$-module isomorphism $f:\mathcal{M}_1\to\mathcal{M}_2$ such that $\theta_2(f(v_1),f(v_2),f(v_3))=\theta_1(v_1,v_2,v_3)$ for all sections v_1,v_2,v_3 of $\mathcal{M}_1$ over the same open subset of X.

1.6. **Cubic forms with adjoint and base point.** ([Pu4], [McC2]) Let $(X,\mathcal{O}_X)$ be a locally ringed space such that $2,3\in H^0(X,\mathcal{O}_X^\times)$. Let $\mathcal{W}$ be an $\mathcal{O}_X$-module. A tripel $(N,\sharp,1)$ is a *cubic form with adjoint and base point* on $\mathcal{W}$ if $N:\mathcal{W}\to\mathcal{O}_X$ is a cubic form, $\sharp:\mathcal{W}\to\mathcal{W}$ a quadratic map and $1\in H^0(X,\mathcal{W})$, such that

$$\begin{aligned}
&x^{\sharp\,\sharp}=N(x)x,\\
&T(x^\sharp,y)=D_yN(x)\text{ for }T(x,y)=T(x)T(y)-6N(1,x,y),\\
&N(1)=1,\ 1^\sharp=1,\\
&1\times y=T(y)1-y\text{ with }T(y)=T(y,1),\quad x\times y=(x+y)^\sharp-x^\sharp-y^\sharp
\end{aligned}$$

for all sections x,y in $\mathcal{W}$ over the same open subset of X.

Here, $D_yN(x)$ denotes the directional derivative of N in the direction y, evaluated at x, see for instance [P-R]. Since we assume that $2,3\in H^0(X,\mathcal{O}_X^\times)$, this means that the quadratic map $D_yN(x)$ is the coefficient $N(x;y)$ of the indeterminant Z in the expansion

$$N(x+Zy)=N(x)+ZN(x;y)+Z^2N(y;x)+Z^3N(y),$$

i.e. that $T(x^\sharp,y)=3N(x,x,y)$.

Every cubic form with adjoint and base point $(N, \sharp, 1)$ on a locally free $\mathcal{O}_X$-module $\mathcal{W}$ of finite rank defines a unital Jordan algebra structure $\mathcal{J}(N, \sharp, 1) = (\mathcal{W}, U, 1)$ on $\mathcal{J}$ via

$$U_x(y) = T(x, y)x - x^\sharp \times y$$

for all sections x, y in $\mathcal{W}$, where the identities given in [P-R, p. 213] (resp., in [McC2, Section 1.3.8]) hold for all sections in $\mathcal{W}$.

1.7. **Some facts on proper schemes.** Let X be a proper scheme over k and l/k an algebraic field extension. The Theorem of Krull-Schmidt holds for vector bundles over X, i.e., every vector bundle on X can be decomposed as a direct sum of indecomposable vector bundles, unique up to isomorphisms and order of sumands [AEJ1, p. 1324]. Moreover, non-isomorphic vector bundles on X extend to non-isomorphic vector bundles on $X_l = X \times_k l$, for every separable algebraic field extension l/k [AEJ1, p. 1325].

For a vector bundle $\mathcal{N}$ on X_l, the direct image $\pi_*\mathcal{N}$ of $\mathcal{N}$ under the projection morphism $\pi \colon X_l \to X$ is a vector bundle on X denoted by $tr_{l/k}(\mathcal{N})$.

If X is any curve over k and l/k is a finite algebraic field extension of degree r, then

$$tr_{l/k}(\mathcal{O}_{X_l}) = \pi_*\mathcal{O}_{X_l} \cong \mathcal{O}_X^r$$

as $\mathcal{O}_X$-modules.

2. Composition algebras

2.1. Let R be a ring.

Proposition 1. *Let $(C, \star, N)$ be a composition algebra over R.*
(i) C has dimension 1, 2, 4 or 8.
(ii) Suppose there is an element $a \in C$ such that $N(a) \in R^\times$. Then there exists a multiplication $(x, y) \to x \diamond y$ on C such that $(C, \diamond, N)$ is a Hurwitz algebra.
(iii) Let $(C', \star', N')$ be another composition algebra over R. Suppose there exists an element $u \in C$ such that $N(u) = 1$. Then any automorphism of algebras $\alpha : (C, \star) \to (C', \star')$ such that $N'(\alpha(u)) = 1$ induces an isometry $(C, N) \to (C', N')$.
In particular, if both C and C' are Hurwitz algebras then any automorphism of algebras $\alpha : (C, \star) \to (C', \star')$ induces an isometry $(C, N) \to (C', N')$.

For Hurwitz algebras, (i) was proved in [P1, 1.7].

Proof. (i) For all $P \in \operatorname{Spec} R$, the residue class algebra $(C(P), N(P))$ is a composition algebra over the residue class field $k(P)$ and therefore has one of the claimed dimensions by [KMRT, VIII.(33.28)].
(ii) Let $a \in C$ such that $N(a) \in R^\times$ and let $u = N(a)^{-1}a^2$, so that $N(u) = 1$. The linear maps $L_u : x \to u \star x$ and $R_u : x \to x \star u$ are compatible with the norm, since $N(L_u(x)) = N(ux) = N(u)N(x) = N(x)$ and $N(R_u(x)) = N(xu) = N(x)N(u) = N(x)$ for all $x \in C$. The maps on the residue class algebras induced by L_u and R_u are bijective for all $P \in \operatorname{Spec} R$. Thus, by Nakayama's Lemma, their localizations $(L_u)_P$ resp. $(R_u)_P$

are bijective over the local rings R_P for all $P \in X$. This shows that L_u and R_u are both bijective, hence isometries. Therefore $v = u^2$ is the identity for the multiplication

$$x \diamond y = (R_u^{-1}x) \star (L_u^{-1}y)$$

and $N(x \diamond y) = N(x)N(y)$ as in [KMRT, VIII.(33.27)(1)].
(iii) As in [KMRT, VIII.(33.27)(2)], we observe that $N' \circ \alpha$ is a multiplicative quadratic form on C, so that we have $N(x \star y) = N(x)N(y)$ and $N'(\alpha(x \star y)) = N'(\alpha(x))N'(\alpha(y))$. Since there exists an element $u \in C$ such that $N(u) = N'(\alpha(u)) = 1$, we obtain a new multiplication $\diamond$ as in (ii) with a unit element 1 with respect to which both N and $N' \circ \alpha$ are multiplicative, so that the uniqueness of the norm of a unital composition algebra implies $N = N' \circ \alpha$.
If both C and C' are Hurwitz algebras then we choose $u = 1_C$. □

Hence, over rings, the nondegenerate multiplicative form N on a non-unital composition algebra C does not always seem to be uniquely determined by the multiplicative structure of the algebra anymore (for fields it is, cf. [KMRT, VIII.(33.27)]).

Note from the above proof that there is an element $a \in C$ such that $N(a) \in R^\times$ if and only if there is an element $u \in C$ such that $N(u) = 1$.

Corollary 1. *Let $(C, \star, N)$, $(C', \star', N')$ be two composition algebra without identity over a domain R.*
(i) Suppose there exists an element $u \in C$ such that $N(u) = 1$ and $\alpha : (C, \star) \to (C', \star')$ is an automorphism of algebras such that $N'(\alpha(u)) \in R^\times$ then α induces an isometry $(C, N) \to (C', N')$.
(ii) Suppose there exists an element $u \in C$ such that $N(u) = 1$. If N' is anisotropic, every automorphism of algebras $\alpha : (C, \star) \to (C', \star')$ induces an isometry $(C, N) \to (C', N')$.

Proof. (i) To find an element $\widetilde{u} \in C$ as needed in Proposition 1 (iii), we use that L_u is a bijective isometry of (C, N) as in the proof of Proposition 1 (ii). Hence there is an element $\widetilde{u} \in C$ such that $u \star \widetilde{u} = u$, implying $N(u \star \widetilde{u}) = 1$ and $N(\widetilde{u}) = 1$. Since we assume that $N(\alpha(u))$ is invertible in R, we also obtain that $N'(\alpha(u)) = N'(\alpha(u))N'(\alpha(\widetilde{u}))$ and thus that $N'(\alpha(\widetilde{u})) = 1$. Proposition 1 (iii) now yields the assertion.
(ii) We choose the element $\widetilde{u} \in C$ needed to apply Proposition 1 (iii) as in (i) such that $u \star \widetilde{u} = u$. Then $N(\widetilde{u}) = 1$. Since $u \neq 0$, $\alpha(u) \neq 0$ and since N' is anisotropic, $N'(\alpha(u)) \neq 0$. Now $N'(\alpha(u)) = N'(\alpha(\widetilde{u} \star u)) = N'(\alpha(\widetilde{u}))N'(\alpha(u))$ implies $0 = N'(\alpha(u))(1 - N'(\alpha(\widetilde{u})))$ and hence $N'(\alpha(\widetilde{u})) = 1$. □

Remark 1. If $2 \in R^\times$ then we can write a Hurwitz algebra C as the direct sum of R-modules $C = R \oplus C_0$ where $C_0 = \ker t = \{x \in C \mid t(x) = 0\}$.

Lemma 1. *For a composition algebra $(C, \star, N)$ over a local ring R there always exists an element $a \in C$ such that $N(a) \in R^\times$. In particular, there exists a new multiplication $\diamond$ such that $(C, \diamond, N)$ is a unital composition algebra over R.*

Proof. Change scalars to the residue class field k of R, then this is known: There exists an element $a' \in C \otimes_R k$ such that $(N \otimes_R k)(a') \neq 0$. Lifting a' to C yields an element $a \in C$ such that $N(a) \in R^\times$ and thus proves the assertion. □

2.2. Let $(X, \mathcal{O}_X)$ be a locally ringed space.

Lemma 2. *(i) Let $\mathcal{C}$ be an algebra over X, not necessarily with identity, and $N : \mathcal{C} \to \mathcal{O}_X$ a quadratic form. Then $\mathcal{C}$ is a composition algebra over X with multiplicative form N if and only if for each $P \in X$, $\mathcal{C}_P$ is a composition algebra over $\mathcal{O}_{P,X}$ with a multiplicative form N_P.*
(ii) Composition algebras are invariant under base change: If $\sigma : X' \to X$ is a morphism of locally ringed spaces and $\mathcal{C}$ a composition algebra over X then $\sigma^*\mathcal{C} = \mathcal{C} \otimes \mathcal{O}'_X$ is a composition algebra over X'.
(iii) If $(\mathcal{C}, N)$ is a symmetric composition algebra over X then $(\mathcal{C}_P, N_P)$ is a symmetric composition algebra over $\mathcal{O}_{P,X}$ for all $P \in X$.
(iv) Let $(\mathcal{C}, \star, N)$ be a composition algebra over X. Then its dimension is 1, 2, 4 or 8.

For Hurwitz algebras, (i), (ii) and (iv) were proved in [P1, 1.7].

Proof. The proof of (i) and (ii) works analogously to the one of [P1, 1.7 (a), (b)].
(iii) is trivial.
(iv) For all $P \in X$, the residue class algebra $(\mathcal{C}(P), \star, N(P))$ is a composition algebra over the residue class field $k(P)$ and therefore of one of the claimed dimensions by Proposition 1 (i). □

Lemma 3. *Let $(\mathcal{C}, \star, N)$, $(\mathcal{C}', \star', N')$ be two composition algebras over X, not necessarily with identity. Then any automorphism of algebras $\alpha : (\mathcal{C}, \star) \to (\mathcal{C}', \star')$ induces an isometry $N(P) \cong N'(P)$ for all $P \in X$.*

Proof. Let $\alpha : (\mathcal{C}, \star) \to (\mathcal{C}', \star')$ be an $\mathcal{O}_X$-algebra automorphism, then $\alpha(P) : (\mathcal{C}, \star) \otimes k(P) \to (\mathcal{C}', \star') \otimes k(P)$ is an $k(P)$-algebra automorphism and hence an isometry $N(P) \cong N'(P)$ by [KMRT, VIII.(33.27)]. □

Lemma 4. *Let X be a scheme over the affine scheme $Y = \operatorname{Spec} R$ and suppose $H^0(X, \mathcal{O}_X) = R$. Then a composition algebra $\mathcal{A}$ over X is defined over R provided it is globally free as an $\mathcal{O}_X$-module.*

The proof is analogous to the one of [P1, 1.10].

3. Symmetric composition algebras

3.1. Let (A, N) be an R-algebra together with a quadratic form $N\colon A \to R$. Let $\star$ be the multiplication of A. N is called *associative* if

$$N(x \star y, z) = N(x, y \star z)$$

for all $x, y, z \in A$. A composition algebra together with a nondegenerate associative multiplicative form N is called a *symmetric composition algebra.* A symmetric composition algebra of rank ≥ 2 does not have a unit element. This is a direct consequence from the analogous situation over fields described in [KMRT, p. 464].

Lemma 5. *Let $N : A \to R$ be a form on an R-algebra A with full support.*
(i) If N is nondegenerate, multiplicative and associative then

(1) $x \star (y \star x) = N(x)y = (x \star y) \star x$

for all $x, y \in A$. In particular, $(A, \star)$ is a flexible algebra.
(ii) If $2 \in R^\times$ and N is associative and satisfies (1) for all $x, y \in A$ then N is multiplicative.
(iii) Let R be a domain and N satisfy (1) for all $x, y \in A$. Then N is multiplicative and associative.

Proof. Adapting the proof of [KMRT, VIII.(34.1)] we get:
(i) We linearize $N(x \star y) = N(x)N(y)$ which yields the assertion, since N is nondegenerate.
(ii) is trivial.
(iii) Linearizing (1) implies

(2) $x \star (y \star z) + z \star (y \star x) = N(x, z)y = (x \star y) \star z + (z \star y) \star x$

and hence

$$N(x \star y, z)y = N(x, y \star z)y$$

for all $x, y, z \in A$. Thus $(N(x \star y, z) - N(x, y \star z))y = 0$. Since A is a projective R-module, thus torsion free as an R-module, this implies $N(x \star y, z) = N(x, y \star z)$. Moreover, (1) implies $(N(x \star y) - N(x)N(y))y = 0$, which analogously yields $N(x \star y) = N(x)N(y)$. □

In particular, if $(A, \star, N)$ is a symmetric composition algebra over R, then

$$x \star (x \star x) = N(x)x = (x \star x) \star x$$

and

$$x \star (x \star (x \star x)) = N(x)x \star x$$

for all $x \in A$ as two special cases of (1). Together with (2) this implies that

(3) $(x \star x) \star (x \star x) = N(x, x \star x)x - N(x)x \star x$

as in [KMRT, p. 464]. In general, a symmetric composition algebra is not power-associative [KMRT, p. 464].

If $(C, \diamond, N)$ is a Hurwitz algebra, then N permits composition with respect to the new multiplication

$$x \star y = \bar{x} \diamond \bar{y}$$

and N is associative with respect to $\star$, making $(C, \star, N)$ into a symmetric composition algebra. Such an algebra is called the *para-Hurwitz algebra* associated to the Hurwitz algebra $(C, \diamond, N)$ [KMRT, VIII.34.A]. By definition, para-Hurwitz algebras live on faithful R-modules.

Remark 2. If R is a field, the para-Hurwitz algebras of dimension 4 are the only symmetric composition algebras of dimension 4, and those of dimension 2 and their forms constitute all the symmetric composition algebras of dimension 2. Examples 2 and 3 below, translated into the setting of algebras over rings, show that this is not true any more for symmetric composition algebras of rank 2 over rings.

The two-dimensional composition algebras over fields were classified by Petersson [P3]: For any such algebra $(A, \star)$, there exists a Hurwitz algebra defined on the same vector space, with multiplication $\diamond$ and canonical involution $^{-}$, such that $\star$ is given by one of the following equations (for some $u \in A$ with $N(u) = 1$):

$$\text{(i) } x \star y = x \diamond y \text{ (ii) } x \star y = \bar{x} \diamond y \text{ (iii) } x \star y = x \diamond \bar{y} \text{ (iv) } x \star y = u\bar{x} \diamond \bar{y}.$$

Each multiplication of the above type also yields a two-dimensional composition algebra over a ring. However, not all two-dimensional composition algebra over a ring can be obtained this way, see Examples 2 and 3. In general, given any Hurwitz algebra $(C, \diamond, N)$ of rank greater or equal to 2 over R, with canonical involution $^{-}$, we can define a new algebra on the R-module C with respective multiplications (i), (ii), (iii) or (iv) as above. These are called the *standard composition algebras* associated to C. Since only in case (i) there is an identity element, only in case (ii) there is a left but not a right identity element, and only in case (iii) there is a right but not a left identity element, the four standard composition algebras cannot be isomorphic [E-P2, p. 378].

Proposition 2. *Let $(C_1, \diamond, N_1)$ and $(C_2, \diamond, N_2)$ be Hurwitz algebras over R.*
(i) Any isomorphism of Hurwitz algebras $(C_1, \diamond) \to (C_2, \diamond)$ is an isomorphism of the corresponding para-Hurwitz algebras.
(ii) If α is an isomorphism $(C_1, \star) \to (C_2, \star)$ of para-Hurwitz algebras with $\alpha(1_{C_1}) = 1_{C_2}$ then α is an isomorphism of Hurwitz algebras $(C_1, \diamond) \to (C_2, \diamond)$.

Proof. (cf. [KMRT, VIII.(34.4)] for base fields instead of rings)
(i) If α is an isomorphism of Hurwitz algebras then $\alpha(\overline{x}) = \overline{\alpha(x)}$ by the uniqueness of the quadratic generic polynomial of a Hurwitz algebra over a ring [P1, 1.2]. Thus α is an isomorphism of para-Hurwitz algebras.
(ii) Let α be an isomorphism of para-Hurwitz algebras with $\alpha(1_{C_1}) = 1_{C_2}$. Since equation (1) holds in both C_1 and C_2, we have

$$N_2(\alpha(x))\alpha(y) = \alpha(x) \star (\alpha(y) \star \alpha(x)) = \alpha(x \star (y \star x)) = \alpha(N_1(x)y) = N_1(x)\alpha(y)$$

for all $x, y \in C_1$. Thus $[N_2(\alpha(x)) - N_1(x)]\alpha(y) = 0$ for all y, in particular, $[N_2(\alpha(x)) - N_1(x)]\alpha(1) = [N_2(\alpha(x)) - N_1(x)]1_{C_2} = 0$. Since C_2 is the underlying module of a Hurwitz algebra and so faithful as an R-module (see 1.1), $N_2(\alpha(x)) = N_1(x)$ for all x. Thus α is an isometry between (C_1, N_1) and (C_2, N_2).

Moreover, it follows from $T_{C_1}(x) = N_1(x, 1_{C_1})$ and $\alpha(1_{C_1}) = 1_{C_2}$ that $T_{C_2}(\alpha(x)) = N_2(\alpha(x), 1_{C_2}) = N_1(x, 1_{C_1}) = T_{C_1}(x)$ for all $x \in C_1$. Therefore, $\alpha(\overline{x}) = \overline{\alpha(x)}$ for all $x \in C_1$ and α is even an isomorphism of Hurwitz algebras over R. □

Remark 3. (i) Let $(S, \star, N)$ be a symmetric composition algebra over R. A non-zero idempotent $e \in S$ is called a *para-unit* of S if $N(e) = 1$ and

$$e \star x = x \star e = -x \text{ for all } x \in S \text{ with } N(e, x) = 0.$$

Every para-Hurwitz algebra C has the unit element 1_C of the associated Hurwitz algebra as a para-unit. Analogoulsy as shown in [KMRT, VIII.(34.8)], a symmetric composition algebra defined on a faithful R-module (resp., over a domain R) is para-Hurwitz if and only if it contains a para-unit.

(ii) Let $(C, \diamond, N)$ be a Hurwitz algebra over R. Given two isometries f and g of its norm, the new multiplication defined via

$$x \star y = f(x)g(y)$$

yields a composition algebra $(C, \star, N)$.

3.2. **Petersson algebras over R.** Let $(C, \diamond, N)$ be a Hurwitz algebra over R. Given an R-automorphism φ of C such that $\varphi^3 = 1$, we define another new multiplication on C via

$$x \star y = \varphi(\bar{x}) \diamond \varphi^2(\bar{y})$$

and call the resulting composition algebra (denoted $(C_\varphi, \star, N)$ or simply C_φ) a *Petersson algebra* (cf. [P2] or [KMRT, 34.B, p. 466]). The unit element of $(C, \diamond, N)$ is a nonzero idempotent of C_φ. φ is an automorphism of C_φ. If $\varphi = 1$ then C_φ is para-Hurwitz.

Obviously, each $C_\varphi(P)$ is a symmetric composition algebra over the residue class field $k(P)$ by [KMRT, 34.B, p. 467]. It is straightforward to check that, if R is a domain, equation (1) from Lemma 5 holds, so that C_φ is a symmetric composition algebra over R.

Lemma 6. *Let $(S, \star, N)$ be a symmetric composition algebra over R.*
(i) If there is an isotropic element $0 \neq x \in S$ of the cubic form $N(x \star x, x)$ such that $N(x) \in R^\times$, then $(S, \star)$ contains an idempotent.
(ii) Let S be a faithful R-module. For any non-trivial idempotent $e \in S$, we have $N(e) = 1$.

Proof. (i) We follow [KMRT, VIII.(34.10) (1)]: Let $0 \neq x \in S$ be such that $N(x \star x, x) = 0$ and $N(x) \in R^\times$. Since $(x \star x) \star (x \star x) = -N(x)(x \star x)$ by (3), the element $e = -N(x)^{-1} x \star x$ is an idempotent.
(ii) is proved as in [KMRT, VIII.(34.10) (2)]. □

Proposition 3. *Let $(S, \star, N)$ be a symmetric composition algebra over R and let $e \in S$ be a non-trivial idempotent. Let S be faithful as an R-module.*
(i) The product $x \diamond y = (e \star x) \star (y \star e)$ makes S into a Hurwitz algebra over R with identity e, norm N, and conjugation $\bar{x} = N(x, e)e - x$.
(ii) The map

$$\varphi(x) = e \star (e \star x) = N(e, x)e - x \star e = \bar{x} \star e$$

is an automorphism of $(S, \diamond)$ (and also of $(S, \star)$) such that $\varphi^3 = id$ and $(S, \star) = (S, \diamond)_\varphi$ is a Petersson algebra over R.

The proof is analogous to the one given in [E-P1, 2.5] or [KMRT, VIII.(34.9)]:

Proof. (i) By Lemma 6 (ii), $N(e) = 1$. Hence as in the proof of Proposition 1 (ii), the linear maps L_e and R_e are bijective and $L_e = R_e^{-1}$. Now Equation (2) implies that

$$(x \star e) \star e + e \star x = N(e, x)e.$$

Multiply this by e on the right to obtain $R_e^3(x) = N(e, x)e - x$. The new multiplication

$$x \diamond y = R_e^{-1}(x) \star L_e^{-1}(x) = (e \star x) \star (y \star e)$$

on S makes S into a Hurwitz algebra with unit element e, norm N and canonical involution R_e^3.
(ii) Equation (2) yields

$$x \diamond y = N(e,x)y - e \star (y \star (e \star y)))$$

which analogously as in [KMRT, VIII.(34.9)] shows that φ is an automorphism of $(S, \diamond)$. (The fact that φ is bijective can also be deduced from the fact that the induced morphisms $\varphi(P)$ are bijective by [KMRT, VIII.(34.9)].) Moreover $\varphi^3(x) = x$ and $x \star y = \varphi(\bar{x}) \diamond \varphi^2(\bar{y})$. □

Thus, if R is a domain, symmetric composition algebras with nontrivial idempotents defined on finitely generated projective R-modules are precisely the Petersson algebras S_φ.

Theorem 1. *Let C be a quaternion algebra over R with C a faithful R-module, and let τ be an inner automorphism of C with $\tau^3 = 1$. Then C_τ is a para-Hurwitz algebra.*

Proof. The proof follows [E-P1, 3.2]: There is an invertible element $a \in C$ such that $\tau(x) = a^{-1}xa$ for all $x \in C$ and $N(a) \in R^\times$. Since $\tau^3 = 1$ we get $a^3 = \alpha 1$ for some $\alpha \in R$. Put $w = a^2/N(a)$ then $w^3 = 1$, $N(w) = 1$ and $\overline{w} = w^2$. Thus $\tau(x) = wxw^2$ for all $x \in C$. As in [E-P1, 3.2], in C_τ we have

$$x \star w = w\bar{x}w = N(w,x)w - x,$$

so, in particular, $w^2 = w$ and w is a para-unit in C_τ. Thus C_τ is para-Hurwitz by Remark 3. □

If $\operatorname{Pic} R = 0$ then each automorphism of a quaternion algebra over R is inner [K, III.(5.2)]. It is not clear what happens if τ is not an inner automorphism.

3.3. **Symmetric composition algebras over X.** Let $\mathcal{S}$ be an $\mathcal{O}_X$-algebra with full support and $N\colon \mathcal{S} \to \mathcal{O}_X$ a quadratic form. Let $\star$ be the multiplication of $\mathcal{S}$. N is called *associative* if

$$N(x \star y, z) = N(x, y \star z)$$

for all sections x, y, z in $\mathcal{S}$ over the same open subset of X. A composition algebra over X with a nondegenerate associative norm N is called a *symmetric composition algebra*.

There is a canonical equivalence between the category of composition algebras (resp., Hurwitz algebras, symmetric composition algebras) over the affine scheme $Z = \operatorname{Spec} R$ and the category of composition algebras (resp., Hurwitz algebras, symmetric composition algebras) over R given by the global section functor $C \longrightarrow H^0(Z, C)$ and the functor $C \longrightarrow \widetilde{C}$.

4. Cubic separable alternative algebras over X

From now on, let $(X, \mathcal{O}_X)$ be a locally ringed space such that $2, 3 \in H^0(X, \mathcal{O}_X^\times)$. Let $\mathcal{A}$ be a unital $\mathcal{O}_X$-algebra of constant rank together with a nondegenerate cubic form $N : \mathcal{A} \to \mathcal{O}_X$ permitting composition; i.e., $N(xy) = N(x)N(y)$ for all sections x, y of $\mathcal{A}$ over the same open subset of X. Then $\mathcal{A}$ is an alternative algebra over X. Let θ be the trilinear form associated with N. Then

$$\theta(xy, xy, xy) = \theta(x, x, x)\theta(y, y, y)$$

for all sections x, y of $\mathcal{A}$ over the same open subset of X. Let $1 = 1_{\mathcal{A}} \in H^0(X, \mathcal{O}_X)$ be the unit element of $\mathcal{A}$ and assume that $N(1) = 1$. Define the *trace* $T : \mathcal{A} \to \mathcal{O}_X$ of $\mathcal{A}$ as the linear form

$$T(x) = 3\theta(x, 1, 1)$$

and a quadratic form $S : \mathcal{A} \to \mathcal{O}_X$ via

$$S(x) = 3\theta(x, x, 1)$$

for all sections x, y of $\mathcal{A}$ over the same open subset of X. We have

$$\mathcal{A} \cong \mathcal{O}_X \oplus \mathcal{A}_0$$

with $\mathcal{A}_0 = \ker T$. The symmetric bilinear form $T(x, y) = T(xy)$ on $\mathcal{A}$ is associative and nondegenerate.

We will consider the following unital alternative $\mathcal{O}_X$-algebras of constant rank which admit a nondegenerate cubic form N permitting composition such that $N(1) = 1$ (it is not clear if these are all of them, see [Pu2]):

(1) A (commutative associative) cubic étale algebra $\mathcal{A}$, such that $\mathcal{A}^+$ is a first or second Tits construction of constant rank 3, and its norm N.
(2) $\mathcal{A} = \mathcal{O}_X \oplus \mathcal{C}$, $N(a + x) = aN_{\mathcal{C}}(x)$, for a Hurwitz algebra $\mathcal{C}$ of constant rank over X with norm $N_{\mathcal{C}}$.
(3) An Azumaya algebra over X of constant rank 9 and its norm.

Every section x of $\mathcal{A}$ satisfies

$$x^3 - T(x)x^2 + S(x)x - N(x)1 = 0$$

[Pu2]. Define

$$x^\sharp = x^2 - T(x)x + S(x)1; \tag{1}$$

$$S(x, y) = S(x + y) - S(x) - S(y); \tag{2}$$

$$x \times y = (x + y)^\sharp - x^\sharp - y^\sharp. \tag{3}$$

Then $(N, \sharp, 1)$ is a cubic form with adjoint and base point on $\mathcal{A}$ and $\mathcal{A}^+ = \mathcal{J}(N, \sharp, 1)$.

Remark 4. Let $\mathcal{A}$ be such that, for every $P \in X$, there is an element $u \in \mathcal{A}(P)$ such that $1, u, u^2$ are linearly independent over $k(P)$. Then the cubic, quadratic and linear maps N, S and T satisfying $x^3 - T(x)x^2 + S(x)x - N(x)1 = 0$ are unique [Ach, 1.12]. Uniqueness of N, S and T therefore holds, if $\mathcal{A}$ is an Azumaya algebra over X, if $\mathcal{A}^+$ is a first Tits construction of constant rank 3, or if X is a k-scheme and k has infinitely many elements [Ach, 1.13].

As in [F, p. 1027] (see also [McC1], [R, p. 95] or [Ach, 1.8]), we obtain the following identities for the Jordan algebra $\mathcal{A}^+ = \mathcal{J}(N, \sharp, 1)$:

Lemma 7. (1) $N(xy) = N(x)N(y)$,
(2) $S(1) = T(1) = 3$,
(3) $S(x) = T(x^\sharp)$,
(4) $S(x, y) = T(x \times y)$,
(5) $S(x, 1) = 2T(x)$
(6) $T(x)T(y) = T(xy) + T(x \times y)$,
(7) $2S(x) = T(x)^2 - T(x^2)$,
(8) $T(xy) = T(yx)$,
(9) $x \times y = xy + yx - T(x)y - T(y)x + (T(x)T(y) - T(xy))1$,
(10) $x^{\sharp\sharp} = N(x)x$,
(11) $(x \times y)^\sharp + x^\sharp \times y^\sharp = T(x^\sharp y)y + T(xy^\sharp)x$,
(12) $(xy)^\sharp = y^\sharp x^\sharp$,
(13) $S(xy) = S(yx)$.

5. Finding composition algebras in alternative algebras

From now on, let $(X, \mathcal{O}_X)$ be a locally ringed space such that $2, 3 \in H^0(X, \mathcal{O}_X^\times)$ and such that $H^0(X, \mathcal{O}_X)$ contains a primitive third root of unity ω and let R be a ring such that $2, 3 \in R^\times$ containing a primitive third root of unity ω. We keep the assumptions from Section 4 on the algebra $\mathcal{A}$. Lemma 5 yields:

Lemma 8. *Let $N : \mathcal{A} \to \mathcal{O}_X$ be a quadratic form on an $\mathcal{O}_X$-algebra $\mathcal{A}$ with full support.*
(i) If N is multiplicative and associative then $x \star (y \star x) = N(U)(x)y = (x \star y) \star x$ for all $x, y \in \mathcal{A}(U)$, for all open subsets $U \subset X$ for which $N(U)$ is nondegenerate.
(ii) Let N be nondegenerate. If N is multiplicative and associative then $x \star (y \star x) = N(P)(x)y = (x \star y) \star x$ for all $x, y \in \mathcal{A}(P)$ and for all $P \in X$.
(iii) If N is associative and satisfies $x \star (y \star x) = N(x)y = (x \star y) \star x$ for all $x, y \in \mathcal{A}$, then N is multiplicative.
(iv) Let $(X, \mathcal{O}_X)$ be an integral scheme. If N satisfies $x \star (y \star x) = N(x)y = (x \star y) \star x$ for all $x, y \in \mathcal{A}$, then N is multiplicative and associative.

Theorem 2. *Let $\mathcal{A}$ be a unital alternative $\mathcal{O}_X$-algebra of constant rank together with a cubic form $N : \mathcal{A} \to \mathcal{O}_X$ permitting composition. Let $\mathcal{A}(P)$ have degree 3 for all $P \in X$. For $x \in \mathcal{A}$, let $x_0 = x - \frac{1}{3}T(x)1$. Define a new product on $\mathcal{A}_0$ via*

$$\begin{aligned} u \star v =& (\omega uv - \omega^2 vu)_0 \\ =& \omega uv - \omega^2 vu - \frac{1}{3}[\omega - \omega^2]\, T(uv)1. \end{aligned}$$

(a) The quadratic form $S|_{\mathcal{A}_0}$ satisfies

$$S(u \star v) = S(u)S(v)$$

for all $u, v \in \mathcal{A}_0$ and $(\mathcal{A}_0, \star)$ is a flexible algebra.
(b) If N is nondegenerate, $(\mathcal{A}_0, \star, S)$ is a flexible symmetric composition algebra.
(c) If N is nondegenerate, $(\mathcal{A}_0, \star, S)$ has rank 2, 4, or 8.

Proof. (a) We have $\mathcal{A} = \mathcal{O}_X \oplus \mathcal{A}_0$. Using the equations proved in Lemma 7 we see, analogously as in the proof of [F, Theorem], that

$$S(u)S(v) = -S(uv) - S(uv, vu) + T(uv)^2 = S(u \star v)$$

for all u, v in $\mathcal{A}_0$. Moreover, we have

$$\begin{aligned} &u \star v \\ &= \omega uv - \omega^2 vu - \tfrac{1}{3}T(\omega uv - \omega^2 vu)1 \\ &= \omega uv - \omega^2 vu - \tfrac{1}{3}[\omega T(uv) - \omega^2 T(vu)1] \\ &= \omega uv - \omega^2 vu - \tfrac{1}{3}[\omega T(uv) - \omega^2 T(uv)]1 \\ &= \omega uv - \omega^2 vu - \tfrac{1}{3}[\omega - \omega^2]\, T(uv)1. \end{aligned}$$

Analogously as in [E-M, Proposition 4.1, (i)], it follows that $(\mathcal{A}_0, \star)$ is flexible.

(b) The residue class forms $S(P)$ are nondegenerate for all $P \in X$ [F, Theorem], hence S is nondegenerate. $T(x, y)$ is associative on $\mathcal{A}$ and, since N is nondegenerate, $T(x, y)$ is nondegenerate (5.1). Now $S(u, v) = -T(uv)$ for all u, v in $\mathcal{A}_0$ by Lemma 7, (4) and (6), thus S is associative on $\mathcal{A}_0$.

(c) follows from (b). □

Symmetric algebras $(\mathcal{A}_0, \star, S)$ of rank 8 constructed out of Azumaya algebras of rank 9 as in Theorem 2 are called *Okubo algebras* over X.

Proposition 4. *In the situation of Theorem 2,*
(i) each residue class algebra $\mathcal{A}_0(P)$ contains an element c (depending on P), such that c and $c \star c$ are linearly independent;
(ii) the cubic, quadratic and linear maps N, S and T on $\mathcal{A}$ satisfying $x^3 - T(x)x^2 + S(x)x - N(x)1 = 0$ are unique.

Proof. (i) This follows immediately from [E-M, Proposition 4.1, (iv)], since $(\mathcal{A}_0(P), \star)$ is obtained by applying [E-M, Main Theorem] to $\mathcal{A}(P)$ and hence as an alternative algebra of degree 3 over $k(P)$ satisfies the conditions of [E-M, Proposition 4.1].

(ii) For every $P \in X$, there is an element $u \in \mathcal{A}(P)$ such that $1, u, u^2$ are linearly independent over $k(P)$, since the degree of $\mathcal{A}(P)$ is 3. The assertion thus follows from Remark 4. □

Remark 5. (i) For all $x \in \mathcal{A}_0$, $S(x) = -\frac{1}{2}T(x^2)$.

(ii) Let X be an integral curve over a field k with function field $K = K(X) = \mathcal{O}_{\xi,X}$ (ξ the generic point of X). Let $\mathcal{A}$ and $\mathcal{D}$ be Azumaya algebras of constant rank 9 over X such that $\mathcal{A}_\xi \not\cong \mathcal{D}_\xi$. Then $(\mathcal{A}_{0,\xi}, \star) \not\cong (\mathcal{D}_{0,\xi}, \star)$ [KMRT, VIII.(34.25)] and thus also $(\mathcal{A}_0, \star) \not\cong (\mathcal{D}_0, \star)$.

(iii) If $\mathcal{A}$ is an Azumaya algebra over X of constant rank 9, if $\mathcal{A}^+$ is a first Tits construction of constant rank 3 as in Remark 4, or if X is a k-scheme and k has infinitely many elements, then for every $P \in X$, there is an element $u \in \mathcal{A}(P)$ such that $1, u, u^2$ are linearly independent over $k(P)$, see Remark 4. We may even assume $u \in \mathcal{A}_0(P)$ since the degree of $\mathcal{A}(P)$ is 3, see [E-M, p. 2489], and hence that $u, u \star u$ are linearly independent.

Corollary 2. *In the situation of Theorem 2, let $a \in H^0(X, \mathcal{A}_0)$ such that $T(a^2) \in H^0(X, \mathcal{O}_X)$ is invertible. Then there exists a multiplication $\diamond$ on $\mathcal{A}_0$ such that $(\mathcal{A}_0, \diamond, S)$ is a Hurwitz algebra over X.*

In particular, if $\mathcal{A}$ is an Azumaya algebra over X of constant rank 9, then $(\mathcal{A}_0, \diamond, S)$ is an octonion algebra.

Proof. Since $S(a) = -\frac{1}{2}T(a^2) \in H^0(X, \mathcal{O}_X)$ is invertible by assumption, we may put $u = S(a)^{-1}a^2 \in H^0(X, \mathcal{A}_0)$. Then $S(u) = 1$ and both the left-multiplication $L_u : \mathcal{A}_0 \to \mathcal{A}_0$ and the right-multiplication $R_u : \mathcal{A}_0 \to \mathcal{A}_0$ with u are bijective, since they are bijective over the residue class fields. The element $e = u \star u$ is the identity for the new multiplication defined via

$$x \diamond y = (R_u^{-1}x) \star (L_u^{-1}y)$$

which satisfies $S(x \diamond y) = S(x)S(y)$ by Proposition 1 (ii). S is nondegenerate by Theorem 2. □

Remark 6. Let $(\mathcal{C}, \cdot, N_\mathcal{C})$ be a Hurwitz algebra over X of constant rank. Then $\mathcal{A} = \mathcal{O}_X \oplus \mathcal{C}$ is a unital alternative algebra together with a nondegenerate cubic form $N : \mathcal{A} \to \mathcal{O}_X$, $N(x) = rN_\mathcal{C}(u)$ for $x = (r, u) \in \mathcal{A}$ (r in $\mathcal{O}_X$, u in $\mathcal{C}$) which permits composition. We have $1_\mathcal{A} = (1, 1_\mathcal{C})$, $(r, u)^\sharp = (N_\mathcal{C}(u), r\overline{u})$, $S(x) = rT_\mathcal{C}(u) + N_\mathcal{C}(u)$ and $T(x) = r + T_\mathcal{C}(u)$ [P-R, p. 219]. Identify $\mathcal{A}_0 = \{x \in \mathcal{A} \,|\, T(x) = 0\} = \ker(T)$ with $\mathcal{C}$ via $w = (-T_\mathcal{C}(w), w)$, then $S|_\mathcal{C}(w) = N_\mathcal{C}(w) - T_\mathcal{C}(w)^2$. We observe that S permits composition on $\mathcal{A}_0$ with respect to the new multiplication

$$u \star v = (\omega u \cdot v - \omega^2 v \cdot u)_0$$

as in Theorem 2. If there is an element $a \in H^0(X, \mathcal{A}_0)$ such that $S(a) \in H^0(X, \mathcal{O}_X)$ then we can find a new multiplication $\diamond$ on $(\mathcal{A}_0, \star, S)$ such that $(\mathcal{A}_0, \diamond, S)$ is isomorphic to the Hurwitz algebra $(\mathcal{C}, \cdot, N_\mathcal{C})$ [F, p. 1028].

Remark 7. Let $\mathcal{A}$ be an Azumaya algebra over X of constant rank 9 with cubic norm N. The residue class form $S(P) : \mathcal{A}_0(P) \to k(P)$ of the form S restricted to $(\mathcal{A}_0, \star)$ is hyperbolic; i.e., $S|_{\mathcal{A}_0}(P) \cong \langle\langle -1, -1, -1 \rangle\rangle$, for all $P \in X$ [KMRT, VIII.(34.25)]. Hence the Okubo algebra $(\mathcal{A}_0, \star)$ constructed in Theorem 2 has a multiplicative quadratic form whose residue class forms are all hyperbolic. Therefore every octonion algebra $\mathcal{C}$ over X, for which there is a point $P \in X$ such that the residue class algebra $\mathcal{C}(P)$ does not split, cannot be obtained as the trace zero elements of some Azumaya algebra as described in Corollary 2.

It remains to be investigated if this construction yields new octonion algebras over X which cannot be constructed by means of a general Cayley-Dickson doubling or by using a hermitian form as described in 1.4, 1.5.

Theorem 3. *Let $\mathcal{A}$ be a unital alternative $\mathcal{O}_X$-algebra of constant rank together with a nondegenerate cubic form $N : \mathcal{A} \to \mathcal{O}_X$ permitting composition. Let $\mathcal{B}$ be a unital alternative subalgebra of $\mathcal{A}$ of constant rank with nondegenerate cubic norm $N' = N|_\mathcal{B} : \mathcal{B} \to \mathcal{O}_X$. Let $\mathcal{A}(P)$ and $\mathcal{B}(P)$ have degree 3 for all $P \in X$. Then $(\mathcal{B}_0, \star)$ is a symmetric composition algebra, which is a subalgebra of the symmetric composition algebra $(\mathcal{A}_0, \star, S)$ from Theorem 2, with the product $\star$ on $\mathcal{A}_0$ given by*

$$u \star v = \omega uv - \omega^2 vu - \frac{1}{3}[\omega - \omega^2]\, T(uv)1.$$

The proof is straightforward.

Translated into the setting of rings, Theorem 2 and Corollary 2 become:

Corollary 3. *Let A be a unital R-algebra of constant rank together with a nondegenerate cubic form $N : A \to R$ permitting composition. Let the residue class algebra $A(P) = A \otimes_R k(P)$ have degree 3 for all $P \in \operatorname{Spec} R$. Define a new product on A_0 via*

$$u \star v = \omega uv - \omega^2 vu - \frac{1}{3}(\omega - \omega^2)T(uv)1.$$

(a) $(A_0, \star, S)$ is a flexible symmetric composition algebra over R.
(b) Suppose there exists an element $a \in A_0$ such that $T(a^2) \in R^\times$. Then there exists a multiplication $\diamond$ on A_0, such that $(A_0, \diamond, S)$ is a Hurwitz algebra over R. In particular, if A is an Azumaya algebra over R of constant rank 9, then $(A_0, \diamond, S)$ is an octonion algebra.

If $(A_0, \star, S)$ has rank 8, it is called an *Okubo algebra*.

Example 1. Let $(X, \mathcal{O}_X)$ be a scheme over R such that $R = H^0(X, \mathcal{O}_X)$.
(i) If $\mathcal{A} = \mathrm{Mat}_3(\mathcal{O}_X)$ then $N(x) = \det(x)$ and $S(x) = tr(\mathrm{adj}(x))$. Let $\rho = (\omega - \omega^2)^{-1} \in H^0(X, \mathcal{O}_X)$ and

$$u = \mathrm{diag}(2\rho, -\rho, -\rho) \in \mathrm{Mat}_3(H^0(X, \mathcal{O}_X)).$$

Then $S(u) = tr(\mathrm{adj}(u)) = tr(\mathrm{diag}(\rho^2, -2\rho^2, -2\rho^2)) = 1$ [F]. Therefore $e = u \star u$ is an identity for the new multiplication

$$x \diamond y = (R_u^{\star -1} x) \star (L_u^{\star -1} y)$$

on $\mathcal{A}_0$ and $N(x \diamond y) = N(x)N(y)$ as in Proposition 1 (ii). Hence $(\mathcal{A}_0, \diamond, S)$ is an octonion algebra over X.
(ii) More generally, let $\mathcal{A} = \mathcal{E}nd_X(\mathcal{E})$ where $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module of constant rank 3. Suppose that

$$\begin{bmatrix} H^0(X, \mathcal{O}_X) & * & * \\ * & H^0(X, \mathcal{O}_X) & * \\ * & * & H^0(X, \mathcal{O}_X) \end{bmatrix} \subset H^0(X, \mathcal{E}nd_X(\mathcal{E}))$$

where the entries denoted by $*$ can be anything. Let

$$u = \mathrm{diag}(2\rho, -\rho, -\rho) \in H^0(X, \mathcal{E}nd_X(\mathcal{E})).$$

Then again $S(u) = tr(\mathrm{adj}(u)) = 1$ and $\mathcal{A}_0$ together with the new multiplication

$$v \diamond w = (R_u^{\star -1} x) \star (L_u^{\star -1} y)$$

is an octonion algebra with unit $e = u \star u$. Due to the construction, $(\mathcal{A}_0(P), \diamond, S(P))$ splits for all $P \in X$.

Lemma 9. *Let $\mathcal{A}$, $\mathcal{A}'$ be two isomorphic $\mathcal{O}_X$-algebras of constant rank together with nondegenerate cubic forms $N : \mathcal{A} \to \mathcal{O}_X$, $N' : \mathcal{A}' \to \mathcal{O}_X$ permitting composition. Let $\mathcal{A}(P)$ and $\mathcal{A}'(P)$ have degree 3 for all $P \in X$. Then the flexible symmetric composition algebras $(\mathcal{A}_0, \star, S)$ and $(\mathcal{A}'_0, \star, S')$ constructed in Theorem 2 are isomorphic.*

Proof. Let $f : \mathcal{A} \to \mathcal{A}'$ be an algebra isomorphism, then due to the uniqueness of the maps T, S and N we have $S(x,y) = S'(f(x), f(y))$ for all $x, y \in \mathcal{A}_0$ and $T(xy) = T'(f(x)f(y))$, since $S(x,y) = -T(xy)$. Using the multiplication $\star$ on $\mathcal{A}_0$, $\mathcal{A}_0'$ this implies the assertion. □

6. On the converse of Theorem 2

Theorem 4. *Let $(\mathcal{B}, \star)$ be an algebra over X of constant rank with an associative quadratic form $S_0 : \mathcal{B} \to \mathcal{O}_X$ such that*

$$(1) \quad (x \star y) \star x = x \star (y \star x) = S_0(x) y$$

for all sections $x, y \in \mathcal{B}$. Assume that, for all $P \in X$, the residue class algebra $\mathcal{B}(P)$ contains an element c such that c and $c \star c$ are linearly independent.
(i) There exists a unital alternative algebra $\mathcal{A}$ of constant rank over X together with linear, quadratic, resp. cubic maps T, S and N from $\mathcal{A}$ to $\mathcal{O}_X$ satisfying

$$x^3 - T(x)x^2 + S(x)x - N(x)1 = 0,$$

$S|_{\mathcal{A}_0} = S_0$ and $(\mathcal{B}, \star)$ is isomorphic to the flexible algebra $(\mathcal{A}_0, \star)$ defined in Theorem 2.
(ii) If S_0 is nondegenerate then $(\mathcal{B}, \star) \cong (\mathcal{A}_0, \star)$ is a flexible symmetric composition algebra and N is a nondegenerate cubic form.

This was proved in [E-M, Proposition 4.2] for algebras over fields.

Proof. (i) By Lemma 8 (iii), S_0 is multiplicative. Define a multiplication on $\mathcal{A} = \mathcal{O}_X \oplus \mathcal{B}$ via $1x = x1 = x$ for any $x \in \mathcal{A}$ and

$$ab = -\frac{S_0(a,b)}{3} 1 + \frac{1}{3}((\omega^2 - 1) a \star b - (\omega - 1) b \star a)$$

for all sections $a, b \in \mathcal{B}$, with $S_0(a,b) = S_0(a+b) - S_0(a) - S_0(b)$. Let $x = s1 + a \in \mathcal{A}$ with $s \in \mathcal{O}_X$ and $a \in \mathcal{B}$. Define $\rho = -\frac{1}{3}(2\omega + 1)$. The multiplication in $\mathcal{A}$ together with (1) and the associativity of $S_0(U)$ for all open sets $U \subset X$ yields

$$a^2 b = a(ab) \text{ and } ba^2 = (ba)a$$

for all $a, b \in \mathcal{B}(U)$ analogously as demonstrated in [E-M, p. 2491], thus $\mathcal{A}$ is alternative. We also obtain

$$x^3 - 3sx^2 + (3s^2 + S_0(a))x - (s^3 + S_0(a)s - \frac{1}{3}\rho S_0(a \star a, a))1 = 0$$

as in the proof of [E-M, Proposition 4.2]. Let

$$T(x) = 3s, \; S(x) = 3s^2 + S_0(a) \text{ and } N(x) = s^3 + S_0(a)s - \frac{1}{3}\rho S_0(a \star a, a)$$

then these are linear, quadratic, resp. cubic maps from $\mathcal{A}$ to $\mathcal{O}_X$ satisfying

$$x^3 - T(x)x^2 + S(x)x - N(x)1 = 0.$$

Furthermore, $\mathcal{B} = \{x \in \mathcal{A} \,|\, T(x) = 0\}$ since $T(x) = 0$ iff $x = 0 + a \in \mathcal{B}$. For all $P \in X$, the residue class algebra $\mathcal{B}(P)$ contains an element c such that $\{c, c \star c\}$ is linearly independent, hence $\{1, c, c^2\}$ is linearly independent over $k(P)$. Thus $\mathcal{A}(P)$ is of degree 3 and N, S

and T are uniquely determined by Remark 4. As the norm of an alternative algebra, N is multiplicative. Since $S(a,b) = -T(ab)$,

$$a \star b = \omega ab - \omega^2 ba + \rho T(ab)1$$

for all $a, b \in \mathcal{A}_0$, so that we have proved that $(\mathcal{B}, \star)$ is isomorphic to $(\mathcal{A}_0, \star)$.
(ii) If S_0 is nondegenerate, then $(\mathcal{B}, \star, S_0)$ is a flexible symmetric composition algebra over X. For all $P \in X$, $S(P)$ is nondegenerate and hence so is the cubic form $N(P)$, by [E-M, Main Theorem I]. □

Proposition 5. *(a) Let $(\mathcal{B}^1, \star)$ and $(\mathcal{B}^2, \star)$ be two isomorphic flexible symmetric composition algebras over X of constant rank, satisfying the assumptions of Theorem 4. Then the unital alternative algebras $\mathcal{A}^1$ and $\mathcal{A}^2$ of constant rank over X constructed in Theorem 4 are isomorphic.*
(b) Let $\mathcal{A}^1$ and $\mathcal{A}^2$ be two unital alternative algebras over X of constant rank, satisfying the assumptions of Theorem 2. If $\mathcal{A}^1 \cong \mathcal{A}^2$ then the flexible symmetric composition algebras $(\mathcal{A}^1_0, \star)$ and $(\mathcal{A}^2_0, \star)$ are isomorphic.

Proof. (a) The isomorphism $f : \mathcal{B}^1 \to \mathcal{B}^2$ canonically induces a homomorphism $F : \mathcal{A}^1 \to \mathcal{A}^2$ which is an isomorphism, since each residue class morphism $F(P)$ is bijective by [E-M, p. 2492].
(b) In the situation of Theorem 2, the cubic, quadratic and linear maps N_i, S_i and T_i on $\mathcal{A}^i$ satisfying $x^3 - T_i(x)x^2 + S_i(x)x - N_i(x)1 = 0$ are unique by Proposition 4. Given an isomorphism $F : \mathcal{A}^1 \to \mathcal{A}^2$, we thus have $S_1(x,y) = S_2(F(x), F(y))$ and $T_1(xy) = T_2(F(x)F(y))$ for $x, y \in \mathcal{A}^1_0$. The definition of the multiplication $\star$ on $\mathcal{A}^i_0$ shows that this implies $(\mathcal{A}^1_0, \star) \cong (\mathcal{A}^2_0, \star)$. □

Corollary 4. *Let $(B, \star)$ be an algebra over R with a nondegenerate associative quadratic form $S_0 : B \to R$ such that*

$$(1) \quad (x \star y) \star x = x \star (y \star x) = S_0(x)y$$

for all $x, y, z \in B$. Assume that, for all $P \in \operatorname{Spec} R$, the residue class algebra $B(P)$ contains an element c such that c and $c \star c$ are linearly independent. Then there exists a separable unital alternative algebra A over R together with linear, quadratic, resp. cubic maps T, S and N from A to R satisfying

$$x^3 - T(x)x^2 + S(x)x - N(x)1 = 0,$$

$S|_{A_0} = S_0$ and $(B, \star)$ is isomorphic to the flexible symmetric composition algebra $(A_0, \star)$ defined in Corollary 3.

7. Petersson algebras over X

Given a Hurwitz algebra $(\mathcal{C}, \diamond, N)$ over X and an automorphism $\varphi : \mathcal{C} \to \mathcal{C}$ such that $\varphi^3 = 1$, we define a new multiplication on $\mathcal{C}$ via

$$x \star y = \varphi(\bar{x}) \diamond \varphi^2(\bar{y})$$

and call the resulting composition algebra (denoted $\mathcal{C}_\varphi$) a *Petersson algebra* over X. For all $P \in X$, $\mathcal{C}_\varphi(P)$ is a symmetric composition algebra. Let X be an integral scheme over a domain R and suppose $H^0(X, \mathcal{O}_X) = R$. Then $\mathcal{C}_\varphi$ is a symmetric composition algebra.

Proposition 6. *Let X be an integral scheme over a domain R and suppose $H^0(X, \mathcal{O}_X) = R$. Let $(\mathcal{S}, \star, N)$ be a symmetric composition algebra over X and let $e \in H^0(X, \mathcal{O}_X)$ be a non-trivial idempotent.*
(i) The product $x \diamond y = (e \star x) \star (y \star e)$ makes $\mathcal{S}$ into a Hurwitz algebra over X with identity e, norm N, and conjugation $\bar{x} = N(x, e)e - x$.
(ii) The map

$$\varphi(x) = e \star (e \star x) = N(e, x)e - x \star e = \bar{x} \star e$$

is an automorphism of $(\mathcal{S}, \diamond)$ (as well as of $(\mathcal{S}, \star)$) such that $\varphi^3 = id$ and $(\mathcal{S}, \star) \cong (\mathcal{S}, \diamond)_\varphi$ is a Petersson algebra over X.

Proof. (i) Since $H^0(X, \mathcal{O}_X) = R$ is a domain and $H^0(X, \mathcal{A})$ is a finitely generated projective R-module, $H^0(X, \mathcal{A})$ is torsion free, hence faithful. In particular, this means $N(e) = 1$. The linear maps L_e and R_e are bijective, since their residue class maps are by [KMRT, VIII.(34.9)], and $L_e = R_e^{-1}$. The rest of the proof is analogous to the proof of Proposition 3 (i).
(ii) Equation (2) yields

$$x \diamond y = N(e, x)y - e \star (y \star (e \star y)))$$

and φ is an automorphism of $(\mathcal{S}, \diamond)$, since its residue class morphisms are bijective by [KMRT, VIII.(34.9)]. Moreover $\varphi^3(x) = x$ and $x \star y = \varphi(\bar{x}) \diamond \varphi^2(\bar{y})$ as in Proposition 3 (ii). □

Note that the fact that the quadratic form $N(U)$ may be degenerate for some open sets U does not create any problems.

Proposition 7. *Let k be a field such that $2, 3 \in k^\times$ containing a primitive third root of unity ω and X be a proper integral scheme over k with function field $K = k(X) = \mathcal{O}_{\xi,X}$. Let $\mathcal{A}$ be an Azumaya algebra over X of constant rank 9. Let $e \in H^0(X, \mathcal{A}_0)$ be a non-trivial idempotent in the Okubo algebra $(\mathcal{A}_0, \star)$ obtained as in Theorem 2. Then the following holds:*
(i) $\mathcal{A}_\xi \cong \mathrm{Mat}_3(K)$.
(ii) The Okubo algebra becomes an octonion algebra under the new multiplication

$$x \diamond y = (e \star x) \star (y \star e).$$

Moreover, $(\mathcal{A}_0, \star) = (\mathcal{A}_0, \diamond)_\varphi$ is a Petersson algebra over X.

Proof. (i) follows from [E-M, p. 2502].
(ii) follows by Proposition 5. □

Proposition 8. *Let $\mathcal{A} = \mathcal{E}nd_X(\mathcal{F} \oplus \mathcal{L})$ be an Azumaya algebra of constant rank 9 over X, with $\mathcal{F}$ an $\mathcal{O}_X$-module of rank 2 and $\mathcal{L}$ a line bundle on X. Let $\mathcal{B} = \mathcal{O}_X \oplus \mathcal{E}nd_X(\mathcal{F})$. Then the Okubo algebra $(\mathcal{A}_0, \star)$ obtained via the construction in Theorem 2 contains the flexible symmetric composition algebra $(\mathcal{B}_0, \star)$ as a subalgebra.*

Proof. The algebra $\mathcal{B} = \mathcal{O}_X \oplus \mathcal{E}nd(\mathcal{F})$ is a unital alternative algebra over X with nondegenerate cubic norm $N(a+x) = aN_0(x)$ permitting composition, where N_0 is the norm of $\mathcal{E}nd(\mathcal{F})$. $\mathcal{B}$ is a subalgebra of $\mathcal{A}$: the inclusion is given by

$$a + M \to \begin{bmatrix} M & & 0 \\ & & 0 \\ 0 & 0 & a \end{bmatrix},$$

for $a \in \mathcal{O}_X$, $M \in \mathcal{E}nd_X(\mathcal{F})$. Now use Theorem 3. □

Proposition 9. *Let k be a field such that $2, 3 \in k^\times$ containing a primitive third root of unity and X a proper integral scheme over k. Let $\mathcal{F}$ be a vector bundle on X of rank 2. Let $\mathcal{A} = \mathcal{E}nd(\mathcal{F} \oplus \mathcal{O}_X)$ and let $(\mathcal{A}_0, \star)$ be the Okubo algebra obtained via the construction in Theorem 2.*
(i) Let

$$e = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \in H^0(X, \mathcal{A}_0).$$

The product

$$x \diamond y = (e \star x) \star (y \star e)$$

makes $\mathcal{A}_0$ into an octonion algebra over X with identity e and norm S.
(ii) $(\mathcal{A}_0, \star) = (\mathcal{A}_0, \diamond)_\varphi$ is a Petersson algebra over X.
(iii) The octonion algebra $(\mathcal{A}_0, \diamond)$ is a Cayley-Dickson doubling of $\mathcal{D} = \mathcal{E}nd_X(\mathcal{F})$, where the right $\mathcal{D}$-module of rank one used in the doubling process is, as $\mathcal{O}_X$-module, isomorphic to the vector bundle $\mathcal{F} \oplus \mathcal{F}^\vee$.

Proof. Since 2 is invertible in $H^0(X, \mathcal{O}_X)$, $\mathcal{E}nd(\mathcal{F}) \cong \mathcal{O}_X \oplus \mathcal{F}'$, where $\mathcal{F}'$ is the subspace of the endomorphisms of trace 0. Analogously as described in [KMRT, VIII.(34.30)], we view the elements of $\mathcal{A} = \mathcal{E}nd(\mathcal{F} \oplus \mathcal{O}_X)$ of trace zero as block matrices

$$\begin{bmatrix} \Phi & v \\ f & -tr(\Phi) \end{bmatrix} \in \begin{bmatrix} \mathcal{E}nd(\mathcal{F}) & \mathcal{F} \\ \mathcal{F}^\vee & \mathcal{O}_X \end{bmatrix}$$

with the product given by

$$\begin{bmatrix} \Phi & v \\ f & -tr(\Phi) \end{bmatrix} \begin{bmatrix} \Phi' & v' \\ f' & -tr(\Phi') \end{bmatrix} = \begin{bmatrix} \Phi \circ \Phi' + v \circ f' & \Phi(v') - tr(\Phi')v \\ f \circ \Phi' - tr(\Phi)f' & f(v') + tr(\Phi)tr(\Phi') \end{bmatrix}$$

where $(v \circ f')(x) = vf'(x)$. With $\star$ as defined in Theorem 2, $(\mathcal{A}_0, \star, S)$ is a symmetric composition algebra with underlying vector bundle

$$(\mathcal{F} \otimes \mathcal{F}^\vee) \oplus \mathcal{F} \oplus \mathcal{F}^\vee$$

containing a non-trivial idempotent

$$e = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \in H^0(X, \mathcal{A}_0).$$

[KMRT, VIII.(34.30)]. By Proposition 6, the product

$$x \diamond y = (e \star x) \star (y \star e)$$

makes $\mathcal{A}_0$ into an octonion algebra over X with identity e, norm S and conjugation $\bar{x} = S(x,e)e - x$. Furthermore, $(\mathcal{A}_0, \star) = (\mathcal{A}_0, \diamond)_\varphi$ is a Petersson algebra over X. This settles (i) and (ii).

Let $\mathcal{B} = \mathcal{O}_X \oplus \mathcal{E}nd(\mathcal{F})$. Using the inclusion given in Proposition 8, $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ and thus $(\mathcal{B}_0, \star)$ is a symmetric composition subalgebra of $(\mathcal{A}_0, \star)$. We also have $e \in H^0(X, \mathcal{B}_0)$. Now $(\mathcal{B}_0, \diamond)$ is isomorphic to the quaternion algebra $\mathcal{D} = \mathcal{E}nd(\mathcal{F})$ by Remark 6. Therefore the octonion algebra $(\mathcal{A}_0, \diamond)$ is a Cayley-Dickson doubling of $\mathcal{D}$, where the right $\mathcal{D}$-module of rank one used in the doubling is, as $\mathcal{O}_X$-module, isomorphic to the vector bundle $\mathcal{F} \oplus \mathcal{F}^\vee$, proving (iii). □

If $(\mathcal{C}, \diamond, N)$ is a Hurwitz algebra over X, then $(\mathcal{C}, \star, N)$ with

$$x \star y = \bar{x} \diamond \bar{y}$$

is a symmetric composition algebra over X, called the *para-Hurwitz algebra* associated to the Hurwitz algebra $(\mathcal{C}, \diamond, N)$ (the associativity is proved as in [KMRT, VIII.34.A]).

Lemma 10. *Let k be a field such that $2, 3 \in k^\times$ containing a primitive third root of unity. Let X be an integral scheme over k and $\mathcal{A}$ an Azumaya algebra of constant rank 9 over X such that $\mathcal{A}_\xi$ is a division algebra. Then there are no para-Hurwitz algebras of constant rank 2 or 4 over X which can be embedded into the associated Okubo algebra $(\mathcal{A}_0, \star)$.*

Proof. This is a direct consequence of [E-M, 8.2], applied to $\mathcal{A}_\xi$. □

7.1. Let us denote the Okubo algebra we obtain by applying Theorem 2 to the Azumaya algebra $\mathrm{Mat}_3(\mathcal{O}_X)$ by $P_8(\mathcal{O}_X)$, and the one we obtain by applying Corollary 3 to the Azumaya algebra $\mathrm{Mat}_3(R)$ by $P_8(R)$, analogously as in [E-P1] or [E, p. 285]. Note that the algebra $P_8(\mathbb{C})$, discovered by [O] in 1978, was the first Okubo algebra which was known.

The original definition of Okubo algebras required the base field to be of characteristic not 2 or 3. A new definition avoiding this restriction was given in [E-P1]. This definition generalizes easily to our setting of algebras over rings (or locally ringed spaces) as follows: let R be an arbitrary ring. Let T be a projective R-module of constant rank 3 such that $\bigwedge^3(T) \cong R$ and $\mathrm{Zor}(T, \alpha)$ be a split octonion algebra over R [P1, 3.2]. Take $id \neq \varphi \in \mathrm{End}_R(T)$ of order 3. Let $\varphi^* : T^\vee \to T^\vee$ be the adjoint of φ with respect to the canonical pairing

$$\langle\, , \,\rangle : T \times T^\vee \to R, \qquad \langle u, \check{v}\rangle = \check{v}(u),$$

so $\langle \varphi(u), \check{v}\rangle = \langle u, \varphi^*(\check{v})\rangle$ for all $u \in T$, $\check{v} \in T^\vee$. φ^* is an automorphism, since the residue class morphism $\varphi^*(P)$ is an automorphism, for all $P \in \operatorname{Spec} R$. Define

$$\tau(\begin{bmatrix} a & u \\ \check{u} & b \end{bmatrix}) = \begin{bmatrix} a & \varphi(u) \\ (\varphi^{*\,-1})(\check{u}) & b \end{bmatrix}.$$

Then τ is an automorphism of $\mathrm{Zor}(T, \alpha)$ of order 3. Obviously, we have

$$\mathrm{Zor}(R)_\tau \cong P_8(R).$$

The Petersson algebra $\mathrm{Zor}(T,\alpha)_\tau$ can be viewed as a natural generalization of the Okubo algebra $P_8(R)$ over R: for all $P \in \operatorname{Spec} R$, the residue class algebra $\mathrm{Zor}(T,\alpha)_\tau(P)$ is isomorphic to the Okubo algebra $P_8(k(P))$.

If we take the attached algebra $(\mathrm{Zor}(T,\alpha)_\tau)^-$ with multiplication

$$[x,y] = xy - yx,$$

we obtain examples of Lie algebras over X, whose residue class algebras are central simple Lie algebras of type A_2 which arise from central simple associative algebras of degree 3 [E-M, p. 2499].

$\mathrm{Zor}(T,\alpha)_\tau$ is a composition algebra and, if R is a domain, a symmetric composition algebra over R by 3.2. However, it does not seem to be clear if $\mathrm{Zor}(T,\alpha)_\tau$ is, indeed, always an Okubo algebra over the ring R. So the question remains if we can obtain $\mathrm{Zor}(T,\alpha)_\tau$ as the trace zero elements of some Azumaya algebra A over R of constant rank 3 by applying Corollary 3. A must have residue class algebras $A(P)$ isomorphic to $\mathrm{Mat}_3(k(P))$ for all $P \in \operatorname{Spec} R$. Note that, in general, we will not be able to choose the split Azumaya algebra $\mathrm{End}_R(T)$ as a candidate for A here, see Proposition 11.

The other remaining question is if, in case this construction really yields an Okubo algebra and R is a domain, all Okubo algebras are isomorphic to an algebra of the type $\mathrm{Zor}(T,\alpha)_\tau$ for suitable T and α. This can be negated immediately, though: we will show in Proposition 9 that there is an Okubo algebra over an elliptic curve over a field k with underlying $\mathcal{O}_X$-module structure

$$\mathcal{F}_3 \oplus \mathcal{F}_5,$$

where $\mathcal{F}_3$ is an absolutely indecomposable vector bundle of rank 3 and $\mathcal{F}_5$ an absolutely indecomposable vector bundle of rank 5. This algebra cannot be isomorphic to $\mathrm{Zor}(\mathcal{T},\alpha)_\tau$ for any suitable vector bundle $\mathcal{T}$ of rank 3 as in the above construction (when generalized to locally ringed spaces), since that one only yields algebras with underlying $\mathcal{O}_X$-module structure $\mathcal{O}_X^2 \oplus \mathcal{T} \oplus \mathcal{T}^\vee$ for some vector bundle $\mathcal{T}$ of constant rank 3 with trivial determinant.

8. Symmetric composition algebras of rank 2

Let k be a field such that $2,3 \in k^\times$ containing a primitive third root of unity ω. Let X be a proper integral scheme over k. Let $\mathcal{E}$ be a separable commutative associative $\mathcal{O}_X$-algebra of rank 3 with norm N. The multiplication $\star$ on the trace zero elements $\mathcal{E}_0$ of $\mathcal{E}$ reduces to

$$u \star v = (\omega - \omega^2)(uv - \frac{1}{3}T(uv)1) = (\omega - \omega^2)(uv)_0.$$

By [KMRT, VIII.(34.28)], for $P \in X$, the residue class algebra

$$(\mathcal{E}_0(P), \star, S(P))$$

is a para-Hurwitz algebra iff $\mathcal{E}(P)$ is not a field, e.g., if $\mathcal{E} = \mathcal{O}_X \times \mathcal{T}$ for a quadratic étale algebra $\mathcal{T}$ over X.

Example 2. Let k' be a quadratic field extension of k. Put $X' = X \times_k k'$ and let $\pi : X' \to X$ be the canonical projection morphism. Let $\mathcal{P} \in \operatorname{Pic} X'$ be a line bundle of order 3 carrying a nondegenerate cubic form N over X'. N is uniquely determined up to some invertible scalar in $H^0(X', \mathcal{O}_{X'})$. Let $\star$ be the involution on $\mathcal{O}_{X'}$ induced by the nontrivial element in the Galois group of k'/k. The Tits process

$$\mathcal{E} = \mathcal{J}(\pi_*\mathcal{O}_{X'}, \mathcal{O}_X, \mathcal{P}, N, \star) = \mathcal{O}_X \oplus \mathcal{P}$$

using $\mathcal{P}$ and N yields a a separable commutative associative $\mathcal{O}_X$-algebra of constant rank 3 with norm

$$N((a,w)) = a^3 + N(w) + N(w)^\star - 3a\langle w, w^\star\rangle$$

and trace

$$T((a,w),(c,v)) = 3ac + 3\langle w, v^\star\rangle + 3\langle v, w^\star\rangle$$

for all $a, c \in \mathcal{O}_X$, $v, w \in \mathcal{P}$ [Pu3, Pu4]. Since the Theorem of Krull-Schmidt for vector bundles holds over X,

$$\mathcal{E}_0 \cong \mathcal{P}$$

as $\mathcal{O}_X$-module. Moreover,

$$S(w) = -\frac{1}{2}T(w^2) = -3\langle w, w^\star\rangle$$

for $w \in \mathcal{P}$. $(\mathcal{P}, \star, S)$ is a flexible symmetric composition algebra over X. If $\mathcal{P}$ does not contain $\mathcal{O}_X$ as a direct summand as a vector bundle over X, then $(\mathcal{P}, \star, S)$ cannot be a Hurwitz algebra for any new multiplication by the Theorem of Krull-Schmidt.

Example 3. Every first Tits construction over X starting with $\mathcal{O}_X$ is isomorphic to

$$\mathcal{E} = \mathcal{J}(\mathcal{O}_X, \mathcal{N}, N) \cong \mathcal{O}_X \oplus \mathcal{N} \oplus \mathcal{N}^\vee$$

for some line bundle $\mathcal{N} \in \mathrm{Pic}X$ of order 3 and some nondegenerate cubic form N on $\mathcal{N}$ which is uniquely determined up to a scalar in $H^0(X, \mathcal{O}_X^\times)$. Let $\mathcal{N} \times \mathcal{N}^\vee \to \mathcal{O}_X$, $\langle w, \check{w}\rangle = \check{w}(w)$ be the canonical pairing. There exists a uniquely determined cubic form $\check{N} : \mathcal{N}^\vee \to \mathcal{O}_X$ and uniquely determined adjoints $\sharp : \mathcal{N} \to \mathcal{N}^\vee$ and $\check{\sharp} : \mathcal{N}^\vee \to \mathcal{N}$ such that $\langle w, w^\sharp\rangle = N(w)1$; $\langle \check{w}^{\check{\sharp}}, \check{w}\rangle = \check{N}(\check{w})1$ and $w^{\sharp\check{\sharp}} = N(w)w$ for v, w in $\mathcal{N}$, $\check{v}, \check{w}$ in $\mathcal{N}^\vee$. $\mathcal{E}^+ = \mathcal{J}(\widetilde{N}, \widetilde{\sharp}, 1)$ with the following cubic norm structure:

$$\widetilde{N}(a, w, \check{w}) = a^3 + N(w) + \check{N}(\check{w}) - 3a\langle w, \check{w}\rangle,$$
$$(a, w, \check{w})^{\widetilde{\sharp}} = (a^2 - \langle w, \check{w}\rangle, \check{w}^{\check{\sharp}} - aw, w^\sharp - \check{w}a),$$
$$\widetilde{T}((a, w, \check{w}), (c, v, \check{v})) = 3ac + 3\langle w, \check{v}\rangle + 3\langle v, \check{w}\rangle$$

for $a, c \in \mathcal{O}_X$, $v, w \in \mathcal{N}$, $\check{v}, \check{w} \in \mathcal{N}^\vee$ [Pu3, Pu4]. $\mathcal{E}$ is a separable commutative associative $\mathcal{O}_X$-algebra of constant rank 3. By the Theorem of Krull-Schmidt for vector bundles,

$$\mathcal{E}_0 \cong \mathcal{N} \oplus \mathcal{N}^\vee$$

as $\mathcal{O}_X$-modules and

$$S((w, \check{w})) = -3\langle w, \check{w}\rangle$$

for all $(w, \check{w}) \in \mathcal{E}_0$. $(\mathcal{N} \oplus \mathcal{N}^\vee, \star, S)$ is a flexible symmetric composition algebra, but cannot be made into a Hurwitz algebra for any new multiplication.

If $\mathcal{N}$ is not isomorphic to $\mathcal{O}_X$, then $(\mathcal{N} \oplus \mathcal{N}^\vee, S) = H(\mathcal{N})$ is a hyperbolic quadratic space and not defined over k.

If $\mathcal{N} \cong \mathcal{O}_X$, then $(\mathcal{N} \oplus \mathcal{N}^\vee, S)$ is defined over k.

Suppose we have two non-isomorphic line bundles $\mathcal{N}, \mathcal{M}$ of order 3 over X such that $\mathcal{N} \not\cong$

$\mathcal{M}^\vee$. Then the symmetric composition algebras $(\mathcal{N}\oplus\mathcal{N}^\vee, \star, S)$ and $(\mathcal{M}\oplus\mathcal{M}^\vee, \star, S')$ obtained as before are not isomorphic and neither are the hyperbolic spaces $H(\mathcal{N}) = (\mathcal{N}\oplus\mathcal{N}^\vee, S)$ and $H(\mathcal{M}) = (\mathcal{M}\oplus\mathcal{M}^\vee, S')$. However, the residue class forms satisfy

$$(\mathcal{N}(P)\oplus\mathcal{N}(P)^\vee, S(P)) \cong \langle 1, -1\rangle \cong (\mathcal{M}(P)\oplus\mathcal{M}(P)^\vee, S'(P))$$

for all $P \in X$.

Corollary 5. *(i) Let k' be a quadratic field extension of k. Put $X' = X \times_k k'$. Let $\star$ be the involution on $\mathcal{O}_{X'}$ induced by the nontrivial element in the Galois group of k'/k. Let $\mathcal{P} \in \operatorname{Pic} X'$ have order 3. Suppose that $\mathcal{P}^\star \cong \mathcal{P}^\vee$. Then there exists a multiplication which makes the rank 2 space $(\mathcal{P}, S)$ into a symmetric composition algebra over X with*

$$S(w) = -3\langle w, w^\star\rangle.$$

(ii) Let $\mathcal{N} \in \operatorname{Pic} X$ be a line bundle of order 3. Then there exists a multiplication which makes the quadratic space (which is hyperbolic for non-trivial $\mathcal{N}$)

$$(\mathcal{N}\oplus\mathcal{N}^\vee, S),\ S((w, \check{w})) = -3\check{w}(w)$$

into a symmetric composition algebra over X, which cannot be a Hurwitz algebra for any new multiplication unless $\mathcal{N} \cong \mathcal{O}_X$.

Note that, over locally ringed spaces, it does not need to be true that every symmetric composition algebra of rank 2 is isomorphic to the trace zero elements $(\mathcal{E}_0, \star)$ of some alternative algebra $\mathcal{E}$ of rank 3. This is true over fields of characteristic not 3 [KMRT, VIII.(34.28)].

Example 4. Let $X = \mathbb{P}^2_k$ be the projective plane over k and $X_0 = \mathbb{A}^2_k$ the affine plane. Let (x, y, z) denote the homogeneous coordinates of $\mathbb{P}^2_k$. Identify the affine plane $\mathbb{A}^2_k$ with the open subscheme $\mathbb{A}^2_k = D(z) = \mathbb{P}^2_k - V(z)$ in $\mathbb{P}^2_k$.

Every first Tits construction over X starting with $\mathcal{O}_X$ is defined over k and every first Tits construction over $k[x, y]$ starting with $k[x, y]$ is defined over k. Hence each symmetric composition algebra of rank 2 over X (resp., over $k[x, y]$) obtained from a first Tits construction through the construction from Theorem 2 is defined over k.

If A is an Azumaya algebra of rank 9 over $\mathbb{A}^2_k$ then A can be extended to an Azumaya algebra $\mathcal{A}$ over $\mathbb{P}^2_k$. If its reduced norm n_A is anisotropic, this extension is unique up to isomorphism [K-Pa-S, Theorem 7.1]. Thus A_0 extends to $\mathcal{A}_0$ over $\mathbb{P}^2_k$ as well, implying that each octonion algebra obtained through the construction of Theorem 2 is extended from $\mathbb{A}^2_k$. It is uniquely extended if n_A is anisotropic. It was already shown in [Pa-S-T, 4.6] that every octonion algebra over $\mathbb{A}^2_k$ with anisotropic norm can be uniquely extended to an octonion algebra $\mathcal{A}$ over $\mathbb{P}^2_k$.

9. Curves of genus zero

9.1. Let X be a curve of genus zero over k; i.e. a geometrically integral, complete, smooth scheme of dimension one over k. Let $P_0 \in X$ be a closed point of minimal degree and $\mathcal{L}(mP_0)$ the line bundle over X associated with the divisor mP_0. The isomorphism $\mathbb{Z} \cong \operatorname{Pic} X$ is given by the map $m \to \mathcal{L}(mP_0)$.

If X is rational, P_0 has degree 1 and $\mathcal{L}(mP_0) \cong \mathcal{O}_X(m)$. In that case, let $\mathbf{h}(m)$ denote the hyperbolic plane given by the symmetric bilinear form $((a,b),(c,d)) \to ad+bc$ on $\mathcal{O}_X(m) \oplus \mathcal{O}_X(-m)$.

If X is nonrational, let D_0 be the quaternion division algebra associated to X. If k'/k is a finite separable field extension which is a maximal subfield of D_0, then for $X' = X \times_k k'$ we have $X' \cong \mathbb{P}^1_{k'}$. In that case, let $\mathcal{E}_0 = tr_{k'/k}(\mathcal{O}_{X'}(1))$ be the indecomposable vector bundle of rank 2 with $D_0 = \mathrm{End}(\mathcal{E}_0)$ described in [P1, 4.3]. Moreover, $\mathcal{E}_0^\vee = tr_{k'/k}(\mathcal{O}_{X'}(-1))$. All vector bundles of rank at least 3 over X are decomposable. The indecomposable vector bundles of rank 2 over X are isomorphic to $\mathcal{E}_0 \otimes \mathcal{L}(mP_0)$, where $m \in \mathbb{Z}$ is unique.

Remark 8. Let k'/k be a quadratic field extension, $X' = X \times_k k'$ and $\pi : X' \to X$ the canonical projection. Then, for $m \in \mathbb{Z}$,

(2) $\pi^*\mathcal{O}_X(m) = \mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_{X'} \cong \mathcal{O}_{X'}(m)$ and $\pi_*\mathcal{O}_{X'}(m) = tr_{k'/k}(\mathcal{O}_{X'}(m)) \cong \mathcal{O}_X(m)^2$ if X is rational,

(3) $\pi^*\mathcal{L}(mP_0) = \mathcal{L}(mP_0) \otimes_{\mathcal{O}_X} \mathcal{O}_{X'} \cong \mathcal{O}_{X'}(2m)$, $\pi_*\mathcal{O}_{X'}(2m) = tr_{k'/k}(\mathcal{O}_{X'}(2m)) \cong \mathcal{L}(mP_0)^2$ and
$\pi_*\mathcal{O}_{X'}(2m+1) = tr_{k'/k}(\mathcal{O}_{X'}(2m+1)) \cong \mathcal{E}_0 \otimes \mathcal{L}(mP_0)$ if X' is rational but X is not,

(4) $\pi^*\mathcal{L}(mP_0) = \mathcal{L}(mP_0) \otimes_{\mathcal{O}_X} \mathcal{O}_{X'} \cong \mathcal{L}(mP_0')$ and $\pi_*\mathcal{L}(mP_0') = tr_{k'/k}(\mathcal{L}(mP_0')) \cong \mathcal{L}(mP_0)^2$ if X' is non-rational and P_0' has (minimal) degree 2.

Let $\mathcal{C}$ be an octonion algebra over X. Then $\mathcal{C}$ is defined over k, split, or X is nonrational and $\mathcal{C} \cong \mathrm{Cay}(\mathcal{D}, \mathcal{P}, N_{\mathcal{P}})$, where $\mathcal{D} = D_0 \otimes \mathcal{O}_X$, $\mathcal{P}$ is a locally free right $\mathcal{D}$-module of rank one and norm one, and $N_{\mathcal{P}}$ is a norm on it. Then we know that $\mathcal{P} = \mathcal{P}_1 \oplus \mathcal{P}_2$ with $\mathcal{P}_1 = \mathcal{L}(mP_0) \otimes \mathcal{E}_0^\vee$ and $\mathcal{P}_2 = \mathcal{L}((-m+1)P_0) \otimes \mathcal{E}_0^\vee$ for some integer $m \geq 0$ uniquely determined by $\mathcal{C}$ [P1].

Let k be a field of characteristic not 2 or 3 containing a primitive third root of unity ω.

Lemma 11. *(i) Every symmetric composition algebra obtained from a first Tits construction over X using Theorem 2 is defined over k.*
(ii) Let l/k be a separable quadratic field extension with $\mathrm{Gal}(l/k) = \langle \sigma \rangle$. *Let $X_l = X \times_k l$ and let $\pi : X_l \to X$ be the canonical projection. Every every symmetric composition algebra obtained from a Tits process over X using Theorem 2, starting with $\mathcal{B} = \pi_*\mathcal{O}_{X_l}$ and $*_{\mathcal{B}} = \sigma$, is defined over k.*

The proof is straightforward and uses the fact that the corresponding Tits constructions/Tits processes are algebras over X which are defined over k [Pu4, Lemma 1].

We cannot exclude the possibility that there are commutative associative algebras $\mathcal{J}(N, \sharp, 1)$ of rank 3 over X with nondegenerate norm form, which do not arise from a first Tits construction or a Tits process. The underlying module structure of such algebras must be $\mathcal{O}_X \oplus \mathcal{L}(mP_0) \oplus \mathcal{L}(-mP_0)$ $(m > 0)$ [Pu4, Remark 2], so these algebras would yield a symmetric composition algebra structure on the vector bundle $\mathcal{L}(mP_0) \oplus \mathcal{L}(-mP_0)$.

9.2. Let X be nonrational with associated quaternion division algebra D_0. For every locally free $\mathcal{O}_X$-module $\mathcal{E}$ of constant rank 3, $\mathcal{A} = \mathcal{E}nd_X(\mathcal{E})$ is an Azumaya algebra of rank 9. We have the following possibilities for $\mathcal{E}$:

Proposition 10. *(i) If $\mathcal{E}=\mathcal{L}(m_1P_0)\otimes\mathcal{E}_0\oplus\mathcal{L}(m_2P_0)$ for some $m_i\in\mathbb{Z}$ then*

$$\mathcal{A}\cong\begin{bmatrix} D\otimes\mathcal{O}_X & \mathcal{L}(-(m_2-m_1)P_0)\otimes\mathcal{E}_0\\ \mathcal{L}((m_2-m_1)P_0)\otimes\mathcal{E}_0^\vee & \mathcal{O}_X\end{bmatrix}$$

and

$$\mathcal{A}_0\cong\mathcal{O}_X^4\oplus\mathcal{L}(-(m_2-m_1)P_0)\otimes\mathcal{E}_0\oplus\mathcal{L}((m_2-m_1)P_0)\otimes\mathcal{E}_0^\vee.$$

(ii) If $\mathcal{E}=\mathcal{L}(n_1P_0)\oplus\mathcal{L}(n_2P_0)\oplus\mathcal{L}(n_3P_0)$ for some $n_i\in\mathbb{Z}$ then

$$\mathcal{A}\cong\begin{bmatrix} \mathcal{O}_X & \mathcal{L}((n_2-n_1)P_0) & \mathcal{L}((n_1-n_3)P_0)\\ \mathcal{L}(-(n_2-n_1)P_0) & \mathcal{O}_X & \mathcal{L}((2n_1-n_3-n_2)P_0)\\ \mathcal{L}(-(n_1-n_3)P_0) & \mathcal{L}((n_2-2n_1+n_3)P_0) & \mathcal{O}_X\end{bmatrix}$$

and

$$\mathcal{A}_0\cong\mathcal{O}_X^2\oplus\mathcal{L}((n_2-n_1)P_0)\oplus\mathcal{L}((n_1-n_3)P_0)\oplus\mathcal{L}(-(n_2-n_1)P_0)\oplus$$
$$\mathcal{L}((2n_1-n_3-n_2)P_0)\oplus\mathcal{L}(-(n_1-n_3)P_0)\oplus\mathcal{L}((n_2-n_1-n_1+n_3)P_0).$$

In both cases, $(\mathcal{A}_0,\star,S)$ is an Okubo algebra with $\star$ as in Theorem 2.

In case (i), $(\mathcal{A}_0,\star)$ contains the flexible symmetric composition subalgebra $(\mathcal{B}_0,\star)$ with $\mathcal{B}=\mathcal{O}_X\oplus\mathcal{E}nd_X(\mathcal{L}(m_1P_0)\otimes\mathcal{E}_0)$ (Proposition 8). $\mathcal{B}_0$ is a free $\mathcal{O}_X$-module, hence $(\mathcal{B}_0,\star)$ is defined over k. $\mathcal{A}_0$ can be made into an octonion algebra via a suitable multiplication $\diamond$, since it is of the type discussed in Example 1. By the Theorem of Krull-Schmidt and the classification of octonion algebras in [P1, 4.4], $(\mathcal{A}_0,\diamond)$ is a Cayley-Dickson doubling of $\mathcal{D}=D_0\otimes\mathcal{O}_X$. (This also follows immediately by applying Theorem 3 and Remark 6.)

In case (ii), $(\mathcal{A}_0,\star)$ contains for instance the flexible symmetric composition subalgebra $(\mathcal{B}_0,\star)$ with $\mathcal{B}=\mathcal{O}_X\oplus\mathcal{E}nd_X(\mathcal{L}(m_1P_0)\oplus\mathcal{L}(m_2P_0))$ (Proposition 8). $\mathcal{A}_0$ can be made into an octonion algebra, since it is of the type discussed By the classification Theorem [P1, 4.4], if $\mathcal{A}_0$ is globally free as $\mathcal{O}_X$-module, it must be isomorphic to $\mathrm{Zor}(k)\otimes\mathcal{O}_X$, and if $\mathcal{A}_0$ is not globally free as $\mathcal{O}_X$-module, it must be a split octonion algebra over X which is not defined over k.

Note that every Azumaya algebra over X which is defined over k, i.e. $\mathcal{A}\cong A\otimes\mathcal{O}_X$, yields a Hurwitz algebra over X which is defined over k.

9.3. Let X be rational. It is well-known that each Azumaya algebra over X of constant rank 9 is either defined over k or isomorphic to $\mathcal{A}=\mathcal{E}nd_X(\mathcal{O}_X(n_1)\oplus\mathcal{O}_X(n_2)\oplus\mathcal{O}_X(n_3))$ with $n_i\in\mathbb{Z}$; i.e.,

$$\mathcal{A}\cong\begin{bmatrix} \mathcal{O}_X & \mathcal{O}_X(n_1-n_2) & \mathcal{O}_X(n_1-n_3)\\ \mathcal{O}_X(-(n_1-n_2)) & \mathcal{O}_X & \mathcal{O}_X(-n_3+n_2)\\ \mathcal{O}_X(-(n_1-n_3)) & \mathcal{O}_X(-n_2+n_3) & \mathcal{O}_X\end{bmatrix}.$$

The Okubo algebra $(\mathcal{A}_0,\star)$ contains for instance the flexible symmetric composition subalgebra $(\mathcal{B}_0,\star)$ with $\mathcal{B}=\mathcal{E}nd_X(\mathcal{O}_X(n_1P)\oplus\mathcal{O}_X(n_2))$ (Proposition 8).

Each Okubo algebra $(\mathcal{A}_0,\star)$ is either defined over k or $\mathcal{A}_0\cong\mathcal{O}_X^2\oplus\mathcal{T}\oplus\mathcal{T}^\vee$ with

$$\mathcal{T}=\mathcal{O}_X(n_1-n_2)\oplus\mathcal{O}_X(-(n_1-n_3))\oplus\mathcal{O}_X(-n_3+n_2)$$

not globally free. Note that $n_1 - n_2 - (n_1 - n_3) + (-n_3 + n_2) = 0$, so that $\mathcal{T}$ has trivial determinant. It remains to be checked if $\mathcal{A}_0$ is perhaps isomorphic to an algebra of the type $\mathrm{Zor}(\mathcal{S}, \alpha)_\tau$ for suitable $\mathcal{S}$ and α as described in 7.1.

$(\mathcal{A}_0, \star)$ contains an idempotent e (Proposition 9), so it can be made into an octonion algebra. If $\mathcal{A}_0$ is globally free as $\mathcal{O}_X$-module, $(\mathcal{A}_0, \diamond)$ is isomorphic to $\mathrm{Zor}\, k \otimes \mathcal{O}_X$. The classification in [P1, 4.4] shows that if $\mathcal{A}_0$ is not globally free as $\mathcal{O}_X$-module, $(\mathcal{A}_0, \diamond)$ is a split octonion algebra over X which is not defined over k. As in 9.2, our results show that $(\mathcal{A}_0, \diamond)$ contains the quaternion subalgebra $(\mathcal{B}_0, \diamond)$ which is split by [P1, 4.4] (and then hence so is $(\mathcal{A}_0, \diamond)$), if its underlying vector bundle is not globally free.

10. Curves of genus one

The advantage of working over elliptic curves instead of curves of genus zero is that there are also bundles of degree higher than 2 which are indecomposable. These contribute to more interesting examples of symmetric composition algebras. We will use the results and terminology from Atiyah [At] and Arason, Elman and Jacob [AEJ1].

For simplicity, we assume from now on that k is a field of characteristic zero. An elliptic curve X/k can be described by a Weierstraß equation of the form

$$y^2 = x^3 + b_2x^2 + b_1x + b_0 \qquad (b_i \in k)$$

with the infinite point as base point O. Let $q(x) = x^3 + b_2x^2 + b_1x + b_0$ be the defining polynomial in $k[x]$. The k-rational points of order 2 on X are the points $(a, 0)$, where $a \in k$ is a root of $q(x)$. Let $K = k(X) = k(x, \sqrt{q(x)})$ be the function field of X. We distinguish three different cases (cf. [AEJ3]).

Case I. X has three k-rational points of order 2 which is equivalent to ${}_2\mathrm{Pic}\,(X) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Write $q(x) = (x-a_1)(x-a_2)(x-a_3)$ and ${}_2\mathrm{Pic}\,(X) = \{\mathcal{O}_X, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3\}$ where $\mathcal{L}_i$ corresponds to the point $(a_i, 0)$ for $i = 1, 2, 3$.

Case II. X has one k-rational point of order 2 which is equivalent to ${}_2\mathrm{Pic}\,(X) \cong \mathbb{Z}_2$. Write $q(x) = (x - a_1)q_1(x)$ and ${}_2\mathrm{Pic}\,(X) = \{\mathcal{O}_X, \mathcal{L}_1\}$ with $\mathcal{L}_1$ corresponding to $(a_1, 0)$. Define $l_2 = k(a_2)$ with a_2 a root of q_1.

Case III. X has no k-rational point of order 2 which is equivalent to ${}_2\mathrm{Pic}\,(X) = \{\mathcal{O}_X\}$. Define $l_1 = k(a_1)$ with a_1 a root of the irreducible polynomial $q(x)$ and let $\Delta(q) = (a_1 - a_2)^2(a_1 - a_3)^2(a_2 - a_3)^2$ be the discriminant of q.

Correspondingly, X/k is called *of type I, II* or *III.* Let $\overline{k}$ be an algebraic closure of k and let $\overline{X} = X \times_k \overline{k}$.

For any integer r, there exists an absolutely indecomposable vector bundle of rank r and degree 0 on X we call $\mathcal{F}_r$, which is unique up to isomorphism, such that $\mathcal{F}_r$ has nontrivial global sections [At, Theorem 5]. Each $\mathcal{F}_r$ is selfdual. In particular, we know that $\mathcal{F}_1 = \mathcal{O}_X$. Furthermore, if $\mathcal{M}$ is an absolutely indecomposable vector bundle of rank r and degree 0 on X, there is a line bundle $\mathcal{L} \in \mathrm{Pic}\, X$ of degree 0, such that $\mathcal{M} \cong \mathcal{L} \otimes \mathcal{F}_r$. This line bundle is unique up to isomorphism.

Let $\mathcal{N}_i$ denote a line bundle of order 3 on X. Let $\beta : \mathcal{N}_i \otimes \mathcal{N}_i \otimes \mathcal{N}_i \to \mathcal{O}_X$ be an isomorphism. Then

$$N : \mathcal{N}_i \to \mathcal{O}_X,\ w \to N(w) = \beta(w \otimes w \otimes w)$$

is a nondegenerate cubic form on $\mathcal{N}_i$. The nondegenerate cubic forms on $\mathcal{N}_i$ are uniquely determined up to an invertible factor in k [Pu3, Lemma 1].

Following [AEJ1], for any separable field extension l/k, we denote the selfdual line bundles $\mathcal{L}_i \otimes_{\mathcal{O}_X} \mathcal{O}_{X_l}$ on $X_l = X \times_k l$ also by $\mathcal{L}_i$, $i = 1, 2, 3$, to avoid complicated terminology. We do the same for the line bundles $\mathcal{N}_i \otimes_{\mathcal{O}_X} \mathcal{O}_{X_l}$ of order 3. This abuse of notation is justified by the fact that the natural map $\operatorname{Pic} X \to \operatorname{Pic} X_l$ is injective (1.7). Recall that ${}_3\operatorname{Pic}(\overline{X}) = \{\mathcal{N}_i \mid 0 \leq i \leq 8\}$ where $\mathcal{N}_0 = \mathcal{O}_{\overline{X}}$ [At, Lemma 22]. Hence ${}_3\operatorname{Pic}(X) = \{\mathcal{N}_i \mid 0 \leq i \leq m\}$ for some even integer m, $0 \leq m \leq 8$, where $\mathcal{N}_0 = \mathcal{O}_X$.

Let k contain a primitive third root of unity ω.

Example 5. Every first Tits construction over X starting with $\mathcal{O}_X$, which is not defined over k, is isomorphic to $\mathcal{A} = \mathcal{J}(\mathcal{O}_X, \mathcal{N}_i, N)$ where $\mathcal{N}_i \in {}_3\operatorname{Pic}X$ is nontrivial and N is a nondegenerate cubic form on $\mathcal{N}_i$ [Pu4]. We have

$$\mathcal{A}_0 \cong \mathcal{N}_i \oplus \mathcal{N}_i^\vee.$$

Note that $H^0(X, \mathcal{A}_0) = 0$. For all $x = (w, \check{w}) \in \mathcal{A}_0$,

$$S(x) = -3\langle w, \check{w}\rangle,$$

see Example 3. $(\mathcal{N}_i \oplus \mathcal{N}_i^\vee, \star, S)$ is a symmetric composition algebra over X by Theorem 2, which is not defined over X and which cannot be a Hurwitz algebra for any new multiplication.

By the Theorem of Krull-Schmidt, $\mathcal{N}_i \oplus \mathcal{N}_i^\vee \cong \mathcal{N}_j \oplus \mathcal{N}_j^\vee$ iff $\mathcal{N}_i \cong \mathcal{N}_j$ or $\mathcal{N}_i \cong \mathcal{N}_j^\vee$. Hence if $m = 8$ (e.g. if k is algebraically closed) there are at least 4 non-isomorphic symmetric composition algebras of rank 2 which are not defined over k.

Example 6. Let l/k be a quadratic field extension with $\operatorname{Gal}(l/k) = \langle\sigma\rangle$. Let $X_l = X \times_k l$ and let $\pi : X_l \to X$ be the canonical projection. Define $\mathcal{B} = \pi_* \mathcal{O}_{X_l}$.
(i) If X has type I or III, or type II and $l \ncong l_2$, then every Tits process over X starting with $\mathcal{B}$ and $*_\mathcal{B} = \sigma$ is defined over k [Pu4, Lemma 6] and thus the construction from Theorem 2 yields a symmetric composition algebra of rank 2 which is defined over k.
(ii) Let X be of type II and $l \cong l_2$. If there is a line bundle $\mathcal{N}_i$ over X_l of order 3 which is not defined over X and satisfies ${}^\sigma\mathcal{N}_i \cong \mathcal{N}_i^\vee$, then there is a Tits process $\mathcal{J} = \mathcal{J}(\mathcal{B}, \mathcal{O}_X, \mathcal{N}_i, N, *) \cong \mathcal{O}_X \oplus \mathcal{N}_i$ which is not defined over X [Pu4, Lemma 6] and thus the construction from Theorem 2 yields a symmetric composition algebra of rank 2 over X on the indecomposable vector bundle $\mathcal{N}_i$ with norm

$$S(w) = -\frac{1}{2}T(w^2) = -3\langle w, w^\star\rangle,$$

see Example 2.

Otherwise every Tits process starting with $\mathcal{B}$ and $*_\mathcal{B} = \sigma$ is defined over X [Pu4, Lemma 6] and thus the construction from Theorem 2 yields a symmetric composition algebra of rank 2 which is defined over k as well.

For every locally free $\mathcal{O}_X$-module $\mathcal{E}$ of constant rank 3, $\mathcal{A} = \mathcal{E}nd_X(\mathcal{E})$ is an Azumaya algebra of rank 9. We look at some examples:

Proposition 11. *(i) If $\mathcal{E}$ is absolutely indecomposable and $\mathcal{E} = \mathcal{M}\otimes\mathcal{F}_3$ for some line bundle $\mathcal{M} \in \operatorname{Pic} X$, then*

$$\mathcal{A} \cong \mathcal{O}_X \oplus \mathcal{F}_3 \oplus \mathcal{F}_5$$

and

$$\mathcal{A}_0 \cong \mathcal{F}_3 \oplus \mathcal{F}_5$$

as $\mathcal{O}_X$-modules.
(ii) If $\mathcal{E}$ is absolutely indecomposable and $\mathcal{E} \in \Omega(3,d)$, $\gcd(3,d) = 1$ then, if $m < 8$,

$$\mathcal{A} \cong \mathcal{O}_X \oplus \mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_m \oplus tr_{l_1/k}(\mathcal{N}_{m+1}) \cdots \oplus tr_{l_j/k}(\mathcal{N}_j)$$

and

$$\mathcal{A}_0 \cong \mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_m \oplus tr_{l_1/k}(\mathcal{N}_{m+1}) \cdots \oplus tr_{l_j/k}(\mathcal{N}_j)$$

as $\mathcal{O}_X$-module, where the line bundles $\mathcal{N}_{m+1}$ over $X_{l_1}, \ldots, \mathcal{N}_j$ over X_{l_j} are not defined over X. If $m = 8$ (e.g., if k is algebraically closed),

$$\mathcal{A}_0 \cong \mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_8$$

as $\mathcal{O}_X$-module.

Proof. This follows from [At, Theorem 8, Lemma 22]. In particular, if $\mathcal{E}$ is absolutely indecomposable and $\mathcal{E} \in \Omega(3,d)$, $\gcd(3,d) = 1$ then

$$\overline{\mathcal{A}} \cong \mathcal{N}_0 \oplus \mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_8$$

over $\overline{X}$ and there is a suitable integer m, $1 \leq m \leq 8$ depending on X such that ${}_3\mathrm{Pic}\,(X) = \{\mathcal{N}_0, \ldots, \mathcal{N}_m\}$, which implies (iii). □

If k contains a primitive third root of unity ω, then in both (i) and (ii), $\mathcal{A}_0$ becomes an Okubo algebra over X via the new multiplication

$$u \star v = \omega uv - \omega^2 vu - \frac{1}{3}[\omega^2 - \omega]T(uv)1$$

by Theorem 2. Due to the structure of the underlying vector bundle, it is obvious that in both cases there does not exist any multiplication which would make $\mathcal{A}_0$ into an octonion algebra, or otherwise $\mathcal{A}_0$ would have $\mathcal{O}_X$ as a direct summand. In case (ii), even $H^0(X, \mathcal{A}_0) = 0$.

Moreover, case (i) is an example of an Okubo algebra which is the direct sum of only two absolutely indecomposable vector bundles of rank 3 and rank 5 and which, over each residue class field, is isomorphic to $P_8(k(P))$.

Recall that for X of type III, the elliptic curve $X_1 = X \times_k l_1$ is of type I and the selfdual line bundle $\mathcal{L}_1$ over X_1 is not defined over X. The vector bundle $tr_{l_1/k}(\mathcal{L}_1)$ is indecomposable of rank 3 and $tr_{l_1/k}(\mathcal{L}_1) \otimes \mathcal{O}_{X_1} \cong \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$.

For X of type II, the elliptic curve $X_2 = X \times_k l_2$ is of type I and the selfdual line bundles $\mathcal{L}_2$ and $\mathcal{L}_3$ on X_2 are not defined over X. The vector bundle $tr_{l_2/k}(\mathcal{L}_2) \cong tr_{l_2/k}(\mathcal{L}_3)$ is indecomposable of rank 2 and $tr_{l_2/k}(\mathcal{L}_3) \otimes \mathcal{O}_{X_2} \cong \mathcal{L}_2 \oplus \mathcal{L}_3$.

Proposition 12. *(i) If $\mathcal{E}$ is indecomposable, but not absolutely so, then there is a suitable cubic field extension l of k and a line bundle $\mathcal{N}$ over $Y = X \times_k l$, such that $\mathcal{E} = tr_{l/k}(\mathcal{N})$ and $\mathcal{A} \cong tr_{l/k}(\mathcal{N}) \otimes tr_{l/k}(\mathcal{N}^\vee)$. If l/k is Galois with* $\mathrm{Gal}(l/k) = \{id, \sigma_1, \sigma_2\}$ *then*

$$\mathcal{A} \cong \mathcal{O}_X^3 \oplus tr_{l/k}(\mathcal{N} \otimes {}^{\sigma_1}\mathcal{N}^\vee) \oplus tr_{l/k}(\mathcal{N} \otimes {}^{\sigma_2}\mathcal{N}^\vee)$$

and

$$\mathcal{A}_0 \cong \mathcal{O}_X^2 \oplus tr_{l/k}(\mathcal{N} \otimes^{\sigma_1} \mathcal{N}^\vee) \oplus tr_{l/k}(\mathcal{N} \otimes {}^{\sigma_2}\mathcal{N}^\vee)$$

as $\mathcal{O}_X$-module. In particular, if X is of type III, we may choose $\mathcal{E} = tr_{l_1/k}(\mathcal{L}_1)$ and get

$$\mathcal{A}_0 \cong \mathcal{O}_X^2 \oplus [tr_{l_1/k}(\mathcal{L}_1)]^2$$

as $\mathcal{O}_X$-module.
(ii) If $\mathcal{E} = \mathcal{M}_1 \oplus \mathcal{M}_2 \otimes \mathcal{F}_2$ for some line bundles $\mathcal{M}_i \in \mathrm{Pic}\, X$, then

$$\mathcal{A} \cong \begin{bmatrix} \mathcal{O}_X & \mathcal{H}om(\mathcal{M}_1, \mathcal{M}_2 \otimes \mathcal{F}_2) \\ \mathcal{H}om(\mathcal{M}_2 \otimes \mathcal{F}_2, \mathcal{M}_1) & \mathcal{E}nd(\mathcal{F}_2) \end{bmatrix}$$
$$\cong \mathcal{O}_X \oplus \mathcal{M}_1 \otimes \mathcal{M}_2^\vee \otimes \mathcal{F}_2 \oplus \mathcal{M}_1^\vee \otimes \mathcal{M}_2 \otimes \mathcal{F}_2 \oplus \mathcal{O}_X \oplus \mathcal{F}_3$$

as $\mathcal{O}_X$-module and

$$\mathcal{A}_0 \cong \mathcal{O}_X \oplus \mathcal{M}_1 \otimes \mathcal{M}_2^\vee \otimes \mathcal{F}_2 \oplus \mathcal{M}_1^\vee \otimes \mathcal{M}_2 \otimes \mathcal{F}_2 \oplus \mathcal{F}_3$$

as $\mathcal{O}_X$-module.
(iii) If $\mathcal{E}$ is the direct sum of a line bundle and an indecomposable (but not absolutely indecomposable) vector bundle of rank 2 then there is a quadratic field extension l/k with $\mathrm{Gal}(l/k) = \{id, \sigma\}$ *and a line bundle $\mathcal{N}$ over $X_l = X \times_k l$, not defined over X, such that $\mathcal{E} = \mathcal{M} \oplus tr_{l/k}(\mathcal{N})$ and*

$$\mathcal{A} \cong \begin{bmatrix} \mathcal{O}_X & \mathcal{H}om_X(\mathcal{M}, tr_{l/k}(\mathcal{N})) \\ \mathcal{H}om_X(tr_{l/k}(\mathcal{N}), \mathcal{M}) & \mathcal{E}nd_X(tr_{l/k}(\mathcal{N})) \end{bmatrix}$$

with

$$\mathcal{E}nd_X(tr_{l/k}(\mathcal{N})) \cong \mathcal{O}_X^2 \oplus tr_{l/k}(\mathcal{N} \otimes {}^{\sigma}\mathcal{N}^\vee).$$

Hence

$$\mathcal{A}_0 \cong \mathcal{O}_X^2 \oplus tr_{l/k}(\mathcal{N} \otimes {}^{\sigma}\mathcal{N}^\vee) \oplus (\mathcal{M} \otimes tr_{l/k}(\mathcal{N}^\vee)) \oplus (\mathcal{M}^\vee \otimes tr_{l/k}(\mathcal{N}))$$

as $\mathcal{O}_X$-module. In particular, if X is of type II, we may choose $\mathcal{E} = \mathcal{M} \oplus tr_{l_2/k}(\mathcal{L}_2)$ and get

$$\mathcal{A}_0 \cong \mathcal{O}_X^2 \oplus \mathcal{L}_1^2 \oplus (\mathcal{M} \otimes tr_{l_2/k}(\mathcal{L}_2)) \oplus (\mathcal{M}^\vee \otimes tr_{l_2/k}(\mathcal{L}_2))$$

as $\mathcal{O}_X$-module.
(iv) If $\mathcal{E}$ is the direct sum of line bundles $\mathcal{E} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3$ then

$$\mathcal{A} \cong \begin{bmatrix} \mathcal{O}_X & \mathcal{H}om_X(\mathcal{M}_1, \mathcal{M}_2) & \mathcal{H}om_X(\mathcal{M}_1, \mathcal{M}_3) \\ \mathcal{H}om_X(\mathcal{M}_2, \mathcal{M}_1) & \mathcal{O}_X & \mathcal{H}om(\mathcal{M}_2, \mathcal{M}_3) \\ \mathcal{H}om_X(\mathcal{M}_3, \mathcal{M}_1) & \mathcal{H}om_X(\mathcal{M}_3, \mathcal{M}_2) & \mathcal{O}_X \end{bmatrix}$$

and, as $\mathcal{O}_X$-module,

$$\mathcal{A}_0 \cong \mathcal{O}_X^2 \oplus \mathcal{M}_1^\vee \otimes \mathcal{M}_2 \oplus \mathcal{M}_1^\vee \otimes \mathcal{M}_3 \oplus \mathcal{M}_2^\vee \otimes \mathcal{M}_1 \oplus \mathcal{M}_2^\vee \otimes \mathcal{M}_3 \oplus \mathcal{M}_3^\vee \otimes \mathcal{M}_1 \oplus \mathcal{M}_3^\vee \otimes \mathcal{M}_2$$

is a direct sum of line bundles.

Proof. This follows from [At, Theorem 8, Lemma 22], (i) uses [AEJ3, 2.2]. □

If k contains a primitive third root of unity ω, then in all of the above cases, $\mathcal{A}_0$ becomes an Okubo algebra over X via the new multiplication

$$u \star v = \omega uv - \omega^2 vu - \frac{1}{3}[\omega^2 - \omega]T(uv)1$$

by Theorem 2.

By Proposition 8, we know that in case (ii), the Okubo algebra $(\mathcal{A}_0, \star)$ contains the flexible symmetric composition subalgebra $(\mathcal{B}_0, \star)$ with $\mathcal{B} = \mathcal{O}_X \oplus \mathcal{E}nd_X(\mathcal{M}_2 \otimes \mathcal{F}_2)$ and in case (iii), $(\mathcal{A}_0, \star)$ contains the flexible symmetric composition subalgebra $(\mathcal{B}_0, \star)$ with $\mathcal{B} = \mathcal{O}_X \oplus \mathcal{E}nd_X(tr_{l/k}(\mathcal{N}))$.

Corollary 6. *Suppose that k contains a primitive third root of unity.*
(i) If $\mathcal{M}_1 \cong \mathcal{O}_X$ in Proposition 11 (ii), then

$$\mathcal{A}_0 \cong \mathcal{O}_X \oplus \mathcal{M}_2^\vee \otimes \mathcal{F}_2 \oplus \mathcal{M}_2 \otimes \mathcal{F}_2 \oplus \mathcal{F}_3$$

can be made into a (non-split) octonion algebra which does not contain any quadratic étale algebra and is a Cayley-Dickson doubling of the quaternion algebra $\mathcal{E}nd_X(\mathcal{F}_2)$.
(ii) If $\mathcal{M} \cong \mathcal{O}_X$ in Proposition 11 (ii), then

$$\mathcal{A}_0 \cong \mathcal{O}_X^2 \oplus tr_{l/k}(\mathcal{N} \otimes {}^\sigma\mathcal{N}^\vee) \oplus tr_{l/k}(\mathcal{N}) \oplus tr_{l/k}(\mathcal{N}^\vee)$$

can be made into an octonion algebra which is a Cayley-Dickson doubling of the quaternion algebra $\mathcal{E}nd_X(tr_{l/k}(\mathcal{N}))$.

Proof. In both cases $\mathcal{A}_0$ is of the type discussed in Proposition 9, hence contains an idempotent $e \in H^0(X, \mathcal{A})$
(i) If $\mathcal{M}_1 \cong \mathcal{O}_X$ then

$$\mathcal{A} \cong \begin{bmatrix} \mathcal{O}_X & \mathcal{H}om_X(\mathcal{O}_X, \mathcal{M}_2 \otimes \mathcal{F}_2) \\ \mathcal{H}om_X(\mathcal{M}_2 \otimes \mathcal{F}_2, \mathcal{O}_X) & \mathcal{E}nd_X(\mathcal{F}_2) \end{bmatrix}.$$

The octonion algebras $(\mathcal{A}_0, \diamond)$ are Cayley-Dickson doublings of the quaternion algebra $\mathcal{E}nd_X(\mathcal{F}_2)$ by Proposition 9. The rest of the assertion is obvious from the module structure.
(ii) If $\mathcal{M} \cong \mathcal{O}_X$ then

$$\mathcal{A} \cong \begin{bmatrix} \mathcal{O}_X & tr_{l/k}(\mathcal{N}) \\ tr_{l/k}(\mathcal{N}^\vee) & \mathcal{E}nd_X(tr_{l/k}(\mathcal{N})) \end{bmatrix}.$$

The algebras $(\mathcal{A}_0, \diamond)$ are Cayley-Dickson doublings of the quaternion algebra $\mathcal{E}nd_X(tr_{l/k}(\mathcal{N}))$. □

For a list of all the possible Cayley-Dickson doublings of $\mathcal{E}nd_X(\mathcal{F}_2)$, the reader is referred to [Pu1, 4.2 (b)].

It would be desirable to find an example of an Okubo algebra which is defined on an indecomposable vector bundle. Due to the behaviour of the vector bundles over elliptic curves, such an example cannot arise out of the trace zero elements of a split Azumaya algebra over X.

Remark 9. Let $\mathcal{S}$ be an Okubo algebra over X. Then $\mathcal{S}^-$ with multiplication $[x,y] = xy - yx$ is a Lie algebra over X of constant rank 8, whose residue class algebras are central simple Lie algebras of type A_2 [E-M, p. 2499]. We have thus found Lie algebras of this type, which can be constructed on an elliptic curve over a field of characteristic 0 containing a primitive third root of unity, for example on the following vector bundles (with the $\mathcal{M}_i$'s line bundles):

(1) $\mathcal{A}_0 \cong \mathcal{F}_3 \oplus \mathcal{F}_5$,
(2) $\mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_8$,
(3) $\mathcal{A}_0 \cong \mathcal{O}_X^2 \oplus [tr_{l_1/k}(\mathcal{L}_1)]^2$, if X is of type III,
(4) $\mathcal{O}_X \oplus \mathcal{M}_1 \otimes \mathcal{M}_2^\vee \otimes \mathcal{F}_2 \oplus \mathcal{M}_1^\vee \otimes \mathcal{M}_2 \otimes \mathcal{F}_2 \oplus \mathcal{F}_3$,
(5) $\mathcal{O}_X^2 \oplus \mathcal{L}_1^2 \oplus (\mathcal{M} \otimes tr_{l_2/k}(\mathcal{L}_2)) \oplus (\mathcal{M}^\vee \otimes tr_{l_2/k}(\mathcal{L}_2))$, if X is of type II,
(6) $\mathcal{O}_X^2 \oplus \mathcal{M}_1^\vee \otimes \mathcal{M}_2 \oplus \mathcal{M}_1^\vee \otimes \mathcal{M}_3 \oplus \mathcal{M}_2^\vee \otimes \mathcal{M}_1 \oplus \mathcal{M}_2^\vee \otimes \mathcal{M}_3 \oplus \mathcal{M}_3^\vee \otimes \mathcal{M}_1 \oplus \mathcal{M}_3^\vee \otimes \mathcal{M}_2$

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