STRUCTURABLE ALGEBRAS OF SKEW-RANK 1 OVER THE AFFINE PLANE

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Abstract. Let k be a field of characteristic not 2 or 3. Infinitely many mutually non-isomorphic structurable algebras of rank 20 over $k[X,Y]$ are constructed whose fibre is a given structurable algebra over k of skew-rank 1.

Introduction

Let R be a ring such that $1/6 \in R$ and k a field of characteristic not 2 or 3. Let A be a unital nonassociative algebra over R with an involution $^-$. The pair $(A, ^-)$ is called a *structurable algebra* if

$$\{x,y,\{z,w,q\}\} - \{z,w,\{x,y,q\}\} = \{\{x,y,z\},w,q\} - \{z,\{y,x,w\},q\}$$

for $x,y,z,w,q \in A$, where

$$\{x,y,z\} = (x\bar{y})z + (z\bar{y})x - (z\bar{x})y.$$

Structurable algebras were introduced by Allison [A1]: An analogue of the Köcher-Kantor-Tits functor gives a correspondence between a structurable algebra and a Lie algebra. Using this functor all classical simple isotropic Lie algebras can be obtained [A2].

In [Pa-Sr-T], non-trivial Albert algebra bundles over the affine plane were constructed whose associated principal F_4 bundle admits no reduction of the structure group to any proper connected reductive subgroup. Over a field, every Albert algebra arises from the first or second Tits construction and the associated F_4 bundle admits reduction of the structure group to $\mathrm{SL}_1(B)$ for a central simple algebra B either over k or to $\mathrm{SU}(B,\sigma)$ for a central simple algebra B over a quadratic field extension of k, σ an involution of the second type. Hence the patched Albert algebras over the affine plane arise neither from a first nor a second Tits construction.

In the present paper we employ the patching arguments from [Pa-Sr-T] to obtain infinitely many structurable algebras M_i of rank 20 over the affine plane $\mathbb{A}^2_k$, which are not extended from k and mutually non-isomorphic and whose fibre is a given matrix algebra over k (Theorem 4). In order to achieve this, we show that the matrix algebra $M(T,N,N^\vee)$ over $k[X,Y]$ admits a unique extension to a matrix algebra over $\mathbb{P}^2_k$ in Section 2. In Section 3, we look at forms of these matrix algebras. For a nonfree projective left $D[X,Y]$-module P of rank one, the structurable algebra $S(D,\sigma,P,N)$ over $k[X,Y]$ admits a unique extension to a

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structurable algebra $S(\mathcal{D}, \sigma, \widetilde{P}, N)$ over $\mathbb{P}^2_k$, where $\widetilde{P}$ is an indecompsable vector bundle. We use this result to construct infinitely many mutually non-isomorphic structurable algebras A^i over $\mathbb{A}^2_k$ such that $A^i \otimes_k K \cong M_i$ where K is a separable quadratic field extension of k (Theorem 9). In Section 4, some general results on extending structurable algebras from affine to projective space are obtained.

We use the results and terminology from [Ach] (see also [Pu1, 2, 3]) and [Pa-Sr-T]. The approach in [Pa-Sr-T] is mostly functorial and formulated for base rings R which are domains with $1/6 \in R$, the one in [Ach] works instead for arbitrary base rings. Both were originally developed to generalize the first and second Tits construction for Jordan algebras over rings.

For the standard terminology on Jordan algebras, the reader is referred to the books by McCrimmon [M], Jacobson [J] and Schafer [Sch].

1. Preliminaries

1.1. **Algebras over R.** For $P \in \operatorname{Spec} R$, let R_P be the local ring of R at P and m_P the maximal ideal of R_P. The corresponding residue class field is denoted by $k(P) = R_P/m_P$. For an R-module F the localization of F at P is denoted by F_P. The *rank* of F is defined to be $\sup\{\operatorname{rank}_{R_P} F_P \mid P \in \operatorname{Spec} R\}$. The term "$R$-algebra" always refers to nonassociative R-algebras which are unital and finitely generated projective of finite constant rank as R-modules.

An anti-automorphism $\sigma : A \to A$ of order 2 is called an *involution* on A. Define $H(A, \sigma) = \{a \in A \mid \sigma(a) = a\}$ and $\mathrm{S}(A, \sigma) = \{a \in A \mid \sigma(a) = -a\}$. Then $A = H(A, \sigma) \oplus \mathrm{S}(A, \sigma)$.

1.2. **Structurable algebras.** An *algebra with involution* is a pair $(A, ^-)$ consisting of an R-algebra A and an involution $^- : A \to A$. A *structurable algebra* is an algebra with involution $(A, ^-)$ satisfying

$$\{x, y, \{z, w, q\}\} - \{z, w, \{x, y, q\}\} = \{\{x, y, z\}, w, q\} - \{z, \{y, x, w\}, q\}$$

for all elements $x, y, z, w, q \in A$, where

$$\{x, y, z\} = (x\bar{y})z + (z\bar{y})x - (z\bar{x})y$$

[A1, (3) and Cor. 5]. If B is an R-submodule if A which is closed under multiplication, we call B a subalgebra of A. If, additionally, $\overline{B} = B$ we call $(B, ^-)$ a subalgebra of $(A, ^-)$.

An *isotopy* from $(A, ^-) \to (A', ^{-\prime})$ is an R-linear bijective map $a : A \to A'$ such that

$$a\{x, y, z\} = \{ax, \hat{a}y, az\}$$

for all $x, y, z \in A$ and some R-linear map $\hat{a} : A \to A'$. Two structurable algebras (A^-) and $(A, ^-)$ are *isotopic* if there exists an isotopy from A to A'. This is equivalent to $(A', ^{-\prime}) \cong (A, ^-)^{\langle u \rangle}$ for some invertible $u \in A$. Every isomorphism between structurable algebras is an isotopy.

In the following, we will only deal with structurable algebras $(A, ^-)$ over R whose residue class algebras $A(P) = A_P \otimes_{R_P} k(P)$ are central simple structurable algebras of skew-dimension 1.

1.3. Let W and W' be two finitely generated projective R-modules of constant rank with cubic forms $N : W \to R$ and $N' : W' \to R$, paired by a nondegenerate bilinear form $T : W \times W' \to R$. That is, T induces R-module isomorphisms

$$T : W \to \mathrm{Hom}_R(W', R), \quad x \mapsto T(x, \cdot)$$

and

$$T : W' \to \mathrm{Hom}_R(W, R), \quad y' \mapsto T(\cdot, y').$$

We say that the triple (T, N, N') *is defined on* (W, W'). Let $N(x, y, z)$ denote the trilinear form associated with N and $N'(x', y', z')$ the trilinear form associated with N'. Let $x \in W$, $x' \in W'$ and define quadratic maps $\sharp : W \to W'$ and $\sharp' : W' \to W$ via

$$D_y N(x) = T(y, x^\sharp) \text{ and } D_{y'} N'(x') = T(x'^{\sharp'}, y')$$

for all elements $x, y \in W$, $x', y' \in W'$, i.e.,

$$3N(x, x, y) = T(y, x^\sharp) \text{ and } 3N'(x', x', y') = T(x'^{\sharp'}, y')$$

for all elements $x, y \in W$, $x', y' \in W'$. The triple (T, N, N') *satisfies the adjoint identities* if

$$(x^\sharp)^{\sharp'} = N(x)x \text{ and } (x'^{\sharp'})^\sharp = N'(x')x'.$$

If $N = 0$ and $N' = 0$ these identities are trivially satisfied. If $N \neq 0$ or $N' \neq 0$ then both N and N' are nonzero and (T, N, N') is called *non-trivial.*
Let (T, N, N') be a triple defined on (W, W'). Define symmetric bilinear maps $\times : W \times W \to W'$ and $\times' : W' \times W' \to W$ via

$$x \times y = (x + y)^\sharp - x^\sharp - y^\sharp, \quad x' \times' y' = (x' + y')^{\sharp'} - x'^{\sharp'} - y'^{\sharp'}.$$

Then

$$x^\sharp = \frac{1}{2} x \times x, \quad x'^{\sharp'} = \frac{1}{2} x' \times' x',$$

$$N(x, y, z) = T(x, y \times z), \quad N'(x', y', z') = T(x' \times' y', z').$$

If the triple (T, N, N') satisfies the adjoint identities then the matrix algebra

$$A = M(T, N, N') = \begin{bmatrix} R & W \\ W' & R \end{bmatrix},$$

$$\begin{bmatrix} a & x \\ x' & b \end{bmatrix} \begin{bmatrix} c & y \\ y' & d \end{bmatrix} = \begin{bmatrix} ac + T(x, y') & ay + dx + x' \times' y' \\ cx' + by' + x \times y & bd + T(y, x') \end{bmatrix}$$

with involution

$$\overline{\begin{bmatrix} a & x \\ x' & b \end{bmatrix}} = \begin{bmatrix} b & x \\ x' & a \end{bmatrix}$$

is a structurable algebra ([A-F1, p. 194], [Pu3, Theorem 1]). We have $S(A, ^-) = s_0 R$ with

$$s_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

invertible and $s_0^2 = 1 \in R^\times$ and the residue class algebras $A(P) = A_P \otimes k(P)$ are central simple structurable algebras of skew-dimension 1 over $k(P)$ ([A-F1], [Pu3]). Let

$$u = \begin{bmatrix} a & x \\ x' & b \end{bmatrix} \text{ and } v = \begin{bmatrix} c & y \\ y' & d \end{bmatrix}$$

with $a, b, c, d \in R$ and $x, y \in W$, $x', y' \in W'$. The *(conjugate) norm* $\nu : M(T, N, N') \to R$ is given by

$$\nu(u) = 4aN(x) + 4bN'(x') - 4T(x'^{\sharp'}, x^\sharp) + (ab - T(x, x'))^2$$

and is isotropic since $\nu(u) = 0$ for

$$u = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The *trace* $\chi : M(T, N, N') \times M(T, N, N') \to R$ is defined by

$$\chi(u, v) = 2(ad + bc + T(x, y') + T(y, x')).$$

Note that $\chi(u, u) = 0$.

1.4. Let B be an Azumaya algebra over R of degree 3, $B^+ = (N_B, \sharp_B, 1)$ with $(N_B, \sharp_B, 1)$ a cubic form with adjoint and base point (cf. for instance [Pu3, 1.4]). Let $\mathrm{Pic}_l B$ denote the set of isomorphism classes of locally free left B-modules of rank 1. Let $P \in \mathrm{Pic}_l B$ such that $N_B(P) \cong R$ and let $N : P \to R$ be a norm on P. Let $N^\vee : P^\vee \to R$ be the uniquely determined norm and $\sharp : P \to P^\vee$, $\check{\sharp} : P^\vee \to P$ be the uniquely determined adjoints satisfying equations

(1) $\langle w, w^\sharp \rangle = N(w)1$;
(2) $\langle \check{w}^{\check{\sharp}}, \check{w} \rangle = N^\vee(\check{w})1$;
(3) $w^{\sharp\check{\sharp}} = N(w)w$

for all $w \in P$, $\check{w} \in P^\vee$ (these are (7), (8), (9) in [Pu3]). Let $\times : P^\vee \times P^\vee \to P$ denote the bilinear map associated to the quadratic map $\sharp$ and $\check{\times} : P^\vee \times P^\vee \to P$ the bilinear map associated to the quadratic map $\check{\sharp}$ (cf. for instance [Pu3, 3.2]). Define $T : P \times P^\vee \to R$ via

$$T(w, \check{w}) = T_B(\langle w, \check{w} \rangle).$$

For any $\mu \in R^\times$, the triple $(\mu T, \mu N, \mu^2 N^\vee)$ satisfies the adjoint identities [Pu3, Theorem 6], hence

$$M = M(\mu T, \mu N, \mu^2 N^\vee) = \begin{bmatrix} R & P \\ P^\vee & R \end{bmatrix}$$

is a structurable algebra over R with automorphism group isomorphic to the semi-direct product of $\mathbb{Z}/2$ and the group of bijective norm isometries of P. [Pu3, Corollary 7, Theorem 18].

The group $\mathrm{Inv}(M)$ defined in Section 4 is an absolutely almost simple linear algebraic group which is connected except in the case that M has rank 9. In that case its connected component is a subgroup of index 2 in $\mathrm{Inv}(M)$ [Krutelevich, p. 941 ff.].

1.5. Let R' be a ring and B be a unital separable associative algebra over R'. Let $*: R' \to R'$ be an involution on R' and $*_B$ an involution on B such that $*_B|_{R'} = *$. Let $(N_B, \sharp_B, 1)$ be a cubic form with adjoint and base point on B such that $B^+ = J(N_B, \sharp_B, 1)$ with 1 the unit element in B and

$$xyx = T_B(x,y)x - x^{\sharp_B} \times_B y,$$
$$N_B(xy) = N_B(x)N_B(y)$$

and

$$N(x^{*_B}) = N(x)^{*_B}$$

for all $x, y \in B$ (these are identities (1), (2), (3) in [Pu3]) and let $(H(B, *_B), H(R', *_B))$ be a B-ample pair. Define $R = H(R', *_B)$. Let $P \in \mathrm{Pic}_l\, B$ such that $N_B(P) \cong R'$ and such that there is a nondegenerate *hermitian* form $h : P \times P \longrightarrow B$ satisfying

$$h(w,w) \in H(B, *_B) \text{ and } N_B(h(w,w)) = N(w)N(w)^{*_B}$$

for $w \in P$. Denote the $H(B, *_B)$-admissible involution $j_h : P \to \overline{P^\vee}$ on P induced by h by $*$. Let $N : P \to R'$ be a norm on P. Let $N^\vee : P^\vee \to R'$ be the uniquely determined norm and $\sharp : P \to P^\vee$, $\check{\sharp} : P^\vee \to P$ be the uniquely determined adjoints satisfying equations (1), (2), (3). We can also write

$$\langle u, v^* \rangle = h(u,v), \quad v^* = j_h(v) \text{ and } \check{v}^{\check{*}} = j_h^{-1}(\check{v})$$

for $j_h : P \to \overline{P^\vee}$ induced by h. The R-module $S(B, *_B, P, N, h) = R' \oplus P$ together with the multiplication

$$(a,u)(b,v) = (ab + T_B(\langle u, v^* \rangle), b^{*_B}u + av + (u \times v)^{\check{*}})$$

and the involution

$$\overline{(a,u)} = (\bar{a}, u)$$

for $a, b \in R'$, $u, v \in P$ is a structurable algebra over R which is a form of the structurable algebra $M(T, N, N^\vee)$ [Pu3, Theorem 20]. We define the (conjugate) *norm* $\nu : S(B, *_B, P, N, h) \to R$ of $S(B, *_B, P, N, h)$ via

$$\nu((\lambda, w)) = N_B(\lambda\lambda^* - h(w,w)).$$

If R' is a field this definition coincides with [A-F2, Theorem 6.1]. ν is a quartic form. Even if B is a division algebra and R' is a field, the norm is isotropic: then $\nu((\lambda, w)) = 0$ if and only if (λ, w) is an admissible scalar, i.e. $\mu \in R'^\times$, $w \in H(B, *_B)^\times$ and $N_B(w) = \mu\mu^*$.

If R' is a quadratic étale ring extension of the ring R then $R' = \mathrm{Cay}(R, P, N)$ with $L \in \mathrm{Pic}\, R$ of order 2, since $2 \in R^\times$. For $A = S(B, *_B, P, N, h)$ this means $S(A, {}^-) = \{(r, 0) \,|\, r \in S(R', *)\} = L$. If R is a domain and $R' = \mathrm{Cay}(R, c) = R(\sqrt{c})$ then $S(A, {}^-) = (\sqrt{c}, 0)R$ and $s_0 = (\sqrt{c}, 0)$ satisfies $s_0^2 = (c, 0) = c1_A$ with $c \in R^\times$. This means we can define the (conjugate) norm $\nu : A \to R$ also by

$$\nu(x) = \frac{1}{12c}\chi(s_0 x, \{x, s_0 x, x\}),$$

and also a *trace* $\chi : A \times A \to R$ on A by

$$\chi(x,y) = \frac{2}{c}\psi(s_0 x, y)s_0 = \frac{2}{c}(V^\delta_{y,x} s_0)s_0,$$

analogously as in [A-F1, 2], where $\psi(x,y)=x\bar{y}-y\bar{x}$ [A-F2, 5.4]. χ is a nondegenerate symmetric bilinear form independent of the choice of s_0 and $\chi(1,1)=4$. (Nondegeneracy follows from [A-F1, Proposition 2.5] applied to the residue class forms.)

2. Non-trivial structurable algebras over the affine plane which locally are matrix algebras

2.1. We mostly use the results and notation of [Pa-Sr-T, Section 4]. Occasionally, we also use the notation of [Pu1]: In the notation of [Pa-Sr-T], the map $\times$ in [Pu1] or 1.4 is denoted by ϕ and the map $\check{\times}$ in 1.4 by ϕ_*. There is the obvious notion of a structurable algebra over a locally ringed space, see [Pu3]. We identify structurable algebras over $k[X,Y]$ and over $\mathbb{A}^2_k$ using the canonical equivalence described in [Pu3, 6.2]. Let $X=\mathbb{P}^2_k$.

Remark 1. Let D be a central simple algebra over k of degree 3. Once we have picked a locally free left $D[X,Y]$-module of rank 1 with $N_{D[X,Y]}(P)\cong k[X,Y]$, the choice of a norm $N:P\to k[X,Y]$ automatically determines $N^\vee$ and the adjoints $\sharp$ and $\check{\sharp}$, see [Pu3, 3.2]. This fact is expressed in [Pa-Sr-T] by explicitly choosing a trivialization $\widetilde{\mu}:N_{D[X,Y]}(P)\to k[X,Y]$ which in turn determines uniquely the choice of N, hence of $N^\vee$, $\sharp$ and $\check{\sharp}$. Recall that the norm N is uniquely determined up to a scalar $\mu\in k^\times$. For any $\mu\in k^\times$, the adjoint belonging to μN is $\mu\,\sharp$ and $(\mu N)^\vee=\mu^2N^\vee$, $(\mu\,\sharp)^\vee=\mu^2\,\sharp$.

2.2. Let D be a central division algebra over k of degree 3. Let De be a free module of rank 1 over D with e as a basis element such that $N_D(De)\cong k$ and let $\mu_0:N_D(De)\to k$ be such an isomorphism. Let $\{g_i\}$ be an infinite family of mutually coprime polynomials in $k[X]$. Then there exist non-free projective left modules P_i of rank 1 over $D[X,Y]$ and polynomials $f_i\in k[X]$ with $(f_i,f_j)=1$ for $i\neq j$, $(f_i,g_j)=1$ for all i,j, such that $P_i\otimes k[X]_{f_i}[Y]$ is free for each i. Further, there exists $\widetilde{\mu_i}:N_{D[X,Y]}(P_i)\to k[X,Y]$ such that

$$(P_i,\widetilde{\mu_i})\text{ modulo }Y\text{ is }(De,\mu_0)\otimes_k k[X]$$

[Pa-Sr-T, 4.1]. The P_i are mutually non-isomorphic $D[X,Y]$-modules [Pa-Sr-T, 4.2].

2.3. Let P be a non-free projective $D[X,Y]$-module such that $N_{D[X,Y]}(P)\cong k[X,Y]$, the isomorphism given by the trivialization $\widetilde{\mu}:N_{D[X,Y]}(P)\to k[X,Y]$ of the reduced norm. Then the pair $(P,\widetilde{\mu})$ is a principal $\mathrm{SL}_1(D)$-bundle over $\mathbb{A}^2_k$ which admits an extension $(\widetilde{P},\widetilde{\mu})$ to $\mathbb{P}^2_k$; the bundle $\widetilde{P}$ is simply an extension of the $D[X,Y]$-module P [Pa-Sr-T, p. 31] (by abuse of notation, we denote both $\widetilde{\mu}$ and its extension by the same name). Let $N:P\to k[X,Y]$ be the norm on P determined by the choice of the trivialization $\widetilde{\mu}$. The choice of $\widetilde{\mu}$ also determines the maps $\times:P\times P\to P^\vee$, $\check{\times}:P^\vee\times P^\vee\to P$ and $N^\vee$, hence also $\sharp$ and $\check{\sharp}$, see [Pa-Sr-T, p. 16]. Take $T(u,\check{v})=T_{D[X,Y]}(\langle u,\check{v}\rangle)$. The adjoints satisfy the adjoint identities [Pa-Sr-T, 1.2].

Analogously, $\widetilde{\mu}$ determines extensions $\widetilde{N}:\widetilde{P}\to\mathcal{O}_X$ of N, $\widetilde{N^\vee}:\widetilde{P}^\vee\to\mathcal{O}_X$ of $N^\vee$, $\widetilde{\sharp}$ and $\widetilde{\check{\sharp}}$ of $\sharp$ and $\check{\sharp}$, which satisfy the adjoint identities. Let $\widetilde{T}(u,\check{v})=T_{D\otimes\mathcal{O}_X}(\langle u,\check{v}\rangle)$.

Proposition 2. *The matrix algebra*

$$M(T,N,N^\vee)=\begin{bmatrix} k[X,Y] & P\\ P^\vee & k[X,Y]\end{bmatrix}$$

over $k[X,Y]$ admits a unique extension to a matrix algebra

$$M(\widetilde{T},\widetilde{N},\widetilde{N}^\vee)=\begin{bmatrix}\mathcal{O}_X & \widetilde{P}\\ \widetilde{P}^\vee & \mathcal{O}_X\end{bmatrix}$$

over $\mathbb{P}^2_k$. The vector bundles $\widetilde{P}$ and $\widetilde{P}^\vee$ are indecomposable and $\widetilde{P}$ and $\widetilde{P}^\vee$ are not isomorphic as vector bundles on X.

Proof. There is a unique extension $\widetilde{P}$ over $X=\mathbb{P}^2_k$ of P of norm one which is a locally free right $D\otimes\mathcal{O}_X$-module: by [Pa-Sr-T, p. 29], P extends to a vector bundle $\widetilde{P}$ which is unique up to a line bundle $\mathcal{L}$. Since we require $\widetilde{P}$ to be of norm one this implies $\mathcal{L}^3\cong\mathcal{O}_X$, hence $\mathcal{L}=\mathcal{O}_X$ and the extension is unique. Let $N:P\to k[X,Y]$ be the norm on P determined by the choice of the trivialization $\widetilde{\mu}$. Two extensions $\widetilde{N}:\widetilde{P}\to\mathcal{O}_X$ and, say $\widetilde{N'}:\widetilde{P}\to\mathcal{O}_X$ of N, can only differ by a scalar $\lambda\in k^\times$. Being its extension, the algebra $M(\widetilde{T},\widetilde{N},\widetilde{N}^\vee)$ restricts to the structurable matrix algebra

$$M(T,N,N^\vee)=\begin{bmatrix}k[X,Y] & P\\ P^\vee & k[X,Y]\end{bmatrix}$$

over $\mathbb{A}^2_k$. Therefore $\widetilde{N}|_{\mathbb{A}^2_k}=N=\widetilde{N'}|_{\mathbb{A}^2_k}$ implies that $\lambda=1$. Thus the maps $\widetilde{N}:\widetilde{P}\to\mathcal{O}_X$, $\widetilde{N^\vee}:\widetilde{P}^\vee\to\mathcal{O}_X$, $\widetilde{\sharp}$ and $\widetilde{\check{\sharp}}$ which are the extensions of the maps N, $N^\vee$, $\sharp$ and $\check{\sharp}$ from $\mathbb{A}^2_k$ to $\mathbb{P}^2_k$ determined by the trivializations $\widetilde{\mu}$ and μ are uniqley determined as well.

The proof of the second statement follows from [Pa-Sr-T, 3.2]. □

More precisely, by [Pa-Sr-T, Remark] and [AEJ1], $\widetilde{P}\cong tr_{l/k}(\mathcal{P}_0)$ for some cubic field extension l/k and a suitable vector bundle $\mathcal{P}_0$ over $\mathbb{P}^2_l$ which is absolutely indecomposable and of rank 3.

2.4. Let J be an Albert algebra over k which is a first Tits construction and a division algebra. Choose two cyclic division algebras D_1,D_2 of degree 3 over k such that the Jordan algebras D_1^+ and D_2^+ are subalgebras of J with $D_1^+\cap D_2^+=k$. By [Pa-Sr-T, 4.3], these can be even chosen such that $D_2^+=\Phi(D_1^+)$ for a suitable automorphism Φ of J, i.e. we may and will assume that additionally we have $D_1^+\cong D_2^+$. Then $J=J(D_1,e_1,\mu_1)=J(D_2,e_2,\mu_2)$ for some $e_i\in J$ and isomorphisms $\mu_i:N(D_ie_i)\to k$. Again, the choice of μ_i determines a norm $N_i:D_i\to k$, (a scalar multiple of N_{D_i}) and an adjoint $\sharp_i:D_i\to D_i$ (a scalar multiple of $\sharp_{D_i}$), so with $T_i(a,b)=T_{D_i}(ab)$ we obtain the structurable algebra

$$M=M(T_1,N_1,N_1)\cong M(T_2,N_2,N_2)$$

over k. By 2.2, for every $i\geq 1$ there exists a pair $(P_i^1,\widetilde{\mu_i^1})$, where P_i^1 is a non-free projective $D_1[X,Y]$-module of rank 1 and $\widetilde{\mu_i^1}$ a trivialization of its reduced norm and a polynomial $f_i\in k[X]$ such that:

(4) The polynomials f_i and f_j are coprime for $i\neq j$ and $(P_i^1)_{f_i}$ is free.
(5) The reduction of $(P_i^1,\widetilde{\mu_i^1})$ modulo Y is $(D_1e_1,\mu_1)\otimes k[X]$.

Similarly, for every $i\geq 1$, there is a pair $(P_i^2,\widetilde{\mu_i^2})$, where P_i^2 is a non-free projective $D_2[X,Y]$-module of rank 1 and $\widetilde{\mu_i^2}$ a trivialization of its reduced norm and a polynomial $g_i\in k[X]$ such that:

(6) The polynomials g_i and g_j are coprime for $i \neq j$, the polynomials f_i and g_j are coprime for all i, j, and $(P_i^2)_{g_i}$ is free.
(7) The reduction of $(P_i^2, \widetilde{\mu_i^2})$ modulo Y is $(D_2e_2, \mu_2) \otimes k[X]$.

To each pair $(P_i^j, \widetilde{\mu_i^j})$, $j = 1, 2$, let $N_i^j : P_i^j \to k[X, Y]$ be the norm on P_i^j induced by $\widetilde{\mu_i^j}$, $T_i^j : P_i^j \times (P_i^j)^\vee \to k[X, Y], T_i^j(u, \check{v}) = T_{D_j}(\langle u, \check{v}\rangle)$ the usual trace and $\sharp_i^j$ the induced adjoint.

Define matrix algebras

$$M_i^1 = M(T_i^1, N_i^1, {N_i^1}^\vee) = \begin{bmatrix} k[X,Y] & P_i^1 \\ (P_i^1)^\vee & k[X,Y] \end{bmatrix}$$

and

$$M_i^2 = M(T_i^2, N_i^2, {N_i^2}^\vee) = \begin{bmatrix} k[X,Y] & P_i^2 \\ (P_i^2)^\vee & k[X,Y] \end{bmatrix}$$

of rank 20. Then $\{M_i^j \mid j = 1, 2, i \geq 1\}$ is a family of structurable algebras over $k[X, Y]$ such that $M_i^j = M \otimes k[X]$ modulo Y and

$$M_i^1 \otimes k[X]_{f_i}[Y] \cong M \otimes k[X]_{f_i}[Y], \quad M_i^2 \otimes k[X]_{g_i}[Y] \cong M \otimes k[X]_{g_i}[Y],$$

with $(f_i, f_j) = 1 = (g_i, g_j)$ for $i \neq j$, $(f_i, g_j) = 1$ for all i, j. As in [Pa-S-T, 4.5] we can then conclude:

Proposition 3. *The matrix algebras M_i^1, respectively M_i^2, over $k[X, Y]$ are mutually non-isomorphic.*

Proof. Suppose there are $i \neq j$ such that $M_i^1 \cong M_j^1$. Since M_i^1 and M_j^1 are extended after inverting f_i and f_j, respectively, and since $(f_i, f_j) = 1$, M_i^1 is extended from $M \otimes k[X]$. Let $\tau : X \to k$ be the structure morphism. Since the extension $\widetilde{M_i^1}$ of M_i^1 to $\mathbb{P}_k^2$ is unique, it must be thus isomorphic to $\tau^*(M)$. Therefore, the underlying vector bundles must be isomorphic, i.e.

$$\mathcal{O}_X^2 \oplus \widetilde{P_i^1} \oplus \widetilde{P_i^1}^\vee \cong \mathcal{O}_X^{20}.$$

This is a contradiction, since $\widetilde{P_i^1}$ is an indecomposable vector bundle by [Pa-Sr-T, 3.2]. $\square$

2.5. Let

$$\pi_i^1 : (P_i^1, \widetilde{\mu_i^1}) \otimes k[X]_{f_i}[Y] \to (D_1e_1, \mu_1) \otimes k[X]_{f_i}[Y]$$

and

$$\pi_i^2 : (P_i^2, \widetilde{\mu_i^2}) \otimes k[X]_{g_i}[Y] \to (D_1e_2, \mu_2) \otimes k[X]_{g_i}[Y]$$

be isomorphisms such that $\overline{\pi_i^j}$ =id, $j = 1, 2$ (we may assume this by [Pa-S-T, 6.1]). These canonically induce isomorphisms

$$M(\pi_i^1) : M_i^1 \otimes k[X]_{f_i}[Y] \to M \otimes k[X]_{f_i}[Y]$$

and

$$M(\pi_i^2) : M_i^2 \otimes k[X]_{g_i}[Y] \to M \otimes k[X]_{g_i}[Y]$$

with $\overline{M(\pi_i^j)}$ =id, $j = 1, 2$. Let M_i be the structurable algebra obtained by patching M_i^1 on $k[X]_{g_i}[Y]$ and M_i^2 on $k[X]_{f_i}[Y]$ over $k[X]_{f_ig_i}[Y]$ by $\phi_i = M(\pi_i^2)^{-1}M(\pi_i^1)$.

We obtain an involution $^- : M_i \to M_i$ by analogously patching the involutions of M_i^1 on $k[X]_{g_i}[Y]$ and of M_i^2 on $k[X]_{f_i}[Y]$ over $k[X]_{f_ig_i}[Y]$ by $\phi_i = M(\pi_i^2)^{-1}M(\pi_i^1)$.

Since $\overline{M_i^j} = M$ modulo Y and $\overline{M(\pi_i^j)}$ =id, we get $\overline{\phi_i}$ =id and $\overline{M_i} = M \otimes k[X]$ modulo Y. By construction,

$$M_i \otimes k[X]_{f_i g_i}[Y] \cong M \otimes k[X]_{f_i g_i}[Y]$$

and the polynomials $r_i := f_i g_i$ are mutually coprime. The algebras M_i are mutually non-isomorphic by the same argument as given in [Pa-Sr-T, p. 33] using Proposition 4 and thus we can conclude:

Theorem 4. *The structurable algebras M_i on $\mathbb{A}_k^2$ have the following properties:*
(i) $\overline{M_i} = M \otimes k[X]$ modulo Y.
(ii) There are mutually coprime polynomials $r_i \in k[X]$ such that $M_i \otimes k[X]_{r_i}[Y] \cong M \otimes k[X]_{r_i}[Y]$.
(iii) The algebras M_i are non-extended and mutually non-isomorphic.

Proof. By construction, we have

$$M_i \otimes k[X]_{f_i g_i}[Y] \cong M \otimes k[X]_{f_i g_i}[Y]$$

and the polynomials $r_i = f_i g_i$ are mutually coprime. To show that the algebras M_i are mutually non-isomorphic, suppose that there are $i \neq j$ such that $M_i \cong M_j$. Then both $(M_i)_{r_i}$ and $(M_i)_{r_j}$ are extended from M. Since $(r_i, r_j) = 1$, $M_i \cong M \otimes k[X, Y]$. Restrict M_i to $k[X]_{g_i}[Y]$. This yields that $M_i^1 \otimes k[X]_{g_i}[Y]$ and $M_i^1 \otimes k[X]_{f_i}[Y]$ are extended. Since $(f_i, g_i) = 1$, M_i^1 is extended from M. This contradicts Proposition 3. $\square$

Note that all the ingredients for the above proofs have been provided in [Pa-Sr-T, Section 4].

It is is not clear that these structurable algebras are again matrix algebras. We are not able to say if the corresponding principal G-bundle P_{M_i} admits reduction of the structure group to a proper reductive subgroup of G or not. They are subalgebras of a 56-dimensional matrix algebra:

2.6. Let J_i^1 (resp. J_i^2) be the infinitely many mutually non-isomorphic Albert algebras over $k[X, Y]$ used in [Pa-Sr-T, Proposition 4.5]. These give rise to infinitely many matrix algebras

$$M(J_i^1) = \begin{bmatrix} k[X,Y] & J_i^1 \\ J_i^1 & k[X,Y] \end{bmatrix},$$

resp.

$$M(J_i^2) = \begin{bmatrix} k[X,Y] & J_i^2 \\ J_i^2 & k[X,Y] \end{bmatrix}$$

over $k[X, Y]$ of rank 56 which contain the mutually non-isomorphic subalgebras

$$M_i^1 = M(T_i^1, N_i^1, {N_i^1}^\vee) = \begin{bmatrix} k[X,Y] & P_i^1 \\ (P_i^1)^\vee & k[X,Y] \end{bmatrix},$$

resp.

$$M_i^2 = M(T_i^2, N_i^2, {N_i^2}^\vee) = \begin{bmatrix} k[X,Y] & P_i^2 \\ (P_i^2)^\vee & k[X,Y] \end{bmatrix}$$

of rank 20 which are stable under the involution $^-$. They also contain the subalgebra

$$M(D_1) = M(T_{D_1}, N_{D_1}, N_{D_1}) = \begin{bmatrix} k[X,Y] & D_1 \\ D_1 & k[X,Y] \end{bmatrix},$$

resp.

$$M(D_2) = M(T_{D_2}, N_{D_2}, N_{D_2}) = \begin{bmatrix} k[X,Y] & D_2 \\ D_2 & k[X,Y] \end{bmatrix}$$

of rank 20 which is again stable under the involution $^-$ [Pu3, Theorem 10].

Let J_i be the Jordan algebra we get if we patch J_i^1 on $k[X]_{g_i}[Y]$ and J_i^2 on $k[X]_{f_i}[Y]$ over $k[X]_{f_i g_i}[Y]$ using the isomorphisms $J(\pi_i^1)$, resp. $J(\pi_i^2)$, which are canonically induced by the π_i^j, $j = 1, 2$, as described in [Pa-Sr-T, p. 32]. The algebras J_i are non-extended, mutually non-isomorphic and no longer a first Tits construction starting with some Azumaya algebra of degree 3 [Pa-Sr-T, 6.3]. The matrix algebra

$$M(J_i) = \begin{bmatrix} k[X,Y] & J_i \\ J_i & k[X,Y] \end{bmatrix}.$$

can then be also viewed as obtained from the matrix algebras

$$M(J_i^1) = \begin{bmatrix} k[X,Y] & J_i^1 \\ J_i^1 & k[X,Y] \end{bmatrix} \text{ and } M(J_i^2) = \begin{bmatrix} k[X,Y] & J_i^2 \\ J_i^2 & k[X,Y] \end{bmatrix}$$

by patching them using the obvious induced isomorphisms. Call them $S(\pi_i^j)$, $j = 1, 2$.

By construction, M_i is then clearly a subalgebra of the matrix algebra $M(J_i)$ (the isomorphisms used to patch it are restrictions of the $S(\pi_i^j)$) and there are mutually coprime polynomials $r_i \in k[X]$ with $M(J_i) \otimes k[X]_{r_i}[Y] \cong M(J) \otimes k[X]_{r_i}[Y]$ and $M_i \otimes k[X]_{r_i}[Y] \cong M \otimes k[X]_{r_i}[Y]$, where $M \cong M(D_1^+) \cong M(D_2^+) \subset M(J)$.

Remark 5. We observe independently of this that be the infinitely many mutually non-isomorphic reduced Albert algebras A_i over $k[X,Y]$ constructed in [Pa-S-T, Step I and 6.2] also give rise to matrix algebras

$$H_i = \begin{bmatrix} k[X,Y] & A_i \\ A_i & k[X,Y] \end{bmatrix}$$

over $k[X,Y]$ of rank 56 which are mutually non-isomorphic, which is proved analogously to [Pa-S-T, 6.2].

3. Structurable algebras over $\mathbb{A}_k^2$ which are forms of matrix algebras

Remark 6. Let T be a quadratic étale algebra over $k[X,Y]$ with anisotropic norm. As in [Pa-S-T, 4.6] one can see that T extends uniquely to a quadratic étale algebra $\mathcal{T} = \mathrm{Cay}(\mathcal{O}_X, \mathcal{L}, N)$ over $X = \mathbb{P}_k^2$. Since $\operatorname{Pic} X = \mathbb{Z}$, $\mathcal{L} \cong \mathcal{O}_X$ and $\mathcal{T}$ is defined over k, thus so is T. We conclude that every quadratic étale algebra over $k[X,Y]$ with anisotropic norm is of the kind $K \otimes_k k[X,Y] \cong K[X,Y]$ with $K = k(\sqrt{c})$ a separable quadratic field extension. As a consequence, every quadratic étale ring extension R' of $k[X,Y]$ satisfies $R' = k(\sqrt{c})[X,Y]$ and every form of a matrix algebra of the type $S(B, *, P, N, h)$, B a central simple algebra over R' has $S(A, ^-) = (\sqrt{c}, 0)R$.

3.1. Let K be a separable quadratic field extension of k. Let D be a central division algebra over K of degree 3 with an involution σ of the second kind over K/k. Let $X = \mathbb{P}^2_k$, $X' = X \otimes_k K = \mathbb{P}^2_K$ and $\mathcal{D} = D \otimes_K \mathcal{O}_{X'}$.

Proposition 7. *Let P be a nonfree projective left $D[X,Y]$-module of rank one. The structurable algebra $S(D,\sigma,P,N) = K[X,Y] \oplus P$ over $k[X,Y]$ admits a unique extension to a structurable algebra $S(\mathcal{D},\sigma,\widetilde{P},N) = \mathcal{O}_{X'} \oplus \widetilde{P}$ over $X = \mathbb{P}^2_k$. The vector bundle $\widetilde{P}$ over X' is indecomposable.*

Proof. There is a unique extension of the quadratic étale algebra $K[X,Y]$ over $k[X,Y]$ to a quadratic étale algebra $\mathcal{O}_{X'} = K \otimes_k \mathcal{O}_X$ over X. There is a unique extension $\widetilde{P}$ over $X' = \mathbb{P}^2_K$ of P of norm one which is a locally free left $\mathcal{D}$-module: by [Pa-Sr-T, p. 29], P extends to a vector bundle $\widetilde{P}$ over X' which is unique up to a line bundle $\mathcal{L} \in \operatorname{Pic} X'$. Since we require $\widetilde{P}$ to be of norm one this implies $\mathcal{L}^3 \cong \mathcal{O}_{X'}$, hence $\mathcal{L} = \mathcal{O}_{X'}$ and the extension is unique. More precisely, by [Pa-Sr-T, Remark] and [AEJ1], $\widetilde{P} \cong tr_{L'/K'}(\mathcal{P}_0)$ for some cubic field extension L'/K' and a suitable vector bundle $\mathcal{P}$ over $\mathbb{P}^2_{L'}$ which is absolutely indecomposable and must have rank 3. In particular, N and h can be extended as well.

The algebra $S(\mathcal{D},\sigma,\widetilde{P},N) = \mathcal{O}_{X'} \oplus \widetilde{P}$ restricts to the structurable algebra $S(D,\sigma,P,N) = K[X,Y] \oplus P$ over $\mathbb{A}^2_k$. The second statement follows from [Pa-Sr-T, 3.2]. □

3.2. Let K be a separable quadratic field extension of k. Let D be a central division algebra over K of degree 3 with an involution σ of the second kind over K/k. Let (u,μ) be an admissible scalar, i.e. $\mu \in K^\times$, $c \in H(B,*_B)^\times$ and $N_B(c) = \mu\mu^*$. By [Pa-Sr-T, p. 33], there exists a projective left $D[X,Y]$-module P of rank 1 together with a nondegenerate hermitian form $h : P \times P \to D[X,Y]$ and a trivialization $\widetilde{\mu} : \operatorname{disc}(h) \to (K[X,Y], \langle 1\rangle)$ such that:

(8) The reduction of $(P,h,\widetilde{\mu})$ modulo Y is isomorphic to $(D,\langle u\rangle,\mu)$, where $\langle u\rangle$ denotes the hermitan form $a \to au\sigma(a)$ and μ is treated as a trivialization of the discriminant of $\langle u\rangle$. Moreover, $(De,u_e,\mu_e) \otimes k[X] = (P,h,\widetilde{\mu})$ modulo Y, where De is the free module of rank one over D with e a basis element, u_e the hermitian form on De given by $u_e(xe,ye) = xu\sigma(y)$ and $\mu_e N_D(e) = \mu$.

(9) There exists $f \in k[X]$, $f(0) \neq 0$, such that $(P,h,\widetilde{\mu}) \otimes k[X]_f[Y] \cong (D,\langle u\rangle,\mu) \otimes k[X]_f[Y]$.

(10) The principal $SU(D,\sigma)$-bundle on $\mathbb{A}^2_k$ associated to $(P,h,\widetilde{\mu})$ admits no reduction of the structure group to any proper connected reductive subgroup of $SU(D,\sigma)$. In particular, $(P,h,\widetilde{\mu})$ is not extended from $(D,\langle u\rangle,\mu)$.

Now let J be an Albert division algebra over k which is a second Tits construction but not a first one. We may write

$$J = J(D^1e_1, u_{e_1}, \mu_{e_1}) = J(D^2e_2, u_{e_2}, \mu_{e_2})$$

where D^1, D^2 are two isomorphic central simple algebras of degree 3 over a quadratic extension F/k with involution σ^1, σ^2 of the second kind and norms N_1 and N_2, such that $H(D^1,\sigma^1) \cap H(D^2,\sigma^2) = k$ [Pa-Sr-T, 5.2].

Define the structurable algebra

$$A = S(D^1, \sigma^1, D^1, N_1, u_{e_1}) \cong S(D^2, \sigma^2, D^2, N_2, u_{e_2}).$$

By [Pa-Sr-T, p. 35], there exist non-trivial hermitian spaces $(P_1^i, h_1^i, \widetilde{\mu_1^i})$ over $(D^1[X,Y], \sigma^1)$ and $(P_2^i, h_2^i, \widetilde{\mu_2^i})$ over $(D^2[X,Y], \sigma^2)$ of rank 1, and $f_i, g_i \in k[X]$ such that:

(11) $(P_1^i, h_1^i, \widetilde{\mu_1^i})$ modulo Y reduces to $(D^1 e_1, u_{e_1}, \mu_{e_1})$, $(P_2^i, h_2^i, \widetilde{\mu_2^i})$ modulo Y reduces to $(D^2 e_2, u_{e_2}, \mu_{e_2})$.

(12) $(P_1^i, h_1^i, \widetilde{\mu_1^i}) \otimes k[X]_{f_i}[Y]$ is isomorphic to $(D^1 e_1, u_{e_1}, \mu_{e_1}) \otimes k[X]_{f_i}[Y]$ and $(P_2^i, h_2^i, \widetilde{\mu_2^i}) \otimes k[X]_{g_i}[Y]$ is isomorphic to $(D^2 e_2, u_{e_2}, \mu_{e_2}) \otimes k[X]_{g_i}[Y]$ with $(f_i, f_j) = 1 = (g_i, g_j)$ for all $i \neq j$ and $(f_i, g_j) = 1$ for all i, j.

(13) The vector bundles (P_1^i, h_1^i) and (P_2^i, h_2^i) are not extended from D^1 and D^2, respectively.

Let $N_j^i : P_j^i \to D_j[X,Y]$ denote the norm on P_j^i determined by the choice of $\widetilde{\mu_j^i}$, $j = 1, 2$. We define two families of structurable algebras

$$A_1^i = S(D^1, \sigma^1, P_1^i, N_1^i, h_1^i) \text{ and } A_2^i = S(D^2, \sigma^2, P_2^i, N_2^i, h_2^i)$$

over $k[X,Y]$ with underlying modules structures

$$A_1^i \cong K[X,Y] \oplus P_1^i \text{ and } A_2^i \cong K[X,Y] \oplus P_2^i.$$

Let

$$\pi_1^i : (P_1^i, h_1^i, \widetilde{\mu_1^i})_{f_i} \to (D^1 e_1, u_{e_1}, \mu_{e_1}) \otimes k[X]_{f_i}[Y]$$

and

$$\pi_2^i : (P_2^i, h_2^i, \widetilde{\mu_2^i})_{g_i} \to (D^2 e_2, u_{e_2}, \mu_{e_2}) \otimes k[X]_{g_i}[Y]$$

be isometries such that $\overline{\pi_j^i}$ =id for $j = 1, 2$. These isometries induce isomorphisms

$$A(\pi_1^i) : A_1^i \otimes k[X]_{f_i}[Y] \to A \otimes k[X]_{f_i}[Y]$$

and

$$A(\pi_1^i) : A_2^i \otimes k[X]_{g_i}[Y] \to A \otimes k[X]_{g_i}[Y]$$

which reduce to the identity map modulo Y.

Proposition 8. *The structurable algebras A_1^i and A_2^i over $k[X,Y]$ have the following properties:*
(i) A_1^i and A_2^i modulo Y reduce to A.
(ii) $A_1^i \otimes k[X]_{f_i}[Y]$ is extended from $A \otimes k[X]_{f_i}[Y]$ and $A_2^i \otimes k[X]_{g_i}[Y]$ is extended from $A \otimes k[X]_{g_i}[Y]$, with $(f_i, f_j) = 1 = (g_i, g_j)$ for $i \neq j$, $(f_i, g_j) = 1$ for all i, j.
(iii) $A_j^i \otimes_k K \cong M(T_j^i, N_j^i, N_j^{i\vee}) = M_j^i$ for $j = 1, 2$, where the matrix algebras M_j^i are the ones constructed in Section 2.4.
(iv) All the A_1^i are mutually non-isomorphic and all the A_2^i are mutually non-isomorphic.

Proof. The properties (i) and (ii) are immediate consequences of the properties of $(P_j^i, h_j^i, \widetilde{\mu_j^i})$. Property (iii) follows from the construction of the algebras. We use the identification from the proof of [Pu3, Theorem 20].
(iv) Since $A_1^i \otimes_k K \cong M(T_j^i, N_j^i, N_j^i)$ is not extended from $M = M(T_{D^1}, N^1, N^1)$ and P_1^i

is not free, it follows that A_1^i is not extended from A by [BCW]. Thus the algebras A_1^i are mutually non-isomorphic. The same argument holds for the A_2^i. □

We now patch the structurable algebras $(A_1^i)_{g_i}$ over $k[X]_{g_i}[Y]$ and $(A_2^i)_{f_i}$ over $k[X]_{f_i}[Y]$ over $k[X]_{f_ig_i}[Y]$ and their involutions using the isomorphism

$$\psi_i : A_1^i \otimes k[X]_{f_ig_i}[Y] \to A_2^i \otimes k[X]_{f_ig_i}[Y],$$

$$\psi_i = A(\pi_2^i)^{-1}A(\pi_1^i).$$

This way we obtain a structurable algebra A^i over $k[X,Y]$.

Theorem 9. *The structurable algebras A^i over $\mathbb{A}_k^2$ have the following properties:*

(1) $\overline{A^i} = A \otimes k[X]$ *modulo* Y.
(2) *There exists* $\pi^i : A^i \otimes k[X]_{s_i}[Y] \to A \otimes k[X]_{s_i}[Y]$ *such that* $\overline{\pi^i} = id$, *for some* $s_i \in k[X]$ *with* $(s_i, s_j) = 1$ *for* $i \neq j$.
(3) *The* A^i *are mutually non-isomorphic.*
(4) $A^i \otimes_k K \cong M_i$ *with the* M_i *as constructed in 2.5.*

Proof. Since A_j^i reduces modulo Y to A and $\overline{\psi_i} = id$, A^i reduces modulo Y to A. By construction,

$$A^i \otimes k[X]]_{f_ig_i}[Y] \cong A \otimes k[X]_{f_ig_i}[Y]$$

and the polynomials $s_i := f_ig_i$ satisfy $(s_i, s_j) = 1$ for $i \neq j$. As in the proof of Theorem 4, it follows that the A^i are mutually non-isomorphic. □

Again, the ingredients for the results were provided by [Pa-Sr-T, Section 5].

4. On extending structurable algebras from the affine to the projective plane

We conclude with some general results about extending structurable algebras from the affine to the projective plane, imitating the techniques used in [Pa-S-T, 4.1, 4.2, 4.3]. Let R be a domain with $1/6 \in R$.

4.1. For a structurable algebra $(A, ^-)$, an *isotopy* from $(A, ^-)$ to $(A, ^-)$ is an element $\alpha \in \mathrm{GL}(A)$ such that

$$\alpha\{x, y, z\} = \{\alpha(x), \widehat{\alpha}(y), \alpha(z)\}$$

for all $x, y, z \in A$ and some $\widehat{\alpha} \in \mathrm{GL}(A)$. $\widehat{\alpha}$ is uniquely determined by α. The *structure group* $\Gamma(A, ^-)$ of $(A, ^-)$ is the subgroup of $\mathrm{GL}(A)$ which consists of all isotopies of $(A, ^-)$ onto itself.

Let $(A, ^-)$ be a structurable algebra of skew-rank one such that $S(A, ^-) = s_0R$ for some $s_0 \in S(A, ^-)$ which is conjugate invertible, which means that left multiplication L_{s_0} with s_0 is invertible. Since $\widehat{s}_0 \in S(A, ^-)$ for its conjugate inverse $\widehat{s}_0$, there is $\beta \in R$, $\beta \neq 0$, such that $\widehat{s}_0 = \beta s_0$ and since $s_0\widehat{s}_0 = -1_A$ we obtain $\beta s_0^2 = -1_A$. Assume that $\beta \in R^\times$ and denote $c = \beta^-1$. Then $s_0^2 = c1_A$ with $c \in R^\times$. Suppose in addition that the invertible elements in $(A, ^-)$ are Zariski dense in A. Then we can define a *(conjugate) norm* $\nu : A \to R$ on A via

$$\nu(x) = \frac{1}{12c}\chi(s_0x, \{x, s_0x, x\}),$$

a *trace* $\chi : A \times A \to R$ on A by

$$\chi(x,y) = \frac{2}{c}\psi(s_0 x, y)s_0 = \frac{2}{c}(V^{\delta}_{y,x}s_0)s_0$$

and a nondegenerate skew-symmetric bilinear form on A

$$\langle x, y\rangle = \psi(x,y)s_0 = \frac{1}{2}\chi(s_0 x, y)$$

analogously as in [A-F1, 2] where $\psi(x,y) = x\bar{y} - y\bar{x}$ [A-F2, 5.4]. (The nondegeneracy of $\langle\, ,\, \rangle$ follows from [A-F1, p. 192] applied to the residue class forms.) ν is a quartic form such that $\nu(1_A) = 1$. χ is a nondegenerate symmetric bilinear form independent of the choice of s_0 and $\chi(1_A, 1_A) = 4$. (Nondegeneracy follows from [A-F1, Proposition 2.5] applied to the residue class forms.) Note that if desired, A can be viewed as a Freudenthal triple system as explained in [A-F1, 2.18] in this setting. An element $x \in A$ is conjugate invertible iff $\nu(x) \neq 0$ [A-F2, 4.4]. So if the norm is anisotropic, every non-zero element of A is conjugate invertible and, if R is a field, $(A, ^-)$ a conjugate division algebra [A-F1, 2.11]. The norm ν is a semi-invariant for the structure group $\Gamma(A, ^-)$ which is proved analogously as in [A-F2, 4.7]. Denote the group of all invertible linear transformations on A that preserve the norm and the skew-symmetric bilinear form $\langle\, ,\, \rangle$ by $\mathrm{Inv}(A)$.

Theorem 10. *Let $(A_1, ^-)$ and $(A_2, ^-)$ be structurable algebras of skew-rank one over R. Suppose that $(A_2, ^-)$ satisfies all of the criteria in 4.1, i.e. carries a conjugate norm, and that the conjugate norm of $(A_2 \otimes R/(p), ^-)$ is anisotropic. Let*

$$\alpha : (A_1 \otimes R[1/p], ^-) \to (A_2 \otimes R[1/p], ^-)$$

be an isotopy of structurable algebras. Then α extends uniquely to an isotopy

$$\tilde{\alpha} : (A_1, ^-) \to (A_2, ^-).$$

In particular, every isomorphism $\alpha : (A_1 \otimes R[1/p], ^-) \to (A_2 \otimes R[1/p], ^-)$ of the structurable algebras $(A_1, ^-)$ and $(A_2, ^-)$ extends uniquely to an isomorphism $\tilde{\alpha} : (A_1, ^-) \to (A_2, ^-)$.

Proof. We show that $\alpha(A_1) = A_2$ which is sufficient: let $x \in A_1$ and assume that $\alpha(x) \notin A_2$. Let n be the least integer such that $y = p^n\alpha(x) \in A_2$ and $p^{n-1}\alpha(x) \notin A_2$. Then $n \geq 1$. ν is a semi-invariant for the structure group of $(A, ^-)$, i.e. there is $0 \neq r \in R$ such that $\nu(\alpha(x)) = r\nu(x)$ for all $x \in A_1$. Thus we obtain $\nu(y) = rp^{4n}\nu(x)$. Hence $\nu(y) = 0$ modulo p and $y \neq 0$ modulo p. This contradicts the assumption that the norm $\nu \otimes R/(p)$ of $(A_2 \otimes R/(p), ^-)$ is anisotropic. □

4.2. There is the obvious notion of a structurable algebra over a locally ringed space, cf. [Pu3, Section 6]. Let $(A, ^-)$ be a structurable algebra of skew-rank one over $X = \mathbb{P}^n_k$ such that $S(A, ^-) = s_0\mathcal{O}_X$ for some $s_0 \in H^0(X, S(A, ^-)) = k$ which is conjugate invertible, which means that left multiplication L_{s_0} with s_0 is invertible. Since $\widehat{s}_0 \in H^0(X, S(A, ^-))$ for its conjugate inverse $\widehat{s}_0$, there is $c \in k^\times$, such that $\widehat{s}_0 = -c^{-1}s_0$ and since $s_0\widehat{s}_0 = -1_A$ we obtain $s_0^2 = c1_A$. Suppose in addition that the invertible elements in $H^0(U, (A, ^-))$ are Zariski dense in $H^0(U, A)$ for every open subset $U \subset X$. Then we can define a *(conjugate) norm* $\nu : A \to \mathcal{O}_X$ via

$$\nu(x) = \frac{1}{12c}\chi(s_0 x, \{x, s_0 x, x\}),$$

a *trace* $\chi : A \times A \to \mathcal{O}_X$ on A by

$$\chi(x,y) = \frac{2}{c}\psi(s_0 x, y)s_0 = \frac{2}{c}(V^{\delta}_{y,x}s_0)s_0$$

and a nondegenerate skew-symmetric bilinear form $\nu : A \times A \to \mathcal{O}_X$

$$\langle x,y\rangle = \psi(x,y)s_0 = \frac{1}{2}\chi(s_0 x, y)$$

analogously as in 4.1, $\psi(x,y) = x\bar{y} - y\bar{x}$. ν is a quartic form such that $\nu(1_A) = 1$. χ is a nondegenerate symmetric bilinear form independent of the choice of s_0 and $\chi(1_A, 1_A) = 4$.

Theorem 10 now implies:

Corollary 11. *Let* $(\mathcal{A}_1, {}^{-}{}_1)$, $(\mathcal{A}_2, {}^{-}{}_2)$ *be two structurable algebras of skew-rank one over* $\mathbb{P}^n_k$ *which satisfy the assumptions of 4.2. Suppose that the restrictions* $(\mathcal{A}_1)_\xi$ *and* $(\mathcal{A}_2)_\xi$ *to the generic point* ξ *have anisotropic norms. Then every isotopy* $\alpha : \mathcal{A}_1 \to \mathcal{A}_2$ *over* $\mathbb{A}^n_k$ *extends uniquely to an isotopy* $\widetilde{\alpha} : \mathcal{A}_1 \to \mathcal{A}_2$ *over* $\mathbb{P}^n_k$. *In particular, every isomorphism* $\alpha : \mathcal{A}_1 \to \mathcal{A}_2$ *over* $\mathbb{A}^n_k$ *extends uniquely to an isomorphism* $\widetilde{\alpha} : \mathcal{A}_1 \to \mathcal{A}_2$ *over* $\mathbb{P}^n_k$.

The proof is verbatim the proof of [Pa-S-T, 4.3], substituting 'isotopy' respectively 'isomorphism' for 'isometry' throughout.

From Corollary 12 and [Pa-S-T, 4.5] we obtain:

Corollary 12. *Let* k *have characteristic 0. Let* $(\mathcal{A}, {}^{-})$ *be a structurable algebra of skew-rank one over* $\mathbb{A}^2_k$ *satisfying the conditions of 4.1, such that its restriction* $\mathcal{A}_\xi$ *to the generic point* ξ *has an anisotropic norm. Then* $(\mathcal{A}, {}^{-})$ *extends uniquely to an algebra* $(\mathcal{A}, {}^{-})$ *over* $\mathbb{P}^2_k$.
If $H = \mathrm{Inv}(A)$ *is a connected reductive algebraic group defined over* k *then every* H*-bundle over* $\mathbb{A}^2_k$ *extends to* $\mathbb{P}^2_k$ *as an* H*-bundle.*
If the structurable algebra bundle has rank 56 and admits a reduction of the structure group to a proper connected reductive subgroup of E_7*, its corresponding extension to* $\mathbb{P}^2_k$ *has the same property.*

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