An elemental characterization of strong primeness in Lie algebras

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ABSTRACT: In this paper we prove that a Lie algebra L is strongly prime if and only if $[x, [y, L]] \neq 0$ for every nonzero elements $x, y \in L$. As a consequence, we give an elementary proof, without the classification theorem of strongly prime Jordan algebras, that a linear Jordan algebra or Jordan pair T is strongly prime if and only if $\{x, T, y\} \neq 0$ for every $x, y \in T$. Moreover, we prove that the Jordan algebras at nonzero Jordan elements of strongly prime Lie algebras are strongly prime.

Introduction

It is well know that an associative algebra R is prime if and only if $aRb \neq 0$ for arbitrary nonzero elements $a, b \in R$ and is semiprime if and only if is nondegenerate, i.e., $aRa \neq 0$ for every nonzero element $a \in R$. In the non-associative setting these characterizations are not so easy, mainly due to the difficulty of building ideals. Moreover, it is known that there exist semiprime, even prime, alternative, Jordan, or Lie algebras that are degenerate (there even exist simple finite dimensional Lie algebras which are degenerate). Nevertheless, there are some characterizations of strong primeness by elements in Jordan systems or alternative algebras, see [**3**] (even for quadratic systems [**2**]):

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(i) An alternative algebra A is strongly prime if and only if $(aA)b = 0$ (or $a(Ab) = 0$), $a, b \in A$, implies $a = 0$ or $b = 0$.

(ii) A linear Jordan algebra or Jordan pair T is strongly prime if and only if $\{a, T, b\} = 0$, $a, b \in T$, implies $a = 0$ or $b = 0$.

We want to point out that (ii) is not true for quadratic Jordan algebras or Jordan pairs (see [**2**]), but in this case we have a "quadratic" characterization of strong primeness by elements:

(iii) A Jordan algebra or Jordan pair T is strongly prime if and only if $U_a U_T U_b T = 0$, $a, b \in T$, implies $a = 0$ or $b = 0$.

In this paper we give a characterization of strong primeness by elements in Lie algebras over an arbitrary ring of scalars:

(iv) *A Lie algebra L is strongly prime if and only if $[a, [b, L]] = 0$, $a, b \in L$, implies $a = 0$ or $b = 0$.*

This result will allow to transfer the strong primeness between a Lie algebra and its Jordan algebras at nonzero Jordan elements (see [**4**] for definitions). Moreover it gives rise to an alternative proof of the elemental characterization of strong primeness in linear Jordan algebras [**3**] without using the classification theorem of strongly prime Jordan algebras (given by Zelmanov in 1983).

1. **Main result**

We will be dealing with Lie algebras over an arbitrary ring of scalars Φ. As usual, $[x, y]$ will denote the Lie bracket, with ad_x the adjoint map determined by x. Given a Lie algebra L, $x \in L$ is an *absolute zero divisor* of L if $\mathrm{ad}_x^2 = 0$, L is *nondegenerate* if it has no nonzero absolute zero divisors, and *prime* if $[I, J] = 0$ implies $I = 0$ or $J = 0$, for ideals I, J of L. We say that a Lie algebra is *strongly prime* if it is prime and nondegenerate.

In this section we are going to prove the main result of the paper: A Lie algebra L is strongly prime if and only if $[x, [y, L]] = 0$ implies $x = 0$ or $y = 0$. As in the Jordan setting it is easy to prove the " if " part of the theorem. In order to prove the " only if " part we are going to study some properties of elements x, y in a Lie algebra L such that $[x, [y, L]] = 0$.

1.1 Proposition. *Let L be a Lie algebra and let $x, y \in L$ such that $[x, [y, L]] = 0 = [x, y]$. Then, $[y, [x, L]] = 0$ and $\mathrm{ad}_x \mathrm{ad}_a \mathrm{ad}_y = -\mathrm{ad}_y \mathrm{ad}_a \mathrm{ad}_x$. Moreover, if L is nondegenerate, $[x, [y, L]] = 0$ implies $[x, y] = 0$.*

Proof. For every $a \in L$, $[y,[x,a]] = [[y,x],a] + [x,[y,a]] = 0$. Now, for any $b \in L$

$$\begin{aligned}[x,[a,[y,b]]] &= [x,[[a,y],b]] + [x,[y,[a,b]]] = [[a,y],[x,b]] + 0 \\ &= [[a,[x,b]],y] + [a,[y,[x,b]]] = -[y,[a,[x,b]]] + 0,\end{aligned}$$

i.e., $\mathrm{ad}_x \mathrm{ad}_a \mathrm{ad}_y = -\mathrm{ad}_y \mathrm{ad}_a \mathrm{ad}_x$. Moreover,

$$\mathrm{ad}^2_{[x,y]} = \mathrm{ad}_x \mathrm{ad}_y \mathrm{ad}_x \mathrm{ad}_y - \mathrm{ad}_x \mathrm{ad}^2_y \mathrm{ad}_x - \mathrm{ad}_y \mathrm{ad}^2_x \mathrm{ad}_y + \mathrm{ad}_y \mathrm{ad}_x \mathrm{ad}_y \mathrm{ad}_x = 0,$$

so, if L is nondegenerate, $[x,y] = 0$. ■

In the following propositions and in theirs proofs, we will use capital letters to denote the adjoint operators, i.e., $X := \mathrm{ad}_x$; $A := \mathrm{ad}_a$.

1.2 Lemma. *Let L be a nondegenerate Lie algebra, $x, y \in L$ such that $XY = 0$. Then any product $\mathrm{ad}_{z_1} \mathrm{ad}_{z_2} \ldots \mathrm{ad}_{z_n}$ is zero if the number of $z_i = x$ (call it s) and the number of $z_j = y$ (call it r) satisfy $s \neq 0 \neq r$ and $n + 1 < 2(r + s)$.*

Proof. We are going to give a proof by induction on n: if $n = 2$ then $\mathrm{ad}_x \mathrm{ad}_y = 0$ by definition, and $\mathrm{ad}_y \mathrm{ad}_x = 0$ by (1.1). Let us suppose that the result is true for $2, \ldots, n-1$ and let us prove it for n. Let $\mathrm{ad}_{z_1} \mathrm{ad}_{z_2} \ldots \mathrm{ad}_{z_n}$ be a product such that the number of $z_i = x$ (call it s) and the number of $z_j = y$ (call it r) satisfy that $s \neq 0 \neq r$ and $n + 1 < 2(r + s)$.

(a). Let us suppose that there exist $k_1 < k_2 < k_3 \leq n$ such that $z_{k_1} = z_{k_3} = x$ and $z_{k_2} = y$. Then, let s' be the number of $z_i = x$ and r' the number of $z_i = y$ between z_1 and z_{k_2}, and s'' the number of $z_i = x$ and r'' the number of $z_i = y$ between z_{k_2} and z_n. Note that $s' = s - s''$ and $r' = r - r'' + 1$. So $s', s'', r', r'' \neq 0$ and either $k_2 + 1 < 2(s' + r')$ (in this case by the induction hypothesis $\mathrm{ad}_{z_1} \mathrm{ad}_{z_2} \ldots \mathrm{ad}_{z_{k_2}} = 0$) or $k_2 + 1 \geq 2(s' + r')$ and then (note that $n - k_2 + 1$ is the number of elements between z_{k_2} and z_n)

$$\begin{aligned}(n - k_2 + 1) + 1 = n + 3 - (k_2 + 1) \leq n + 3 - 2s' - 2r' = n + 3 - 2s + 2s'' \\ - 2r + 2r'' - 2 = n + 1 - 2(s + r) + 2s'' + 2r'' < 2s'' + 2r'',\end{aligned}$$

which implies by the induction hypothesis that $\mathrm{ad}_{z_{k_2}} \ldots \mathrm{ad}_{z_{n-1}} \mathrm{ad}_{z_n} = 0$. In any case, $\mathrm{ad}_{z_1} \mathrm{ad}_{z_2} \ldots \mathrm{ad}_{z_n} = 0$.

Note that if $z_{k_1} = z_{k_3} = y$ and $z_{k_2} = x$, the meanings of s, s' and s'' can be exchanged with r, r' and r'' respectively to also obtain that $\mathrm{ad}_{z_1} \mathrm{ad}_{z_2} \ldots \mathrm{ad}_{z_n} = 0$.

(b). There exists $k < k' \in \{1, 2, \ldots, n\}$ such that $z_k = x$ and if $z_i = x$, then $i \leq k$, and $z_{k'} = y$ and if $z_i = y$, then $i \geq k'$. Note that, since $n > 2$, either r or s are bigger than 1. Let us suppose that $s > 1$ (if $s = 1$ we exchange the roles of x and y) and in this case there exists $i \in \{1, 2, \ldots, k-1\}$ such that $z_i = x$. Then,

$$\mathrm{ad}_{z_1} \mathrm{ad}_{z_2} \ldots \mathrm{ad}_{z_{k-1}} \mathrm{ad}_{[[x, z_{k+1}], z_{k+2}]} \mathrm{ad}_{z_{k+3}} \ldots \mathrm{ad}_{z_n} = 0$$

because if z_{k+1} or $z_{k+2} = y$ then $\mathrm{ad}_{[[x,z_{k+1}],z_{k+2}]} = 0$ by (1.1), and if $z_{k+1}, z_{k+2} \neq y$ the result follows by induction since $s > 1$ and the number of elements in the above formula is $n - 2$. So $\mathrm{ad}_{[x,z_{k+1}]}$ and $\mathrm{ad}_{z_{k+2}}$ commute. In a similar way $\mathrm{ad}_{[x,z_{k+1}]}$ commutes with $\mathrm{ad}_{z_{k+3}}$ and if we follow this way $\mathrm{ad}_{[x,z_{k+1}]}$ meets and crosses with an ad_y, and we return to case (a). Then by (a) $\mathrm{ad}_{z_1} \mathrm{ad}_{z_2} \ldots \mathrm{ad}_{z_n} = 0$. ■

1.3 PROPOSITION. *Let L be a nondegenerate Lie algebra and $n \in \mathbb{N}$. Let us consider $x, y \in L$ such that $XA_1A_2 \ldots A_nY = 0$ for every $a_1, a_2, \ldots, a_n \in L$. Then $XA_1A_2 \ldots A_kY = 0$ and $YA_1A_2 \ldots A_kX = 0$ for any $a_1, a_2, \ldots, a_k \in L$, $k \in \{0, 1, 2, \ldots, n\}$. Moreover $[x, y] = 0$.*

Proof. Firstly, let us prove that $\mathrm{ad}_x \mathrm{ad}_{a_1} \ldots \mathrm{ad}_{a_k} \mathrm{ad}_y \mathrm{ad}_{a_{k+1}} \ldots \mathrm{ad}_{a_n}(b) = 0$, for every $a_1, a_2, \ldots, a_n, b \in L$. Notice that

$$\begin{aligned}\mathrm{ad}_y \mathrm{ad}_{a_{k+1}} \ldots \mathrm{ad}_{a_n} b &= \mathrm{ad}_{[y,a_{k+1}]} \mathrm{ad}_{a_{k+2}} \ldots \mathrm{ad}_{a_n} b + \mathrm{ad}_{a_{k+1}} \mathrm{ad}_y \mathrm{ad}_{a_{k+2}} \ldots \mathrm{ad}_{a_n} b \\ &= -\mathrm{ad}_{(\mathrm{ad}_{a_{k+2}} \ldots \mathrm{ad}_{a_n} b)} \mathrm{ad}_y a_{k+1} + \mathrm{ad}_{a_{k+1}} \mathrm{ad}_y \mathrm{ad}_{a_{k+2}} \ldots \mathrm{ad}_{a_n} b.\end{aligned}$$

So if $k = n - 1$,

$$\begin{aligned}\mathrm{ad}_x \mathrm{ad}_{a_1} \ldots \mathrm{ad}_{a_{n-1}} \mathrm{ad}_y \mathrm{ad}_{a_n}(b) = &-\mathrm{ad}_x \mathrm{ad}_{a_1} \ldots \mathrm{ad}_{a_{n-1}} \mathrm{ad}_b \mathrm{ad}_y(a_n) \\ &+ \mathrm{ad}_x \mathrm{ad}_{a_1} \ldots \mathrm{ad}_{a_{n-1}} \mathrm{ad}_{a_n} \mathrm{ad}_y(b) = 0,\end{aligned}$$

and if the result is true for any k,

$$\begin{aligned}&\mathrm{ad}_x \mathrm{ad}_{a_1} \ldots \mathrm{ad}_{a_{k-1}} \mathrm{ad}_y \mathrm{ad}_{a_k} \ldots \mathrm{ad}_{a_n}(b) \\ &\quad = -\mathrm{ad}_x \mathrm{ad}_{a_1} \ldots \mathrm{ad}_{a_{k-1}} \mathrm{ad}_{(\mathrm{ad}_{a_{k+1}} \ldots \mathrm{ad}_{a_n}(b))} \mathrm{ad}_y(a_k) \\ &\qquad + \mathrm{ad}_x \mathrm{ad}_{a_1} \ldots \mathrm{ad}_{a_{k-1}} \mathrm{ad}_{a_k} \mathrm{ad}_y \mathrm{ad}_{a_{k+1}} \ldots \mathrm{ad}_{a_n}(b) = 0.\end{aligned}$$

The first summand is zero because it begins by an ad_x, it ends by an ad_y and if we span the terms in the middle, it contains n elements of the form ad_{a_i} or ad_b, while the second summand is zero by the induction hypothesis.

Now, for every $b \in L$, let us denote by $z = \operatorname{ad}_x \operatorname{ad}_{a_1} \dots \operatorname{ad}_{a_{n-1}} \operatorname{ad}_y(b)$ and by $t = \operatorname{ad}_{a_1} \dots \operatorname{ad}_{a_{n-1}} \operatorname{ad}_y(b)$. Then, since $\operatorname{ad}_x \operatorname{ad}_t = \operatorname{ad}_x \operatorname{ad}_{\operatorname{ad}_{a_1} \dots \operatorname{ad}_{a_{n-1}} \operatorname{ad}_y(b)} = 0$ by the previous formula, $\operatorname{ad}_z^2 = \operatorname{ad}_{[x,t]}^2 = (\operatorname{ad}_x \operatorname{ad}_t - \operatorname{ad}_t \operatorname{ad}_x)^2 = 0$. Moreover, since L is nondegenerate, $z = 0$, i.e., $XA_1 \dots A_{n-1}Y = 0$. The remaining equalities follow by induction. Moreover, by (1.1), $[x, y] = 0$.

Let us show that also $YA_1 \dots A_kX = 0$ for any $a_1, \dots, a_k \in L$, $k \in \{0, \dots, n\}$: We have $XY = 0$ which implies, by (1.1), that $YX = 0$ and if $n \geq 1$, $YAX = -XAY = 0$ for every $a \in L$. Let us suppose that $YA_1 \dots A_sX = 0$, for any $a_1, \dots, a_s \in L$, $s \in \{0, 1, \dots, k\}$, $k < n$. Then if we take arbitary $a_1, \dots, a_{k+1} \in L$:

$$\begin{aligned} YA_1A_2 \dots A_{k+1}X = YA_1A_2 \dots [A_{k+1}, X] &= Y[A_1, [A_2, [\dots, [A_{k+1}, X]]]] \\ &= [Y, [A_1, [A_2, [\dots, [A_{k+1}, X]]]]] + [A_1, [A_2, [\dots, [A_{k+1}, X]]]]Y \\ &= [A_1, [A_2, [\dots, [A_{k+1}, X]]]]Y = (-1)^{k+1} XA_{k+1}A_k \dots A_1Y = 0. \end{aligned}$$

(Notice that $[Y, [A_1, [\dots, [A_{k+1}, X]]]] = 0$ because $[Y, [A_1, [\dots, [A_{k+1}, X]]]] = -\operatorname{ad}_{[y,[a_1,[\dots,[x,a_{k+1}]]]]} = -\operatorname{ad}_{YA_1 \dots A_kX(a_{k+1})}$ and $YA_1 \dots A_kX = 0$ by the induction hypothesis.) ∎

1.4 Proposition. *Let L be a nondegenerate Lie algebra and $n \in \mathbb{N}$. Let us consider $x, y \in L$ such that $XA_1A_2 \dots A_nY = 0$, for every $a_1, a_2, \dots, a_n \in L$ (if $n = 0$ we understand $XY = 0$). Then there exist $x' \neq 0$ and $y' \neq 0$ such that $X'A_1A_2 \dots A_{n+1}Y' = 0$ for every $a_1, a_2, \dots, a_{n+1} \in L$.*

Proof. Let us prove the case $n = 0$. Without loss of generality we can suppose that $\operatorname{ad}_x^3 L = 0 = \operatorname{ad}_y^3 L$: If there exists $a \in L$ such that $\operatorname{ad}_x^3 a \neq 0$, then, by (1.2), $x' = \operatorname{ad}_x^3 a$, $y' = y$ satisfy $X'Y' = 0$ and $X'A_1Y' = 0$ for every $a_1 \in L$ (respectively, if there exists $b \in L$ such that $\operatorname{ad}_y^3 b \neq 0$, consider $x' = x$ and $y' = \operatorname{ad}_y^3 b$).

Now, let us prove some equalities: For every $e, f, g \in L$, $X^2EY^2 = XEY^2 = X^2EFY^2 = X^2EY = 0$ and $XEFY^2 = XFEY^2$, by (1.2). Moreover,

$$\begin{aligned} X^2EFY = X[X, E]FY + XEXFY &= X[[X, E], F]Y + XF[X, E]Y \\ &+ XEXFY = -Y[[X, E], F]X + XFXEY + XEXFY \\ &= YEXFX + YFXEX - YEFX^2 + XFXEY + XEXFY \\ &= 2YEXFX + 2YFXEX - YEFX^2, \end{aligned}$$

$XEFY^2 = 2XFYEY + 2XEYFY - Y^2FEX$ follows symmetrically, and

$$\begin{aligned} X^2EFGY^2 &= X[X,E]FGY^2 + XEXFGY^2 = X[[X,E],F]GY^2 \\ &+ XF[X,E]GY^2 + XEXFGY^2 = XG[[X,E],F]Y^2 \\ &+ XFXEGY^2 + XEXFGY^2 = XGXEFY^2 + XFXEGY^2 \\ &+ XEXFGY^2 = 2XGXEYFY + 2XGXFYEY + 2XFXEYGY \\ &+ 2XFXGYEY + 2XEXFYGY + 2XEXGYFY. \end{aligned}$$

Now, using the formulas above:

$$\begin{aligned} &\mathrm{ad}_{\mathrm{ad}_x^2 a} \mathrm{ad}_b \mathrm{ad}_{\mathrm{ad}_y^2 c} = (X^2A + AX^2 - 2XAX)B(Y^2C + CY^2 - 2YCY) \\ &= X^2ABY^2C + AX^2BY^2C - 2XAXBY^2C + X^2ABCY^2 + AX^2BCY^2 \\ &- 2XAXBCY^2 - 2X^2ABYCY - 2AX^2BYCY + 4XAXBYCY \\ &= X^2ABCY^2 - 2XAXBCY^2 - 2X^2ABYCY + 4XAXBYCY \\ &= 2XAXBYCY + 2XAXCYBY + 2XBXAYCY + 2XBXCYAY \\ &+ 2XCXAYBY + 2XCXBYAY - 4XAXBYCY - 4XAXCYBY \\ &- 4XAXBYCY - 4XBXAYCY + 4XAXBYCY = -2XAXBYCY \\ &+ 2XCXBYAY - 2XAXCYBY + 2XCXAYBY - 2XBXAYCY \\ &+ 2XBXCYAY. \end{aligned}$$

Note that in the last expression the roles of a and c are skew-symmetrical. So if we exchange a and c we obtain: $\mathrm{ad}_{\mathrm{ad}_x^2 a} \mathrm{ad}_b \mathrm{ad}_{\mathrm{ad}_y^2 c} = -\mathrm{ad}_{\mathrm{ad}_x^2 c} \mathrm{ad}_b \mathrm{ad}_{\mathrm{ad}_y^2 b}$.

Therefore, if we take $a = \mathrm{ad}_u^2 \mathrm{ad}_x^2 v$ and $c = \mathrm{ad}_{u'}^2 \mathrm{ad}_y^2 v'$, for $u, u', v, v' \in L$:

$$\begin{aligned} \mathrm{ad}_{\mathrm{ad}_x^2 a} \mathrm{ad}_b \mathrm{ad}_{\mathrm{ad}_y^2 c} &= \mathrm{ad}_{\mathrm{ad}_x^2 \mathrm{ad}_u^2 \mathrm{ad}_x^2 v} \mathrm{ad}_b \mathrm{ad}_{\mathrm{ad}_y^2 \mathrm{ad}_{u'}^2 \mathrm{ad}_y^2 v'} \\ &= -\mathrm{ad}_{\mathrm{ad}_x^2 \mathrm{ad}_{u'}^2 \mathrm{ad}_y^2 v'} \mathrm{ad}_b \mathrm{ad}_{\mathrm{ad}_y^2 \mathrm{ad}_u^2 \mathrm{ad}_x^2 v} = 0 \end{aligned} \tag{$*$}$$

because, by (1.2), $\mathrm{ad}_x^2 \mathrm{ad}_{u'}^2 \mathrm{ad}_y^2 v' = 0$. Finally, since x and y are nonzero and L is nondegenerate there exist $u, u' \in L$ such that $\mathrm{ad}_x^2 u \neq 0 \neq \mathrm{ad}_y^2 u'$ and again there exist $v, v' \in L$ such that $\mathrm{ad}_{\mathrm{ad}_x^2 u}^2 v \neq 0 \neq \mathrm{ad}_{\mathrm{ad}_y^2 u'}^2 v'$. Moreover, since at the beginning of the proof we showed that x and y can be taken with the extra hypothesis $\mathrm{ad}_x^3 = \mathrm{ad}_y^3 = 0$ we have, see [**1**], that

$$x' = \mathrm{ad}_{\mathrm{ad}_x^2 u}^2 v = \mathrm{ad}_x^2 \mathrm{ad}_u^2 \mathrm{ad}_x^2 v \neq 0 \quad \text{and} \quad y' = \mathrm{ad}_{\mathrm{ad}_y^2 u'}^2 v' = \mathrm{ad}_y^2 \mathrm{ad}_{u'}^2 \mathrm{ad}_y^2 v' \neq 0$$

satisfy the case $n = 0$, see $(*)$.

Let us prove the case $n \geq 1$: Let us denote by S_{n+1} the group of permutations on $n+1$ elements and let $\sigma \in S_{n+1}$. Then, by (1.3):

$$\begin{aligned}
XA_1A_2\dots A_{n+1}Y &= XA_{\sigma(1)}\dots A_{\sigma(n)}A_{\sigma(n+1)}Y\\
XA_1A_2\dots A_{n+1}Y &= XA_1A_2\dots[A_{n+1},Y] = X[A_1,[A_2,[\dots,[A_{n+1},Y]]]]\\
&=[X,[A_1,[A_2,[\dots,[A_{n+1},Y]]]]] + [A_1,[A_2,[\dots,[A_{n+1},Y]]]]X\\
&=[A_1,[A_2,[\dots,[A_{n+1},Y]]]]X = (-1)^{n+1}YA_{n+1}A_n\dots A_1X\\
X^2A_1A_2\dots A_{n+2}Y &= X[X,A_1]A_2\dots A_{n+2}Y + XA_1XA_2\dots A_{n+2}Y\\
&=X[[X,A_1]A_2]A_3\dots A_{n+2}Y + XA_2[X,A_1]A_3\dots A_{n+2}Y\\
&\quad + (-1)^{n+1}XA_1YA_2\dots A_{n+2}X = XA_3\dots A_{n+2}[[X,A_1]A_2]Y\\
&=XA_3\dots A_{n+2}XA_1A_2Y = XA_3\dots A_{n+2}YA_1A_2X = 0\\
XA_1A_2\dots A_{n+2}Y^2 &= 0, \quad \text{follows simetrically.}
\end{aligned}$$

Finally,

$$X^2A_1A_2\dots A_{n+3}Y^2 = X[X,A_1]A_2\dots A_{n+3}Y^2 = X[[X,A_1],A_2]\dots A_{n+3}Y^2 = 0$$

Let us denote by $\mathcal{B} := A_1A_2\dots A_nA_{n+1}$. Then

$$\begin{aligned}
\mathrm{ad}_{\mathrm{ad}_x^2 a}\mathcal{B}\,\mathrm{ad}_{\mathrm{ad}_y^2 c} &= (X^2A + AX^2 - 2XAX)\mathcal{B}(Y^2C + CY^2 - 2YCY)\\
&= X^2A\mathcal{B}Y^2C + AX^2\mathcal{B}Y^2C - 2XAX\mathcal{B}Y^2C + X^2A\mathcal{B}CY^2 + AX^2\mathcal{B}CY^2\\
&\quad - 2XAX\mathcal{B}CY^2 - 2X^2A\mathcal{B}YCY - 2AX^2\mathcal{B}YCY + 4XAX\mathcal{B}YCY = 0,
\end{aligned}$$

which finishes the proof. ■

Given a subset S of L, the *annihilator* or *centralizer* of S in L, $\mathrm{Ann}_L(S)$, consists of the elements $x \in L$ such that $[x,S] = 0$. By the Jacobi identity, $\mathrm{Ann}_L(S)$ is a subalgebra of L, and also an ideal whenever S is so. Moreover, if L is semiprime and I is an ideal of L, $I \cap \mathrm{Ann}_L(I) = 0$ which implies that in a semiprime Lie algebra L an ideal I is essential if and only if $\mathrm{Ann}_L(I) = 0$.

1.5 PROPOSITION. *Let L be a nondegenerate Lie algebra and let $x, y \in L$ such that $\mathrm{ad}_x \mathrm{ad}_a \mathrm{ad}_b \mathrm{ad}_y = 0$ for every $a, b \in L$. Then $y \in \mathrm{Ann}_L(\mathrm{Id}_L(x))$ and $x \in \mathrm{Ann}_L(\mathrm{Id}_L(y))$.*

Proof. Let us prove that $[[x,a],[b,[c,[y,d]]]] = 0$ for every $a,b,c,d \in L$. In this case the set

$$I = \{s \in L \mid [s,[b,[c,[y,d]]]] = 0 \quad \text{for every } b,c,d \in L\}$$

is an ideal of L. Moreover, by (1.3), $[I, y] = 0$ and therefore $y \in \mathrm{Ann}_L(I) \subset \mathrm{Ann}_L(\mathrm{Id}_L(x))$. Now, if we exchange the roles of x and y we obtain that $x \in \mathrm{Ann}_L(\mathrm{Id}_L(y))$.

In what follows we use, without mentioning it, that $XY = YX = XAY = YAX = XABY = YABX = 0$ and $XABCY = -YABCX$ for every $a, b, c \in L$, see (1.3).

$$\begin{aligned} \mathrm{ad}_{[[x,a],[b,[c,[y,d]]]]} = [[X, A], [B, [C, [Y, D]]]] &= XABCYD - XABCDY \\ &+ XABDYC + XACDYB + AXBCDY + YDCBXA \\ &- BYDCAX - CXDBAX - DYCBAX. \end{aligned}$$

An therefore, since in each summand of $\mathrm{ad}^2_{[[x,a],[b,[c,[y,d]]]]}$ we always have a product of the form $\ldots XE_1E_2Y \ldots$ (which is equal to 0), we have that $\mathrm{ad}^2_{[[x,a],[b,[c,[y,d]]]]} = 0$ and, since L is nondegenerate, $[[x, a], [b, [c, [y, c]]]] = 0$. ∎

Now we show the main result of this paper.

1.6 Theorem. *A Lie algebra L is strongly prime if and only if for every $x, y \in L$ such that $[x, [y, L]] = 0$ we have that $x = 0$ or $y = 0$.*

Proof. Let us suppose that L is not strongly prime. If L has a nonzero zero divisor $x \in L$, then $[x, [x, L]] = 0$ for $0 \neq x \in L$, and if L is nondegenerate but not prime, there exist two nonzero ideals I, J of L such that $[I, J] = 0$, so given $0 \neq x \in I$ and $0 \neq y \in J$, $[x, [y, L]] \subset [I, J] = 0$.

Conversely, if L is strongly prime and there exist two nonzero elements $x, y \in L$ such that $[x, [y, L]] = 0$, since L is, in particular, nondegenerate, there exist $0 \neq x' \in L$ and $0 \neq y' \in L$ such that $X'CDY' = 0$ for any $c, d \in L$ (1.4), i.e., $\mathrm{ad}_{x'} \mathrm{ad}_c \mathrm{ad}_d \mathrm{ad}_{y'} = 0$, which implies, by (1.5), that $0 \neq y' \in \mathrm{Ann}_L(\mathrm{Id}_L(x')) = 0$, since L is strongly prime, a contradiction. ∎

2. Some consequences

Now we are going to study the transfer of strong primeness between Lie algebras and some related structures.

2.1 We say that an element x in a Lie algebra L over Φ is a *Jordan element* if $\mathrm{ad}_x^3 = 0$. When $\frac{1}{2}, \frac{1}{3}$ belong to Φ, every Jordan element gives rise to a Jordan algebra, called the *Jordan algebra of L at x*, see [**4**]: Let L be a Lie algebra and let $x \in L$ be a Jordan element. Then L with the new product given by $a \bullet b := \frac{1}{2}[[a,x],b]$ is an algebra such that

$$\ker(x) := \{z \in L \mid [x,[x,z]] = 0\}$$

is an ideal of $(L, \bullet)$. Moreover, $L_x := (L/\ker(x), \bullet)$ is a Jordan algebra. In this Jordan algebra the U-operator has this very nice expression:

$$U_{\overline{a}}\overline{b} = \frac{1}{8}\overline{\mathrm{ad}_a^2\,\mathrm{ad}_x^2\,b}, \qquad \text{for all } a,b \in L, \qquad \text{and}$$
$$\{\overline{a},\overline{b},\overline{c}\} = -\frac{1}{4}\overline{[a,[\mathrm{ad}_x^2\,b,c]]} \qquad \text{for all } a,b,c \in L$$

A Lie algebra is nondegenerate if and only if L_x is nonzero for every Jordan element $x \in L$. Moreover, in this case, L_x is a nondegenerate Jordan algebra [**4**, 2.15(i)].

Now we show the inheritance of strong primeness by the Jordan algebras of a strongly prime Lie algebra. We also give a sufficient condition for the lifting of strong primeness from the Jordan algebras at Jordan elements to the Lie algebra.

2.2 THEOREM. *Let L be a nondegenerate Lie algebra over a ring of scalars with $\frac{1}{2}, \frac{1}{3}$. Then*

(i) if L is a strongly prime Lie algebra, every Jordan algebra of L at a nonzero Jordan element is strongly prime.

Conversely, if for every ideal $I \neq 0$ of L such that $\mathrm{Ann}_L(I) \neq 0$ we have that $\mathrm{Ann}_L(I)$ contains a nonzero Jordan element, then

(ii) L is strongly prime if every Jordan algebra of L at a nonzero Jordan element is strongly prime.

Proof. (i) Let us suppose that L is strongly prime and let $x \in L$ be a Jordan element of L with Jordan algebra L_x. Let $\overline{a}, \overline{b} \in L_x$ be such that $\overline{\{a,y,b\}} = \overline{0}$ for every $y \in L$. Then, by [**4**, 2.3(vi)]

$$0 = [x,[x,[[a,\mathrm{ad}_x^2(y)],b]]] = [\mathrm{ad}_x^2\,a,[y,\mathrm{ad}_x^2\,b]] \quad \text{for every } y \in L,$$

so $[\mathrm{ad}_x^2\,a,[\mathrm{ad}_x^2\,b,y]] = 0$ for every $y \in L$, which implies, by (1.6), that $\mathrm{ad}_x^2\,a = 0$ or $\mathrm{ad}_x^2\,b = 0$, i.e., $\overline{a} = \overline{0}$ or $\overline{b} = \overline{0}$.

(ii) Let us suppose that every ideal $I \neq \{0\}$ of L with $\mathrm{Ann}_L(I) \neq 0$ contains a nonzero Jordan element, and let x', y' be two nonzero elements of L such that

$[x', [y', L]] = 0$. By (1.4), we can suppose that $[x', [a, [b, [y', c]]]] = 0$ for every $a, b, c \in L$. Now, by (1.5), $0 \neq y' \in \mathrm{Ann}_L(\mathrm{Id}_L(x'))$ and, by hypothesis, there exists an ad-nilpotent element $0 \neq y \in L$ of index 3 such that $y \in \mathrm{Ann}_L(\mathrm{Id}_L(x'))$. So $[x', [a, [b, [y, c]]]] = 0$ for every $a, b, c \in L$ and if we repeat the argument, $x' \in \mathrm{Ann}_L(\mathrm{Id}_L(y))$, and there exists an ad-nilpotent element $0 \neq x \in L$ of index 3 such that $x \in \mathrm{Ann}_L(\mathrm{Id}_L(y))$. Finally, since by (1.3) $\mathrm{ad}_x \mathrm{ad}_y = \mathrm{ad}_y \mathrm{ad}_x = 0$ and $[x, y] = 0$, we have that $x + y$ is a Jordan element.

Let us prove that L_{x+y} is not strongly prime: take $a, b \in L$ such that $\mathrm{ad}_x^2 a \neq 0$ and $\mathrm{ad}_y^2 b \neq 0$ and $a', b' \in L$ such that

$$0 \neq \mathrm{ad}^2_{\mathrm{ad}_x^2 a} a' = \mathrm{ad}_x^2 \mathrm{ad}_a^2 \mathrm{ad}_x^2 a' \quad \text{and} \quad 0 \neq \mathrm{ad}^2_{\mathrm{ad}_y^2 b} b' = ad_y^2 \mathrm{ad}_b^2 \mathrm{ad}_y^2 b'.$$

Then, $\bar{0} \neq \overline{\mathrm{ad}_a^2 \mathrm{ad}_x^2 a'} \in L_{x+y}$ and $\bar{0} \neq \overline{\mathrm{ad}_b^2 \mathrm{ad}_y^2 b'} \in L_{x+y}$:

$$\begin{aligned} \mathrm{ad}_{x+y}^2 \mathrm{ad}_a^2 \mathrm{ad}_x^2 a' &= \mathrm{ad}_x^2 \mathrm{ad}_a^2 \mathrm{ad}_x^2 a' \neq 0, \\ \mathrm{ad}_{x+y}^2 \mathrm{ad}_b^2 \mathrm{ad}_y^2 b' &= \mathrm{ad}_y^2 \mathrm{ad}_b^2 \mathrm{ad}_y^2 b' \neq 0, \end{aligned}$$

because $x \in \mathrm{Ann}_L(\mathrm{Id}_L(y))$, and, for every $d \in L$, see (2.1),

$$\{\mathrm{ad}_a^2 \mathrm{ad}_x^2 a', d, \mathrm{ad}_b^2 \mathrm{ad}_y^2 b'\}_{x+y} = -\frac{1}{4}\overline{[\mathrm{ad}_a^2 \mathrm{ad}_x^2 a', [\mathrm{ad}_{x+y}^2 d, \mathrm{ad}_b^2 \mathrm{ad}_y^2 b']]} = \bar{0},$$

since $x \in \mathrm{Ann}_L(\mathrm{Id}_L(y))$, which shows that L_{x+y} is not strongly prime, a contradiction. ■

The next theorem gives an alternative proof of the characterization of strong primeness by elements for Jordan algebras [**3**] without the use of Zelmanov classification of strongly prime Jordan systems. An element x in a Jordan algebra or pair T is called an *absolute zero divisor* if $U_x = 0$. Thus T is said to be *nondegenerate* if it has no nonzero absolute zero divisors, and *prime* if $\{B, C, B\} = 0$ implies $B = 0$ or $C = 0$, for ideals B, C of T. We say that T is *strongly prime* if it is prime and nondegenerate.

2.3 THEOREM. *Let T be a Jordan algebra or a Jordan pair over a ring of scalars with $\frac{1}{2}$. Then T is strongly prime if and only if for every $x, y \in T$ we have that $\{x, T, y\} = 0$ implies $x = 0$ or $y = 0$.*

Proof. The "if" part follows as in (1.6). Now, if $T = (T^+, T^-)$ is a strongly prime Jordan pair, the Tits-Kantor-Koecher algebra $\mathrm{TKK}(T)$ is a strongly prime

Lie algebra, see [**5**, 2.6]. Moreover, if $x, y \in T^\sigma$ satisfy that $0 = \{x, T^{-\sigma}, y\} = [x, [y, T^{-\sigma}]]$, then, by the grading, $[x, [y, \mathrm{TKK}(T)]] = 0$ which implies that $x = 0$ or $y = 0$. If T is a strongly prime Jordan algebra, (T, T) is a strongly prime Jordan pair, see [**2**, 1.12], and the result follows from the above. ■

The next proposition is an alternative proof of [**6**, Lemma 3.1], which was proved by means of a strong result related with the Kostrikin radical of a Lie algebra. Recall that for every Lie algebra with a $(2n+1)$-grading $L = L_{-n} \oplus \ldots \oplus L_0 \oplus \ldots \oplus L_n$, the pair of modules (L_{-n}, L_n) is a Jordan pair with product $\{x, y, z\} = [[x, y], z]$ for every $x, z \in L_{\sigma n}$, $y \in L_{-\sigma n}$, as soon as $\frac{1}{2}, \frac{1}{3}$ belong to the ring of scalars Φ. The pair (L_{-n}, L_n) is called the *associated Jordan pair* of L.

2.4 PROPOSITION. *Let L be a strongly prime Lie algebra with a $(2n+1)$-grading over a ring of scalars with $\frac{1}{2}, \frac{1}{3}$. Then its associated Jordan pair (L_{-n}, L_n) is strongly prime.*

Proof. If $x, y \in L_{\sigma n}$, with $\sigma = \pm$, are elements such that $0 = \{x, L_{-\sigma n}, y\} = [x, [y, L_{-\sigma n}]]$ we have by the grading that $[x, [y, L]] = 0$ and therefore, since L is strongly prime, $x = 0$ or $y = 0$, which proves that (L_{-n}, L_n) is strongly prime itself. ■

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