LIE INNER IDEALS ARE NEARLY JORDAN INNER IDEALS

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Dedicated to Professor Georgia Benkart

Abstract. In this note we extend the Lie inner ideal structure of simple Artinian rings developed by Benkart to centrally closed prime algebra A. New Lie inner ideals (which we call non-standard) occur when making this extension. A necessary and sufficient condition for A to have a non-standard inner ideal is the existence in A of a zero square element which is not regular von Neumann.

1. Introduction

Let A be an associative algebra (not necessarily with a unit element) over a ring of scalars Φ. By a *Lie inner ideal* of A we mean an abelian inner ideal B of the Lie algebra A^- in the sense of [2], i.e., B is a Φ-submodule of A such that $[B,[B,A]] \subseteq B$ and $[B,B] = 0$.

Suppose now that $\frac{1}{2} \in \Phi$. By a *Jordan inner ideal* of A we mean an inner ideal V of the Jordan algebra A^+, i.e., $vAv \subseteq V$ for all $v \in V$. It is easy to see that if V is a Φ-submodule of A such that $VV = 0$, then V is a Jordan inner ideal if and only if it is a Lie inner ideal. In this case, V will be called a *Jordan-Lie inner ideal.* If V is a Jordan-Lie inner ideal, then for any Φ-module Ω of $Z(A)$, $V + \Omega$ is a Lie inner ideal. Any Lie inner ideal of this form, $B = V + \Omega$, will be called *standard*.

In this note we study conditions under which a Lie inner ideal of a semiprime Lie algebra is standard, give a construction of non-standard inner ideals, and classify the Lie inner ideals of any centrally closed

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prime algebra of characteristic not 2 or 3. As a consequence we obtain the following:

Corollary 4.6. *Let A be a centrally closed prime algebra of characteristic not* 2 *or* 3. *If A is non-unital, then every Lie inner ideal of A is standard. If A is unital, then the following conditions are equivalent:*

(i) *Every zero square element of A is von Neumann regular.*
(ii) *Every Lie inner ideal of A is standard.*

Since any simple algebra is centrally closed over its centroid, we obtain as a consequence of the above result that any Lie inner ideal of a semiprime associative algebra A coinciding with its socle is standard. In fact, we describe its Lie inner ideals and refine the description in the case that A is Artinian. The reader is referred to [2, Theorem 5.1] and [4, Theorem 2.5] for related results.

In the last section we adopt a different approach in the study of the Lie inner ideals of an associative algebra A ($\frac{1}{6} \in \Phi$ is required). We prove that if A is semiprime and B is a Lie inner ideal of A such that its image $\overline{B}$ in the Lie algebra $A^-/Z(A)$ is von Neumann regular in the Lie sense, then B is standard. Then using the Jordan structure theory for Lie algebras developed in [5] and [6], we prove that the von Neumann regularity of $\overline{B}$ is guaranteed when it has finite length.

2. Associative algebras, Lie algebras and Jordan systems

Throughout this section, and unless otherwise specified, we will be dealing with (not necessarily unital) associative algebras A, with product xy; Lie algebras L, with $[x, y]$ denoting the Lie bracket and ad_x the adjoint map determined by x; Jordan pairs $V = (V^+, V^-)$, with triple products $\{x, y, z\}$, for $x, z \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$, and quadratic operators $Q_x y = \frac{1}{2}\{x, y, x\}$; and Jordan algebras J, with product $x \cdot y$, quadratic operator $U_x y = 2x \cdot (x \cdot y) - x^2 \cdot y$ and triple product $\{x, y, z\} = U_{x+z}y - U_x y - U_z y$, over a ring of scalars Φ containing $\frac{1}{6}$. So Jordan pairs and Jordan algebras considered here are *linear*. Since any Jordan algebra can be regarded as a Jordan pair, any definition for Jordan pairs makes sense for Jordan algebras. The reader is referred to [8] as a general reference for Jordan pairs.

2.1. Any associative algebra A gives rise to:

(i) a Lie algebra A^-, with Lie bracket $[x, y] = xy - yx$,

(ii) a Jordan pair (A, A), with Jordan triple products given by $\{x, y, z\} = xyz + zyx$.

(iii) a Jordan algebra A^+, with Jordan product $x \cdot y = \frac{1}{2}(xy + yx)$, quadratic operator $U_x y = xyx$ and triple product $\{x, y, z\} = xyz + zyx$.

2.2. Given a Jordan pair $V = (V^+, V^-)$, an *inner ideal* of V is any Φ-submodule B of V^σ such that $\{B, V^{-\sigma}, B\} \subseteq B$. Similarly, an *inner ideal* of a Lie algebra L is a Φ-submodule B of L such that $[[B, L], B] \subseteq B$. An *abelian inner ideal* is an inner ideal B which is also an abelian subalgebra, i.e., $[B, B] = 0$.

2.3. Let A be an associative algebra. An abelian inner ideal of A^- will be called a *Lie inner ideal* of A. Similarly, an inner ideal of the Jordan algebra A^+ will be called a *Jordan inner ideal* of A.

2.4. An ad-nilpotent element $x \in L$ of index of nilpotency ≤ 3 is called a *Jordan element.*

(i) Clearly, any element of an abelian inner ideal is a Jordan element. Conversely, by [3, 1.8], if A is 3-torsion free, then any Jordan element $b \in L$ yields the *principal* abelian inner ideal $\mathrm{ad}_b^2 L$.

(ii) Any zero square element x in an associative algebra A is a Jordan element of the Lie algebra A^-. Indeed, $x^2 = 0$ implies $\mathrm{ad}_x^2 y = -2xyx$ for all $y \in A$, and hence $\mathrm{ad}_x^3 = 0$.

2.5. Recall that an element x in an associative algebra A is *von Neumann regular* if $x \in xAx$. Similarly, an element $x \in V^\sigma$, $\sigma = \pm$, in a Jordan pair V is *von Neumann regular* if $x \in Q_x V^{-\sigma}$.

2.6. A Jordan element e in a Lie algebra L is called *von Neumann regular* if $e \in \mathrm{ad}_e^2 L$. It is easy to see that if x is an element of an associative algebra A such that $x^2 = 0$, then x is von Neumann regular in the associative sense if and only if it is von Neumann regular in the Lie sense.

2.7. Let $V = (V^+, V^-)$ be a Jordan pair. An element $x \in V^\sigma$, $\sigma = \pm$, is called an *absolute zero divisor* if $Q_x = 0$. A Jordan pair V is said to be *nondegenerate* if it has no nonzero absolute zero divisors. Similarly, given a Lie algebra L, $x \in L$ is an *absolute zero divisor* of L if $\mathrm{ad}_x^2 = 0$, and L is said to be *nondegenerate* if it has no nonzero absolute zero divisors.

2.8. Let $B \subseteq V^+$ be an inner ideal of a Jordan pair V. Following [9], the *kernel* of B is the set $\mathrm{Ker}_V B = \{y \in V^- \mid Q_B y = 0\}$. Then $(0, \mathrm{Ker}_V B)$ is an ideal of the Jordan pair (B, V^-), and the quotient $\mathrm{Sub}_V B = (B, V^-)/(0, \mathrm{Ker}_V B) = (B, V^-/\mathrm{Ker}_V B)$ is a Jordan pair called the *subquotient* of B. The kernel and the corresponding subquotient of an inner ideal $B \subseteq V^-$ are defined in a similar way.

The analogues of all these results hold for abelian inner ideals of a Lie algebra, if we replace the Jordan triple product $\{x, y, z\}$ by the left double commutator $[[x, y], z]$ as we describe next.

2.9. Let M be an abelian inner ideal of a Lie algebra L.

(i) The *kernel* of M is the set $\mathrm{Ker}_L M := \{y \in L \mid [M, [M, y]] = 0\}$.

(ii) The pair of Φ-modules $\mathrm{Sub}_L M := (M, L/\mathrm{Ker}_L M)$ with the triple products given by

$$\begin{aligned} \{m, \overline{a}, n\} &:= [[m, a], n] \quad \text{for every } m, n \in M \text{ and } a \in L \\ \{\overline{a}, m, \overline{b}\} &:= \overline{[[a, m], b]} \quad \text{for every } m \in M \text{ and } a, b \in L, \end{aligned}$$

where $\overline{x}$ denotes the coset of x relative to the submodule $\mathrm{Ker}_L M$, is a Jordan pair called the *subquotient of* M [6, Lem. 3.2].

(iii) A Φ-submodule B of M is an inner ideal of L if and only if it is an inner ideal of $\mathrm{Sub}_L M$ [6, 3.5 (i)].

Definition 2.10. Let B and C be abelian inner ideals of a Lie algebra L. We will say that B and C are *isomorphic* ($B \cong C$) if their subquotients $\mathrm{Sub}_L B$ and $\mathrm{Sub}_L C$ are isomorphic as Jordan pairs.

In [5] a Jordan algebra was attached to any Jordan element of a Lie algebra. Many properties of a Lie algebra can be transferred to its Jordan algebras, as well as the nature of the Jordan element in question is reflected on the structure of the attached Jordan algebra. These facts turn out to be crucial for applications of Jordan theory to Lie algebras.

2.11. Let a be a Jordan element of a Lie algebra L over a field Φ of characteristic $\neq 2, 3$. Then L with the new product defined by $x \cdot_a y := \frac{1}{2}[[x, a], y]$ is a nonassociative algebra denoted by $L^{(a)}$, such that

(i) $\mathrm{Ker}_L a := \{x \in L : [a, [a, x]] = 0\}$ is an ideal of $L^{(a)}$.
(ii) $L_a := L^{(a)}/\mathrm{Ker}_L a$ is a Jordan algebra, called the Jordan algebra of L at a.

3. STANDARD INNER IDEALS

Let A be an associative algebra A over a ring of scalars Φ containing $\frac{1}{2}$, and let $Z(A)$ denote the centre of A.

3.1. If V be a Φ-submodule of A such that $VV = 0$, then V is Jordan inner ideal if and only if it is a Lie inner ideal: for $u, v \in V$ and $x \in A$, $VV = 0$ implies $[[u, x], v] = uxv + vxu = \{u, x, v\}$. In this case, V will be called a *Jordan-Lie inner ideal.*

Definition 3.2. A Lie inner ideal B of A is said to be *standard* if $B = \Omega + V$, where Ω is a Φ-module of $Z(A)$ and V is a Jordan-Lie inner ideal of A.

Note that if A is semiprime, then the sum $\Omega + V$ is direct and the intersection of any family of standard inner ideals of A is a standard inner ideal.

Notation 3.3. Given a Lie inner ideal B of A, we denote by V_B (or simply by V when there is no risk of confusion) the subset of all zero square elements of the commutative set $B + Z(A)$.

Lemma 3.4. *Let A be semiprime algebra over a ring of scalars Φ containing $\frac{1}{2}$, and let B be a Lie inner ideal of A.*

(i) *If $B \subseteq V_B + Z(A)$, then V_B is a Jordan-Lie inner ideal of A, the sum $V_B + Z(A)$ is direct, and $\{V_B, A, V_B\} \subseteq B$,*

(ii) *If in addition* $V_B \subseteq B$ *(in particular, if* $Z(A) \subseteq B$*), then* $B = V_B \oplus (B \cap Z(A))$ *is standard.*

Proof. (i) Let N be the set of all nilpotent elements of $B+Z(A)$. Then $V = V_B \subseteq N$, and since $B + Z(A)$ is commutative, we have

$$V + V \subseteq N + N \subseteq N \subseteq B + Z(A) \subseteq V + Z(A),$$

which implies $V + V \subseteq V$, because $Z(A)$ does not contain any nonzero nilpotent element since A is semiprime. Then, for any $u, v \in V$, we have $0 = (u + v)^2 = 2uv$, and hence $\{u, x, v\} = uxv + vxu = [[u, x], v] \in B$ with $\{u, x, v\}^2 = 0$. This proves that V has the required properties.

(ii) Suppose in addition that $V \subseteq B$. Then the Modular Law applied to the inclusion $B \subseteq V \oplus Z(A)$ yields $B = V \oplus (Z(A) \cap B)$. Note that $Z(A) \subseteq B$ implies $V_B \subseteq B+Z(A) \subseteq B$, which completes the proof. □

Theorem 3.5. *Let* A *be semiprime algebra over a ring of scalars* Φ *containing* $\frac{1}{2}$*, and let* B *be a Lie inner ideal of* A*. Then* B *is standard if and only the following condition holds:*

$$V_B \subseteq B \subseteq V_B + Z(A). \tag{ST}$$

Proof. By Lemma 3.4, condition (ST) is sufficient for B to be standard. Suppose then that $B = V \oplus \Omega$ is standard. Clearly, $V \subseteq V_B$, and $V_B \subseteq B + Z(A) \subseteq V \oplus Z(A)$ implies $V_B = V$, which proves that B satisfies (ST). □

We now show a way of constructing non-standard inner ideals.

3.6. Let A be a unital semiprime algebra A over a field Φ of characteristic not 2 such that $Z(A) = \Phi 1$. Let V be a Jordan-Lie inner ideal of A and suppose that $V = \Phi u \oplus V_0$ where V_0 is a hyperplane of V such that $[V, [V, A]] \subseteq V_0$. (Note that $u \in V$ cannot be von Neumann regular.) Define the functional f of V by putting $f(u) = 1$ and $f(V_0) = 0$.

Theorem 3.7. *The set* $B = \{v + f(v)1 : v \in V\}$ *is a Lie inner ideal of* A *which is not standard.*

Proof. (1) B is a Lie inner ideal. Indeed,

$$[B, [B, A]] \subseteq [V, [V, A]] \subseteq V_0 = \mathrm{Ker}(f) \subseteq B$$

and

$$[B, B] \subseteq [V, V] \subseteq VV = 0.$$

(2) $B \cap Z(A) = B \cap \Phi 1 = 0$, since $v + f(v)1 = \alpha 1$ implies $v = (f(v) - \alpha)1$ and hence $v = 0$ and $\alpha = f(v) = 0$.

(3) $V_B = V$. By definition, $V \subseteq B \oplus \Phi 1$, and since $VV = 0$ we have $V \subseteq V_B$. Conversely, let $b + \alpha 1 \in V_B$, with $b = v + f(v)1$. Then

$$0 = (b + \alpha 1)^2 = (v + (f(v) + \alpha)1)^2 = 2(f(v) + \alpha)v + (f(v) + \alpha)^2 1$$

implies $\alpha + f(v) = 0$, so $b + \alpha 1 = v \in V$, which proves that $V_B \subseteq V$.

(4) Since $V \cap B = \mathrm{Ker}(f) = 0$, $f(u) = 1$ implies that u does not belong to B. Thus $V_B = V$ is not contained in B. So B is not standard by Theorem 3.5. □

3.8. The above non-standard inner ideal B has been constructed from a triple (V, V_0, u), where V is a Jordan-Lie inner ideal, V_0 is a hyperplane of V and u is distinguished element of V such $V = \Phi u \oplus V_0$ and $[V, [V, A]] \subseteq V_0$; equivalently, there exists a nonzero $f \in V^*$ such that $[V, [V, A]] \subseteq \mathrm{Ker} f$ and $f(u) = 1$. A such triple (V, V_0, u) will be called *special*. We will see now that the non-standard inner ideal B constructed from the special triple (V, V_0, u) is independent of the choice of u; equivalently, from the choice of the functional f such that $\mathrm{Ker} f = V_0$. For the time being, set $B = \mathrm{Inn}(V, V_0, u) = \mathrm{Inn}(V, V_0, f)$.

Lemma 3.9. *Let A be a unital semiprime algebra A over a field Φ of characteristic not 2 such that $Z(A) = \Phi 1$, let V be a Jordan-Lie inner ideal of A, and let V_0 be a hyperplane of V such that $[V, [V, A]] \subseteq V_0$. If f, g are functional of V such that $\mathrm{Ker} f = \mathrm{Ker} g = V_0$, then the Lie inner ideals $\mathrm{Inn}(V, V_0, f)$ and $\mathrm{Inn}(V, V_0, g)$ are isomorphic.*

Proof. Note first that $\mathrm{Ker}_{A^-} B = \mathrm{Ker}_{A^-} C = \mathrm{Ker}_{(A,A)} V$. Now we have that the pair of linear mappings (φ, Id), where $\varphi : B \to C$ is defined by $\varphi(v + f(v)) = v + g(v)$, $v \in V$, and Id is the identity mapping on the vector space $A/\mathrm{Ker}_{A^-} B$, is a Jordan pair isomorphism of $\mathrm{Sub}_{A^-}\mathrm{Inn}(V, V_0, f)$ onto $\mathrm{Sub}_{A^-}\mathrm{Inn}(V, V_0, g)$. □

Corollary 3.10. *Let A be a unital semiprime algebra A over a field Φ of characteristic not 2 such that $Z(A) = \Phi 1$, and let $x \in A$ be a* *zero square element which is not von Neumann regular. Then* $(\Phi x \oplus xAx, xAx, x)$ *is a special triple and therefore it gives rise to the non-standard inner ideal* $\mathrm{Inn}((\Phi x \oplus xAx, xAx, x)$.

4. Lie inner ideals of central closed prime algebras

4.1. A prime associative algebra A over a field Φ is *centrally closed* over Φ if the *extended centroid* (see [1] for definition) of A is Φ itself. Clearly, any simple associative algebra is centrally closed over its centroid.

The following Lemma, which was proved in [2, Lemma 3.10] for simple associative algebras, is a refinement of a more general result by Martindale and Miers for prime algebras.

Lemma 4.2. *Let A be a centrally closed prime associative algebra over a field Φ of characteristic not 2 or 3 and let a be a Jordan element of A^-. Then there exists $z \in Z(A)$, necessarily unique, such that $(a-z)^2 = 0$.*

Proof. By [10, Corollary 1], there exists $\lambda \in \Phi$ such that $(a-\lambda 1)^2 = 0$, the formula making sense in the unital hull $\hat{A} = A + \Phi 1$ of A. If A is unital, then $\lambda 1 \in Z(A)$. If A is non-unital, then $a^2 - 2\lambda a = \lambda^2 1$ implies $\lambda = 0$. □

Proposition 4.3. *Let A be a centrally closed prime associative algebra over a field Φ of characteristic not 2 or 3, and let B be a Lie inner ideal of A. In any of the following situations B is standard:*

(i) *A is non-unital.*
(ii) *Every zero square element of A is von Neumann regular.*
(iii) *$Z(A) \subseteq B$.*
(iv) *B is a maximal Lie inner ideal.*

In case (i), $B = V_B$ is in fact a Jordan-Lie inner ideal.

Proof. (i) Since A is non-unital, $Z(A) = 0$. Hence $V_B \subseteq B$ (by definition) and $B \subseteq V_B$ by Lemma 4.2. Now we have by Lemma 3.4(i) that $B = V_B$ is a Jordan-Lie inner ideal.

(ii) By Lemma 4.2, $B \subseteq V_B + Z(A)$. Hence V_B is Jordan-Lie inner of A such that $\{V_B, A, V_B\} \subseteq B$ by Lemma 3.4(i). Since every zero square element of A is von Neumann regular, $V_B = \{V_B, A, V_B\} = [[V_B, A], V_B] \subseteq B$, which implies that B is standard by Lemma 3.4(ii).

(iii) The inclusion $Z(A) \subseteq B$ implies $V_B \subseteq B$. Then the proof follows as in (i).

(iv) It follows from (iii) since $B + Z(A)$ is a Lie inner ideal. □

Corollary 4.4. *Let X be a right vector space over a division ring Δ of characteristic not 2 or 3. Then every Lie inner ideal of $\mathrm{End}_\Delta(X)$ (regarded as a $Z(\Delta)$-algebra) is standard.*

Proof. The ring $\mathrm{End}_\Delta(X)$ is prime (in fact, primitive) and von Neumann regular. Moreover, it is centrally closed over $Z(\Delta)$ by [1, Theorem 4.3.7(ix)]. Thus we can apply Proposition 4.3 to get that any Lie inner ideal of $\mathrm{End}_\Delta(X)$ is standard. □

Theorem 4.5. *Let A be a centrally closed prime associative algebra over a field Φ of characteristic not 2 or 3. If B is a Lie inner ideal of A, then either*

(i) *$B = V$ where V is a Jordan-Lie inner ideal of A, or*
(ii) *$B = V \oplus \Phi 1$ where V is as in (i), or*
(iii) *$B = \mathrm{Inn}(V, V_0, u)$ where (V, V_0, u) is a special triple.*

Note that in cases (ii) and (iii), A is necessarily unital.

Proof. If A is non-unital, then every Lie inner ideal of A is a Jordan-Lie inner ideal by Proposition 4.3(i). Suppose that A has a unit element, and therefore $Z(A) = \Phi 1$. If $1 \in B$, then $B = V \oplus \Phi 1$ is standard by Proposition 4.3(iii). Suppose then that $1 \notin B$. Then, again by Proposition 4.3(iii), $B \oplus \Phi 1 = V \oplus \Phi 1$ for some Jordan-Lie inner ideal V of A. We also have that if $v + \alpha 1$ and $v + \beta 1 \in B$, then $\alpha = \beta$ (otherwise 1 would belong to B, what has been discarded). Thus there exists $f \in V^*$ such that $B = \{v + f(v)1 : v \in V\}$. If $f = 0$, then $B = V$ and we have in case (i). Suppose then that f is non-zero and let $u \in V$ such that $f(u) = 1$. Then $V = \Phi u \oplus V_0$, with $V_0 = \mathrm{Ker} f$. Clearly, $V \cap B = V_0$, and since $[V, [V, A]] = [B, [B, A]]$, we have $[V, [V, A]] \subseteq V \cap B = V_0$. Thus (V, V_0, u) is a special triple and $B = \mathrm{Inn}(V, V_0, u)$, which completes the proof. □

Corollary 4.6. *Let A be a centrally closed prime associative algebra over a field Φ of characteristic not 2 or 3. If A is non-unital, then every Lie inner ideal of A is standard. If A is unital, then the following conditions are equivalent:*

(i) *Every zero square element of A is von Neumann regular.*
(ii) *Every Lie inner ideal of A is standard.*

Proof. It follows from Theorem 4.5 together with Corollary 3.10. □

The following example provides a primitive algebra which is not centrally closed but which still enjoys the property that all its Lie inner ideals are standard.

Example 4.7. Let X be an infinite-dimensional complex vector space, let $\mathcal{F}(X)$ be the simple complex associative algebra of all finite rank operators on X, and set $A = \mathcal{F}(X) \oplus \mathbb{R}\mathrm{Id}_X$. Then A is a real primitive algebra which is not centrally closed, its extended centroid being the complex field. However, every Lie inner ideal B of A is standard. In fact, either $B = V$, where V is a Jordan-Lie inner ideal, or $B = \mathbb{R}\mathrm{Id}_X \oplus V$.

5. Lie inner ideals of semiprime algebras with finiteness conditions

In this section we see that in a semiprime algebra coinciding with its socle every Lie inner ideal is standard. In fact, we describe its Lie inner ideas in associative terms.

Theorem 5.1. *Let A be semiprime and 6-torsion free and let B be a Lie inner ideal of A.*

(i) *If A coincides with its socle, then $B = \Omega \oplus RL$, where Ω is a Φ-submodule of $Z(A)$, R is a right ideal of A, and L is a left ideal of A with $LR = 0$ and $RL = R \cap L$.*
(ii) *If A is actually Artinian, then $B = \Omega \oplus eAf$, where Ω is a Φ-submodule of $Z(A)$ and e, f are idempotents of A such that $fe = 0$.*

Proof. (i) By the structure of the socle, $A = \oplus M_i$ is a direct sum of minimal ideals, each of which is a simple (and therefore centrally closed over its centroid) algebra of characteristic not 2 or 3 coinciding with its socle. Let $b \in B$. Then $b = \sum b_i$ where each $b_i \in M_i$ and $b_i = 0$ up to a finite subset of indexes. By Lemma 4.2, for any b_i there exists a unique $z_i \in Z(M_i)$ such that $(b_i - z_i)^2 = 0$ (with $z_i = 0$ if $b_i = 0$). Set $z = \sum z_i$. Then $z \in Z(A)$ and $(b - z)^2 = 0$, which proves that B satisfies condition (i) of Lemma 3.4. Moreover, since A is von Neumann regular (because it coincides with its socle), B also satisfies condition (ii) of of Lemma 3.4, so $B = V \oplus (B \cap Z(A))$ is standard. Using again the von Neumann regularity of A^+, we get that $V = \oplus V_i$, where each V_i is Jordan-Lie inner ideal of M_i. By [7, Theorem 3(ii)], for each index i, $V_i = R_i L_i$ where R_i is a right ideal of M_i and L_i is a left ideal. Then $V = RL$ where $R = \oplus R_i$ and $L = \oplus L_i$, with $LR = 0$ since $VV = 0$. The equality $RL = R \cap L$ follows because A is von Neumann regular.

(ii) If A is Artinian, then it coincides with its socle. Hence, by (i), $B = RL \oplus (B \cap Z(A))$ is standard. Now we can apply the structure of one-sided ideals of semiprime Artinian rings to get that $R = eA$ and $L = Af$, where e, f are idempotents of A with $fe = 0$, or use [11, Theorem 1] to get $V = eAf$ directly. $\square$

6. STANDARD INNER IDEALS BY A JORDAN APPROACH

Let A be an associative algebra A (over a ring of scalars Φ) and let $Z(A)$ be its centre. We will denote by $\pi : x \mapsto \overline{x}$ the canonical homomorphism of Lie algebras of A^- onto $\overline{A} := A^-/Z(A)$.

Proposition 6.1. *Let A be a semiprime and let b be an element of A.*

(i) *If $\mathrm{ad}_b^n A \subseteq Z(A)$, then $\mathrm{ad}_b^n A = 0$ for all $n \geq 1$, and*

$$n(\mathrm{ad}_b^{n-1} x)(\mathrm{ad}_b^{n-1} y) = 0$$

for all $x, y \in A$ and $n \geq 2$.

(ii) *If A is 2-torsion free, then the Lie algebra $\overline{A}$ is nondegenerate.*

(iii) *If A is 3-torsion free, then every principal inner ideal of B of A^- satisfies $BB = 0$. Therefore it is a Jordan-Lie inner ideal of A.*

(iv) *If A is 3-torsion free and $\overline{b}$ is von Neumann regular in $\overline{A}$, then $b = v + z$ where $v \in [[b, A], b]$, $v^2 = 0$, v is von Neumann regular in A, and $z \in Z(A)$.*

Proof. (i) Since ad_b is a derivation of the associative algebra A, we have by Leibniz rule that $\mathrm{ad}_b^n(xb) = \mathrm{ad}_b^n(x)b \in Z(A)$ for all $x \in A$ and $n \geq 1$. Hence, by Jacobi identity, $0 = [\mathrm{ad}_b^n(x)b, y] = \mathrm{ad}_b^n(x)[b, y]$ for all $y \in A$. Taking $y = \mathrm{ad}_b^{n-1}x$, we obtain $\mathrm{ad}_b^n(x)^2 = 0$, which implies that $\mathrm{ad}_b^n(x) = 0$ for all $x \in A$, since $\mathrm{ad}_b^n A \subseteq Z(A)$ and semiprime algebras does not contain nonzero nilpotent central elements. If $n \geq 2$, again by Leibniz rule, we have $0 = \mathrm{ad}_b^n(x\mathrm{ad}_b^{n-2}y) = n(\mathrm{ad}_b^{n-1}x)(\mathrm{ad}_b^{n-1}y)$, as required.

(ii) Let $a \in A$ be such that $\mathrm{ad}_a^2 A \subseteq Z(A)$. By (i), $\mathrm{ad}_a^2 A = 0$, and hence, for all $x, y \in A$, $0 = \mathrm{ad}_a^2(xy) = 2\mathrm{ad}_a(x)\mathrm{ad}_a(y)$, which implies $\mathrm{ad}_a(x)\mathrm{ad}_a(y) = 0$ since A is 2-torsion free. Then $\mathrm{ad}_a(xy)\mathrm{ad}_a(x) = \mathrm{ad}_a(x)y\mathrm{ad}_a(x) = 0$ implies by semiprimeness of A that $\mathrm{ad}_a x = 0$ for all $x \in A$, i.e., $a \in Z(A)$, which proves that $\overline{A}$ is nondegenerate.

(iii) Let B be a principal inner ideal of A^-, i.e., $B = \mathrm{ad}_x^2 A$ where x is a Jordan element of A^-. Since A is 3-torsion free, it follows from (i) that $BB = 0$. Hence $\{B, A, B\} = [[B, A], B] \subseteq B$, as required.

(iv) Von Neumann regularity of $\bar{b}$ in $\overline{A}$ means that $\bar{b}$ is a Jordan element of $\overline{A}$ and there exist $x \in A$ and $z \in Z(A)$ such that $b = [[b, x], b] + z$. Since we are assuming that A is 3-torsion free, $[[b, x], b]^2 = 0$ by (iii). The von Neumann regularity of $v = [[b, x], b]$ (in the usual associative sense) is proved as follows: By [5, Lemma 3.2(ii)], we have that the Jordan algebras A_v^+ and $\overline{A}_{\bar{v}} = \overline{A}_{\bar{b}}$ are isomorphic. Since the latter is unital by [5, Proposition 2.15(ii)], A_v^+ is unital, equivalently, v is von Neumann regular in the associative sense. □

Theorem 6.2. *Let A be semiprime and 6-torsion free and let B be a Lie inner ideal of A. If every element $\bar{b}$ of $\overline{B}$ is von Neumann regular, then B is standard.*

Proof. It follows from Proposition 6.1(iv) and Lemma 3.4. □

6.3. Let L be a Lie algebra and V a Jordan pair.

(i) The *length* of an inner ideal B of L (respectively V) is defined as the supremum of the lengths of the chains $0 \subset B_1 \subset B_2 \subset \cdots \subset B_n$ of inner ideals of L (respectively V) contained in B.

(ii) L (respectively V) is said to be *Artinian* if it satisfies the descending chain condition on all inner ideals.

Corollary 6.4. *Let A be a semiprime associative algebra over a ring of scalars Φ containing $\frac{1}{6}$, and let B be an abelian inner ideal of A^- such that $\overline{B}$ has finite length. Then $B = \Omega \oplus V$, where Ω is a Φ-submodule of $Z(A)$ of finite length, and V is a Jordan-Lie inner ideal of A of finite length.*

Proof. By Proposition 6.1(iii) the Lie algebra $\overline{A}$ is nondegenerate, and since $\overline{B}$ has finite length, we have by [6, Proposition 3.5(iii),(v)] that $\mathrm{Sub}_{\overline{A}}\overline{B}$ is a nondegenerate Artinian Jordan pair, so it is von Neumann regular by [8, Theorem 10.17], which clearly implies that $\overline{B}$ is von Neumann regular in $\overline{A}$. Then it follows from Theorem 6.2 that $B = V \oplus \Omega$ is standard. Since $V \cap Z(A) = 0$, any n-length chain $0 \subset U_1 \subset U_2 \subset \cdots \subset U_n$ of Jordan inner ideals of A contained in V gives rise to the n-length $\overline{U}_1 \subset \overline{U}_2 \subset \cdots \subset \overline{U}_n$ of Lie inner ideal of $\overline{A}$ contained in $\overline{B}$, so V has finite length. Similarly, Ω has finite length, as a Φ-module. $\square$

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