Banach-Lie algebras spanned by extremal elements

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Abstract

Any nondegenerate Banach-Lie algebra which is spanned by extremal elements has finite dimension.

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1 Introduction

A finite-dimensional Lie algebra L over an algebraically closed field $\mathbb{F}$ of characteristic 0 is semisimple if and only if it is nondegenerate, $[x,[x,L]] = 0$ implies $x = 0$ for all x in L, and is spanned by its extremal elements, i.e., elements e in L such that $[e,[e,L]] = \mathbb{F}e$. Recently [4], we have described the infinite-dimensional strongly prime Banach Lie algebras containing extremal elements. Using that description and socle theory [3], we prove here that any nondegenerate Banach-Lie algebra which is the linear span of its extremal elements is necessarily finite-dimensional.

2 Preliminaries

2.1. Throughout this section we will be dealing with Lie algebras L, with $[x,y]$ denoting the Lie bracket and ad_x the adjoint map determined by x, over a field $\mathbb{F}$ of characteristic 0 [8]. Any associative algebra A gives rise to a Lie algebra A^- with Lie bracket $[x,y] = xy - yx$. If A has an involution $*$, then $\mathrm{Skew}(A,*)$ is a subalgebra of A^- and therefore it is a Lie algebra.

2.2. An element $x \in L$ is an *absolute zero divisor* if $\mathrm{ad}_x^2 = 0$; L is *nondegenerate* if it has no nonzero absolute zero divisors, *semiprime* if $[I,I] = 0$ implies $I = 0$, and *prime* if $[I,J] = 0$ implies $I = 0$ or $J = 0$, for any ideals I, J of L. A Lie algebra is *strongly prime* if it is prime and nondegenerate, and *simple* if it is nonabelian and contains no proper ideals. Any simple Lie algebra can be considered as an algebra over its centroid and the latter is central simple. Nondegeneracy and strong primeness are inherited by ideals [9, Lemma 4] and [7, (0.4), (1.5)].

2.3. The *annihilator* or *centralizer* of a subset S of L is the set $\operatorname{Ann}_L S$ consisting of the elements $x \in L$ such that $[x, S] = 0$. By the Jacobi identity, $\operatorname{Ann}_L S$ is a subalgebra of L and an ideal whenever S is so. Clearly, $\operatorname{Ann}_L L = Z(L)$, the center of L. If L is semiprime, then $I \cap \operatorname{Ann}_L I = 0$ for any ideal I of L, so an ideal is essential if and only if has zero annihilator. If L is nondegenerate, then $\operatorname{Ann}_L I = \{a \in L \mid [a, [a, I]] = 0\}$ [5, (2.5)].

2.4. An *inner ideal* of a Lie algebra L is a subspace B of L such that $[B, [B, L]] \subset B$ [1]. An *abelian inner ideal* is an inner ideal which is also an abelian subalgebra. An element $x \in L$ is said to be *extremal* if it generates a one-dimensional inner ideal, that is, $\operatorname{ad}_x^2 L = \mathbb{F}x$.

2.5. The *socle* of a nondegenerate Lie algebra L is defined as the sum of all minimal inner ideals of L. By [3, Theorem 2.5], $\operatorname{Soc} L = \bigoplus_\alpha M_\alpha$ is a direct sum of minimal ideals, each of which is a simple nondegenerate Lie algebra coinciding with its socle.

2.6. An element x in L is called a *Jordan element* if $\operatorname{ad}_x^3 = 0$. Clearly, any element of an abelian inner ideal is a Jordan element. Conversely, by [1, (1.8)], any Jordan element x yields the abelian inner ideal $\operatorname{ad}_x^2 L$. A good reason for this terminology is the following analogue of the fundamental identity for Jordan algebras [1, (1.7)]:

$$\operatorname{ad}^2_{\operatorname{ad}^2_x y} = \operatorname{ad}^2_x \operatorname{ad}^2_y \operatorname{ad}^2_x$$

for any $y \in L$. ❧

3 Lie algebras with extremal elements

All the vector spaces considered in this section are infinite-dimensional over an algebraically closed field $\mathbb{F}$ of characteristic 0.

3.1. Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a pair of dual vectors spaces over $\mathbb{F}$, i.e., X, Y are vector spaces over $\mathbb{F}$, and $\langle\cdot,\cdot\rangle : X \times Y \to \mathbb{F}$ is a nondegenerate bilinear form. (Notice that any vector space X gives rise to the canonical pair (X, X^*), where X^* is the dual of X.) We associate with $(X, Y, \langle\cdot,\cdot\rangle)$ the following algebras:

(i) The associative algebra of all the linear operators $a : X \to X$ having a (unique) adjoint $a^\# : Y \to Y$, i.e., $\langle ax, y\rangle = \langle x, a^\# y\rangle$ for all $x \in X$, $y \in Y$. Notice that $\mathcal{L}_{X^*}(X) = \operatorname{End} X$.

(ii) The ideal $\mathcal{F}_Y(X)$ of all linear operators $a \in \mathcal{L}_Y(X)$ having finite rank.

(iii) The *general linear algebra* $\mathfrak{gl}_Y(X) := \mathcal{L}_Y(X)^-$.

(iv) The *finitary linear algebra* $\mathfrak{fgl}_Y(X) := \mathcal{F}_Y(X)^-$.

(v) The *special linear algebra* $\mathfrak{fsl}_Y(X) := [\mathfrak{fgl}_Y(X), \mathfrak{fgl}_Y(X)]$. Clearly, $\mathfrak{fgl}_Y(X)$ and $\mathfrak{fsl}_Y(X)$ are ideals of $\mathfrak{gl}_Y(X)$.

3.2. Given $x \in X$ and $y \in Y$, let y^*x denote the linear operator defined by $y^*x(x') = \langle x', y\rangle x$, for all $x' \in X$. It is easy to see that $y^*x \in \mathcal{F}_Y(X)$, with adjoint x^*y. Moreover, $y^*x \in \mathfrak{fsl}_Y(X)$ if and only if $\langle x, y\rangle = 0$ [6, Theorem 1.7].

3.3. Let X be a vector space over $\mathbb{F}$ endowed with a nondegenerate symmetric (respectively, alternate) bilinear form $\langle\cdot,\cdot\rangle$. Then $(X, X, \langle\cdot,\cdot\rangle)$ is a pair of dual vector spaces and the adjoint becomes an involution, denoted by $*$, in the associative algebra $\mathcal{L}(X) := \mathcal{L}_X(X)$, making the ideal $\mathcal{F}(X)$ $*$-invariant. We have the following Lie algebras:

If $\langle\cdot,\cdot\rangle$ is symmetric, then $\mathfrak{o}(X, \langle\cdot,\cdot\rangle) := \mathrm{Skew}(\mathcal{L}(X), *)$ is the *orthogonal algebra*), and $\mathfrak{fo}(X, \langle\cdot,\cdot\rangle) := \mathrm{Skew}(\mathcal{F}(X), *) = [\mathrm{Skew}(\mathcal{F}(X), *), \mathrm{Skew}(\mathcal{F}(X), *)]$ is the *finitary orthogonal algebra*.

If $\langle\cdot,\cdot\rangle$ is alternate, then $\mathfrak{s}p(X, \langle\cdot,\cdot\rangle) := \mathrm{Skew}(\mathcal{L}(X), *)$ is the *symplectic algebra*, and $\mathfrak{fsp}(X, \langle\cdot,\cdot\rangle) := \mathrm{Skew}(\mathcal{F}(X), *) = [\mathrm{Skew}(\mathcal{F}(X), *), \mathrm{Skew}(\mathcal{F}(X), *)]$ is the *finitary symplectic algebra*.

3.4. If $\langle\cdot,\cdot\rangle$ is symmetric, then for any $x, y \in X$ the linear operator $[x, y] := x^*y - y^*x$ belongs to $\mathfrak{fo}(X, \langle\cdot,\cdot\rangle)$. In fact, these operators span $\mathfrak{fo}(X, \langle\cdot,\cdot\rangle)$. If $\langle\cdot,\cdot\rangle$ is alternate, then $\mathfrak{fsp}(X, \langle\cdot,\cdot\rangle)$ is spanned by the operators x^*x.

3.5. Let $\langle\cdot,\cdot\rangle$ be symmetric or alternate. For a *hyperbolic pair* we mean a pair (x, y) of isotropic vectors of X such that $\langle x, y\rangle = 1$. A *hyperbolic plane* is any 2-dimensional subspace of X having a basis consisting of a hyperbolic pair. Since $\mathbb{F}$ is algebraic closed, a 2-dimensional subspace H of X is a hyperbolic plane if and only if it is nondegenerate.

Theorem 3.6. *Let L be an infinite-dimensional Lie algebra over $\mathbb{F}$. Then L is strongly prime and contains extremal elements if and only if it is, up to isomorphism, one of the following:*

(i) $(\mathfrak{fsl}_Y(X) + \mathbb{F}\,\mathrm{Id}_X)/\mathbb{F}\,\mathrm{Id}_X \leq L \leq \mathfrak{gl}_Y(X)/\mathbb{F}\,\mathrm{Id}_X$, *where (X, Y) is an infinite-dimensional pair of dual vector spaces over $\mathbb{F}$.*

(ii) $\mathfrak{fo}(X, \langle\cdot,\cdot\rangle) \leq L \leq \mathfrak{o}(X, \langle\cdot,\cdot\rangle)$, *where X is an infinite-dimensional vector space over $\mathbb{F}$ with a nondegenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$.*

(iii) $\mathfrak{fsp}(X, \langle\cdot,\cdot\rangle) \leq L \leq \mathfrak{s}p(X, \langle\cdot,\cdot\rangle)$, *where X is an infinite-dimensional vector space over $\mathbb{F}$ with a nondegenerate alternate bilinear form $\langle\cdot,\cdot\rangle$.*

Moreover, L is simple if and only if it is either $\mathfrak{fsl}_Y(X)$, $\mathfrak{fo}(X, \langle\cdot,\cdot\rangle)$ or $\mathfrak{fsp}(X, \langle\cdot,\cdot\rangle)$.

Proof. See [4, Theorem 3.10]. ❧

4 Banach-Lie algebras with extremal elements

4.1. Let L be a complex Lie algebra. By an *algebra norm* of L we mean any norm $||\cdot||$ on the complex vector space L making continuous the bracket product, i.e., there exists a positive number k such that $||[x,y]|| \leq k||x||||y||$ for all $x, y \in L$. A *normed* Lie algebra is a complex Lie algebra L endowed with an algebra norm. If the norm is complete, then L is called a Banach-Lie algebra.

4.2. Following [2, Section 27, Definition 1], a *Banach pairing* is a pair of dual vector spaces $(X, Y, \langle\cdot,\cdot\rangle)$ over $\mathbb{C}$ such that both X and Y are endowed with prefixed complete norms making the bilinear form $\langle\cdot,\cdot\rangle$ continuous. An standard application of the closed graph theorem allows us to prove that the complete norms of X and Y making continuous the nondegenerate bilinear form are unique up to equivalence. By another application of the closed graph theorem we obtain that every $a \in \mathcal{L}_Y(X)$ is a norm-continuous operator on X, so $\mathcal{L}_Y(X)$ is a subalgebra of the Banach (associative) algebra $\mathrm{BL}(X)$ of all bounded linear operators on X. Although $\mathcal{L}_Y(X)$ needs not be complete for the operator norm, it has a natural structure of Banach algebra under the norm $|\cdot|'$ defined by $|a|' = \max\{|a|, |a^{\#}|\}$, where $|\cdot|$ denotes the operator norm. As a consequence, we have

4.3. Let $(X, Y, < .,. >)$ be a Banach pairing. Then (i) up to equivalence of norms, there is a unique Banach pairing structure on the pair of dual vector spaces $(X, Y, < .,. >)$, and (ii) $\mathfrak{gl}_Y(X)$ and $\mathfrak{gl}_Y(X)/\mathbb{F}\,\mathrm{Id}_X$ are Banach-Lie algebras for the norm defined by $|a|' = \max\{|a|, |a^{\#}|\}$, with $|\cdot|$ denoting the operator norm, and its quotient norm, also denoted by $|\cdot|'$, respectively.

4.4. A Banach pairing $(X, Y, \langle\cdot,\cdot\rangle)$ with $X = Y$ will be called a *Banach inner product space* and will be denoted by $(X, \langle\cdot,\cdot\rangle)$.

Theorem 4.5. *Let $(L, ||\cdot||)$ be an infinite-dimensional Banach-Lie algebra. Then L is strongly prime and contains extremal elements if and only if any one of the following statements holds:*

(i) *There exists an infinite-dimensional Banach pairing $(X, Y, \langle\cdot,\cdot\rangle)$ such that $(\mathfrak{fsl}_Y(X) + \mathbb{C}\,\mathrm{Id}_X)/\mathbb{C}\,\mathrm{Id}_X \leq L \leq \mathfrak{gl}_Y(X)/\mathbb{C}\,\mathrm{Id}_X$, and the injection $(L, ||\cdot||)$ into $(\mathfrak{gl}_Y(X)/\mathbb{C}\,\mathrm{Id}_X, |\cdot|')$ is continuous.*

(ii) *There exists an infinite-dimensional Banach inner product space $(X, \langle\cdot,\cdot\rangle)$, where $\langle\cdot,\cdot\rangle$ is symmetric, such that $\mathfrak{fo}(X, \langle\cdot,\cdot\rangle) \leq L \leq \mathfrak{o}(X, \langle\cdot,\cdot\rangle)$, and the injection of $(L, ||\cdot||)$ into $(\mathfrak{o}(X, \langle\cdot,\cdot\rangle), |\cdot|)$ is continuous.*

(iii) *There exists an infinite-dimensional Banach inner product space $(X, \langle\cdot,\cdot\rangle)$, where $\langle\cdot,\cdot\rangle$ is alternate, such that $\mathfrak{fsp}(X, \langle\cdot,\cdot\rangle) \leq L \leq \mathfrak{s}p(X, \langle\cdot,\cdot\rangle)$, and the injection of $(L, ||\cdot||)$ into $(\mathfrak{s}p(X, \langle\cdot,\cdot\rangle), |\cdot|)$ is continuous.*

Proof. See [4, Theorem 7.2].

Theorem 4.6 *Any nondegenerate Banach-Lie algebra spanned by extremal elements is finite-dimensional.*

Proof. By (2.5), $L = \mathrm{Soc}\, L = \bigoplus M_\alpha$ is a direct sum of minimal ideals. This allows us to reduce the question to the case where L is simple. First, each M_α is a simple nondegenerate Banach-Lie algebra with extremal elements (M_α is an annihilator ideal and therefore closed, and there exists an extremal element x of L such that $\mathrm{ad}_x^2\, M_\alpha \neq 0$; then it follows from the analogue of the fundamental Jordan identity (2.6) that $\mathrm{ad}_x^2\, y$ is an extremal element of M_α for any $y \in M_\alpha$ such that

$\mathrm{ad}_x^2\, y \neq 0$). And second, there are only finitely many M_α. If $\{M_{\alpha_n}\}$ is an infinite sequence, take a nonzero element a_n in each M_{α_n} and set $a = \sum_{n=1}^{\infty} \frac{a_n}{2^n ||a_n||}$. Since L coincides with its socle, $a \in M_{\beta_1} \oplus \cdots \oplus M_{\beta_r}$ for finitely many M_{β_i}. Let α_m be distinct from all the β_i. Then $[M_{\beta_i}, M_{\alpha_m}] \subset M_{\beta_i} \cap M_{\alpha_m} = 0$, but $[\sum_{n=1}^{\infty} \frac{a_n}{2^n ||a_n||}, M_{\alpha_m}] = [a_m, M_{\alpha_m}] \neq 0$, a contradiction.

Let us then suppose then that L is simple. If L were infinite-dimensional, then L would be either $\mathfrak{fsl}_Y(X)$, $\mathfrak{fo}(X, \langle, \cdot, \rangle)$ or $\mathfrak{fsp}(X, \langle, \cdot, \rangle)$, where in all the cases X is infinite dimensional, Theorem 3.6. We analyze the three cases separately.

Case I. *Let $(X, Y, < .,. >)$ be an infinite-dimensional pair of dual vector spaces over $\mathbb{C}$. If L is a Lie algebra such that $\mathfrak{fsl}_Y(X) \leq L \leq \mathfrak{fgl}_Y(X)$, then L cannot be equipped with a complete algebra norm.*

Suppose on the contrary that L admits a complete algebra norm $|| \cdot ||$. Then it follows from [4, Proposition 4.3] that $(X, Y, < .,. >)$ becomes a Banach pairing and $|| \cdot ||$ majorizes the operator norm $| \cdot |$ of $\mathrm{BL}(X)$. Let $\{x_n\} \subset X$ and $\{y_n\} \subset Y$ be infinite sequences of vectors such that $\langle x_n, y_m \rangle = \delta_{nm}$, and set $a_n = y_n^* x_{n+1}$. Since $< x_{n+1}, y_n >= 0$, $a_n \in \mathfrak{fsl}_Y(X) \subset L$ and the infinite series $\sum_{n=1}^{\infty} \frac{a_n}{2^n ||a_n||}$ converges absolutely to an element $a \in L \subset \mathfrak{fgl}_Y(X)$. But since $|\cdot| \leq k||\cdot||$, we have that $\sum_{n=1}^{\infty} \frac{a_n}{2^n ||a_n||}$ also converges to a with respect to the operator norm. Hence $ax = \sum_{n=1}^{\infty} \frac{a_n x}{2^n ||a_n||}$ for any $x \in X$. Taking $x = x_m$, we get that $ax_m = \frac{x_{m+1}}{2^m ||a_m||}$ for any $m \geq 1$, which is a contradiction since a has finite rank.

Case II. *Let X be an infinite-dimensional complex vector space with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$. Then $\mathfrak{fo}(X, \langle \cdot, \cdot \rangle), | \cdot |)$ cannot be equipped with a complete algebra norm.*

The proof is similar to that of Case I, but taking a_n to be $[y_n, x_n]$ instead of $y_n^* x_{n+1}$, and where $\{(x_n, y_n)\}$ is now an infinite sequence of pairwise orthogonal hyperbolic pairs (3.5).

Case III. *Let X be an infinite-dimensional complex vector space with a nondegenerate alternate bilinear form $\langle \cdot, \cdot \rangle$. Then $\mathfrak{fsp}(X, \langle \cdot, \cdot \rangle)$ cannot be equipped with a complete algebra norm.*

Pick an infinite sequence $\{(x_n, y_n)\}$ up of pairwise orthogonal hyperbolic pairs, and set $a_n = y_n^* y_n$. ■ ❧

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