INNER IDEAL STRUCTURE OF NEARLY ARTINIAN LIE ALGEBRAS

ANTONIO FERNÁNDEZ LÓPEZ, ESTHER GARCÍA, AND MIGUEL GÓMEZ LOZANO

ABSTRACT. In this paper we study the inner ideal structure of nondegenerate Lie algebras with essential socle, and characterize, in terms of the whole algebra, when the socle is Artinian.

INTRODUCTION

Let L be a Lie algebra over a ring of scalars Φ. A Φ-submodule B of L is an inner ideal if $[B,[B,L]] \subset B$, and B is abelian if $[B,B]=0$. The initial motivation to study inner ideals in Lie algebras was due to the fact that inner ideals are closely related to ad-nilpotent elements, and certain restrictions of these elements yield an elementary criterion for distinguishing the nonclassical from classical (finite dimensional) simple Lie algebras over algebraic closed fields of characteristic greater than 5 [2].

In [1], G. Benkart examines the Lie inner ideal structure of semiprime associative rings, and of the skew elements of prime rings with involution. An extension of these results was carried out by the authors in [6], where the inner ideals of infinite dimensional finitary simple Lie algebras were described.

Inner ideals also become a key notion to develop a socle theory for nondegenerate Lie algebras [9], and were used in [8] to construct gradings of Lie algebras: it requires the existence of abelian inner ideals whose subquotient, a Jordan pair, is covered by a finite grid, and it produces a grading of the Lie algebra by the weight lattice of the root system associated to the covering grid.

Very recently, inner ideals, and their associated notions of kernel and complement, have allowed us to obtain [4] a Lie algebra analogue to the module theoretic characterization of semiprime one-sided Artinian associative rings (R is unital and completely reducible as a module), which parallels that due to O. Loos and E. Neher for Jordan systems [13].

Any nondegenerate Artinian Jordan pair agrees with its socle [12]. However, as mentioned in [4], there are examples of nondegenerate Artinian Lie algebras

---

Date: November 8, 2007.

2000 *Mathematics Subject Classification.* Primary 17B05; Secondary 17B60.

Key words and phrases. Artinian Lie algebras, inner ideals, socle, complementation.

The first author was partially supported by the MEC and Fondos FEDER, MTM2007-61978, and by the Junta de Andalucía FQM264.

The second author was partially supported by the MEC and Fondos FEDER, MTM2004-06580-C02-01 and MTM2007-62390, and by the Plan de Investigación del Principado de Asturias FICYT-IB05-017.

The third author was partially supported by the MEC and Fondos FEDER, MTM2007-61978, by the Plan de Investigación del Principado de Asturias FICYT-IB05-017, and by the Junta de Andalucía FQM264.

which do not coincide with their socles, although any nondegenerate Artinian Lie algebra has an essential Artinian socle [9]. In this paper we look into the inner ideal structure of nondegenerate Lie algebras with essential socle, and study what "having an essential Artinian socle" means for the whole algebra. This property will be related to being complemented (the socle) or abelian complemented (the whole algebra). Indeed, we show the following result:

Theorem *For a nondegenerate Lie algebra L over a ring of scalars Φ containing $\frac{1}{210}$ with essential socle S, the following conditions are equivalent:*

(i) *S is Artinian.*
(ii) *S is a complemented Lie algebra and has finitely many ideals.*
(iii) *L is abelian complemented and has finitely many simple ideals.*

1. Lie algebras and Jordan pairs

1.1. Throughout this paper, and at least otherwise specified, we will be dealing with Lie algebras L [10], [14] (with $[x, y]$ denoting the Lie bracket and ad_x the adjoint map determined by x), and Jordan pairs $V = (V^+, V^-)$ [11] (with Jordan triple products $\{x, y, z\}$, for $x, z \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$) over a ring of scalars Φ containing $\frac{1}{6}$. So Jordan pairs considered here are *linear.*

1.2. Let $V = (V^+, V^-)$ be a Jordan pair. An element $x \in V^\sigma$, $\sigma = \pm$, is called an *absolute zero divisor* if $Q_x = 0$. Thus V is said to be *nondegenerate* if it has no nonzero absolute zero divisors, *semiprime* if $Q_{B^\pm} B^\mp = 0$ implies $B = 0$, and *prime* if $Q_{B^\pm} C^\mp = 0$ implies $B = 0$ or $C = 0$, for any ideals $B = (B^+, B^-)$, $C = (C^+, C^-)$ of V. Similarly, given a Lie algebra L, $x \in L$ is an *absolute zero divisor* of L if $\mathrm{ad}_x^2 = 0$, L is *nondegenerate* if it has no nonzero absolute zero divisors, *semiprime* if $[I, I] = 0$ implies $I = 0$, and *prime* if $[I, J] = 0$ implies $I = 0$ or $J = 0$, for any ideals I, J of L. A Jordan pair or Lie algebra is *strongly prime* if it is prime and nondegenerate. We note that any ideal of a nondegenerate Lie algebra is again nondegenerate, see [16, Lemma 4]. A Lie algebra is *simple* if it is nonabelian and contains no proper ideals.

1.3. Given a Jordan pair $V = (V^+, V^-)$, an *inner ideal* of V is any Φ-submodule I of V^σ such that $\{I, V^{-\sigma}, I\} \subset I$. Similarly, an *inner ideal* of a Lie algebra L is a Φ-submodule B of L such that $[B, [B, L]] \subset B$. An *abelian inner ideal* is an inner ideal B which is also an abelian subalgebra, i.e., $[B, B] = 0$.

1.4. The annihilator of an ideal I in a Lie algebra L is defined as $\mathrm{Ann}\, I = \{x \in L \mid [x, I] = 0\}$. If L is nondegenerate, then $\mathrm{Ann}\, I = \{x \in L \mid [x, [I, x]] = 0\}$ and $I \cap \mathrm{Ann}\, I = 0$ [5, 2.5]. In this case, essential ideals of L have zero annihilator and hit nonzero inner ideals: if I is an essential ideal of L and B is an inner ideal of L, then $[B, [B, I]] \subset B \cap I = 0$ would imply $B \subset \mathrm{Ann}\, I = 0$.

1.5. An ad-nilpotent element $x \in L$ of index of nilpotency ≤ 3 is called a *Jordan element.* In this case, $\mathrm{ad}_x^2\, a$ is also a Jordan element for any $a \in L$ [7, 2.3(viii)]. By [2, 1.7(iii)], any Jordan element $x \in L$ satisfies the following analogue of the Jordan identity:

$$\mathrm{ad}^2_{\mathrm{ad}^2_x y} = \mathrm{ad}^2_x\, \mathrm{ad}^2_y\, \mathrm{ad}^2_x$$

for any $y \in L$. Clearly, any element of an abelian inner ideal is a Jordan element. Conversely, [2, 1.8], any Jordan element $b \in L$ yields the abelian inner ideals $[b] := [b, [b, L]]$ and $(b) := \Phi b + [b]$.

Proposition 1.6. *Let L be a nondegenerate Lie algebra ($\frac{1}{6} \in \Phi$), let I be an ideal of L, and let $x \in I$ be a Jordan element of I. Then x is a Jordan element of L.*

Proof. Since $\mathrm{ad}_x^3 I = 0$ and I is an ideal of L, $\mathrm{ad}_x^4 L = 0$. So, for every $a \in L$,

$$\begin{aligned} 0 = \mathrm{ad}_{\mathrm{ad}_x^4 a} &= X^4A - 4X^3AX + 6X^2AX^2 - 4XAX^3 + AX^4 \\ &= -4X^3AX + 6X^2AX^2 - 4XAX^3 \qquad (1) \end{aligned}$$

where capital letters denote the adjoint maps with respect to those elements. Since $\mathrm{ad}_x^2[x,a]$ is a Jordan element of I, for every $y \in I$ we have

$$\begin{aligned} \mathrm{ad}^2_{\mathrm{ad}_x^2[x,a]}\ y &= \mathrm{ad}_x^2\,\mathrm{ad}^2_{[x,a]}\,\mathrm{ad}_x^2\, y = (X^2(XA-AX)^2X^2)\ y \\ &= (-X^2AX^2AX^2 + X^2AXAX^3)\ y = -X^2AX^2AX^2\ y \\ &= (-\frac{2}{3}X^2AX^3AX - \frac{2}{3}X^2AXAX^3)\ y = 0 \text{ (by (1))}. \end{aligned}$$

Then $\mathrm{ad}_x^2[x,a] \in I \cap \mathrm{Ann}\, I = 0$, so x is a Jordan element of L. □

1.7. The *socle* of a nondegenerate Lie algebra L is defined as the sum of all minimal inner ideals of L. By [9, Theorem 2.5], $\mathrm{Soc}\, L$ is an ideal of L and a direct sum $\mathrm{Soc}\, L = \bigoplus_\alpha M_\alpha$ of simple ideals M_α of L. Moreover, each simple component M_α of $\mathrm{Soc}\, L$ is either inner simple or contains an abelian minimal inner ideal [2, Theorem 1.12].

A Lie algebra L is said to be *Artinian* if it satisfies the descending chain condition on all inner ideals. Simple nondegenerate Artinian Lie algebras coincide with their socles.

1.8. A Jordan element $e \in L$ is called *von Neumann regular* if $e \in \mathrm{ad}_e^2 L$. Assume that $\frac{1}{30} \in \Phi$. Then

(i) Any von Neumann regular element e of L can be extended to an *idempotent* (e,f) (see [15, V.8.2] or [9, Proposition 1.18], i.e.,

$$\mathrm{ad}_e^3 = \mathrm{ad}_f^3 = 0,\ [[e,f],e] = 2e \text{ and } [[e,f],f] = -2f.$$

Note that the last two conditions imply that $(e,[e,f],f)$ is a $\mathfrak{sl}(2)$-*triple*.

(ii) Any idempotent (e,f) yields a 5-grading $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ (called the *Peirce decomposition* of (e,f)), where each L_i is the eigenspace of the ad-semisimple element $h := [e,f]$ relative to the eigenvalue i, $i = 0, \pm 1, \pm 2$. Moreover, $L_2 = [e]$ and $L_{-2} = [f]$ [9, 1.18].

Proposition 1.9. *Let L be a nondegenerate Lie algebra ($\frac{1}{30} \in \Phi$), let I be an ideal of L, and let B be an abelian inner ideal of I. If any element of B is von Neumman regular, then B is an inner ideal of L.*

Proof. Since B is an abelian inner ideal of I, any $x \in B$ is a Jordan element of I, so, by 1.6, a Jordan element of L. Let $y \in I$ be such that $x = \mathrm{ad}_x^2 y$. By 1.5, $\mathrm{ad}_x^2 L = \mathrm{ad}^2_{\mathrm{ad}_x^2 y} L = \mathrm{ad}_x^2\,\mathrm{ad}_y^2\,\mathrm{ad}_x^2 L \subset \mathrm{ad}_x^2 I \subset B$. □

1.10. Let M be an inner ideal of a Lie algebra L. The *kernel* of M is the set $\mathrm{Ker}\, M = \{x \in L : [M,[M,x]] = 0\}$. For any abelian inner ideal M of L, the pair of Φ-modules $V = (M, L/\mathrm{Ker}\, M)$ with the triple products given by

$$\begin{aligned} \{m,\overline{a},n\} &:= [[m,a],n] \quad \text{for every } m,n \in M \text{ and } a \in L \\ \{\overline{a},m,\overline{b}\} &:= \overline{[[a,m],b]} \quad \text{for every } m \in M \text{ and } a,b \in L, \end{aligned}$$

where $\overline{x}$ denotes the coset of x relative to the submodule $\operatorname{Ker} M$, is a Jordan pair called the *subquotient of L with respect to M*, [8, Lemma 3.2].

1.11. A *complement* of an inner ideal M of L is another inner ideal N of L such that $L = M \oplus \operatorname{Ker} N = N \oplus \operatorname{Ker} M$. A Lie algebra L is said to be *(abelian) complemented* if any (abelian) inner ideal M of L has an (abelian) complement. It was shown in [4, 3.7(iii)] that complemented Lie algebras are abelian complemented. Moreover, subquotients of abelian complemented Lie algebras are complemented as Jordan pairs [4, 3.4].

2. Lie algebras with essential socle

Lemma 2.1. *Let L be a Lie algebra ($\frac{1}{30} \in \Phi$), let (e, f) be a nontrivial idempotent of L with Peirce decomposition $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$, and let I be an abelian inner ideal of L such that $e \in I$. Then*

(i) *$I = I_0 + I_1 + I_2$, with $I_i = L_i \cap I$ for each index i, and I_0, I_2 are inner ideals of L.*
(ii) *If $0 \neq y_0 \in I_0$ is von Neumann regular, then so is $e' = y_0 + e$, and both e, $y_0 \in [e']$.*
(iii) *If L is nondegenerate and $I_0 = 0$, then I_1 is also an inner ideal of L.*

Proof. (i) Since $e \in [e] = L_2 \subset I$, for any $y = y_{-2} + y_{-1} + y_0 + y_1 + y_2 \in I$ we have $0 = [y, e] = [y_{-2}, e] + [y_{-1}, e] + [y_0, e] \in L_0 \oplus L_1 \oplus L_2$. Hence

$$[y_i, e] = 0, \quad \text{for } i = -2, -1, 0, 1, 2. \tag{1}$$

Then

$$y_{-2} = -\frac{1}{2}[h, y_{-2}] = -\frac{1}{2}[[e, f], y_{-2}] = -\frac{1}{2}[[e, y_{-2}], f] = 0,$$
$$y_{-1} = -[h, y_{-1}] = -[[e, f], y_{-1}] = -[[e, y_{-1}], f] = 0.$$

Therefore, $I \subset L_0 \oplus L_1 \oplus L_2$. Now $y_1 + 2y_2 = [h, y] = [[e, f], y] \in I$. Hence $y_1 = [h, y] - 2y_2 \in I$ and $y_0 = y - y_1 - y_2 \in I$. Thus $I = I_0 \oplus I_1 \oplus I_2$, with $I_i = I \cap L_i$.

Clearly, $I_2 = L_2$ (since $[e] = L_2 \subset I$) is an inner ideal; let us now see that I_0 is also an inner ideal of L. By (1), $[[y_0, f], e] = [[y_0, e], f] - [y_0, h] = 0$, so $0 = [[[y_0, f], e], f] = [[y_0, f], h] = -2[y_0, f]$ and hence

$$[y_0, f] = 0. \tag{2}$$

Now, given $y_0 \in I_0$, we have

$$[y_0, L_2] \subset [I, I] = 0 \,(\text{because } I \text{ is abelian}), \tag{3}$$

$$[y_0, L_{-2}] = [y_0, [f, [f, L_2]]] = [f, [f, [y_0, L_2]]] = 0 \;\; (\text{by (2)}), \tag{4}$$

$$[y_0, [y_0, L_{-1}]] \subset L_{-1} \cap I = L_{-1} \cap (I_0 \oplus I_1 \oplus I_2) = 0, \;\; \text{and}$$
$$[y_0, [y_0, L_1]] \;= [y_0, [y_0, [e, L_{-1}]]] = [e, [y_0, [y_0, L_{-1}]]] = 0 \;\; (\text{by (1) and the fact,}$$

proved in [2, 2.1], that $L_1 = [e, L_{-1}]$). So $[y_0, [y_0, L]] = [y_0, [y_0, L_0]] \subset I_0$, i.e., I_0 is an inner ideal of L.

(ii) Let $0 \neq y_0 \in I_0$ be von Neumann regular. By grading properties, there exists $z_0 \in L_0$ such that $[y_0, [y_0, z_0]] = -2y_0$. Set $e' := e + y_0$. Then, using (1)-(4), it is routine to verify that e' is a Jordan element of L, i.e., $\mathrm{ad}^3_{e'} = 0$. We also have

$$\begin{aligned}[e', [e', f]] &= [e + y_0, [e + y_0, f]] = [e, [e, f]] + [e, [y_0, f]] \\ &+ [y_0, [e, f]] + [y_0, [y_0, f]] = [e, [e, f]] = -2e,\end{aligned} \tag{5}$$

since $[y_0, [e, f]] = [y_0, h] = 0$, and $[y_0, f] = 0$ by (2). We also have

$$\begin{aligned}[e', [e', z_0]] &= [e, [e, z_0]] + [e, [y_0, z_0]] + [y_0, [e, z_0]] \\ &+ [y_0, [y_0, z_0]] = [y_0, [y_0, z_0]] = -2y_0,\end{aligned} \tag{6}$$

since $[e, [e, z_0]] = 0$ by grading properties, and $[e, [y_0, z_0]] = [y_0, [e, z_0]] \in [y_0, L_2] = 0$ by (3). Therefore, $e, y_0 \in [e']$, and $e' = -\frac{1}{2}[e', [e', f + z_0]]$ is von Neumann regular.

(iii) Suppose that $I_0 = 0$. Given $y_1 \in I_1$, we have

$$[y_1, [y_1, L_{-2}]] \subset I \cap L_0 = I_0 = 0. \tag{7}$$

We also have, by grading properties, that

$$[y_1, [y_1, L_i]] = 0, \text{ for i=1, 2.} \tag{8}$$

Since I is abelian, $y_1 \in I$ is a Jordan element, so, by 1.5 and (7),

$$\mathrm{ad}^2_{\mathrm{ad}^2_{y_1} x_0} L = \mathrm{ad}^2_{\mathrm{ad}^2_{y_1} x_0} L_{-2} = \mathrm{ad}^2_{y_1} \mathrm{ad}^2_{x_0} \mathrm{ad}^2_{y_1} L_{-2} = 0,$$

and hence $[y_1, [y_1, L_0]] = 0$ by nondegeneracy of L. Then

$$[y_1, [y_1, L]] \subset [y_1, [y_1, L_{-1}]] \subset I \cap L_1 = I_1,$$

and I_1 is an inner ideal of L. □

The following result extends one by G. Benkart [2, 1.12] for simple nondegenerate Artinian Lie algebras over a field of characteristic 0 or $p > 3$.

Lemma 2.2. *Let L be a simple nondegenerate Lie algebra ($\frac{1}{210} \in \Phi$) containing minimal inner ideals. Then every proper inner ideal of L is abelian.*

Proof. Let $0 \neq B$ be a proper inner ideal of L. Since minimal inner ideals of L are either abelian or inner-simple ideals [2, 1.12], L contains abelian minimal inner ideals. Then, by the structure theorem of the simple Lie algebras containing abelian minimal inner ideals [9, 5.1] (here characteristic greater than 7 is required), there are three possibilities for L. If L is finite-dimensional over its centroid, then it is Artinian and hence any proper inner ideal of L is abelian by [2, 1.13]; if $L = [R, R]/Z(R) \cap [R, R]$ (where R is a simple ring with minimal one-sided ideals), then any proper inner ideal of L is of the form $B/Z(R) \cap [R, R]$, where B is a proper inner ideal of $[R, R]$ containing $Z(R) \cap [R, R]$, but proper inner ideals of $[R, R]$ are abelian by [1, 3.13]; if $L = [K, K]/Z(R) \cap [K, K]$ (where K is the set of skew elements of a simple ring R with minimal one-sided ideals relative to an involution $*$), then any proper ideal of L is of the form $B/Z(R) \cap [K, K]$, where B is a proper inner ideal of $[K, K]$. It follows from [1, 4.21] (if the involution is of the first kind), and from [1, 4.26] (if the involution is of the second kind), that B is abelian. □

Proposition 2.3. *Let L be a nondegenerate Lie algebra ($\frac{1}{210} \in \Phi$), and let $\operatorname{Soc} L = \bigoplus_\alpha M_\alpha$ be the decomposition of $\operatorname{Soc} L$ into its simple components. Then every inner ideal B of $\operatorname{Soc} L$ is an inner ideal of L. In fact, $B = \oplus B_\alpha$, where for each index α*

either $B_\alpha = M_\alpha$ is a simple component of $\operatorname{Soc} L$ *or B_α is an abelian inner ideal of L contained in M_α.*

Proof. For each index α, denote by π_α the projection of $\operatorname{Soc} L = \oplus M_\alpha$ onto M_α. Then $B_\alpha := \pi_\alpha(B)$ is an inner ideal of M_α. If $B_\alpha = M_\alpha$, then $M_\alpha = [M_\alpha, [M_\alpha, M_\alpha]] = [B, [B, M_\alpha]] \subset B$. Suppose then that B_α is a proper inner ideal of M_α. By 2.2, B_α is an *abelian* inner ideal of M_α. Let $b_\alpha = \pi_\alpha(b) \in B_\alpha$. Since by [7, 4.2] b_α is von Neumann regular, B_α is an inner ideal of L by 1.9, and $B_\alpha = [B_\alpha, [B_\alpha, M_\alpha]] = [B, [B, M_\alpha]] \subset B$. Hence $B = \oplus B_\alpha$ and it is an inner ideal of L. □

Proposition 2.4. *Let L be a nondegenerate Lie algebra ($\frac{1}{210} \in \Phi$). If* $\operatorname{Soc} L$ *is an essential ideal of L, then every inner ideal B of L containing no nonzero ideals is abelian.*

Proof. Suppose first that L is simple. Then B is a proper inner ideal of $L = \operatorname{Soc} L$ and hence B is abelian by 2.2.

Consider now the general case. By [9, 2.5], $\operatorname{Soc} L = \oplus M_\alpha$, where the M_α are simple ideals of L coinciding with their socles. Since M_α is not contained in B, $B \cap M_\alpha$ is a *proper* inner ideal of M_α. But M_α is simple , so $B \cap M_\alpha$ is abelian by 2.2. Let $b \in B$, $b' \in B \cap M_\alpha$ and $x \in L$, and set $a := [[b, b'], x]$. We have

$$a = [[b, b'], x] = [[b, x], b'] + [b, [b', x]] \in B \cap M_\alpha, \text{ and}$$
$$[b, a] = [b, [[b, x], b']] + [b, [b, [b', x]]] \in B \cap M_\alpha.$$

Then $[[b, b'], [[b, b'], x]] = [[b, b'], a] = [[b, a], b'] \in [B \cap M_\alpha, B \cap M_\alpha] = 0$, and hence $[b, b'] = 0$ by nondegeneracy of L. Therefore,

$$[B, B \cap M_\alpha] = 0. \tag{9}$$

Now let $b, c \in B$ and $y \in M_\alpha$, and set $a' := [[b, c], y]]$. We have

$$a' = [[b, c], y]] = [[b, y], c] + [b, [c, y]] \in B \cap M_\alpha, \text{ and hence}$$
$$[[b, c], [[b, c], y]] = [[b, c], a'] = [[b, a'], c] + [b, [c, a']] = 0 \text{ by (9).}$$

It follows from 1.4 that $[B, B] \subset \operatorname{Ann} M_\alpha$ for each M_α; but $\operatorname{Soc} L = \oplus M_\alpha$, so $[B, B] \subset \operatorname{Ann}(\operatorname{Soc} L) = 0$, because $\operatorname{Soc} L$ is an essential ideal, and B is abelian as required. □

3. Lie algebras with essential Artinian socle

Proposition 3.1. *Let L be a nondegenerate Lie algebra ($\frac{1}{210} \in \Phi$) with essential Artinian socle. Then L and* $\operatorname{Soc} L$ *share the same abelian inner ideals, so L satisfies the descending chain condition on abelian inner ideals.*

Proof. By 2.3 every abelian inner ideal of $\operatorname{Soc} L$ is an (abelian) inner ideal of L. Conversely, let B be an abelian inner ideal of L. We claim that B is contained in $\operatorname{Soc} L$. We may assume that $B \neq 0$. Since $\operatorname{Soc} L$ is essential, $B \cap \operatorname{Soc} L \neq 0$ by 1.4. Set $M := B \cap \operatorname{Soc} L$ and $V := (M, L/\operatorname{Ker} M)$. Since $\operatorname{Soc} L$ is Artinian, M has finite length and the nondegenerate Jordan pair V is also Artinian, [8, Proposition 3.5(v)]. Therefore, V satisfies the ascending chain condition on principal inner ideals and hence there exists an element $e \in M$ generating a principal inner ideal $[e]$ which is maximal in M. Moreover, since V is Artinian and nondegenerate, e is von Neumann regular in V [11, 10.17], i.e, e is part of an idempotent $(e, \bar{f})$ of V.

Use [8, Proposition 5.4(d)] to extend $(e, \bar{f})$ to an idempotent (e, f) of L, and let $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ be its associated Peirce-decomposition. By 2.1(i), $B = B_0 \oplus B_1 \oplus B_2$, $B_i = B \cap L_i$ and both B_0 and B_2 are abelian inner ideals of L, and B_2 is contained in $\operatorname{Soc} L$, because $e \in \operatorname{Soc} L$. Thus, to prove that the whole B is contained in $\operatorname{Soc} L$, it suffices to show that $B_0 = 0$. Suppose otherwise that $B_0 \neq 0$. By 1.4, $B_0 \cap \operatorname{Soc} L \neq 0$, so it contains a von Neumann regular element b_0 [7, 4.2]. Then, by 2.1(ii), $e' = e + b_0$ is a von Neumann regular element yielding a principal inner ideal $[e']$ greater than $[e]$ in M, which is a contradiction. So $B_0 = 0$ and $B = B_2 + B_1 \subset [e, [e, L]] + [e, L_{-1}] \subset \operatorname{Soc} L$, as required. □

Now we prove the main result of this paper:

Theorem 3.2. *Let L be a nondegenerate Lie algebra ($\frac{1}{210} \in \Phi$) with essential socle S. Then the following conditions are equivalent:*

(i) *S is Artinian.*
(ii) *S is a complemented Lie algebra and has finitely many ideals.*
(iii) *L is abelian complemented and has finitely many simple ideals.*

Proof. Note first that any of the hypothesis (i), (ii) or (iii) implies that S is a *finite* direct sum of simple ideals S_i coinciding with their socles.

(i)⇔(ii). It is [4, 3.7].

(ii)⇒(iii). Let B be an abelian inner ideal of L. By 3.1 $B \subset S$, and since S is abelian complemented [4, 3.7(iii)], B has an abelian complement in S, i.e., there exists an abelian inner ideal C of S such that $S = B \oplus \operatorname{Ker}_S C = C \oplus \operatorname{Ker}_S B$, but C is actually an abelian inner ideal of L 2.3. We also note that B has finite *length* (B does not contain any infinite properly ascending chain of inner ideals of L) because S is Artinian [4, 3.7].

We claim that there exists a Φ-submodule W of L contained in $\operatorname{Ker}_L B \cap \operatorname{Ker}_L C$ such that $L = S \oplus W$. By [8, 3.5(iii)(v)] and [4, 2.7], $V := (B, C) \cong (B, S/\operatorname{Ker}_S B)$ is a nondegenerate Artinian Jordan pair and hence it contains a maximal idempotent (b, c). Note that $b \in [b, [b, S]] = [b, [b, L]]$ and $c \in [c, [c, S]] = [c, [c, L]]$, and since, by 1.6, Jordan elements of S are Jordan elements of L, (b, c) is also an idempotent in L. Let $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ be the Peirce decomposition of (b, c) 1.8. By 2.1(i), $B = B_0 \oplus B_1 \oplus B_2$ and $C = C_{-2} \oplus C_{-1} \oplus C_0$. By maximality of (b, c) in V, $B_0 = C_0 = 0$ (otherwise, taking $0 \neq x \in B_0$, which is regular von Neumann because the subquotient $(B_0, S/\operatorname{Ker}_S B_0)$ is Artinian, and using 2.1(ii) we would obtain as in 3.1 that the principal inner ideal generated by $x + b$ is bigger than $[b, [b, S]]$, a contradiction). Hence it follows from 2.1(iii) that B_1 and C_{-1} are inner ideals of L. Since $L_{-2} \oplus L_{-1} \oplus L_1 \oplus L_2 \subset S$, any subspace W of L_0 such that $L_0 = (S \cap L_0) \oplus W$ satisfies $L = S \oplus W$. Moreover, $W \subset L_0 \subset \operatorname{Ker}_L B \cap \operatorname{Ker}_L C$: $[B, [B, L_0]] = [B_2, [B_2, L_0]] + [B_2, [B_1, L_0]] + [B_1, [B_2, L_0]] = 0$ (by grading properties), and $[B_1, [B_1, L_0]] \subset B_1 \cap L_2 = 0$ (by grading properties and the fact that B_1 is an inner ideal); similarly, $[C, [C, L_0]] = 0$, which proves the claim.

Now $B = B_1 \oplus B_2 \subset S$ implies $\operatorname{Ker}_L C \cap B = \operatorname{Ker}_S C \cap B = 0$, and by above, $L = S \oplus W = B \oplus \operatorname{Ker}_S C \oplus W = B \oplus \operatorname{Ker}_L C$, because $W \subset \operatorname{Ker}_L C$; similarly, $L = C \oplus \operatorname{Ker}_L B$.

(iii)⇒(i). Since S has finitely many simple components, it suffices to show that any simple component M of the socle is Artinian, equivalently, any proper (and therefore abelian) inner ideal B of M (and therefore also of L) has finite length. By

[8, 3.5(v)(vi)], $V = (B, L/\operatorname{Ker}_L B) \cong (B, M/\operatorname{Ker}_M B)$ is a simple nondegenerate Artinian Jordan pair, so B has finite length, as required. □

Remark 3.3. Nondegenerate Artinian Lie algebras have essential Artinian socle, [9, 2.6] and 2.3. However, a nodegenerate Lie algebra can have essential Artinian socle and not be Artinian itself, equivalently (by [4, 3.7], not be complemented. Consider the infinite dimensional division algebra $\Delta = K((t;\sigma))$ given by the σ-twisted Laurent series algebra over a field extension of a base field F by a countable set of indeterminates $K = F(..., x_{-1}, x_0, x_1, ...)$, $\sigma(x_n) = x_{n+1}$. Following the proof of [3, Example 2], Δ is infinite dimensional over $[\Delta,\Delta] + Z(\Delta)$ (none of the indeterminates $\{x_i\}$ belong to $[\Delta,\Delta] + Z(\Delta)$). Take $L = \Delta^{(-)}/Z(\Delta)$. Then L is strongly prime with inner-simple socle $[\Delta,\Delta] + Z(\Delta)/Z(\Delta)$, [1, 3.15] and [2, 2.2]. But L has an infinite descending chain of inner ideals: take $I_i = \operatorname{Soc} L + \sum_{k=i}^{\infty} F \cdot e_k$, where the cosets $\{e_i + [\Delta,\Delta] + Z(\Delta)\}_i$ are linearly independent.

Remark 3.4. One might think that abelian complemented nondegenerate Lie algebras coincide with Lie algebras where complementation holds on their essential socle. But if $\{S_i\}_{i\in\mathbb{N}}$ is an infinite family of simple nondegenerate Artinian Lie algebras containing abelian inner ideals, $L = \prod_{i\in\mathbb{N}} S_i$ has an essential socle $\bigoplus_{i\in\mathbb{N}} S_i$ which is a complemented Lie algebra, but L is not abelian complemented itself. Notice that infinite products of simple nondegenerate Artinian Lie algebras containing abelian inner ideals cannot be abelian complemented: if so, take for every $i \in \mathbb{N}$ a Jordan element $x_i \in S_i$ and consider $x = \pi_{i\in\mathbb{N}}(x_i) \in L$. Then $B = [x, [x, L]]$ is an abelian inner ideal of L whose subquotient $S = (B, L/\operatorname{Ker} B)$ is a complemented Jordan pair [4, 3.4]. By [13, 5.9] S coincides with its socle, so it satisfies the descending chain condition on principal inner ideals, but there exists a infinite descending chain of inner ideals $B_n = \{\pi_{i>n}(x_i), L/\operatorname{Ker} B, \pi_{i>n}(x_i)\}$ contained in S.

References

1. Georgia Benkart, *The Lie inner ideal structure of associative rings*, J. Algebra **43** (1976), no. 2, 561–584. MR MR0435149 (55 #8110)
2. ———, *On inner ideals and ad-nilpotent elements of Lie algebras*, Trans. Amer. Math. Soc. **232** (1977), 61–81. MR MR0466242 (57 #6122)
3. C. L. Chuang and P. H. Lee, *Idempotents in simple rings*, J. Algebra **56** (1979), no. 2, 510–515. MR MR528592 (80h:16013)
4. Antonio Fernández López, Esther García, and Miguel Gómez Lozano, *An Artinian theory for Lie algebras*, J. Algebra, to appear.
5. Antonio Fernández López, Esther García, and Miguel Gómez Lozano, *The Jordan socle and finitary Lie algebras*, J. Algebra **280** (2004), no. 2, 635–654. MR 2089256 (2005i:17007)
6. ———, *Inner ideals of finitary simple Lie algebras*, J. Lie Theory **16** (2006), no. 1, 97–114. MR MR2196417 (2006m:17020)
7. ———, *The Jordan algebras of a Lie algebra*, J. Algebra **308** (2007), no. 1, 164–177. MR 2290916 (2007k:17049)
8. Antonio Fernández López, Esther García, Miguel Gómez Lozano, and Erhard Neher, *A construction of gradings in Lie algebras*, Int. Math. Res. Not. (2007), doi: 10.1093/imrn/rnm051.
9. Draper Fontanals, Antonio Fernández López, Esther García, and Miguel Gómez Lozano, *The socle of a nondegenerate Lie algebra*, J. Algebra, to appear.
10. Nathan Jacobson, *Lie algebras*, Dover Publications Inc., New York, 1979, Republication of the 1962 original. MR MR559927 (80k:17001)
11. Ottmar Loos, *Jordan pairs*, Springer-Verlag, Berlin, 1975, Lecture Notes in Mathematics, Vol. 460. MR 0444721 (56 #3071)

12. ______, *On the socle of a Jordan pair*, Collect. Math. **40** (1989), no. 2, 109–125 (1990). MR MR1094683 (92a:17050)
13. Ottmar Loos and Erhard Neher, *Complementation of inner ideals in Jordan pairs*, J. Algebra **166** (1994), no. 2, 255–295. MR MR1279257 (95f:17028)
14. Erhard Neher, *Lie algebras graded by* 3*-graded root systems and Jordan pairs covered by grids*, Amer. J. Math. **118** (1996), no. 2, 439–491. MR MR1385288 (97a:17029)
15. G. B. Seligman, *Modular Lie algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 40, Springer-Verlag New York, Inc., New York, 1967. MR MR0245627 (39 #6933)
16. E. I. Zelmanov, *Lie algebras with algebraic associated representation*, Mat. Sb. (N.S.) **121(163)** (1983), no. 4, 545–561. MR 716113 (84k:17006)

Departamento de Álgebra, Geometría y Topología, Universidad de Málaga, 29071 Málaga, Spain

E-mail address: emalfer@agt.cie.uma.es, magomez@agt.cie.uma.es

Departamento de Matemática Aplicada, Universidad Rey Juan Carlos, 28933 Móstoles (Madrid), Spain

E-mail address: esther.garcia@urjc.es