SPECIAL IDENTITIES FOR QUASI-JORDAN ALGEBRAS

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ABSTRACT. Velásquez and Felipe defined a (right) quasi-Jordan algebra to be a nonassociative algebra satisfying right commutativity $a(bc) = a(cb)$ and the right quasi-Jordan identity $(ba)a^2 = (ba^2)a$. These identities are satisfied by the product $ab = \frac{1}{2}(a \dashv b + b \vdash a)$ in an associative dialgebra with operations $\dashv$ and $\vdash$ over a field of characteristic $\neq 2, 3$. This product also satisfies the associator-derivation identity $(b, a^2, c) = 2(b, a, c)a$. We use computer algebra to show that there are no new identities for this product in degree ≤ 7, but that six new irreducible identities exist in degree 8. These new identities are quasi-Jordan analogues of the Glennie identities for special Jordan algebras.

1. INTRODUCTION

Loday [17, 18] introduced a new variety of algebras with two binary operations:

Definition 1. An *associative dialgebra* is a vector space with bilinear operations $a \dashv b$ and $a \vdash b$, the *left* and *right* products, satisfying these polynomial identities:

$$(a \vdash b) \vdash c = (a \dashv b) \vdash c, \quad a \dashv (b \dashv c) = a \dashv (b \vdash c),$$
$$(a \dashv b) \dashv c = a \dashv (b \dashv c), \quad (a \vdash b) \vdash c = a \vdash (b \vdash c), \quad (a \vdash b) \dashv c = a \vdash (b \dashv c).$$

Since $(a \dashv b) \vdash c = a \dashv (b \vdash c)$ does not hold, associative dialgebras are a class of algebras that are "nearly associative"; see Shirshov [24], Zhevlakov et al. [30].

Definition 2. A *dialgebra monomial* on the set X of generators is a product $w = \overline{a_1 \cdots a_n}$ where $a_1, \ldots, a_n \in X$ and the bar indicates some placement of parentheses and some choice of operations. We define $c(w)$, the *center* of w, inductively: If $w \in X$ then $c(w) = w$; otherwise $c(w_1 \dashv w_2) = c(w_1)$ and $c(w_1 \vdash w_2) = c(w_2)$.

Lemma 3. (Loday [18], 1.7 Theorem) *If $w = \overline{a_1 \cdots a_n}$ is any dialgebra monomial with $c(w) = a_k$ then*

$$w = (a_1 \vdash \cdots \vdash a_{k-1}) \vdash a_k \dashv (a_{k+1} \dashv \cdots \dashv a_n). \tag{1}$$

Definition 4. The expression on the right side of equation (1) will be called the *normal form* of w and will be abbreviated as

$$w = a_1 \cdots a_{k-1} \widehat{a}_k a_{k+1} \cdots a_n.$$

Lemma 5. (Loday [18], 2.5 Theorem) *The monomials $a_1 \cdots a_{k-1} \widehat{a}_k a_{k+1} \cdots a_n$ with $1 \leq k \leq n$ and $a_1, \ldots, a_n \in X$ form a basis of the free associative dialgebra on the set X of generators.*

Definition 6. We write FD_n for the multilinear subspace of degree n in the free associative dialgebra on n free generators. It is clear from Lemma 5 that $\dim FD_n = n(n!)$.

Just as every associative algebra can be endowed with two nonassociative operations, the Lie bracket and the Jordan product, so every associative dialgebra can be endowed with the Leibniz bracket and the quasi-Jordan product.

Definition 7. (Loday [16]) The *Leibniz bracket* in a dialgebra is this bilinear operation:

$$[a, b] = a \dashv b - b \vdash a.$$

In an associative dialgebra this operation satisfies the *Leibniz identity*:

$$[[a, b], c] = [[a, c], b] + [a, [b, c]].$$

A *Leibniz algebra* is a nonassociative algebra satisfying the Leibniz identity.

Theorem 8. (Loday [18], Section 4) *Every polynomial identity satisfied by the Leibniz bracket in every associative dialgebra follows from the Leibniz identity.*

Definition 9. (Velásquez and Felipe [28, 29]) The *quasi-Jordan product* in a dialgebra over a field of characteristic $\neq 2$ is this bilinear operation:

$$a \triangleleft b = \tfrac{1}{2}(a \dashv b + b \vdash a).$$

If D is a dialgebra, then the *plus algebra* of D is the algebra D^+ with the same underlying vector space but the operation $a \triangleleft b$. In this paper we omit the product symbol $\triangleleft$ as well as the coefficient $\frac{1}{2}$; thus we write $ab = a \dashv b + b \vdash a$.

Definition 10. Consider three identities for an algebra: the right-commutative identity, the right quasi-Jordan identity, and the associator-derivation identity:

$$a(bc) = a(cb), \qquad (ba)a^2 = (ba^2)a, \qquad (b, a^2, c) = 2(b, a, c)a, \tag{2}$$

where $(a, b, c) = (ab)c - a(bc)$. The multilinear forms of the last two identities are:

$$J = (a(bc))d + (a(bd))c + (a(cd))b - (ab)(cd) - (ac)(bd) - (ad)(bc),$$
$$K = ((ab)d)c + ((ac)d)b - (a(bc))d - (a(bd))c - (a(cd))b + a((bc)d).$$

Remark 11. Identities similar to J and K appear in Kolesnikov [13], equations (26) and (27). We thank Raúl Felipe for this reference. See also Pozhidaev [21].

Lemma 12. (Velásquez and Felipe [28], Bremner [1]) *The quasi-Jordan product in an associative dialgebra satisfies the identities* (2)*, and these identities imply every identity of degree* ≤ 4 *for the quasi-Jordan product in an associative dialgebra.*

Definition 13. A *quasi-Jordan algebra* is a nonassociative algebra over a field F of characteristic $\neq 2, 3$ satisfying the identities (2). (Velásquez and Felipe [28] include only the first two identities in their definition of quasi-Jordan algebras.)

Definition 14. A quasi-Jordan algebra is *special* if it is isomorphic to a subalgebra of D^+ for some associative dialgebra D.

Glennie [5, 6] (see also Hentzel [9]) discovered identities satisfied by special Jordan algebras that are not satisfied by all Jordan algebras. In this paper we resolve the corresponding question for quasi-Jordan algebras. We use computer algebra to show that every identity of degree ≤ 7 for the quasi-Jordan product in an associative dialgebra is a consequence of the identities (2). We then demonstrate the existence of identities in degree 8 which do not follow from the identities of lower degree. We show that there are six new irreducible identities in degree 8, and present an explicit identity for which the variables in each term are a permutation of *aaaabbbc*. These new identities in degree 8 are satisfied by all special quasi-Jordan algebras but not satisfied by all quasi-Jordan algebras; they are quasi-Jordan analogues of the Glennie identities for special Jordan algebras.

Our methods depend on computational linear algebra on large matrices over a finite field, together with the representation theory of the symmetric group. Our computations were done with Maple, C and Albert [10].

2. Preliminaries on free nonassociative algebras

2.1. **Free right-commutative algebras.** The simplest identity satisfied by the quasi-Jordan product is right commutativity, $a(bc) = a(cb)$. Our computations depend on basic facts about free right-commutative algebras. As a reference for free nonassociative algebras, we mention Zhevlakov et al. [30, Chapter 1].

Lemma 15. *Let $w = \overline{a_1 \cdots a_n}$ be a nonassociative monomial of degree n on the set X of generators; that is, $a_1, \dots, a_n \in X$ and the bar denotes some placement of parentheses. If we assume right-commutativity, then in any submonomial $x = yz$ we may assume commutativity for the right factor z.*

Proof. By induction on n. For $n \le 2$ the claim is vacuous, and for $n = 3$ it is immediate from right-commutativity. The monomial w has the unique factorization $w = uv$; by the inductive hypothesis we may assume the result for u and v. Any right factor of a submonomial of w is either v, or a right factor of a submonomial of u, or a right factor of a submonomial of v. It therefore suffices to show that we may assume commutativity for v itself. We have the unique factorization $v = xy$, and we may assume commutativity for y. Right-commutativity implies that $uv = u(xy) = u(yx)$; and by induction we may assume commutativity for x. □

Lemma 15 gives an algorithm for generating inductively a complete minimal set of right-commutative association types up to a given degree n.

Algorithm 16. Assume that the right-commutative association types have been generated for degrees $1, \dots, n{-}1$. Any right-commutative association type in degree n has the form $w = uv$ where (for some $i = 1, \dots, n{-}1$) u is a right-commutative association type in degree $n{-}i$ and v is a commutative association type in degree i. This algorithm also induces a total order on the association types.

This gives a formula for the number R_n of right-commutative association types in degree n; we also compute the number C_n of commutative association types.

Lemma 17. *We have $C_1 = R_1 = 1$, and for $n \ge 2$ we have*

$$C_n = \sum_{i=1}^{\lfloor (n-1)/2 \rfloor} C_{n-i} C_i + \binom{C_{n/2}+1}{2}, \qquad R_n = \sum_{i=1}^{n-1} R_{n-i} C_i.$$

(The binomial coefficient only appears for n even.)

Proof. This follows directly from Algorithm 16. □

The following table gives the numbers C_n and R_n for $1 \le n \le 12$, together with the Catalan number K_n of all association types in degree n:

n	1	2	3	4	5	6	7	8	9	10	11	12
C_n	1	1	1	2	3	6	11	23	46	98	207	451
R_n	1	1	2	4	9	20	46	106	248	582	1376	3264
K_n	1	1	2	5	14	42	132	429	1430	4862	16796	58786

The quantities C_n are called the Wedderburn-Etherington numbers in Sloane's *On-Line Encyclopedia of Integer Sequences* [25] (sequence A001190). Consider the generating function:

$$G(x) = \sum_{n=1}^{\infty} C_n x^n = x + x^2 + x^3 + 2x^4 + 3x^5 + 6x^6 + 11x^7 + 23x^8 + \cdots .$$

Sloane [25] (sequence A085748) gives the next result.

Lemma 18. *The generating function of the R_n has the form*

$$\sum_{n=1}^{\infty} R_n x^n = \frac{x}{1-G(x)} = x + x^2 + 2x^3 + 4x^4 + 9x^5 + 20x^6 + 46x^7 + 106x^8 + \cdots$$

Definition 19. The *basic monomial* for an association type in degree n is the monomial in which the variables are the first n letters of the alphabet in lex order.

For $n = 1$ (respectively $n = 2$) we have the single type a (respectively ab) for both commutative and right-commutative algebras. For $n = 3, 4, 5$ each type is represented as follows by the corresponding basic monomial:

n	commutative	right-commutative
3	$(ab)c$	$(ab)c, a(bc)$
4	$((ab)c)d, (ab)(cd)$	$((ab)c)d, (a(bc))d, (ab)(cd), a((bc)d)$
5	$(((ab)c)d)e, ((ab)(cd))e, ((ab)c)(de)$	$(((ab)c)d)e, ((a(bc))d)e, ((ab)(cd))e,$ $(a((bc)d))e, ((ab)c)(de), (a(bc))(de),$ $(ab)((cd)e), a(((bc)d)e), a((bc)(de))$

Problem 20. Every subalgebra of an (absolutely) free nonassociative algebra is also free; see Kurosh [14, 15]. This statement also holds for free commutative and free anticommutative algebras; see Shirshov [22]. A variety of algebras with this property is called a Schreier variety; see Umirbaev [27]. Is the variety of right-commutative algebras a Schreier variety?

2.2. **Multilinear right-commutative monomials.** Throughout most of this paper we consider only multilinear identities: in degree n, the variables in each monomial are a permutation of the first n letters of the alphabet. To obtain a basis for the space of multilinear right-commutative polynomials in degree n, we need a straightening algorithm which replaces each monomial w by the first monomial (in lex order) in its equivalence class $[w]$: the set of all monomials which are equal to w as a consequence of right-commutativity. To straighten a right-commutative monomial, it suffices to determine the symmetries of its association type.

Definition 21. Let v be the basic monomial for a right-commutative association type in degree n. Let x and y be submonomials of v such that (i) x and y have the same degree and the same association type, and (ii) v contains the submonomial xy: $v = \cdots(xy)\cdots$. Let w be the monomial obtained from v by transposing x and y: $w = \cdots(yx)\cdots$. If right-commutativity implies $v = w$ then this identity will be called a *symmetry* of the association type.

Lemma 22. *If a right-commutative association type in degree n has s symmetries, then the number of multilinear monomials with this association type is $n!/2^s$.*

Proof. Each symmetry reduces the number of monomials by a factor of 2. □

We list the symmetries of the right-commutative association types in degree 5:

type 1: $(((ab)c)d)e$ has no symmetries;
type 2: $((a(bc))d)e = ((a(cb))d)e$; type 3: $((ab)(cd))e = ((ab)(dc))e$;
type 4: $(a((bc)d))e = (a((cb)d))e$; type 5: $((ab)c)(de) = ((ab)c)(ed)$;
type 6: $(a(bc))(de) = (a(cb))(de) = (a(bc))(ed)$;
type 7: $(ab)((cd)e) = (ab)((dc)e)$; type 8: $a(((bc)d)e) = a(((cb)d)e)$;
type 9: $a((bc)(de)) = a((cb)(de)) = a((bc)(ed)) = a((de)(bc))$.

These types have (respectively) 0, 1, 1, 1, 1, 2, 1, 1, 3 symmetries, and contain 120, 60, 60, 60, 60, 30, 60, 60, 15 distinct multilinear monomials, for a total of 525.

Definition 23. We write FRC_n for the multilinear subspace of degree n in the free right-commutative algebra on n free generators. As an ordered basis of FRC_n we have the distinct multilinear monomials in degree n, ordered first by association type, and then by lex order of the underlying permutation of the variables.

Lemma 24. *If $s(i)$ is the number of symmetries in association type i then*

$$\dim FRC_n = \sum_{i=1}^{R_n} \frac{n!}{2^{s(i)}}.$$

Proof. This follows directly from Lemma 22. □

Algorithm 25. This is a recursive algorithm to determine the symmetries of a right-commutative association type represented by the basic monomial $w = uv$. The algorithm uses a global variable `symmetrylist`, initially empty. On input w, the primary procedure `findsymmetry` calls itself on input u (the left factor) and then calls the secondary procedure `findcommutativesymmetry` on input v (the right factor). Writing $v = xy$, the secondary procedure calls itself on input x and then on input y; it then checks to see if x and y have the same association type, and if so it appends the symmetry $u(xy) = u(yx)$ to `symmetrylist`. Both procedures do nothing if the input has degree 1; this is the basis of the recursion.

The following table gives the number of right-commutative association types, the total number of symmetries over all association types, and the total number of multilinear right-commutative monomials, for $1 \leq n \leq 9$:

n	1	2	3	4	5	6	7	8	9
types (R_n)	1	1	2	4	9	20	46	106	248
symmetries	0	0	1	3	11	31	89	242	659
monomials	1	2	9	60	525	5670	72765	1081080	18243225

It is easy to verify that for $n \leq 9$ the number of monomials in degree n is given by the following formula from Sloane [25] (sequence A001193).

Conjecture 26. *For all $n \geq 1$ we have*

$$\dim FRC_n \stackrel{?}{=} \frac{n(2n-2)!}{2^{n-1}(n-1)!}.$$

2.3. **The expansion map and the expansion matrix.** In degree n we have the space FRC_n of multilinear right-commutative monomials and the space FD_n of multilinear dialgebra monomials.

Definition 27. We define a linear map $E_n\colon FRC_n \to FD_n$, the *expansion map*, inductively on basis monomials: If w has degree 1 then $E_1(w) = w$; otherwise $w = uv$ where u has degree $n{-}i$ and v has degree i, and

$$E_n(w) = E_{n-i}(u) \dashv E_i(v) + E_i(v) \vdash E_{n-i}(u). \tag{3}$$

The multilinear polynomial identities in degree n satisfied by the quasi-Jordan product are precisely the (nonzero) elements of the kernel of E_n. This kernel includes all the identities of degree n; many of these may be consequences of identities from lower degree. We need to distinguish the "old" from the "new" identities.

Definition 28. With respect to the ordered bases of FRC_n and FD_n, we represent E_n by the *expansion matrix* $[E_n]$ in which every entry is either 0 or 1: we have $[E_n]_{ij} = 1$ if and only if dialgebra monomial i occurs in the expansion of right-commutative monomial j. (The 96×60 matrix $[E_4]$ appears in Bremner [1].)

The sizes of the matrices $[E_n]$ grow very rapidly:

n	1	2	3	4	5	6	7	8	9
rows	1	4	18	96	600	4320	35280	322560	3265920
columns	1	2	9	60	525	5670	72765	1081080	18243225

We can use Maple's `LinearAlgebra` package to compute, using rational arithmetic, a basis of the nullspace of $[E_n]$ for $n \le 5$, and we can use `LinearAlgebra[Modular]` to compute, using modular arithmetic, a basis of the nullspace of $[E_n]$ for $n \le 6$. For $n \ge 7$ we must make the matrices smaller, and for this we use the representation theory of the symmetric group as described in Section 5.

Definition 27 gives a simple recursive algorithm for computing the expansion of a right-commutative monomial. In Maple, we represent a right-commutative monomial as a nested list containing two items, each of which is either a variable or another nested list representing a submonomial; the first n letters of the alphabet are represented by $1, \dots, n$. For example, the basic monomial $((ab)c)(de)$ is represented by $[[[1,2],3],[4,5]]$; applying the expansion algorithm gives a list of 16 dialgebra monomials where $x \dashv y$ and $x \vdash y$ are represented by $[x,L,y]$ and $[x,R,y]$:

$$
\begin{array}{llll}
[& [[[1,L,2],L,3],L,[4,L,5]], & [[4,L,5],R,[[1,L,2],L,3]], & [[[1,L,2],L,3],L,[5,R,4]], \\
 & [[5,R,4],R,[[1,L,2],L,3]], & [[3,R,[1,L,2]],L,[4,L,5]], & [[4,L,5],R,[3,R,[1,L,2]]], \\
 & [[3,R,[1,L,2]],L,[5,R,4]], & [[5,R,4],R,[3,R,[1,L,2]]], & [[[2,R,1],L,3],L,[4,L,5]], \\
 & [[4,L,5],R,[[2,R,1],L,3]], & [[[2,R,1],L,3],L,[5,R,4]], & [[5,R,4],R,[[2,R,1],L,3]], \\
 & [[3,R,[2,R,1]],L,[4,L,5]], & [[4,L,5],R,[3,R,[2,R,1]]], & [[3,R,[2,R,1]],L,[5,R,4]], \\
 & [[5,R,4],R,[3,R,[2,R,1]]] &].
\end{array}
$$

We now convert each dialgebra monomial to its normal form, using equation (1). In Maple we represent a dialgebra monomial in normal form by a nested list:

$$a_1 \cdots a_{k-1} \widehat{a}_k a_{k+1} \cdots a_n \longmapsto [\,[\iota(a_1), \dots, \iota(a_{k-1})],\ \iota(a_k),\ [\iota(a_{k+1}, \dots, \iota(a_n)]\,],$$

where $\iota(a_i)$ is the position of the variable a_i among the first n letters of the alphabet. The previous list of 16 dialgebra monomials becomes the following list:

[[[ ],1,[2,3,4,5]], [[4,5],1,[2,3]], [[ ],1,[2,3,5,4]], [[5,4],1,[2,3]], [[3],1,[2,4,5]],
[[4,5,3],1,[2]], [[3],1,[2,5,4]], [[5,4,3],1,[2]], [[2],1,[3,4,5]], [[4,5,2],1,[3]],
[[2],1,[3,5,4]], [[5,4,2],1,[3]], [[3,2],1,[4,5]], [[4,5,3,2],1,[ ]], [[3,2],1,[5,4]],
[[5,4,3,2],1,[ ]]].

In mathematical notation, we have computed the expansion $E_5(((ab)c)(de))$:

$$\widehat{a}bcde + de\widehat{a}bc + \widehat{a}bced + ed\widehat{a}bc + c\widehat{a}bde + dec\widehat{a}b + c\widehat{a}bed + edc\widehat{a}b$$
$$+ b\widehat{a}cde + deb\widehat{a}c + b\widehat{a}ced + edb\widehat{a}c + cb\widehat{a}de + decb\widehat{a} + cb\widehat{a}ed + edcb\widehat{a}.$$

To initialize the matrix $[E_n]$, we let j go from left to right across the columns, compute the expansion of the corresponding right-commutative monomial, obtain a sum of 2^{n-1} dialgebra monomials, convert each dialgebra monomial to normal form and determine its row index i, and set the (i,j) entry of the matrix to 1.

2.4. **Lifting multilinear identities.** Let $I(x_1,\dots,x_n)$ be a multilinear polynomial identity in degree n; we want to find all its consequences in degree $n{+}1$.

Definition 29. The *T-ideal* generated by $I = I(x_1,\dots,x_n)$ is the smallest ideal containing I which is sent into itself by all algebra homomorphisms.

The homomorphism condition in Definition 29 implies that we must consider the consequences of I obtained by introducing a new variable x_{n+1} and substituting $x_i x_{n+1}$ for x_i. The ideal condition implies that we must consider the consequences of I obtained by multiplying on the left or the right by x_{n+1}.

Lemma 30. *If $I(x_1,\dots,x_n)$ is a multilinear identity in degree n, then the following $n{+}2$ multilinear identities in degree $n{+}1$ generate all the consequences of I in degree $n{+}1$; that is, every consequence of I is a linear combination of permutations of these identities:*

$$I(x_1x_{n+1},x_2,\dots,x_n), \quad \dots, \quad I(x_1,\dots,x_{n-1},x_nx_{n+1}).$$
$$I(x_1,\dots,x_n)x_{n+1}, \quad x_{n+1}I(x_1,\dots,x_n).$$

Definition 31. The identities of Lemma 30 are the *liftings* of I to degree $n{+}1$.

This process can be repeated; an identity I in degree n will produce $(n{+}2)\cdots(n{+}k{+}1)$ liftings in degree $n{+}k$. These liftings may be redundant: a subset will generate all the consequences of I in degree $n{+}k$. We have already seen one example: the symmetries of the right-commutative association types in degree n are the liftings of right-commutativity from degree 3 to degree n. By our choice of association types, we have already eliminated most of the consequences of right-commutativity; only the symmetries within each association type remain.

In this paper, the most important examples of this process are the liftings of the multilinear identities J and K of Definition 10 from degree 4 to degree n. Lifting J and K to degree 5 is the first problem we must consider in the next section.

3. Nonexistence of new identities in degree 5

In this section we provide detailed examples of our methods; for higher degrees the objects we work with—polynomial identities and expansion matrices—become so large that it is impossible to include all the computations.

3.1. **Old identities.** Identities J and K each have six liftings to degree 5. The terms of each lifting must be straightened to lie in the standard basis of FRC_5:

$$
\begin{aligned}
&J(ae,b,c,d)\\
&= ((ae)(bc))d+((ae)(bd))c+((ae)(cd))b-((ae)b)(cd)-((ae)c)(bd)-((ae)d)(bc),\\
&J(a,be,c,d)\\
&= (a((be)c))d+(a((be)d))c+(a(cd))(be)-(a(be))(cd)-(ac)((be)d)-(ad)((be)c),\\
&J(a,b,ce,d)\\
&= (a(b(ce)))d+(a(bd))(ce)+(a((ce)d))b-(ab)((ce)d)-(a(ce))(bd)-(ad)(b(ce))\\
&= (a((ce)b))d+(a(bd))(ce)+(a((ce)d))b-(ab)((ce)d)-(a(ce))(bd)-(ad)((ce)b),\\
&J(a,b,c,de)\\
&= (a(bc))(de)+(a(b(de)))c+(a(c(de)))b-(ab)(c(de))-(ac)(b(de))-(a(de))(bc)\\
&= (a(bc))(de)+(a((de)b))c+(a((de)c))b-(ab)((de)c)-(ac)((de)b)-(a(de))(bc),\\
&J(a,b,c,d)e\\
&= ((a(bc))d)e+((a(bd))c)e+((a(cd))b)e-((ab)(cd))e-((ac)(bd))e-((ad)(bc))e,\\
&eJ(a,b,c,d)\\
&= e((a(bc))d)+e((a(bd))c)+e((a(cd))b)-e((ab)(cd))-e((ac)(bd))-e((ad)(bc)),\\
&= e(((bc)a)d)+e(((bd)a)c)+e(((cd)a)b)-e((ab)(cd))-e((ac)(bd))-e((ad)(bc)),\\
&K(ae,b,c,d)\\
&= (((ae)b)d)c+(((ae)c)d)b-((ae)(bc))d-((ae)(bd))c-((ae)(cd))b+(ae)((bc)d),\\
&K(a,be,c,d)\\
&= ((a(be))d)c+((ac)d)(be)-(a((be)c))d-(a((be)d))c-(a(cd))(be)+a(((be)c)d),\\
&K(a,b,ce,d)\\
&= ((ab)d)(ce)+((a(ce))d)b-(a(b(ce)))d-(a(bd))(ce)-(a((ce)d))b+a((b(ce))d)\\
&= ((ab)d)(ce)+((a(ce))d)b-(a((ce)b))d-(a(bd))(ce)-(a((ce)d))b+a(((ce)b)d),\\
&K(a,b,c,de)\\
&= ((ab)(de))c+((ac)(de))b-(a(bc))(de)-(a(b(de)))c-(a(c(de)))b+a((bc)(de))\\
&= ((ab)(de))c+((ac)(de))b-(a(bc))(de)-(a((de)b))c-(a((de)c))b+a((bc)(de)),\\
&K(a,b,c,d)e\\
&= (((ab)d)c)e+(((ac)d)b)e-((a(bc))d)e-((a(bd))c)e-((a(cd))b)e+(a((bc)d))e,\\
&eK(a,b,c,d)\\
&= e(((ab)d)c)+e(((ac)d)b)-e((a(bc))d)-e((a(bd))c)-e((a(cd))b)+e(a((bc)d)),\\
&= e(((ab)d)c)+e(((ac)d)b)-e(((bc)a)d)-e(((bd)a)c)-e(((cd)a)b)+e(((bc)d)a).
\end{aligned}
$$

We allocate memory for a matrix M of size 645×525 with an upper block (525×525) and a lower block (120×525); 525 is the number of multilinear right-commutative monomials, and 120 is the number of permutations. For each of the 12 lifted and straightened identities L displayed above, we do the following: for each permutation π_j of a, b, c, d, e (enumerated in lex order) we apply π_j to L, straighten the terms, and store the resulting coefficient vector in row $525{+}j$ of M. We compute the row canonical form of M and note the rank; the lower block of M is now zero. Using rational arithmetic with the Maple package `LinearAlgebra`, we obtain the following ranks: 20, 50, 50, 50, 70, 90, 150, 210, 210, 220, 250, 250. The lifted identities which do not increase the rank are redundant, so we consider only numbers 1, 2, 5, 6, 7, 8, 10, 11. (If we do the same computation with modular arithmetic using `LinearAlgebra[Modular]` then we obtain the same ranks approximately 1000 times faster.) The identities in degree 5 which are consequences of identities from lower degree span a 250-dimensional subspace of the 525-dimensional space FRC_5.

3.2. **All identities.** We allocate memory for a matrix E of size 600×525; this is the expansion matrix $[E_5]$. We compute the expansions of the basic monomials for the right-commutative association types; we have already seen one of these:

$$
\begin{aligned}
(((ab)c)d)e \mapsto\; & \widehat{a}bcde + e\widehat{a}bcd + d\widehat{a}bce + ed\widehat{a}bc + c\widehat{a}bde + ec\widehat{a}bd + dc\widehat{a}be + edc\widehat{a}b \\
& + b\widehat{a}cde + eb\widehat{a}cd + db\widehat{a}ce + edb\widehat{a}c + cb\widehat{a}de + ecb\widehat{a}d + dcb\widehat{a}e + edcb\widehat{a}, \\
((a(bc))d)e \mapsto\; & \widehat{a}bcde + e\widehat{a}bcd + d\widehat{a}bce + ed\widehat{a}bc + bc\widehat{a}de + ebc\widehat{a}d + dbc\widehat{a}e + edbc\widehat{a} \\
& + \widehat{a}cbde + e\widehat{a}cbd + d\widehat{a}cbe + ed\widehat{a}cb + cb\widehat{a}de + ecb\widehat{a}d + dcb\widehat{a}e + edcb\widehat{a}, \\
((ab)(cd))e \mapsto\; & \widehat{a}bcde + e\widehat{a}bcd + cd\widehat{a}be + ecd\widehat{a}b + \widehat{a}bdce + e\widehat{a}bdc + dc\widehat{a}be + edc\widehat{a}b \\
& + b\widehat{a}cde + eb\widehat{a}cd + cdb\widehat{a}e + ecdb\widehat{a} + b\widehat{a}dce + eb\widehat{a}dc + dcb\widehat{a}e + edcb\widehat{a}, \\
(a((bc)d))e \mapsto\; & \widehat{a}bcde + e\widehat{a}bcd + bcd\widehat{a}e + ebcd\widehat{a} + \widehat{a}dbce + e\widehat{a}dbc + dbc\widehat{a}e + edbc\widehat{a} \\
& + \widehat{a}cbde + e\widehat{a}cbd + cbd\widehat{a}e + ecbd\widehat{a} + \widehat{a}dcbe + e\widehat{a}dcb + dcb\widehat{a}e + edcb\widehat{a}, \\
((ab)c)(de) \mapsto\; & \widehat{a}bcde + de\widehat{a}bc + \widehat{a}bced + ed\widehat{a}bc + c\widehat{a}bde + dec\widehat{a}b + c\widehat{a}bed + edc\widehat{a}b \\
& + b\widehat{a}cde + deb\widehat{a}c + b\widehat{a}ced + edb\widehat{a}c + cb\widehat{a}de + decb\widehat{a} + cb\widehat{a}ed + edcb\widehat{a}, \\
(a(bc))(de) \mapsto\; & \widehat{a}bcde + de\widehat{a}bc + \widehat{a}bced + ed\widehat{a}bc + bc\widehat{a}de + debc\widehat{a} + bc\widehat{a}ed + edbc\widehat{a} \\
& + \widehat{a}cbde + de\widehat{a}cb + \widehat{a}cbed + ed\widehat{a}cb + cb\widehat{a}de + decb\widehat{a} + cb\widehat{a}ed + edcb\widehat{a}, \\
(ab)((cd)e) \mapsto\; & \widehat{a}bcde + cde\widehat{a}b + \widehat{a}becd + ecd\widehat{a}b + \widehat{a}bdce + dce\widehat{a}b + \widehat{a}bedc + edc\widehat{a}b \\
& + b\widehat{a}cde + cdeb\widehat{a} + b\widehat{a}ecd + ecdb\widehat{a} + b\widehat{a}dce + dceb\widehat{a} + b\widehat{a}edc + edcb\widehat{a}, \\
a(((bc)d)e) \mapsto\; & \widehat{a}bcde + bcde\widehat{a} + \widehat{a}ebcd + ebcd\widehat{a} + \widehat{a}dbce + dbce\widehat{a} + \widehat{a}edbc + edbc\widehat{a} \\
& + \widehat{a}cbde + cbde\widehat{a} + \widehat{a}ecbd + ecbd\widehat{a} + \widehat{a}dcbe + dcbe\widehat{a} + \widehat{a}edcb + edcb\widehat{a}, \\
a((bc)(de)) \mapsto\; & \widehat{a}bcde + bcde\widehat{a} + \widehat{a}debc + debc\widehat{a} + \widehat{a}bced + bced\widehat{a} + \widehat{a}edbc + edbc\widehat{a} \\
& + \widehat{a}cbde + cbde\widehat{a} + \widehat{a}decb + decb\widehat{a} + \widehat{a}cbed + cbed\widehat{a} + \widehat{a}edcb + edcb\widehat{a}.
\end{aligned}
$$

From these basic expansions we obtain the expansions of all 525 multilinear right-commutative monomials corresponding to the columns of E, and set to 1 the appropriate entries of E. We obtain a sparse 0-1 matrix in which each column has exactly 16 nonzero entries. We compute the rank of this matrix and obtain 275. Hence the subspace of FRC_5 consisting of polynomial identities satisfied by the quasi-Jordan product has dimension $525 - 275 = 250$.

Lemma 32. *Every polynomial identity in degree 5 for the quasi-Jordan product follows from the identities of degree ≤ 4: there are no new identities in degree 5.*

Proof. The subspace generated by the lifted identities is contained in the subspace of all identities; since the dimensions are equal, the subspaces are equal. □

4. Nonexistence of new identities in degree 6: first computation

Lemma 33. *In degree 6 there are 20 right-commutative association types:*

$$((((ab)c)d)e)f,\ (((a(bc))d)e)f,\ (((ab)(cd))e)f,\ ((a((bc)d))e)f,\ (((ab)c)(de))f,$$
$$((a(bc))(de))f,\ ((ab)((cd)e))f,\ (a(((bc)d)e))f,\ (a((bc)(de)))f,\ (((ab)c)d)(ef),$$
$$((a(bc))d)(ef),\ ((ab)(cd))(ef),\ (a((bc)d))(ef),\ ((ab)c)((de)f),\ (a(bc))((de)f),$$
$$(ab)(((cd)e)f),\ (ab)((cd)(ef)),\ a((((bc)d)e)f),\ a(((bc)(de))f),\ a(((bc)d)(ef)).$$

Each type has (respectively) 720, 360, 360, 360, 360, 180, 360, 360, 90, 360, 180, 180, 180, 360, 180, 360, 90, 360, 90, 180 monomials, for a total of 5670.

Proof. This follows directly from Lemmas 17 and 22. □

For a matrix with 5670 columns, it is not practical to compute the row canonical form using rational arithmetic. Instead we use modular arithmetic (with $p = 101$) to compute the dimensions of the subspaces of lifted identities and all identities.

4.1. **Old identities.** Our computations in degree 5 showed that we need only 8 of the 12 lifted identities in order to generate the subspace of all lifted identities:

$$J(ae,b,c,d), \qquad J(a,be,c,d), \qquad J(a,b,c,d)e, \qquad eJ(a,b,c,d),$$
$$K(ae,b,c,d), \qquad K(a,be,c,d), \qquad K(a,b,c,de), \qquad K(a,b,c,d)e.$$

Each of these identities produces 7 liftings in degree 6. Altogether we obtain an ordered list of 56 identities in degree 6.

We now follow the same algorithm as described for degree 5, except that in degree 6 the matrix M has size 6390×5670 with an upper block of size 5670×5670 and a lower block of size 720×5670. To each of the 56 lifted identities, we apply all 720 permutations of a, b, c, d, e, f and straighten the terms to obtain monomials in the standard basis of FRC_6; we then store the coefficient vectors of these identities in the rows of the lower block of M, and compute the row canonical form using the Maple package `LinearAlgebra[Modular]`. (Maple allocates 4 bytes of memory for every matrix entry, even if the modulus only requires 1 byte, so the matrix M uses almost 145 megabytes.) We obtain the following list of ranks: 120, 300, 300, 300, 360, 480, 540, 540, 720, 810, 810, 810, 990, 1170, 1170, 1170, 1170, 1170, 1230, 1350, 1410, 1410, 1410, 1410, 1410, 1530, 1626, 1626, 1986, 2346, 2346, 2406, 2586, 2766, 2766, 2766, 3126, 3210, 3330, 3330, 3510, 3510, 3510, 3510, 3510, 3510, 3510, 3570, 3570, 3570, 3570, 3570, 3570, 3570, 3690, 3690. Only 25 lifted identities in the ordered list produce an increase in the rank; these are numbers 1, 2, 5, 6, 7, 9, 10, 13, 14, 19, 20, 21, 26, 27, 29, 30, 32, 33, 34, 37, 38, 39, 41, 48, 55. The dimension of the subspace of FRC_6 consisting of the lifted identities is 3690.

4.2. **All identities.** In degree 6 the expansion matrix $[E_6]$ has size 4320×5670. As described for degree 5, we compute the expansions of the basic monomials in the 20 association types, determine the normal forms of the resulting dialgebra monomials, use these to obtain the expansions of all the multilinear right-commutative monomials, and store the results in the columns of E. We obtain a very sparse 0-1 matrix in which each column has 32 nonzero entries. We compute the rank of this matrix and obtain 1980, which implies that the nullspace has dimension $5670 - 1980 = 3690$. This is also the dimension of the subspace of lifted identities.

Lemma 34. *Every polynomial identity in degree 6 for the quasi-Jordan product (over the field with $p = 101$ elements) follows from the identities of degree ≤ 4: there are no new identities in degree 6.*

In the next two sections we show how to obtain the same result using smaller matrices.

5. Preliminaries on representation theory

5.1. **Representations of semisimple algebras.** Let A be a finite-dimensional semisimple associative algebra over a field F. We make A into a left A-module ${}^*A$, the left regular representation: $a \cdot b = ab$ for $a \in A$ and $b \in {}^*A$. We know that A is the direct sum of r simple two-sided ideals which are orthogonal as subalgebras:

$$A = A_1 \oplus \cdots \oplus A_r, \qquad A_i A_j = \{0\} \ (1 \leq i \neq j \leq r). \tag{4}$$

Each summand A_i is isomorphic to the algebra of $d_i \times d_i$ matrices with entries in some division algebra D_i over F. From equation (4) we obtain the direct sum decomposition of ${}^*A$ into isotypic components:

$${}^*A = {}^*A_1 \oplus \cdots \oplus {}^*A_r. \tag{5}$$

Each isotypic component ${}^*A_i$ decomposes as the direct sum of d_i simple submodules, all of which are isomorphic. Each of these simple submodules is a minimal left ideal in A, and can be regarded as a column in the algebra of $d_i \times d_i$ matrices.

We consider the direct sum of t copies of ${}^*A$, with the diagonal action:

$$({}^*A)^t = ({}^*A)^{[1]} \oplus \cdots \oplus ({}^*A)^{[t]}, \qquad a \cdot (b_1, \ldots, b_t) = (ab_1, \ldots, ab_t). \tag{6}$$

Let $U \subseteq ({}^*A)^t$ be a submodule. In general, U is not homogeneous with respect to the decomposition (6):

$$U \neq \sum_{k=1}^{t} \oplus \Big(U \cap ({}^*A)^{[k]} \Big). \tag{7}$$

We combine decompositions (5) and (6) to obtain a finer decomposition of $({}^*A)^t$, and then transpose the summations:

$$({}^*A)^t = \sum_{k=1}^{t} \oplus \big({}^*A\big)^{[k]} = \sum_{k=1}^{t} \oplus \sum_{i=1}^{r} \oplus \big({}^*A\big)_i^{[k]} = \sum_{i=1}^{r} \oplus \sum_{k=1}^{t} \oplus \big({}^*A\big)_i^{[k]}. \tag{8}$$

This is a direct sum decomposition of $({}^*A)^t$ into components R_i, each of which is the sum over all k of the simple two-sided ideals isomorphic to A_i:

$$({}^*A)^t = \sum_{i=1}^{r} \oplus R_i, \qquad R_i = \sum_{k=1}^{t} \oplus \big({}^*A\big)_i^{[k]}. \tag{9}$$

An arbitrary submodule U is homogeneous with respect to the decomposition (9).

Lemma 35. *For any submodule $U \subseteq ({}^*A)^t$, we have*

$$U = \sum_{i=1}^{r} \oplus \Big(U \cap R_i \Big).$$

Proof. For any $u \in U$, equation (9) shows that $u = u_1 + \cdots + u_r$ where $u_i \in R_i$. It suffices to show that each $u_i \in U$. Let $I_i \in A_i$ be the element corresponding to the $d_i \times d_i$ identity matrix. Equations (4), (6) and (9) imply that $I_i \cdot u = u_i$. □

5.2. **Irreducible representations of the symmetric group.** We apply this general construction to $A = FS_n$, the group algebra over F of the symmetric group S_n on n letters. We assume that either $F = \mathbb{Q}$, or $F = \mathbb{F}_p$ for $p > n$; then by Maschke's theorem the group algebra is semisimple.

We briefly recall the structure theory of FS_n; see James and Kerber [12]. The irreducible representations of S_n are in one-to-one correspondence with the partitions of n. Let $\lambda = (n_1, \dots, n_\ell)$ be a partition: $n = n_1 + \cdots + n_\ell$ with $n_1 \geq \cdots \geq n_\ell \geq 1$. The frame $[\lambda]$ consists of n empty boxes in ℓ rows (left-justified) with n_i boxes in row i. A tableau for λ consists of some placement of $1, \dots, n$ into the boxes of $[\lambda]$. A standard tableau is one in which the numbers increase in each row from left to right and in each column from top to bottom. The number d_λ of standard tableaux with frame $[\lambda]$ is the dimension of the corresponding irreducible representation of S_n. We have the following direct sum decomposition of the group algebra FS_n into orthogonal two-sided ideals isomorphic to simple matrix algebras over F:

$$FS_n \approx \sum_{\lambda} \oplus A_\lambda, \qquad A_\lambda = M_{d_\lambda}(F). \tag{10}$$

This is a special case of equation (4); the sum is over all partitions λ of n.

For us the most important problem is this: Given a permutation $\pi \in S_n$ and a partition λ of n, compute the $d_\lambda \times d_\lambda$ matrix in A_λ representing π; that is, compute the projection of π onto the summand A_λ in equation (10). A simple algorithm for this was found by Clifton [3]. Let $T_1, \dots, T_d$ ($d = d_\lambda$) be the standard tableaux for λ. Let E^λ_π be the matrix defined as follows; we quote [3] with a minor change of notation:

> Apply π to the tableau T_j. If there exist two numbers that appear together in a column of T_i and a row of πT_j, then $(E^\lambda_\pi)_{ij} = 0$. If not, then $(E^\lambda_\pi)_{ij}$ equals the sign of the vertical permutation for T_i which leaves the columns of T_i fixed as sets and takes the numbers of T_i into the correct rows they occupy in πT_j.

The matrix E^λ_{id} corresponding to the identity permutation is not necessarily the identity matrix, but it is always invertible.

Lemma 36. [3] *The matrix representing π in partition λ is equal to $(E^\lambda_{\mathrm{id}})^{-1} E^\lambda_\pi$.*

Since Clifton's algorithm is very important for us, we present it formally in Figure 1, following an idea of Hentzel: the algorithm tries to compute the vertical permutation whose sign gives $(E^\lambda_\pi)_{ij}$ and returns 0 if it fails.

- Input: A permutation $\pi \in S_n$ and a partition $\lambda = (n_1, \dots, n_\ell)$ of n.
- Output: The Clifton matrix E^λ_π.

(1) Compute the standard tableaux $T_1, \dots, T_d$ for λ where $d = d_\lambda$.
(2) For j from 1 to d do:
  (a) Compute πT_j.
  (b) For i from 1 to d do:
    (i) Set `ijentry` $\leftarrow 1$, `number` $\leftarrow 1$, `finished` $\leftarrow$ `false`.
    (ii) While `number` $\leq n$ and not `finished` do:
      - Set `irow`, `icol` $\leftarrow$ row, column indices of `number` in T_i.
      - Set `jrow`, `jcol` $\leftarrow$ row, column indices of `number` in πT_j.
      - If `irow` $\neq$ `jrow` then [`number` *is not in the correct row*]
        – If `icol` $> n_{\texttt{jrow}}$ then
            [*the required position does not exist*]
            set `ijentry` $\leftarrow 0$, `finished` $\leftarrow$ `true`
          else if $(T_i)_{\texttt{jrow,icol}} < (T_i)_{\texttt{irow,icol}}$ then
            [*the required position is already occupied*]
            set `ijentry` $\leftarrow 0$, `finished` $\leftarrow$ `true`
          else
            [*transpose* `number` *into the required position*]
            set `ijentry` $\leftarrow -$`ijentry`,
            interchange $(T_i)_{\texttt{irow,icol}} \leftrightarrow (T_i)_{\texttt{jrow,icol}}$
      - Set `number` $\leftarrow$ `number` $+ 1$
    (iii) Set $(E^\lambda_\pi)_{ij} \leftarrow$ `ijentry`
(3) Return E^λ_π.

FIGURE 1. Hentzel's algorithm to compute the Clifton matrix E^λ_π

5.3. **Polynomial identities and representation theory.** The application of the representation theory of the symmetric group to polynomial identities was initiated independently in 1950 by Malcev [20] and Specht [26]. The implementation of these techniques in computer algebra was initiated by Hentzel [7, 8] in the 1970's.

We first recall that any polynomial identity (not necessarily multilinear or even homogeneous) of degree $\leq n$ over a field F of characteristic 0 or $p > n$ is equivalent to a finite set of multilinear identities; see Zhevlakov et al. [30] (Chapter 1). We therefore consider a multilinear nonassociative polynomial identity $I(x_1, \dots, x_n)$ of degree n. We collect the terms of I which have the same association type, and write $I = I_1 + \cdots + I_t$. In each summand I_k for $1 \leq k \leq t$, all the monomials have association type k: they differ only by the permutation of the variables $x_1, \dots, x_n$. We can therefore regard each I_k as an element of the group algebra FS_n, and the identity I as an element of the direct sum of t copies of FS_n, one for each association type. Following the discussion in the previous two subsections, let U be the submodule of $(FS_n)^t$ generated by I. Every element of U is a linear combination of permutations of I, and hence is a polynomial identity implied by I. By Lemma 35 we know that U is the direct sum of its components corresponding to the irreducible representations of S_n. This allows us to study I and its consequences one representation at a time, and this means that we can break down a large computational problem into much smaller pieces.

$$\left[\begin{array}{ccccc|ccccc}
\rho_\lambda(E_1^1) & \rho_\lambda(E_2^1) & \cdots & \rho_\lambda(E_{n-1}^1) & \rho_\lambda(E_n^1) & -I_d & O & \cdots & O & O \\
\rho_\lambda(E_1^2) & \rho_\lambda(E_2^2) & \cdots & \rho_\lambda(E_{n-1}^2) & \rho_\lambda(E_n^2) & O & -I_d & \cdots & O & O \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\rho_\lambda(E_1^{t-1}) & \rho_\lambda(E_2^{t-1}) & \cdots & \rho_\lambda(E_{n-1}^{t-1}) & \rho_\lambda(E_n^{t-1}) & O & O & \cdots & -I_d & O \\
\rho_\lambda(E_1^t) & \rho_\lambda(E_2^t) & \cdots & \rho_\lambda(E_{n-1}^t) & \rho_\lambda(E_n^t) & O & O & \cdots & O & -I_d
\end{array}\right]$$

TABLE 1. Representation matrix of the dialgebra expansions of the right-commutative association types (partition λ, degree n)

Example 37. To illustrate this, we take a different approach to Example 2 from [2, Section 9]. In a commutative nonassociative algebra we have the Jordan identity, $(a^2b)a - a^2(ba)$. The multilinear form of this identity (divided by 2) is

$$u = ((ac)b)d + ((ad)b)c + ((cd)b)a - (ac)(bd) - (ad)(bc) - (cd)(ba).$$

In degree 4, for a commutative nonassociative operation, there are two association types: $((ab)c)d$ and $(ab)(cd)$. We regard $u = u_1 + u_2$ as an element of the direct sum of two copies of the left regular representation of the group algebra $\mathbb{Q}S_4$:

$$u_1 = acbd + adbc + cdba \quad \text{(type 1)}, \qquad u_2 = -acbd - adbc - cdba \quad \text{(type 2)}.$$

To illustrate the inequality (7) we note that the two components u_1 and u_2 represent identities which are not consequences of the Jordan identity:

$$u_1 \leftrightarrow ((ac)b)d + ((ad)b)c + ((cd)b)a, \qquad u_2 \leftrightarrow -(ac)(bd) - (ad)(bc) - (cd)(ba).$$

To illustrate the equality of Lemma 35 we decompose the submodule $U \subseteq (\mathbb{Q}S_4)^2$ generated by u into components corresponding to the irreducible representations of S_4. We find that the Jordan identity implies (the linearization of) fourth-power associativity (corresponding to $\lambda = 4$) and the identity which says that the commutator of multiplications is a derivation (corresponding to $\lambda = 31$):

$$(a^2a)a - (a^2)^2, \qquad (ab)[M_c, M_d] - (a[M_c, M_d])b - a(b[M_c, M_d]).$$

5.4. **Ranks and multiplicities.** Let $u^{[i]}$ for $1 \le i \le g$ be a set of multilinear nonassociative polynomial identities of degree n over a field F of characteristic 0 or $p > n$. Suppose that the terms of the $u^{[i]}$ involve t association types, and let $U \subseteq (FS_n)^t$ be the submodule generated by the $u^{[i]}$. We fix a partition λ of n and write $d = d_\lambda$ for the dimension of the corresponding irreducible representation. To determine the component of U in representation λ we construct a matrix M_λ with dg rows and dt columns, regarded as a block matrix with g rows and t columns of $d \times d$ blocks. In the block in position (i, j) we put the representation matrix for the terms of $u^{[i]}$ in association type j; this matrix can be computed by repeated application of Lemma 36. We then compute the row canonical form of M_λ.

Definition 38. The number of nonzero rows of the row canonical form of M_λ (that is, the rank of M_λ) is the *rank of the submodule U in partition λ*.

Lemma 39. *The rank of M_λ is the multiplicity of the irreducible representation corresponding to λ in the submodule U.*

A modification of this procedure can be used to determine the structure of the kernel of the expansion map. In degree n, there are $t = R_n$ right-commutative association types and

n dialgebra association types. (The number of right-commutative types is given by Lemma 17; the number of dialgebra types corresponds to the positions of the center.) We choose a partition λ and write d for the dimension of the corresponding irreducible representation. We create a matrix X with td rows and $(n{+}t)d$ columns; see Table 1. We regard X as a $t \times (n{+}t)$ matrix of $d \times d$ blocks, with a left side of size $td \times nd$ and a right side of size $td \times td$. In the right side, in block $(n{+}i, i)$ for $1 \leq i \leq t$, we put $-I_d$, the negative of the $d \times d$ identity matrix; the other blocks of the right side are zero. In the left side, in block (i, j) for $1 \leq i \leq t$ and $1 \leq j \leq n$, we put $\rho_\lambda(E^i_j)$: the representation matrix for partition λ of the terms in dialgebra association type j of the expansion of the basic monomial in right-commutative association type i. The i-th row of blocks represents the polynomial identity which states that the basic monomial for the i-th right-commutative association type equals its expansion in the free associative dialgebra. Since the right side is the negative of the identity matrix, X has full row rank.

We compute the row canonical form of X; there are no zero rows, since X has full row rank. We introduce a division between upper and lower parts of the row canonical form: the upper part contains the rows with leading ones in the left side, and the lower part contains the rows with leading ones in the right side. The lower left block is the zero matrix; the lower right block contains rows representing polynomial identities which are satisfied by the right-commutative association types as a result of dependence relations among the dialgebra expansions of the basic right-commutative monomials.

Definition 40. The number of rows in the lower right block of the row canonical form of the matrix X in Table 1 is the *rank of all identities in the partition* λ.

Lemma 41. *The rank of all identities in partition λ is the multiplicity of the corresponding irreducible representation in the kernel of the expansion map.*

Lemma 42. *Suppose that the module U of lifted identities in degree n for the quasi-Jordan product is generated by identities $u^{[i]}$ for $1 \leq i \leq g$. Let λ be a partition of n, let* $\text{oldrank}(\lambda)$ *be the rank of the submodule U in the partition λ from Definition 38, and let* $\text{allrank}(\lambda)$ *be the rank of all identities in the partition λ from Definition 40. Then* $\text{oldrank}(\lambda) \leq \text{allrank}(\lambda)$ *with equality if and only if there are no new identities corresponding to representation λ.*

5.5. **Rational arithmetic and modular arithmetic.** We prefer to do these computations using rational arithmetic, but this is impractical when the matrices are too large: during the computation of the row canonical form, the numerators and denominators of the matrix entries can become extremely large, even if the original matrix (and its row canonical form) have small integer entries.

To control the amount of memory used, it is sometimes necessary to use modular arithmetic, with a prime p greater than the degree n of the identities. This choice of p guarantees that the group algebra $\mathbb{F}_p S_n$ is semisimple. Furthermore, the structure theory of $\mathbb{Q}S_n$ shows that—referring to the isomorphism (10)—the idempotents in the group algebra which represent the matrix units in the simple ideals A_λ all have coefficients in which the denominators are divisors of $n!$. It follows that the S_n-module $(FS_n)^t$ has the "same" structure over $\mathbb{F}_p$ as over $\mathbb{Q}$ whenever $p > n$. We can therefore be confident that the ranks we obtain using modular arithmetic will be the same as the ranks we would have obtained using rational arithmetic.

This leaves open the question of reconstructing the correct rational results from modular computations. In some cases (as in this paper) the modular results we obtain using (say)

			old identities			all identities			
	λ	d	rows	cols	rank	rows	cols	rank	new
1	6	1	21	20	17	20	26	17	0
2	51	5	105	100	85	100	130	85	0
3	42	9	189	180	153	180	234	153	0
4	411	10	210	200	172	200	260	172	0
5	33	5	105	100	85	100	130	85	0
6	321	16	336	320	274	320	416	274	0
7	3111	10	210	200	176	200	260	176	0
8	222	5	105	100	85	100	130	85	0
9	2211	9	189	180	157	180	234	157	0
10	21111	5	105	100	91	100	130	91	0
11	111111	1	21	20	19	20	26	19	0

Table 2. Degree 6: matrix ranks for all representations

$p = 101$ include only coefficients for which the corresponding rational numbers are easy to reconstruct: for example, 1, 2, 3, 49, 50, 51, 52, 98, 99, 100 in $\mathbb{F}_{101}$ represent 1, 2, 3, $-3/2$, $-1/2$, $1/2$, $3/2$, -3, -2, -1 in $\mathbb{Q}$. (In other cases we have to use many different primes and the Chinese Remainder Theorem to reconstruct the rational results.)

6. Nonexistence of new identities in degree 6: second computation

The ranks in Table 2 are from modular arithmetic with $p = 101$. There are no new identities, confirming our earlier computations without representation theory.

When we use representation theory, we have two types of lifted identities: the 31 identities which represent the symmetries of the association types (the liftings of the right-commutative identity from degree 3) and the 56 liftings of the identities J and K from degree 4. In order to process these identities for partition λ, we create a matrix with td columns (t is the number of association types and d is the dimension of the irreducible representation) and $td+d$ rows. We include the identities one at a time in the bottom $d \times td$ part; after each fill of the bottom part we compute the row canonical form. In Table 2, we have $t = 20$: under the heading "old identities", the column labeled "rows" contains $td + d = 21d$ and the column labeled "cols" contains $td = 20d$. The column labeled "rank" contains the rank of the matrix.

Out of the complete list of 56 identities obtained by lifting the known identities from degree 5 to degree 6, we retain only those identities which increase the rank in at least one representation. We recover the same list of 25 generators that we obtained earlier without representation theory.

To compute all the identities for a given partition λ, we create a matrix with td rows and $nd + td$ columns: the left block of size $td \times nd$ corresponds to the dialgebra expansions, and the right block of size $td \times td$ corresponds to the basic right-commutative monomials. We obtain $20d$ rows and $26d$ columns; these are the numbers labeled "rows" and "cols" under "all identities" in Table 2. The column labeled "rank" contains the number of nonzero rows in the row canonical form which have leading ones in the lower right block of the matrix: these rows represent identities satisfied by the quasi-Jordan product.

When the two ranks are the same for partition λ, it follows that there are no new identities for the corresponding representation. We checked these results by verifying that the two

			old identities			all identities			
	λ	d	rows	cols	rank	rows	cols	rank	new
1	7	1	47	46	42	46	53	42	0
2	61	6	282	276	255	276	318	255	0
3	52	14	658	644	594	644	742	594	0
4	511	15	705	690	641	690	795	641	0
5	43	14	658	644	595	644	742	595	0
6	421	35	1645	1610	1490	1610	1855	1490	0
7	4111	20	940	920	859	920	1060	859	0
8	331	21	987	966	895	966	1113	895	0
9	322	21	987	966	892	966	1113	892	0
10	3211	35	1645	1610	1499	1610	1855	1499	0
11	31111	15	705	690	651	690	795	651	0
12	2221	14	658	644	598	644	742	598	0
13	22111	14	658	644	607	644	742	607	0
14	211111	6	282	276	265	276	318	265	0
15	1111111	1	47	46	45	46	53	45	0

TABLE 3. Degree 7: matrix ranks for all representations

resulting matrices are in fact equal. More precisely, let r be the common rank for partition λ. The first matrix has size $r \times td$; these are the nonzero rows of the row canonical form of the matrix for the lifted identities. The second matrix has the same size; it contains the rows—of the row canonical form of the matrix for the expansion identities—with leading ones in the lower right block.

7. Nonexistence of new identities in degree 7

See Table 3: there are no new identities in degree 7. The computations in this degree are similar to those for degree 6, except that the matrices are larger. The lifted ("old") identities consist of 89 symmetries and 200 lifted identities. There are $t = 46$ association types for the right-commutative monomials and $n = 7$ association types for the dialgebra monomials, so the matrix of "old identities" has size $47d \times 46d$, and the matrix of "all identities" has size $46d \times 53d$. A subset of 55 identities suffices to generate the lifted identities. The ranks in Table 3 were computed using modular arithmetic with $p = 101$.

8. New identities in degree 8

The computations in degree 8 are similar to those for degree 7, except that the matrices are larger. In Table 4 the ranks are equal in all representations except numbers 9, 10, 13, 14, 15 corresponding to partitions 431, 422, 332, 3311, 3221 where the differences between allrank(λ) and oldrank(λ) are 1, 1, 2, 1, 1. The lifted ("old") identities consist of 242 symmetries of the association types together with 495 liftings of J and K. There are $t = 106$ association types for the right-commutative monomials and $n = 8$ association types for the dialgebra monomials, so the matrix of "old identities" has size $107d \times 106d$, and the matrix of "all identities" has size $106d \times 114d$. We find that it suffices to consider only 186 of the liftings of J and K. The ranks in Table 4 were computed using modular arithmetic

			old identities			all identities				
	λ	d	rows	cols	rank	rows	cols	rank	new	
1	8	1	107	106	102	106	114	102	0	
2	71	7	749	742	714	742	798	714	0	
3	62	20	2140	2120	2040	2120	2280	2040	0	
4	611	21	2247	2226	2145	2226	2394	2145	0	
5	53	28	2996	2968	2856	2968	3192	2856	0	
6	521	64	6848	6784	6532	6784	7296	6532	0	
7	5111	35	3745	3710	3582	3710	3990	3582	0	
8	44	14	1498	1484	1428	1484	1596	1428	0	
9	431	70	7490	7420	7142	7420	7980	7143	1	$\leftarrow$
10	422	56	5992	5936	5712	5936	6384	5713	1	$\leftarrow$
11	4211	90	9630	9540	9199	9540	10260	9199	0	
12	41111	35	3745	3710	3594	3710	3990	3594	0	
13	332	42	4494	4452	4284	4452	4788	4286	2	$\leftarrow$
14	3311	56	5992	5936	5722	5936	6384	5723	1	$\leftarrow$
15	3221	70	7490	7420	7149	7420	7980	7150	1	$\leftarrow$
16	32111	64	6848	6784	6565	6784	7296	6565	0	
17	311111	21	2247	2226	2169	2226	2394	2169	0	
18	2222	14	1498	1484	1429	1484	1596	1429	0	
19	22211	28	2996	2968	2870	2968	3192	2870	0	
20	221111	20	2140	2120	2065	2120	2280	2065	0	
21	2111111	7	749	742	729	742	798	729	0	
22	11111111	1	107	106	105	106	114	105	0	

TABLE 4. Degree 8: matrix ranks for all representations

with $p = 101$. But we can recover rational results from modular results, and use rational arithmetic to verify the results; see the next section for the case $\lambda = 431$.

Definition 43. We say that a polynomial identity in degree n is *irreducible* if its complete linearization in FRC_n generates an irreducible representation of S_n.

Theorem 44. *There are six new irreducible identities for the quasi-Jordan product in degree 8: one each for partitions 431, 422, 3311, 3221 and two for partition 332.*

Corollary 45. *There exist exceptional (non-special) quasi-Jordan algebras.*

Example 46. Consider the subvariety $\mathcal{N}$ of quasi-Jordan algebras defined by the identities $\overline{x_1 \cdots x_9} = 0$, where the bar denotes any placement of parentheses. Let X be the free algebra in $\mathcal{N}$ on the generators a, b, c. The quasi-Jordan polynomial displayed in Tables 5 and 6 below is nonzero in X, but vanishes in every special quasi-Jordan algebra; see the next section for details.

Remark 47. The existence of exceptional quasi-Jordan algebras also follows from the existence of special identities for Jordan algebras, as follows. There is a canonical map from special quasi-Jordan algebras to special Jordan algebras obtained by identifying the dialgebra operations $\dashv$ and $\vdash$. Any element in the inverse image of the Glennie identity of degree 8 will be a special identity of degree 8 for quasi-Jordan algebras. (We thank Ivan Shestakov for pointing this out.)

9. A special quasi-Jordan identity for partition 431

Since the rank has increased by 1 for partition 431, we expect there to be a new identity with 4 a's, 3 b's and 1 c: every monomial consists of a right-commutative association type applied to a permutation of $aaaabbbc$. There are 106 association types and 280 permutations, so an upper bound for the number of distinct monomials is 29680. (The other partitions which have new identities give even larger numbers.) However, right-commutativity implies that many of these monomials are equal; we only count those which are equal to their own straightened forms, and we obtain 12131 distinct monomials. When we consider dialgebra monomials with the same variables, we have 8 association types and 280 permutations, for a total of 2240 distinct monomials. The expansion of each nonassociative monomial is a linear combination of 128 of these 2240 dialgebra monomials.

In step 1, we create a matrix of size 12411×12131 with an upper block of size 12131×12131 and a lower block of size 280×12131. For each of the 186 multilinear generators of the lifted identities in degree 8, we apply all 280 substitutions of 4 a's, 3 b's and 1 c into the terms of the generator, store these nonlinear identities in the lower block of the matrix, and compute the row canonical form using arithmetic modulo $p = 101$. After this process is complete, the rank of the matrix is 11020.

In step 2, we create a matrix of size 2240×12131, initialize it with the coefficients of the expansions, and compute the row canonical form using arithmetic modulo $p = 101$. The rank is 1110 and the nullspace has dimension $12131 - 1110 = 11021$.

The difference between the rank for step 1 and the nullity for step 2 is exactly 1, as expected from Table 4. The row space from step 1 is a subspace of the nullspace from step 2. We need to find a nullspace vector which is not in the row space.

In step 3, we compute the canonical basis of the nullspace from step 2. We sort the basis vectors by increasing number of distinct coefficients. For our purposes, the most natural notion of "length" for a vector over a finite field is the number of distinct coefficients. We include the basis vectors one at a time as a new bottom row of the matrix from step 1 until we find the first vector that increases the rank. This vector contains the coefficients of the 296-term identity in Tables 5 and 6. (More precisely, we multiply the coefficients by 2 and then reduce modulo 101 using symmetric representatives; in this way all the coefficients become small integers.) We expand the identity using rational arithmetic and verify that it collapses to zero in the free associative dialgebra; hence it is a special identity over $\mathbb{Q}$.

The special identity that we have discovered has the property that it involves three variables and is linear in one of them. This implies that the obvious generalization of Macdonald's theorem [19] to quasi-Jordan algebras is not true, since our identity is satisfied by all special quasi-Jordan algebras but not by all quasi-Jordan algebras. If we reduce our identity by assuming commutativity and collecting terms, we obtain an identity with 191 terms in the free commutative nonassociative algebra. This commutative identity involves three variables, is linear in one of them, and is satisfied by all special Jordan algebras (since every special Jordan algebra is a special quasi-Jordan algebra corresponding to an associative dialgebra in which the two operations coincide). Therefore, by Macdonald's theorem, this commutative identity is satisfied by all Jordan algebras, and hence must be satisfied by the Albert algebra (the exceptional simple Jordan algebra). Therefore we cannot use the Albert algebra to give a short proof that the identity of Tables 5 and 6 is not satisfied by all quasi-Jordan algebras.

$2((((((aa)a)a)b)b)c)b$	$-2((((((aa)a)b)a)b)c)b$	$+2((((((aa)a)b)b)a)b)c$
$-2((((((aa)a)b)b)b)a)c$	$+2((((((aa)a)b)b)b)c)a$	$-2((((((aa)a)b)b)c)a)b$
$-2((((((aa)a)b)b)c)b)a$	$+4((((((aa)a)b)c)b)b)a$	$+2((((((aa)a)c)b)a)b)b$
$-4((((((aa)a)c)b)b)a)b$	$-2((((((aa)b)a)a)b)b)c$	$+4((((((aa)b)a)b)a)b)c$
$-2((((((aa)b)a)b)b)a)c$	$-2((((((aa)b)a)b)c)a)b$	$+2((((((aa)b)a)b)c)b)a$
$+2((((((aa)b)a)c)a)b)b$	$-2((((((aa)b)a)c)b)a)b$	$+2((((((aa)b)b)c)b)a)a$
$-2((((((aa)b)c)b)a)a)b$	$+2((((((aa)b)c)b)a)b)a$	$-2((((((aa)b)c)b)b)a)a$
$+2((((((ab)a)a)a)b)b)c$	$-2((((((ab)a)a)b)a)b)c$	$-2((((((ab)a)a)b)a)c)b$
$-2((((((ab)a)a)b)c)b)a$	$-2((((((ab)a)a)c)a)b)b$	$+2((((((ab)a)a)c)b)a)b$
$-4((((((ab)a)b)a)a)b)c$	$+2((((((ab)a)b)a)a)c)b$	$+4((((((ab)a)b)a)b)a)c$
$+4((((((ab)a)b)a)c)a)b$	$-2((((((ab)a)b)b)a)c)a$	$+2((((((ab)a)b)c)a)a)b$
$-6((((((ab)a)b)c)a)b)a$	$-2((((((ab)a)c)a)a)b)b$	$+4((((((ab)a)c)a)b)a)b$
$-2((((((ab)a)c)a)b)b)a$	$+2((((((ab)a)c)b)a)a)b$	$+2((((((ab)a)c)b)b)a)a$
$+2((((((ab)b)a)b)c)a)a$	$-2((((((ab)b)a)c)a)a)b$	$+2((((((ab)b)a)c)a)b)a$
$-2((((((ab)b)a)c)b)a)a$	$+2((((((ab)b)c)a)b)a)a$	$-2((((((ab)b)c)b)a)a)a$
$-2((((((ab)c)a)b)a)a)b$	$+2((((((ab)c)a)b)a)b)a$	$-2((((((ab)c)a)b)b)a)a$
$+2((((((ab)c)b)a)a)a)b$	$-2((((((ab)c)b)a)a)b)a$	$+2((((((ab)c)b)b)a)a)a$
$+2((((((ac)a)a)b)a)b)b$	$-2((((((ac)a)a)b)b)a)b$	$+2((((((ac)a)a)b)b)b)a$
$-2((((((ac)a)b)a)a)b)b$	$+2((((((ac)a)b)a)b)a)b$	$-4((((((ac)a)b)a)b)b)a$
$+2((((((ac)a)b)b)a)a)b$	$+2((((((ac)a)b)b)a)b)a$	$-2((((((ac)a)b)b)b)a)a$
$-2((((((ac)b)a)b)a)a)b$	$+2((((((ac)b)a)b)a)b)a$	$+2((((((ac)b)a)b)b)a)a$
$-2((((((ac)b)b)a)b)a)a$	$-2(((((a(aa))a)b)b)c)b$	$+2(((((a(aa))b)a)b)c)b$
$-2(((((a(aa))b)b)a)b)c$	$+2(((((a(aa))b)b)b)a)c$	$-2(((((a(aa))b)b)b)c)a$
$+2(((((a(aa))b)b)c)a)b$	$+2(((((a(aa))b)b)c)b)a$	$-4(((((a(aa))b)c)b)b)a$
$-2(((((a(aa))c)b)a)b)b$	$+4(((((a(aa))c)b)b)a)b$	$-2(((((a(ab))a)a)b)b)c$
$-2(((((a(ab))a)a)c)b)b$	$+2(((((a(ab))a)b)a)b)c$	$+4(((((a(ab))a)b)a)c)b$
$-2(((((a(ab))a)b)c)a)b$	$+2(((((a(ab))a)b)c)b)a$	$+4(((((a(ab))a)c)a)b)b$
$-4(((((a(ab))a)c)b)a)b$	$+2(((((a(ab))a)c)b)b)a$	$+4(((((a(ab))b)a)a)b)c$
$-6(((((a(ab))b)a)b)a)c$	$+2(((((a(ab))b)a)b)c)a$	$-4(((((a(ab))b)a)c)a)b$
$-2(((((a(ab))b)a)c)b)a$	$+2(((((a(ab))b)b)a)c)a$	$-2(((((a(ab))b)b)c)a)a$
$-2(((((a(ab))b)c)a)a)b$	$+6(((((a(ab))b)c)a)b)a$	$+2(((((a(ab))c)a)a)b)b$
$-2(((((a(ab))c)a)b)a)b$	$-2(((((a(ab))c)b)a)b)a$	$-4(((((a(ac))a)b)a)b)b$
$+4(((((a(ac))a)b)b)a)b$	$-2(((((a(ac))a)b)b)b)a$	$+2(((((a(ac))b)a)a)b)b$
$-2(((((a(ac))b)b)a)b)a$	$+2(((((a(ac))b)b)b)a)a$	$+(((((a(bb))a)a)a)c)b$
$-(((((a(bb))a)a)c)a)b$	$-(((((a(bb))a)b)a)a)c$	$+(((((a(bb))a)b)a)c)a$
$+(((((a(bb))a)b)c)a)a$	$-(((((a(bb))a)c)b)a)a$	$-(((((a(bb))b)c)a)a)a$
$+(((((a(bb))c)b)a)a)a$	$+2(((((a(bc))a)b)a)a)b$	$-2(((((a(bc))a)b)a)b)a$
$+2(((((a(bc))b)a)b)a)a$	$+2(((((aa)(ab))a)b)b)c$	$-2(((((aa)(ab))a)b)c)b$
$+4(((((aa)(ab))a)c)b)b$	$-2(((((aa)(ab))b)a)b)c$	$-4(((((aa)(ab))b)a)c)b$
$+2(((((aa)(ab))b)b)a)c$	$-2(((((aa)(ab))b)b)c)a$	$+6(((((aa)(ab))b)c)a)b$
$-4(((((aa)(ab))c)a)b)b$	$+4(((((aa)(ab))c)b)a)b$	$-2(((((aa)(ab))c)b)b)a$
$-3(((((aa)(bb))a)a)c)b$	$+2(((((aa)(bb))a)b)a)c$	$-2(((((aa)(bb))a)b)c)a$
$+2(((((aa)(bb))a)c)a)b$	$+2(((((aa)(bb))a)c)b)a$	$-(((((aa)(bb))b)a)c)a$
$+2(((((aa)(bb))c)a)a)b$	$-2(((((aa)(bb))c)a)b)a$	$+(((((aa)(bb))c)b)a)a$
$+2(((((ab)(ac))a)a)b)b$	$-2(((((ab)(ac))a)b)a)b$	$+2(((((ab)(ac))a)b)b)a$
$-2(((((ab)(ac))b)a)a)b$	$+2(((((ab)(ac))b)a)b)a$	$-2(((((ab)(ac))b)b)a)a$
$-2(((((ab)(bc))a)a)a)b$	$+2(((((ab)(bc))a)a)b)a$	$-2(((((ab)(bc))a)b)a)a$
$-2(((((ab)(bc))b)a)a)a$	$+2(((((aa)a)(ab))b)c)b$	$-2(((((aa)a)(ab))c)b)b$
$-3(((((aa)a)(bb))a)b)c$	$+4(((((aa)a)(bb))a)c)b$	$+2(((((aa)a)(bb))b)a)c$
$+(((((aa)a)(bb))c)a)b$	$-2(((((aa)a)(bb))c)b)a$	$-2(((((aa)a)(bc))a)b)b$

TABLE 5. Special identity for partition 431 (terms 1 to 150)

$+4(((((aa)a)(bc))b)a)b$	$+4(((((aa)b)(ac))a)b)b$	$+2(((((aa)b)(ac))b)a)b$
$+2(((((aa)b)(ac))b)b)a$	$+2(((((aa)b)(bc))a)a)b$	$+2(((((aa)b)(bc))a)b)a$
$+2(((((ab)a)(ab))a)b)c$	$-2(((((ab)a)(ab))a)c)b$	$-2(((((ab)a)(ab))b)a)c$
$+2(((((ab)a)(ab))b)c)a$	$-6(((((ab)a)(ab))c)a)b$	$-4(((((ab)a)(ac))a)b)b$
$-6(((((ab)a)(ac))b)a)b$	$-2(((((ab)a)(ac))b)b)a$	$+(((((ab)a)(bb))a)c)a$
$-2(((((ab)a)(bb))c)a)a$	$-4(((((ab)a)(bc))a)a)b$	$-2(((((ab)a)(bc))a)b)a$
$-2(((((ab)a)(bc))b)a)a$	$-2(((((ab)b)(ac))a)b)a$	$+2(((((ab)b)(ac))b)a)a$
$+2(((((ab)b)(bc))a)a)a$	$-4(((((ac)a)(ab))b)a)b$	$+2(((((ac)a)(ab))b)b)a$
$-(((((ac)a)(bb))a)a)b$	$-(((((ac)a)(bb))a)b)a$	$+(((((ac)a)(bb))b)a)a$
$+2((((a(aa))(bb))a)b)c$	$-((((a(aa))(bb))a)c)b$	$-2((((a(aa))(bb))b)a)c$
$+2((((a(aa))(bb))b)c)a$	$-2((((a(aa))(bb))c)a)b$	$+2((((a(aa))(bc))a)b)b$
$-4((((a(aa))(bc))b)a)b$	$-2((((a(ab))(ab))a)b)c$	$+4((((a(ab))(ab))b)a)c$
$-4((((a(ab))(ab))b)c)a$	$+4((((a(ab))(ab))c)a)b$	$+4((((a(ab))(ac))b)a)b$
$+((((a(ab))(bb))a)a)c$	$-2((((a(ab))(bb))a)c)a$	$+((((a(ab))(bb))c)a)a$
$+2((((a(ab))(bc))a)a)b$	$+2((((a(ab))(bc))b)a)a$	$+((((a(ac))(bb))a)b)a$
$-((((a(ac))(bb))b)a)a$	$+(((((aa)a)a)(bb))b)c$	$-2(((((aa)a)a)(bb))c)b$
$+2(((((aa)a)b)(ac))b)b$	$-2(((((aa)a)b)(bc))b)a$	$-2(((((aa)b)a)(ab))b)c$
$+2(((((aa)b)a)(ab))c)b$	$-4(((((aa)b)a)(ac))b)b$	$-(((((aa)b)a)(bb))a)c$
$-4(((((aa)b)a)(bc))a)b$	$-2(((((aa)b)a)(bc))b)a$	$-6(((((aa)b)b)(ac))a)b$
$-4(((((aa)b)b)(ac))b)a$	$-2(((((aa)b)b)(bc))a)a$	$-2(((((aa)c)a)(bb))a)b$
$+(((((aa)c)a)(bb))b)a$	$+2(((((ab)a)a)(ac))b)b$	$+(((((ab)a)a)(bb))a)c$
$-2(((((ab)a)a)(bb))c)a$	$+4(((((ab)a)a)(bc))a)b$	$+6(((((ab)a)b)(ac))a)b$
$+6(((((ab)a)b)(ac))b)a$	$+4(((((ab)a)b)(bc))a)a$	$+2(((((ab)b)a)(ab))c)a$
$+4(((((ab)b)a)(ac))a)b$	$+2(((((ab)b)a)(ac))b)a$	$+2(((((ab)b)a)(bc))a)a$
$+4(((((ab)c)a)(ab))a)b$	$+2(((((ab)c)a)(bb))a)a$	$+(((((ac)a)a)(bb))a)b$
$-(((((ac)a)a)(bb))b)a$	$-(((((ac)b)a)(bb))a)a$	$+((((a(aa))a)(bb))c)b$
$+4((((a(aa))b)(bc))b)a$	$-2((((a(ab))a)(bb))a)c$	$+3((((a(ab))a)(bb))c)a$
$-4((((a(ab))b)(ac))a)b$	$-4((((a(ab))b)(ac))b)a$	$-4((((a(ab))b)(bc))a)a$
$+2(((((aa)a)a)b)(bc))b$	$-2(((((aa)a)b)a)(bb))c$	$-4(((((aa)a)b)a)(bc))b$
$-2(((((aa)a)b)b)(ac))b$	$+2(((((aa)b)a)a)(bb))c$	$+2(((((aa)b)a)a)(bc))b$
$+2(((((aa)b)a)b)(bc))a$	$+4(((((aa)b)b)a)(ac))b$	$+2(((((aa)b)b)a)(bc))a$
$+2(((((aa)b)b)b)(ac))a$	$-2(((((aa)b)c)a)(ab))b$	$-(((((ab)a)a)a)(bb))c$
$+2(((((ab)a)a)b)(ac))b$	$+4(((((ab)a)a)b)(bc))a$	$+2(((((ab)a)b)a)(ab))c$
$-2(((((ab)a)b)a)(ac))b$	$-4(((((ab)a)b)a)(bc))a$	$+4(((((ab)a)c)a)(ab))b$
$+2(((((ab)a)c)a)(bb))a$	$-2(((((ab)b)a)a)(ac))b$	$-2(((((ab)b)a)a)(bc))a$
$-6(((((ab)b)a)b)(ac))a$	$-2(((((ab)b)c)a)(ab))a$	$-2(((((ab)c)a)a)(ab))b$
$-(((((ab)c)a)a)(bb))a$	$+2(((((ac)a)b)a)(ab))b$	$+(((((ac)a)b)a)(bb))a$
$-2((((a(aa))b)b)(bc))a$	$+((((a(ab))a)a)(bb))c$	$-2((((a(ab))a)b)(ac))b$
$-4((((a(ab))a)b)(bc))a$	$-2((((a(ab))b)a)(ab))c$	$+2((((a(ab))b)a)(ac))b$
$+4((((a(ab))b)a)(bc))a$	$+4((((a(ab))b)b)(ac))a$	$-2((((a(ab))c)a)(ab))b$
$-((((a(ab))c)a)(bb))a$	$-2(((((aa)a)a)b)b)(bc)$	$+4(((((aa)a)b)a)b)(bc)$
$-2(((((aa)b)a)b)a)(bc)$	$+2(((((aa)b)a)b)b)(ac)$	$-2(((((aa)b)b)a)b)(ac)$
$+2(((((aa)b)b)c)a)(ab)$	$-(((((aa)b)c)a)a)(bb)$	$-2(((((aa)b)c)b)a)(ab)$
$+(((((aa)c)a)b)a)(bb)$	$+(((((ab)a)a)c)a)(bb)$	$-2(((((ab)a)b)a)b)(ac)$
$-2(((((ab)a)b)c)a)(ab)$	$+2(((((ab)a)c)b)a)(ab)$	$+2(((((ab)b)a)b)a)(ac)$
$-2(((((ab)b)a)c)a)(ab)$	$+2(((((ab)b)c)a)a)(ab)$	$-2(((((ab)c)a)b)a)(ab)$
$-((((a(ab))a)c)a)(bb)$	$+2((((a(ab))b)a)b)(ac)$	$-2((((a(ab))b)b)a)(ac)$
$+2((((a(ab))b)c)a)(ab)$	$+((((a(ac))a)b)a)(bb)$	$-((((a(ac))b)a)a)(bb)$
$-2((((aa)(ab))a)b)(bc)$	$+2((((aa)(ab))b)a)(bc)$	

TABLE 6. Special identity for partition 431 (terms 151 to 296)

10. Conclusion

Our results provide evidence that quasi-Jordan algebras are a natural generalization of Jordan algebras to a noncommutative setting. It would be interesting to find the correct generalizations of well-known classical results on free (special) Jordan algebras to quasi-Jordan algebras. For example, (*i*) the theorem of Cohn [4] that gives a criterion for a quotient of a free special Jordan algebra to be special and implies that special Jordan algebras do not form a variety; (*ii*) the characterization by Cohn [4] of free special Jordan algebras on ≤ 3 generators as symmetric elements in free associative algebras; (*iii*) the theorem of Shirshov [23] (see also Jacobson and Paige [11]) that the free Jordan algebra on two generators is special.

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