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Algebras with scalar involution revisited

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Abstract. We study algebras with scalar involution and, more generally, conic algebras (formerly known as quadratic algebras) over an arbitrary base ring k on a fixed finitely generated and projective k-module X with base point 1_X. By variation of the base ring, these algebras define schemes whose structure is described in detail. They also admit natural group actions under which they are trivial torsors. We determine the quotients by these group actions. This requires a new invariant of conic algebras, an alternating trilinear map on $M = X/k \cdot 1_X$ with values in the second symmetric power of M. An important tool is the coordinatization of conic algebras in terms of a linear form, a cross product and a bilinear form on M, all depending on a choice of unital linear form on X, which replaces the usual description in terms of a vector algebra and a bilinear form in case 2 is a unit in k.

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Introduction

The theory of composition algebras with non-singular norm form is well known over fields. These algebras are automatically alternative. It was H. P. Petersson [**14**] who extended the theory to arbitrary base rings and even base schemes. Much less is known when one drops the non-degeneracy condition on the quadratic form but still requires the algebra to be alternative. Over fields of characteristic $\neq 2$, there is an unpublished dissertation by L. Zagler [**17**], later rediscovered by Kunze and Scheinberg [**8**] and A. Elduque [**5**], and extended to general base rings by K. McCrimmon [**12**].

In the present paper, we continue McCrimmon's work in a different direction. Instead of studying a single algebra of a certain type, we fix a module X and a base point 1_X in X and consider all algebras of a certain type living on X and having unit element 1_X. We assume X to be a finitely generated and projective module of constant rank $n+1$ over an arbitrary base ring k, and the base point to be a unimodular vector. Since all constructions are compatible with arbitrary change of base ring, these algebras give rise to schemes. Specifically, we consider conic algebras and algebras with scalar involution (see below) and determine the structure of the schemes defined by them.

An algebra on X is a bilinear map $A\colon X \times X \to X$, usually written $A(x, y) = xy$, and satisfying $1_X x = x 1_X = x$ for all $x \in X$. Following a terminology proposed by H. P. Petersson, A is called *conic* if there exists a quadratic form N on X, called the *norm*, such that $N(1_X) = 1$ and

$$x^2 - T(x)x + N(x)1_X = 0$$

for all $x \in X$, where the *trace* T is given by $T(x) = N(x+1_X) - N(x) - 1$. These algebras used to be called quadratic or of degree 2 in the literature, but the term "quadratic algebra" has now acquired a different meaning [**11**], and the notion of degree for arbitrary non-associative algebras over rings is problematic, so a new terminology is welcome.

In Section 1 we first establish some basic facts on exterior products and formal differentiation of multilinear maps used throughout the paper. Then we deal with constructions of conic algebras. Given a scalar-valued bilinear form f on X satisfying $f(1_X, 1_X) = 1$, one obtains a conic algebra $f^{\mathfrak{m}}$ by defining the multiplication

$$f^{\mathfrak{m}}(x, y) = f(x, 1_X)y - f(x, y)1_X + f(1_X, y)x, \tag{1}$$

with norm $N(x) = f(x, x)$ and trace $T(x) = f(x, 1_X) + f(1_X, x)$. There are two basic ways of modifying a given conic algebra A. The one is by changing A to

$$A' = A + g^{\mathtt{m}} \tag{2}$$

where $g \in \mathrm{Bil}_0(X)$, the bilinear forms on X vanishing at $(1_X, 1_X)$, and $g^{\mathtt{m}}$ is defined just like $f^{\mathtt{m}}$. To explain the other, let $M = X/k \cdot 1_X$ and denote the canonical map $X \to M$ by $x \mapsto \dot{x}$. Let $\Gamma \in \Omega^2(M, X)$, the set of alternating bilinear maps on M with values in X. Then if A is conic so is

$$A'(x, y) = A(x, y) + \Gamma(\dot{x}, \dot{y}), \tag{3}$$

and any conic algebra on X has the form $A(x, y) = f^{\mathtt{m}}(x, y) + \Gamma(\dot{x}, \dot{y})$, for suitable (not unique) f and Γ.

Denote by $\mathrm{Con}(X)$ the set of conic algebras on X. Our first structure theorem says that $\mathrm{Con}(X)$ is a torsor under the group $\Omega^2(M, X)$ acting as in (3), with quotient the unital quadratic forms on X (Theorem 1.13). As a scheme, $\mathrm{Con}(X)$ is smooth, affine and finitely presented over k with fibres isomorphic to affine space of dimension $n\Big(1 + n + \binom{n}{2}\Big)$.

We begin Section 2 by introducing a cochain complex of alternating p-linear maps on a module M with values in $\mathrm{S}^{p-1}M$, the $(p-1)$st symmetric power of M. If M is finitely generated and projective of rank at least 2, this complex is acyclic (Proposition 2.2). Then we determine necessary and sufficient conditions for a conic algebra A to be of the form (1). The obstruction to this is an element Θ_A of $\mathrm{Z}^3(M)$, the closed 3-forms on M with values in S^2M, called the *canonical* 3*-form* of A. The main result is Theorem 2.8: A conic algebra A has the form $f^{\mathtt{m}}$ if and only if $\Theta_A = 0$. As a consequence (Corollary 2.10), we obtain a second structure theorem for the scheme $\mathrm{Con}(X)$: it is a torsor with group $\mathrm{Bil}_0(X)$ acting as in (2), with base $\mathrm{Z}^3(M)$.

If 2 is a unit in k then $X = k \cdot 1_X \oplus \mathrm{Ker}(T_A)$, and A can be described in terms of a bilinear form and an alternating product on $\mathrm{Ker}(T_A)$. This is the way conic algebras are treated in most of the literature, see for example [**13**, **1**]. In general, since 1_X is unimodular, X admits linear forms α satisfying $\alpha(1_X) = 1$ (unital linear forms) and hence decompositions $X = k \cdot 1_X \oplus \mathrm{Ker}(\alpha)$ but they have all to be treated on an equal footing. In Section 3 we expand on an idea of Petersson's [**15**] and, having chosen a unital linear form α, describe A by a triple (t, K, b) consisting of a linear form t, a bilinear form b and a cross product K on M, called the *α-coordinates of A*. We express the norm, trace and canonical 3-form of a conic algebra as well as the various constructions discussed earlier in terms of these coordinates, and show how the coordinates change when changing α.

By a *transvection* we mean an element of $\varphi \in \mathrm{GL}(X)$ fixing 1_X and inducing the identity on M. The transvections act simply transitively on the set of unital linear forms. They also act on $\mathrm{Con}(X)$ on the right by means of $A^{\varphi}(x, y) = \varphi^{-1}\big(A(\varphi(x), \varphi(y)\big)$. We use α-coordinates to describe the quotient of $\mathrm{Con}(X)$ by transvections (Theorem 3.10).

In any conic algebra, the map $x \mapsto \bar{x} = T(x)1 - x$, called the *conjugation*, has period two. The algebra is said to be *involutive* if the conjugation is an algebra involution. These are the algebras with scalar involution in the sense of [**12**]. We show in Section 4 that there is an alternating 2-form ω_A on M which measures the deviation of a conic algebra from being involutive, and study its behaviour under the various constructions. The equation $\omega_A = 0$, describing the set $\mathrm{Scalin}(X) \subset \mathrm{Con}(X)$ of algebras with scalar involution, amounts to quadratic relations between the α-coordinates (t, K, b). Geometrically, $\mathrm{Scalin}(X)$ is a parabolic cylinder whose generators are given by an action of the group of symmetric bilinear forms on M. We identify the quotient by this action and show that $\mathrm{Scalin}(X)$ is smooth, affine and finitely presented k-scheme, with fibres isomorphic to affine space of dimension $n + n\binom{n}{2} + \binom{n+1}{2}$ (Theorem 4.10).

The theory presented here takes place over an arbitrary commutative ring k, that is to say, over the affine scheme defined by k. Following the precedent of H. P. Petersson [**14**], it is possible to replace k by an arbitrary base scheme. The necessary – and mostly straightforward — modifications are briefly discussed in the final Section 5.

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Throughout, k denotes an arbitrary commutative associative ring with unit element. Unsubscripted tensor products are taken over k. The set of natural numbers including 0 is denoted by $\mathbb{N}$.

1. Conic algebras

1.1. Notation. Let X, Y, Z be k-modules. We denote by $\mathscr{L}^p(X, Y)$ the set of multilinear maps $f\colon X^p \to Y$. Thus $\mathscr{L}^1(X, k) = X^*$ is the dual of X, $\mathscr{L}^1(X, Y) = \mathrm{Hom}(X, Y)$, and $\mathscr{L}^2(X, Y) = \mathrm{Bil}(X, Y)$ are the bilinear maps on X with values in Y. The *transpose* of $f \in \mathrm{Bil}(X, Y)$ is $f^{\mathrm{op}}(x_1, x_2) = f(x_2, x_1)$. The alternating p-linear maps on X with values in Y are denoted $\Omega^p(X, Y)$. If $Y = k$ we simply write $\mathscr{L}^p(X) = \mathscr{L}^p(X, k)$, $\mathrm{Bil}(X) = \mathrm{Bil}(X, k)$ and $\Omega^p(X) = \Omega^p(X, k)$. Let $\psi\colon X \to Y$ be linear and $g \in \mathscr{L}^p(Y, Z)$. Then $\psi^*(g) = g \circ \psi^p$ denotes the pull-back of g to X.

Let $\mathrm{Quad}(X, Y)$ denote the quadratic maps from X to Y and $\mathrm{Quad}(X) = \mathrm{Quad}(X, k)$ the quadratic forms on X. The polarization of a quadratic map q is the symmetric bilinear map $\partial q(x, y) = q(x + y) - q(x) - q(y)$, often simply written $q(x, y) = \partial q(x, y)$.

The category of (commutative associative unital) k-algebras is denoted k-**alg**. If X is a k-module and $R \in k$-**alg**, we write $X_R = X \otimes_k R$ for the R-module obtained by base change from k to R, and denote the R-linear extension of a linear map $f\colon X \to Y$ by f_R. Following [**4**], k-schemes will be considered as k-functors, that is, set-valued functors on k-**alg**. If X is a finitely generated and projective k-module, $X_{\mathbf{a}}$ denotes the affine k-group scheme given by

$$X_{\mathbf{a}}(R) = X_R \tag{1.1.1}$$

for all $R \in k$-**alg**. Its affine algebra is the symmetric algebra over X^*.

1.2. Unital modules. A *unital k-module* is a pair $(X, 1_X)$ where X is a finitely generated and projective k-module and 1_X, the *base point*, is a unimodular vector; i.e., there exist linear forms α on X such that $\alpha(1_X) = 1$. If there is no confusion, we often simply write 1 for the base point and, by abuse of language, refer to X as to a unital module. It is always assumed that X is of constant rank $r = n + 1$, and put $X^\flat = X/k \cdot 1$, which is then projective of rank n. Let $\pi = \pi_X\colon X \to X^\flat$ denote the canonical map. We often use the notation $M = X^\flat$ and $\pi(x) = \dot{x}$ for the image of an element $x \in X$ under π. Thus the sequence

$$0 \longrightarrow k \xrightarrow{1_X} X \xrightarrow{\pi_X} M \longrightarrow 0 \tag{1.2.1}$$

is split-exact. A *morphism* $\varphi\colon (X, 1_X) \to (Y, 1_Y)$ of unital modules is a module homomorphism $\varphi\colon X \to Y$ preserving base points: $\varphi(1_X) = 1_Y$. Then φ induces a unique homomorphism $\varphi^\flat\colon X^\flat \to Y^\flat$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k & \xrightarrow{1_X} & X & \xrightarrow{\pi_X} & X^\flat & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi^\flat} & & \\ 0 & \longrightarrow & k & \xrightarrow[1_Y]{} & Y & \xrightarrow[\pi_Y]{} & Y^\flat & \longrightarrow & 0 \end{array} \tag{1.2.2}$$

commutative. Unital modules form a category, and the assignments $(X, 1_X) \mapsto X^\flat$, $\varphi \mapsto \varphi^\flat$ define a functor $\flat$ from unital k-modules to finitely generated projective k-modules.

Unital modules admit arbitrary base change: Since (1.2.1) is split-exact, it remains so upon tensoring with an arbitrary $R \in k$-**alg**. Hence X_R is unital with base point $1_{X_R} = 1_X \otimes 1_R \in X_R$, and we have a natural isomorphism

$$(X^\flat)_R = X^\flat \otimes_k R \cong (X_R/R \cdot 1_{X_R}) = (X_R)^\flat, \tag{1.2.3}$$

so the functor $\flat$ is compatible with base change.

For $\lambda \in k$ let X^*_λ be the set of all $\alpha \in X^*$ such that $\alpha(1) = \lambda$. Similarly, $\mathrm{Bil}_\lambda(X)$ denotes the set of $f \in \mathrm{Bil}(X)$ with $f(1, 1) = \lambda$ and $\mathrm{Quad}_\lambda(X)$ is the set of quadratic forms $q\colon X \to k$ with $q(1) = \lambda$. The elements of X^*_1 resp. $\mathrm{Bil}_1(X)$, $\mathrm{Quad}_1(X)$ are called *unital linear* (*bilinear*, *quadratic*) forms. Since 1_X is unimodular, X^*_1 is not empty. This easily implies that $\mathrm{Quad}_1(X)$ and $\mathrm{Bil}_1(X)$ are non-empty as well.

1.3. Unital algebras. Let $(X, 1_X)$ be a unital k-module. We denote by $\mathrm{Alg}(X)$ the set of (not necessarily associative) algebra structures on X with unit element $1 = 1_X$, that is, the set of bilinear maps $A\colon X \times X \to X$ such that $A(x,1) = A(1,x) = x$ for all $x \in X$. Sometimes we will also refer to the triple $(X, 1, A)$ as to "the algebra A" and, as long as A is fixed, simply write $xy = A(x,y)$ for the product in A. Note that $\mathrm{Alg}(X)$ is not empty; for example, choosing $\alpha \in X_1^*$, the rule $A(x,y) = \alpha(x)y + \alpha(y)x - \alpha(x)\alpha(y)1$ defines a unital algebra on X, which is even associative and commutative.

It is immediately seen that the additive group of $H(X) := \mathrm{Bil}(X^\flat, X)$ acts simply transitively on the set $\mathrm{Alg}(X)$: If $A \in \mathrm{Alg}(X)$ and $B \in H(X)$ then

$$A'(x,y) = A(x,y) + B(\dot{x}, \dot{y}), \tag{1.3.1}$$

i.e., $A' = A + \pi^*(B)$, defines a unital multiplication on X, and conversely, if $A, A' \in \mathrm{Alg}(X)$ then there exists a unique $B \in H(X)$ such that (1.3.1) holds.

The functor $\mathbf{Alg}(X)\colon R \mapsto \mathrm{Alg}(X_R)$ from k-**alg** to sets is representable, but not in a canonical way, by the scheme $H(X)_{\mathbf{a}}$. Indeed, $H(X_R)$ acts simply transitively on $\mathrm{Alg}(X_R)$ as above, and since all modules involved are finitely generated and projective, $H(X_R) \cong H(X)_R$ canonically. Now it suffices to fix some $A_0 \in \mathrm{Alg}(X)$ and then map $B \in H(X_R)$ to $A_0 + \pi^*(B)$. Thus $\mathbf{Alg}(X)$ is a trivial k-torsor with group $H(X)_{\mathbf{a}}$, but there is no canonical trivialization. In particular, $\mathbf{Alg}(X)$ is a smooth finitely presented k-scheme, with fibres isomorphic to affine space of dimension $n^2(n+1)$.

We introduce next exterior products and a formal differential calculus for multilinear maps.

1.4. Lemma. *Let X be a k-module and B a unital associative k-algebra. Recall the notations $\mathscr{L}^p(X,B)$ and $\Omega^p(X,B)$ introduced in* 1.1. *For $f \in \mathscr{L}^p(X,B)$ and $g \in \mathscr{L}^q(X,B)$, define $f \wedge g \in \mathscr{L}^{p+q}(X,B)$ by*

$$(f \wedge g)(x_1, \ldots, x_{p+q}) = \sum_{\sigma \in \mathfrak{S}_{p,q}} \mathrm{sgn}(\sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(p)}) g(x_{\sigma(p+1)}, \ldots, x_{\sigma(p+q)}), \tag{1.4.1}$$

where $\mathfrak{S}_{p,q} \subset \mathfrak{S}_{p+q}$ is the set of all (p,q)-shuffle permutations: $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$.

(a) *With this product, $\mathscr{L}(X,B) := \bigoplus_{p \geqslant 0} \mathscr{L}^p(X,B)$ is an $\mathbb{N}$-graded associative unital k-algebra, and the direct sum $\Omega(X,B) = \bigoplus_{p \geqslant 0} \Omega^p(X,B)$ is a subalgebra.*

(b) *If B is commutative, then $\mathscr{L}(X,B)$ is an alternating algebra in the sense of* [**2**, III, §4, No. 9, Definition 7]*; i.e., $\mathscr{L}(X,B)$ is anticommutative and the squares of homogeneous elements of odd degree are zero:*

$$f \wedge g = (-1)^{\deg(f)\deg(g)} g \wedge f, \qquad f \wedge f = 0 \ \text{ if } \deg(f) \text{ is odd.}$$

(c) *A homomorphism $\eta\colon B \to C$ of k-algebras induces an algebra homomorphism $\eta_*\colon \mathscr{L}(X,B) \to \mathscr{L}(X,C)$ by composition on the left, and a homomorphism $\varphi\colon X \to Y$ of k-modules induces an algebra homomorphism $\varphi^*\colon \mathscr{L}(Y,B) \to \mathscr{L}(X,B)$ ("pull-back"), by composition on the right.*

Remark. The exterior product defined above is the usual one for alternating multilinear maps, see [**2**, III, §11, No. 2, Exemple 3]. In this case, the lemma is well known. The point here is that it works as well for arbitrary (not necessarily alternating) multilinear maps.

Proof. (a) Clearly, (1.4.1) defines an element $f \wedge g$ of $\mathscr{L}^{p+q}(X,B)$ which depends k-bilinearly on f and g, and $1_B \in B = \mathscr{L}^0(X,B)$ is the unit element for this multiplication. It remains to show associativity. We first rewrite (1.4.1) as follows. Let $[1,n]$ denote the interval $\{1, \ldots, n\}$ in $\mathbb{N}$. For a partition $[1, p+q] = I \,\dot\cup\, J$ where $|I| = p$ and $|J| = q$, say $I = \{i_1, \ldots, i_p\}$ and $J = \{j_1, \ldots, j_q\}$ with $i_1 < \cdots < i_p$ and $j_1 < \cdots < j_q$, we put $x_I = (x_{i_1}, \ldots, x_{i_p})$ and $x_J = (x_{j_1}, \ldots, x_{j_q})$. Then

$$(f \wedge g)(x_1, \ldots, x_{p+q}) = \sum \varrho_{I,J} f(x_I) g(x_J), \tag{1.4.2}$$

where the sum runs over all such partitions of $[1, p+q]$ and where $\varrho_{I,J} = (-1)^\nu$, with ν the number of pairs $(i,j) \in I \times J$ such that $i > j$, cf. [**2**, III, §7, No. 3, Lemma 1].

Let also $h \in \mathscr{L}^r(X, B)$ and put $N = [1, p+q+r]$. Then

$$\begin{aligned}\big((f \wedge g) \wedge h\big)(x_N) &= \sum_{N = L \dot\cup K} \varrho_{L,K} (f \wedge g)(x_L) h(x_K) \\ &= \sum_{N = L \dot\cup K} \varrho_{L,K} \left(\sum_{L = I \dot\cup J} \varrho_{I,J} f(x_I) g(x_J) \right) h(x_K) \\ &= \sum_{N = I \dot\cup J \dot\cup K} \varrho_{I,J}\, \varrho_{I \cup J, K}\, f(x_I) g(x_J) h(x_K),\end{aligned}$$

where L runs over all $p+q$-element subsets of N with complement K and I runs over all p-element subsets of L with complement J. Similarly,

$$\big(f \wedge (g \wedge h)\big)(x_N) = \sum_{N = I \dot\cup J \dot\cup K} \varrho_{I, J \cup K}\, \varrho_{J,K}\, f(x_I) g(x_J) h(x_K).$$

Now the assertion follows from the associativity of the exterior algebra of a free module which has the structure constants $\varrho_{I,J}$ [**2**, III, §7, No. 8, formula (20)]. Finally, if f and g are alternating multilinear maps, then $f \wedge g$ is their usual exterior product which is known to be alternating as well [**2**, III, §11, No. 2].

(b) If I and J are disjoint index sets of p and q elements, respectively, then $\varrho_{I,J} = (-1)^{pq} \varrho_{J,I}$ by [**2**, III, §7, No. 8, formula (21)]. Hence for $f \in \mathscr{L}^p(X, B)$ and $g \in \mathscr{L}^q(X, B)$ it follows from (1.4.2) and commutativity of B that $f \wedge g = (-1)^{pq} g \wedge f$. Now suppose p odd and decompose $[1, 2p] = I \dot\cup \complement I$ with $|I| = p$. Then $\varrho_{I, \complement I} = -\varrho_{\complement I, I}$. Hence for each term $\varrho_{I,\complement I} f(x_I) f(x_{\complement I})$ in (1.4.2), there is a corresponding term $\varrho_{\complement I, I} f(x_{\complement I}) f(x_I) = -\varrho_{I,\complement I} f(x_I) f(x_{\complement I})$ (by commutativity of B), whence $(f \wedge f)(x_1, \ldots, x_{2p}) = 0$.

(c) This is straightforward.

1.5. Definition. Now consider the special case where $B = \mathrm{S}Y$ is the symmetric algebra of a k-module Y. We denote the product in $\mathrm{S}Y$ by $\vee$. Let φ: $X \to Y$ be a linear map. Since $Y = \mathrm{S}^1 Y$, we consider φ as an element of $\mathscr{L}^1(X, \mathrm{S}Y)$. We define the *$\varphi$-differential* of $f \in \mathscr{L}^p(X, \mathrm{S}Y)$ by $\mathrm{d}_p^\varphi f = f \wedge \varphi \in \mathscr{L}^{p+1}(X, \mathrm{S}Y)$. Explicitly,

$$(\mathrm{d}_p^\varphi f)(x_0, \ldots, x_p) = \sum_{i=0}^{p} (-1)^{p+i} f(x_0, \ldots, \hat{x}_i, \ldots, x_p) \vee \varphi(x_i). \tag{1.5.1}$$

In the special case where $Y = X$ and $\varphi = \mathrm{Id}$, we simply write $\mathrm{d}_p f = \mathrm{d}_p^{\mathrm{Id}} f$ and say $\mathrm{d}_p f$ is the differential of f.

It is immediate from Lemma 1.4(b) that

$$\mathrm{d}_{p+1}^\varphi(\mathrm{d}_p^\varphi f) = (f \wedge \varphi) \wedge \varphi = f \wedge (\varphi \wedge \varphi) = 0,$$

so we have a cochain complex

$$0 \longrightarrow \mathscr{L}^0(X, \mathrm{S}Y) \xrightarrow{\mathrm{d}_0^\varphi} \mathscr{L}^1(X, \mathrm{S}Y) \xrightarrow{\mathrm{d}_1^\varphi} \mathscr{L}^2(X, \mathrm{S}Y) \xrightarrow{\mathrm{d}_2^\varphi} \cdots \tag{1.5.2}$$

If f is alternating then so is $\mathrm{d}_p^\varphi f$. Indeed, φ is an alternating 1-form with values in $\mathrm{S}Y$ and by Lemma 1.4(a), the alternating maps form a subalgebra. If f has values in $\mathrm{S}^r Y$ then $\mathrm{d}_p^\varphi f$ has values in $\mathrm{S}^{r+1} Y$. Finally, $\mathscr{L}^p(X, \mathrm{S}Y)$ is a left $\mathrm{S}Y$-module by defining $(s \vee f)(x_1, \ldots, x_p) = s \vee f(x_1, \ldots, x_p)$, and then it follows immediately from (1.5.1) that d_p^φ is $\mathrm{S}Y$-linear:

$$\mathrm{d}_p^\varphi(s \vee f) = s \vee \mathrm{d}_p^\varphi f. \tag{1.5.3}$$

For small degrees and $\varphi = \mathrm{Id}$, we have the explicit formulas

$$(\mathrm{d}_0 s)(x) = s \vee x, \tag{1.5.4}$$
$$(\mathrm{d}_1 h)(x,y) = h(x) \vee y - h(y) \vee x, \tag{1.5.5}$$
$$(\mathrm{d}_2 b)(x,y,z) = b(x,y) \vee z - b(x,z) \vee y + b(y,z) \vee x, \tag{1.5.6}$$

where $s \in \mathrm{S}X = \mathscr{L}^0(X, \mathrm{S}X)$, $h \in \mathscr{L}^1(X, \mathrm{S}X) = \mathrm{Hom}(X, \mathrm{S}X)$, and $b \in \mathscr{L}^2(X, \mathrm{S}X) = \mathrm{Bil}(X, \mathrm{S}X)$. Note that (1.5.5) implies the formula

$$u \circ \mathrm{d}_1 v = v \wedge u \tag{1.5.7}$$

for all $u, v \in X^*$, since $(u \circ \mathrm{d}_1 v)(x,y) = u\big(v(x)y - v(y)x\big) = v(x)u(y) - v(y)u(x)$, for all $x, y \in X$.

Let $\psi\colon Y \to Z$ be a linear map between k-modules and let

$$\mathrm{S}\psi\colon \mathrm{S}Y \to \mathrm{S}Z \tag{1.5.8}$$

denote the induced homomorphism of the symmetric algebras. If there is no danger of confusion, we also write simply $\psi\colon \mathrm{S}Y \to \mathrm{S}Z$. Correspondingly, we use the notation $\psi_*(f) = (\mathrm{S}\psi) \circ f \in \mathscr{L}^p(X, Z)$ for $f \in \mathscr{L}^p(X, Y)$. Then the differential behaves as follows with respect to ψ and φ:

$$\psi_*(\mathrm{d}_p^\varphi f) = \mathrm{d}_p^{\psi\circ\varphi}\big(\psi_*(f)\big), \tag{1.5.9}$$
$$\varphi^*(\mathrm{d}_p^\psi g) = \mathrm{d}_p^{\psi\circ\varphi}\big(\varphi^*(g)\big), \tag{1.5.10}$$

where $f \in \mathscr{L}^p(X, Y)$ and $g \in \mathscr{L}^p(Y, Z)$. The proof is straightforward.

We say $f \in \mathscr{L}^p(X, \mathrm{S}X)$ and $g \in \mathscr{L}^p(Y, \mathrm{S}Y)$ are *φ-related* if $\varphi_*(f) = \varphi^*(g)$. Then $\mathrm{d}_p f$ and $\mathrm{d}_p g$ are φ-related as well. Indeed,

$$\varphi_*(\mathrm{d}_p f) = \mathrm{d}_p^\varphi(\varphi_*(f)) \text{ (by (1.5.9)) } = \mathrm{d}_p^\varphi(\varphi^*(g)) = \varphi^*(\mathrm{d}_p g) \text{ (by (1.5.10))}.$$

We now return to unital algebras on a unital module. The following types of multiplications will play a distinguished role in the sequel.

1.6. Lemma. *Let X be a unital module. For a scalar-valued bilinear form $f \in \mathrm{Bil}(X)$ define a multiplication $f^{\mathtt{m}} \in \mathrm{Bil}(X, X)$ by*

$$f^{\mathtt{m}}(x,y) = (\mathrm{d}_2 f)(x, 1, y) = f(x,1)y - f(x,y)1 + f(1,y)x. \tag{1.6.1}$$

(a) *Then $f^{\mathtt{m}}(1,x) = f^{\mathtt{m}}(x,1) = f(1,1)x$. Hence $f \in \mathrm{Bil}_1(X)$ implies $f^{\mathtt{m}} \in \mathrm{Alg}(X)$, and $f \in \mathrm{Bil}_0(X)$ implies that $f^{\mathtt{m}}(x,y)$ depends only on $\dot{x}$ and $\dot{y}$.*

(b) *$(f^{\mathtt{m}})^{\mathrm{op}} = (f^{\mathrm{op}})^{\mathtt{m}}$, and f is alternating if and only if $f^{\mathtt{m}}$ is alternating.*

(c) *The map $(\ )^{\mathtt{m}}\colon \mathrm{Bil}(X) \to \mathrm{Bil}(X,X)$ is linear and injective if $\mathrm{rk}(X) \neq 2$. In case $\mathrm{rk}(X) = 2$, we have $f^{\mathtt{m}} = 0$ if and only if f is alternating.*

Proof. (a) is immediate from the definition, as is the first statement of (b). If f is alternating then it follows at once from (1.6.1) that $f^{\mathtt{m}}(x,x) = 0$. Conversely, assume $f^{\mathtt{m}}$ is alternating. If X has rank 1 then after identifying $X = k \cdot 1_X$ with k, we have $f^{\mathtt{m}} = f$, so we may assume $\mathrm{rk}\, X \geqslant 2$. Then $0 = f^{\mathtt{m}}(x,x) = \big(f(x,1) + f(1,x)\big)x - f(x,x)1$, so by applying π, it follows that $\big(f(x,1) + f(1,x)\big)\dot{x} = 0$. Since $M = X^\flat$ has rank $\geqslant 1$, this easily implies (for example, by localization and using a basis) that $f(x,1) + f(1,x) = 0$, so we have $f(x,x)1 = 0$ and f is alternating.

We prove (c). Linearity being obvious, assume $f^{\mathtt{m}} = 0$. The case $\mathrm{rk}\, X = 1$, i.e., $X = k \cdot 1_X \cong k$, is clear. Applying π to (1.6.1) shows $f(x,1)\dot{y} + f(1,y)\dot{x} = 0$, and taking the exterior product with $\dot{x}$ resp. $\dot{y}$ yields $0 = f(x,1)\dot{x} \wedge \dot{y}$ and $0 = f(1,y)\dot{x} \wedge \dot{y}$. Hence if X has rank $\geqslant 3$, i.e., M has rank $\geqslant 2$, it follows by localization that $f(x,1) = f(1,y) = 0$ and therefore by (1.6.1) also $f(x,y) = 0$, for all $x, y \in X$.

Now let $\mathrm{rk}\, X = 2$. If $f^{\mathtt{m}} = 0$ then $f^{\mathtt{m}}$ is in particular alternating, hence so is f by (b). Conversely, let f be alternating. After localizing we may assume that $X = k \cdot 1 \oplus k \cdot e$ is free of rank 2. Then $f^{\mathtt{m}}$ is alternating by (b), and $f^{\mathtt{m}}(1,e) = f(1,1)e$ (by (a)) $= 0$, so $f^{\mathtt{m}} = 0$.

Remark. If X has rank 2, every $A \in \mathrm{Alg}(X)$ is of the form $A = f^{\mathtt{m}}$, where $f \in \mathrm{Bil}_1(X)$ is unique up to an alternating bilinear form. This is the well-known parametrization of rank 2 algebras, see [**9**, Proposition 1.6]. These algebras are automatically associative and commutative [**7**, I, (1.3.6)].

1.7. Transvections. The automorphism group of a unital module X is $\mathrm{GL}_1(X)$, the subgroup of all $\varphi \in \mathrm{GL}(X)$ with $\varphi(1) = 1$. By functoriality (cf. (1.2.2)), we have a homomorphism $\mathrm{GL}_1(X) \to \mathrm{GL}(X^\flat)$ sending φ to $\varphi^\flat$. To describe its kernel, let $V = M^*$ be the dual of $M = X^\flat$. Then there is a split exact sequence of groups

$$1 \longrightarrow V \xrightarrow{\tau} \mathrm{GL}_1(X) \xrightarrow{(\ )^\flat} \mathrm{GL}(M) \longrightarrow 1$$

where, for $v \in V$, the *transvection* $\tau_v \in \mathrm{GL}_1(X)$ is defined by

$$\tau_v(x) = x - v(\dot{x}) \cdot 1. \tag{1.7.1}$$

This is easily proved by choosing a splitting of the exact sequence (1.2.1). The group $\mathrm{GL}_1(X)$ acts on the set $\mathrm{Alg}(X)$ on the right by means of

$$A^\varphi(x, y) := \varphi^{-1}\bigl(A(\varphi(x), \varphi(y)\bigr), \tag{1.7.2}$$

and clearly $\varphi\colon A^\varphi \to A$ is an isomorphism of algebras. Explicitly, the action of a transvection τ_v on an algebra A is given by

$$A^v := A^{\tau_v} = A + g_{v,A}^{\mathtt{m}} \qquad (v \in V), \tag{1.7.3}$$

where $g_{v,A} \in \mathrm{Bil}_0(X)$ is

$$g_{v,A}(x, y) = v(\dot{x})v(\dot{y}) - v\bigl(\pi(xy)\bigr). \tag{1.7.4}$$

Indeed, by a straightforward computation,

$$\begin{aligned} A^v(x, y) &= \tau_{-v}\bigl(\tau_v(x)\tau_v(y)\bigr) = \tau_{-v}\bigl((x - v(\dot{x})1)(y - v(\dot{y})1\bigr) \\ &= xy - v(\dot{x})y - xv(\dot{y}) - \bigl(v(\dot{x})v(\dot{y}) - v(\pi(xy))\bigr) \cdot 1 \\ &= (A + g_{v,A}^{\mathtt{m}})(x, y). \end{aligned}$$

We claim that

$$\mathrm{rk}\, X \geqslant 3 \quad \Longrightarrow \quad V \text{ acts freely on } \mathrm{Alg}(X) \text{ by transvections.} \tag{1.7.5}$$

Indeed, assume that $A = A^v$. Then (1.7.3) and Lemma 1.6(c) imply $g_{v,A} = 0$, which by (1.7.4) says that $\pi^*(v) = v \circ \pi\colon X \to k$ is an algebra homomorphism. Since $\pi^*(v)(1) = 0$ and A has unit element 1, this implies $\pi^*(v) = 0$. As π is surjective, this shows $v = 0$.

1.8. Conic algebras. Let $(X, 1_X)$ be a unital module. An algebra $A \in \mathrm{Alg}(X)$ is called *conic* if there exists a unital quadratic form N such that

$$x^2 - T(x)x + N(x)1 = 0 \tag{1.8.1}$$

for all $x \in X$, where $T(x) = N(x, 1)$. By linearization, this implies

$$x \circ y - T(x)y - T(y)x + N(x, y)1 = 0, \tag{1.8.2}$$

where $x \circ y = xy + yx$ is the symmetrized product. Note that algebras of rank two are automatically conic [**7**, I, (1.3.6)]. Also note that

$$T(1) = N(1, 1) = 2N(1) = 2. \tag{1.8.3}$$

There is at most one N satisfying (1.8.1): Indeed, assume N' (with analogously defined T') also satisfies (1.8.1). Then $T(1) - T'(1) = 2 - 2 = 0$, so there exists a unique linear form t on

$M = X^\flat$ such that $t(\dot{x}) = T(x) - T'(x)$. Moreover, $T(x)x - N(x)1 = x^2 = T'(x)x - N'(x)1$ implies, by projecting to M, that $t(\dot{x})\dot{x} = 0$ for all $\dot{x} \in M$. Since M is finitely generated and projective, it follows easily by localization that $t = 0$. Hence $T = T'$ and then also $N = N'$ by (1.8.1). We call $N = N_A$ the *norm* and $T = T_A$ the *trace form* of A. From the definition it is clear that the norm of a conic algebra does not change when passing to the opposite algebra:

$$N_{A^{\mathrm{op}}} = N_A. \tag{1.8.4}$$

We denote by $\mathrm{Con}(X)$ the set of conic algebras on X.

Let $(Y, 1_Y)$ be another unital module and let $A \in \mathrm{Con}(X)$ and $B \in \mathrm{Con}(Y)$ be conic algebras. A *morphism* $\varphi\colon A \to B$ *of conic algebras* is defined to be a morphism of unital modules which preserves products and norms: $\varphi(xy) = \varphi(x)\varphi(y)$ and $N_B\big(\varphi(x)\big) = N_A(x)$, for all $x, y \in X$. The latter property is automatic if φ is injective, but not in general.

Conic algebras admit arbitrary base change: let $R \in k$-**alg** and let A_R be the R-linear extension of A to a bilinear map $A_R\colon X_R \times X_R \to X_R$. Then A_R is a conic algebra on X_R, with norm $N_{A_R} = N_A \otimes_k R$, the base change of the norm of A.

1.9. The conjugation. Let $A \in \mathrm{Con}(X)$ with norm N and trace T. The *conjugation* of A is the linear map $\iota = \iota_A\colon X \to X$ defined by

$$\iota(x) = \bar{x} = T(x) \cdot 1 - x. \tag{1.9.1}$$

From $T(1) = N(1,1) = 2$ it follows that $\iota^2 = \mathrm{Id}_X$ and $\iota(1) = 1$. The defining equation (1.8.1) of a conic algebra can be written as

$$\bar{x}x = N(x) \cdot 1 = x\bar{x}, \tag{1.9.2}$$

which implies by linearization that

$$\bar{x}y + \bar{y}x = N(x,y) \cdot 1 = x\bar{y} + y\bar{x}. \tag{1.9.3}$$

In general, ι is not an involution of the algebra A, cf. Section 4. Note, however, that it preserves norms and traces and is compatible with squaring:

$$N(\bar{x}) = N(x), \quad T(\bar{x}) = T(x), \quad \iota(x^2) = \iota(x)^2. \tag{1.9.4}$$

This follows from (1.9.2) and (1.8.1).

It is useful to introduce the bilinear map $H = H_A \in \mathrm{Bil}(X,X)$ given by

$$H(x,y) = \bar{x}y.$$

Then H has diagonal values in $k \cdot 1$; indeed, (1.9.2) and (1.9.3) imply

$$H(x,x) = N(x) \cdot 1, \tag{1.9.5}$$

$$H(x,y) + H(y,x) = N(x,y) \cdot 1. \tag{1.9.6}$$

Although H has some properties of a hermitian form, it is in general not true that $\iota\big(H(x,y)\big) = H(y,x)$. Rather, this is equivalent to ι being an involution of A, see Lemma 4.2. Since A^{op} has the same norm and trace as A, it has the same conjugation as well, which implies

$$H_{A^{\mathrm{op}}}(x,y) = y\bar{x} = H_A(\bar{y}, \bar{x}). \tag{1.9.7}$$

We also have the relation

$$H_{A^{\mathrm{op}}} = H_A^{\mathrm{op}} + \mathrm{d}_1 T_A. \tag{1.9.8}$$

Indeed, by (1.5.5),

$$\begin{aligned} H_A^{\mathrm{op}}(x,y) + (\mathrm{d}_1 T)(x,y) &= H_A(y,x) + T(x)y - T(y)x \\ &= \bar{y}x + T(x)y - T(y)x = (T(y)1 - y)x + T(x)y - T(y)x \\ &= y(T(x)1 - x) = y\bar{x} = H_{A^{\mathrm{op}}}(x,y). \end{aligned}$$

Let $A \in \mathrm{Con}(X)$ and $B \in \mathrm{Con}(Y)$ be conic algebras, and let $\varphi\colon A \to B$ be a homomorphism of unital algebras. Then it is easy to see that the following conditions are equivalent:

(i) φ is a morphism of conic algebras,
(ii) φ preserves traces: $T_B\big(\varphi(x)\big) = T_A(x)$,
(iii) φ commutes with conjugations: $\varphi\big(\iota_A(x)\big) = \iota_B\big(\varphi(x)\big)$,
(iv) φ preserves H: $\varphi\big(H_A(x,y)\big) = H_B\big(\varphi(x),\varphi(y)\big)$.

We introduce the following notation. For a bilinear form f on X, let f_1 and f_2 be the linear forms on X obtained by substituting 1_X in the first and second variable, respectively:

$$f_1(x) = f(1,x), \quad f_2(x) = f(x,1), \tag{1.9.9}$$

and denote by $[f]$ the quadratic form obtained by contraction: $[f](x) = f(x,x)$.

1.10. Lemma. *Let $(X, 1_X)$ be a unital module and put $M = X^\flat = X/k\cdot 1_X$.*

(a) *Let $f \in \mathrm{Bil}_1(X)$. Then the algebra f^{m} of Lemma 1.6 is conic, with*

$$N_{f^{\mathsf{m}}} = [f], \quad T_{f^{\mathsf{m}}} = f_1 + f_2, \quad H_{f^{\mathsf{m}}} = \mathrm{d}_1 f_1 + f\cdot 1_X. \tag{1.10.1}$$

(b) *Let $g \in \mathrm{Bil}_0(X)$. If A is conic then so is $A + g^{\mathsf{m}}$ with*

$$N_{A+g^{\mathsf{m}}} = N_A + [g], \quad T_{A+g^{\mathsf{m}}} = T_A + g_1 + g_2, \quad H_{A+g^{\mathsf{m}}} = H_A + \mathrm{d}_1 g_1 + g\cdot 1_X. \tag{1.10.2}$$

(c) *If $A \in \mathrm{Con}(X)$ and $\Gamma \in \Omega^2(M,X)$ then $A + \pi^*(\Gamma) \in \mathrm{Con}(X)$ with*

$$N_{A+\pi^*(\Gamma)} = N_A, \quad T_{A+\pi^*(\Gamma)} = T_A, \quad H_{A+\pi^*(\Gamma)} = H_A - \pi^*(\Gamma). \tag{1.10.3}$$

Conversely, given $A, A' \in \mathrm{Con}(X)$ with the same norm, there exists a unique $\Gamma \in \Omega^2(M,X)$ such that $A' = A + \pi^*(\Gamma)$.

Proof. (a) By Lemma 1.6, we have $f^{\mathsf{m}}(x,x) = \big(f(x,1) + f(1,x)\big)x - f(x,x)1$, so f^{m} is conic with the indicated norm and trace. Moreover,

$$\begin{aligned} H_{f^{\mathsf{m}}}(x,y) &= T(x)y - f^{\mathsf{m}}(x,y) \\ &= \big(f_1(x) + f_2(x)\big)y - f_2(x)y - xf_1(y) + f(x,y)\cdot 1 \\ &= f_1(x)y - f_1(y)x + f(x,y)\cdot 1. \end{aligned}$$

(b) Let $A' = A + g^{\mathsf{m}}$. Since A is conic,

$$\begin{aligned} A'(x,x) &= T_A(x)x - N_A(x)\cdot 1 + \big(g(x,1) + g(1,x)\big)x - g(x,x)\cdot 1 \\ &= \big(T_A(x) + g_1(x) + g_2(x)\big)\,x - \big(N_A(x) + g(x,x)\big)\cdot 1, \end{aligned}$$

so A' is conic with the indicated norm and trace. Denote the conjugation of A' by $\iota'(x) = T'(x)1 - x$. Then

$$\begin{aligned} H_{A'}(x,y) &= A'(\iota'(x),y) = T'(x)y - A'(x,y) \\ &= T(x)y - A(x,y) + \big(g(1,x) + g(x,1)\big)y - g^{\mathsf{m}}(x,y) \\ &= \bar{x}y + g_1(x)y - g_1(y)x + g(x,y)\cdot 1 \\ &= H_A(x,y) + (\mathrm{d}_1 g_1)(x,y) + g(x,y)\cdot 1. \end{aligned}$$

(c) From $\pi^*(\Gamma)(x,x) = \Gamma(\dot{x},\dot{x}) = 0$ it is clear that $A + \pi^*(\Gamma)$ is conic and its norm and trace is that of A. Hence A and $A + \pi^*(\Gamma)$ have the same conjugation as well, and it follows that

$$\begin{aligned} H_{A+\pi^*(\Gamma)}(x,y) &= \bar{x}y + \Gamma\big(\pi(\bar{x}),\pi(y)\big) \\ &= H_A(x,y) + \Gamma\big(\pi(T(x)1 - x),\, \pi(y)\big) = H_A(x,y) - \Gamma(\dot{x},\dot{y}). \end{aligned}$$

Now let A and A' be conic with the same norm N and hence trace T. By (1.3.1), $A' = A + \pi^*(\Gamma)$ for a unique $\Gamma \in \mathrm{Bil}(M,X)$. By (1.8.1) we have $\Gamma(\dot{x},\dot{x}) = A'(x,x) - A(x,x) = 0$, so Γ is in fact alternating.

1.11. Proposition. *Every $A \in \mathrm{Con}(X)$ can be written as*

$$A = f^{\mathsf{m}} + \pi^*(\Gamma) \tag{1.11.1}$$

where $f \in \mathrm{Bil}_1(X)$ and $\Gamma \in \Omega^2(M,X)$. If also $A = (f')^{\mathsf{m}} + \pi^*(\Gamma')$ then $f' = f + a$ and $\pi^*(\Gamma') = \pi^*(\Gamma) - a^{\mathsf{m}}$ for a unique $a \in \Omega^2(X)$.

Proof. Let c: $\mathrm{Bil}(X) \to \mathrm{Quad}(X)$, $f \mapsto [f]$, be the contraction map. Since X is finitely generated and projective, the sequence

$$0 \longrightarrow \Omega^2(X) \xrightarrow{\text{inc}} \mathrm{Bil}(X) \xrightarrow{c} \mathrm{Quad}(X) \longrightarrow 0 \tag{1.11.2}$$

is split exact [**6**, 5.1.15]. Hence, if $A \in \mathrm{Con}(X)$ with norm N, there exist bilinear forms f on X such that $N(x) = f(x,x)$, and even $f \in \mathrm{Bil}_1(X)$ since $N(1) = f(1,1) = 1$. We claim that (1.11.1) holds for a suitable Γ. Indeed, $(A - f^{\mathsf{m}})(x,x) = x^2 - \big(f(x,1)x - f(x,x)\cdot 1 + f(1,x)x\big) = x^2 - T(x)x + N(x)\cdot 1 = 0$, so $\tilde{\Gamma} := A - f^{\mathsf{m}}$ is alternating. Moreover, $\tilde{\Gamma}(1,y) = 1y - f^{\mathsf{m}}(1,y) = y - y = 0$, showing that $\tilde{\Gamma}$ induces a unique $\Gamma \in \Omega^2(M,X)$ such that $\pi^*(\Gamma) = \tilde{\Gamma}$.

Now suppose also $A = f'^{\mathsf{m}} + \pi^*(\Gamma')$. Then by (a) and (b), $N_A = [f] = [f']$. Hence there exists a unique $a \in \Omega^2(X)$ such that $f' = f + a$. Furthermore,

$$(\Gamma' - \Gamma)(\dot{x}, \dot{y}) = (f - f')^{\mathsf{m}}(x,y) = -a^{\mathsf{m}}(x,y).$$

1.12. The scheme $\mathbf{Quad}_1(X)$. Since X has rank $r = n+1$, the module $\mathrm{Quad}(X)$ has rank $\binom{n+2}{2}$. Moreover, $\mathrm{Quad}_1(X)$ being not empty, the evaluation map ε: $q \mapsto q(1)$ from $\mathrm{Quad}(X)$ to k is surjective. Hence the sequence

$$0 \longrightarrow \mathrm{Quad}_0(X) \xrightarrow{\text{inc}} \mathrm{Quad}(X) \xrightarrow{\varepsilon} k \longrightarrow 0$$

is exact, so $\mathrm{Quad}_0(X)$ is finitely generated and projective of rank $\binom{n+2}{2} - 1$.

Let $\mathbf{Quad}_1(X)$ denote the k-functor $R \mapsto \mathrm{Quad}_1(X_R)$. Any choice of $q_1 \in \mathrm{Quad}_1(X)$ yields a bijection $\mathrm{Quad}_1(X) \cong \mathrm{Quad}_0(X)$ by $q \mapsto q - q_1$, compatible with base change. Hence

$$\mathbf{Quad}_1(X) \cong \mathrm{Quad}_0(X)_{\mathbf{a}} \tag{1.12.1}$$

(not canonically) is a smooth affine k-scheme with fibres isomorphic to affine space of dimension $\binom{n+2}{2} - 1$.

Since conic algebras admit arbitrary base change (cf. 1.8), we have a set-valued functor $\mathbf{Con}(X)$ on k-**alg** given by $R \mapsto \mathrm{Con}(X_R)$. There is a morphism p: $\mathbf{Con}(X) \to \mathbf{Quad}_1(X)$ given by the norm: $p(A) = N_A$. From Lemma 1.10(c) it is clear that the map $(A, \Gamma) \mapsto A + \pi^*(\Gamma)$ defines an action of the additive group $\Omega^2(M,X)$ on $\mathrm{Con}(X)$. This action is compatible with base change as well, so we have an action of the group functor $\mathbf{G} = \Omega^2(M,X)_{\mathbf{a}}$ on $\mathbf{Con}(X)$. Refer to [**4**, III, §4] for torsors.

1.13. Theorem. *$\mathbf{Con}(X)$ is a trivial torsor with projection p and group $\mathbf{G}$ over the base $\mathbf{Quad}_1(X)$. As a k-scheme, $\mathbf{Con}(X)$ is smooth, affine and finitely presented, with fibres isomorphic to affine space of dimension $n\Big(1 + n + \binom{n}{2}\Big)$.*

Proof. By Lemma 1.10(c), the additive group of $\Omega^2(M,X)$ acts freely on $\mathrm{Con}(X)$ and its orbits are precisely the fibres of p. A section s of p can be obtained as follows. Let σ: $\mathrm{Quad}(X) \to \mathrm{Bil}(X)$ be a splitting of the exact sequence (1.11.2). Then σ maps $\mathrm{Quad}_1(X)$ to $\mathrm{Bil}_1(X)$, and Lemma 1.10(a) shows that $s(q) := \sigma(q)^{\mathsf{m}}$ (for $q \in \mathrm{Quad}_1(X)$) defines a section of p.

Since all this is compatible with arbitrary base change, $\mathbf{Con}(X)$ is a torsor as claimed. A choice of section yields a (non-canonical) isomorphism $\mathbf{Con}(X) \cong \mathbf{Quad}_1(X) \times \mathbf{G}$, and by (1.12.1) an isomorphism $\mathbf{Con}(X) \cong \mathrm{Quad}_0(X)_{\mathbf{a}} \times \mathbf{G}$. The modules $\mathrm{Quad}_0(X)$ and $\Omega^2(M,X)$ are finitely generated and projective of ranks $\binom{n+2}{2} - 1$ and $\binom{n}{2}(n+1)$, respectively. This implies the statement about the fibres of $\mathbf{Con}(X)$.

2. The canonical 3-form of a conic algebra

We begin with a cohomological result on a subcomplex of (1.5.2) which may be of independent interest.

2.1. Lemma. *Let M be a finitely generated and projective k-module of constant rank $n = p+q$. Then for all $\vec{x} = (x_0, \dots, x_p) \in M^{p+1}$, $\vec{y} = (y_1, \dots, y_q) \in M^q$ and $\omega \in \Omega^n(M)$,*

$$\sum_{i=0}^{p} (-1)^{p+i} x_i \, \omega(x_0, \dots, \hat{x}_i, \dots, x_p, \vec{y}) = \sum_{j=1}^{q} (-1)^{j-1} y_j \, \omega(\vec{x}, y_1, \dots, \hat{y}_j, \dots, y_q),$$

Proof. Let $x_0, \dots, x_n \in M$. Since M has rank n the exterior product $x_0 \wedge \cdots \wedge x_n$ vanishes. Hence by [**2**, Chapter III, §7, No. 4, Cor. 3 of Prop. 7]

$$\sum_{i=0}^{n} (-1)^i x_i \, \omega(x_0, \dots, \hat{x}_i, \dots, x_n) = 0.$$

Now the formula follows by renaming $x_{p+i} = y_i$ for $i = 1, \dots, q$ and multiplying with $(-1)^p$.

2.2. Proposition. *Let M be a finitely generated and projective k-module of constant rank $n \geqslant 2$. Let $X = Y = M$ and $\varphi = \mathrm{Id}$ in (1.5.2). Restricting the differential to the submodules $\Omega^p(M, \mathrm{S}^{p-1}M)$, $p \geqslant 0$, we obtain a cochain complex*

$$0 \xrightarrow{\mathrm{d}_0} M^* \xrightarrow{\mathrm{d}_1} \Omega^2(M,M) \xrightarrow{\mathrm{d}_2} \Omega^3(M, \mathrm{S}^2M) \xrightarrow{\mathrm{d}_3} \cdots \xrightarrow{\mathrm{d}_{n-1}} \Omega^n(M, \mathrm{S}^{n-1}M) \xrightarrow{\mathrm{d}_n} 0 \quad (2.2.1)$$

and this complex is split-exact.

Proof. M is in particular flat, so by [**3**, §9.3, Proposition 3], there are acyclic complexes

$$\mathscr{E}_l: \ 0 \longrightarrow \mathrm{S}^0M \otimes \textstyle\bigwedge^l M \xrightarrow{d} \mathrm{S}^1M \otimes \bigwedge^{l-1} M \xrightarrow{d} \cdots \xrightarrow{d} \mathrm{S}^lM \otimes \bigwedge^0 M \longrightarrow 0 \qquad (2.2.2)$$

for $l \geqslant 1$, where d is given by the formula

$$d\big(s \otimes (y_1 \wedge \cdots \wedge y_q)\big) = \sum_{j=1}^{q} (-1)^{j-1} (s \vee y_j) \otimes (y_1 \wedge \cdots \wedge \hat{y}_j \wedge \cdots \wedge y_q),$$

for $s \in \mathrm{S}^{l-q}M$ and $y_1, \dots, y_q \in M$.

Since M is finitely generated and projective of rank n, there is an isomorphism $\varphi\colon \bigwedge^q M \otimes \Omega^n(M) \xrightarrow{\cong} \Omega^p(M)$ given by the inner product:

$$\varphi(y \otimes \omega)(x) = (y \rfloor \omega)(x) = \omega(x \wedge y),$$

for $y \in \bigwedge^q M$, $\omega \in \Omega^n(M)$, $x \in \bigwedge^p M$, and $p + q = n$ [**2**, III, §11, No. 11, Prop. 12]. By tensoring φ with S^rM on the left, this induces an isomorphism, again denoted

$$\varphi : \mathrm{S}^rM \otimes \bigwedge^q M \otimes \Omega^n(M) \xrightarrow{\cong} \mathrm{S}^rM \otimes \Omega^p(M) \cong \Omega^p(M, \mathrm{S}^rM),$$

and given explicitly by $\varphi(s \otimes y \otimes \omega)(x) = s\,\omega(x \wedge y)$. We claim that the diagram

$$\begin{array}{ccc} \mathrm{S}^rM \otimes \bigwedge^q M \otimes \Omega^n(M) & \xrightarrow{d \otimes \mathrm{Id}} & \mathrm{S}^{r+1}M \otimes \bigwedge^{q-1} M \otimes \Omega^n(M) \\ \varphi \downarrow \cong & & \cong \downarrow \varphi \\ \Omega^p(M, S^rM) & \xrightarrow[\mathrm{d}_p]{} & \Omega^{p+1}(M, \mathrm{S}^{r+1}M) \end{array}$$

is commutative. Indeed, let $s \in \mathrm{S}^r M$, $y \in \bigwedge^q M$ and $\omega \in \Omega^n(M)$, and put $\vec{x} = (x_0, \dots, x_p) \in M^{p+1}$. We must show that

$$\varphi\big((d \otimes \mathrm{Id})(s \otimes y \otimes \omega)\big)(\vec{x}) = \mathrm{d}_p\big(\varphi(s \otimes y \otimes \omega)\big)(\vec{x}).$$

It is no restriction of generality to assume $y = y_1 \wedge \cdots \wedge y_q$ decomposable. Then the left hand side is

$$\varphi\Big(\sum_{j=1}^{q}(-1)^{j+1}(s \vee y_j)\otimes(y_1 \wedge \cdots \wedge \hat{y}_j \wedge \cdots \wedge y_q) \otimes \omega\Big)(\vec{x})$$
$$= \sum_{j=1}^{q}(-1)^{j+1}(s \vee y_j)\omega(\vec{x}, y_1, \dots, \hat{y}_j, \dots, y_q).$$

On the other hand, putting $\vec{y} = (y_1, \dots, y_q)$,

$$\begin{aligned}\mathrm{d}_p\big(\varphi(s \otimes y \otimes \omega)\big)(\vec{x}) &= \sum_{i=0}^{p}(-1)^{p+i}x_i \vee \varphi(s \otimes y \otimes \omega)(x_0, \dots, \hat{x}_i, \dots, x_p)\\ &= \sum_{i=0}^{p}(-1)^{p+i}(x_i \vee s)\,\omega(x_0, \dots, \hat{x}_i, \dots, x_p, \vec{y}).\end{aligned}$$

Now the commutativity of the diagram follows from Lemma 2.1 by multiplication with s in the symmetric algebra.

Since $\Omega^n(M)$ is flat, the sequence obtained from (2.2.2) by tensoring with $\Omega^n(M)$ is still exact. By applying the isomorphisms φ we obtain the exact sequence

$$0 \longrightarrow \Omega^{n-l}(M, \mathrm{S}^0 M) \xrightarrow{\mathrm{d}_{n-l}} \Omega^{n-l+1}(M, \mathrm{S}^1 M) \xrightarrow{\mathrm{d}_{n-l+1}} \cdots \xrightarrow{\mathrm{d}_{n-1}} \Omega^n(M, \mathrm{S}^l M) \longrightarrow 0$$

Now the exactness of (2.2.1) follows for $l = n - 1$ (which is still $\geqslant 1$ since $n \geqslant 2$).

To prove that the complex splits, let us put $C^p = \Omega^p(M, \mathrm{S}^{p-1}M)$ and $Z^p = \mathrm{Ker}(\mathrm{d}_p) = \mathrm{Im}(\mathrm{d}_{p-1})$. We must show that Z^p is a direct summand in C^p [**3**, §2, No. 5]. This is done by descending induction on p. Clearly all C^p are finitely generated and projective. For $p = n$ we have $Z^n = C^n$ since $\mathrm{d}_n = 0$. Assume by induction that Z^p is a direct summand in C^p, in particular, that it is finitely generated and projective. Then it follows from the exact sequence

$$0 \longrightarrow Z^{p-1} \longrightarrow C^{p-1} \xrightarrow{\mathrm{d}_{p-1}} Z^p \longrightarrow 0$$

that the same holds true of Z^{p-1}.

2.3. Corollary. *Let M be finitely generated and projective of constant rank $n \geqslant 2$ and put*

$$\mathrm{Z}^3(M) = \mathrm{Ker}\left(\mathrm{d}_3 : \Omega^3(M, \mathrm{S}^2 M) \to \Omega^4(M, \mathrm{S}^3 M)\right).$$

Then the sequence of k-modules

$$0 \longrightarrow M^* \xrightarrow{\mathrm{d}_1} \Omega^2(M, M) \xrightarrow{\mathrm{d}_2} \mathrm{Z}^3(M) \longrightarrow 0 \qquad (2.3.1)$$

is split-exact and $\mathrm{Z}^3(M)$ is finitely generated and projective of rank $n\big(\binom{n}{2} - 1\big)$.

Proof. Immediate.

We now return to the study of conic algebras and let, as in 1.2, $(X, 1_X)$ be a unimodular k-module of constant rank $n + 1$, so $M = X^\flat = X/k \cdot 1_X$ has rank n. The canonical projection $\pi\colon X \to M$ is denoted $\pi(x) = \dot{x}$. More generally, if f is any X-valued map, we put $\dot{f} = \pi \circ f$.

2.4. Lemma and Definition. *Let A be a conic algebra on X and let $H = H_A$ be the bilinear map defined in* 1.9.

(a) *There exists a unique $\Theta \in \mathrm{Z}^3(M)$ such that, for all $x, y, z \in X$,*

$$\Theta(\dot{x}, \dot{y}, \dot{z}) = \sum_{\text{cyc}} \dot{x} \vee \dot{H}(y, z) \tag{2.4.1}$$

$$= \mathrm{S}\pi\big((\mathrm{d}_2 H)(x, y, z)\big), \tag{2.4.2}$$

where $\sum_{\text{cyc}}$ denotes the cyclic sum over x, y, z and $\mathrm{S}\pi$ is the induced map on the symmetric algebras, see (1.5.8). *We call $\Theta = \Theta_A$ the* canonical 3-form *of A. Note that* (2.4.2) *says Θ and $\mathrm{d}_2 H$ are π-related as defined in* 1.5*:*

$$\pi^*(\Theta) = \pi_*(\mathrm{d}_2 H). \tag{2.4.3}$$

(b) *The canonical* 3*-form is compatible with morphisms: let $B \in \mathrm{Con}(Y)$ be a second conic algebra, let $\varphi\colon A \to B$ be a homomorphism of conic algebras as in* 1.8 *and let $\varphi^\flat\colon X^\flat \to Y^\flat$ be the induced module homomorphism. Then Θ_B and Θ_A are $\varphi^\flat$-related: $(\varphi^\flat)^*(\Theta_B) = (\varphi^\flat)_*(\Theta_A)$; explicitly, $\Theta_B\big(\varphi^\flat(\dot{x}), \varphi^\flat(\dot{y}), \varphi^\flat(\dot{z})\big) = (\mathrm{S}\varphi^\flat)\big(\Theta_A(\dot{x}, \dot{y}, \dot{z})\big)$, for all $\dot{x}, \dot{y}, \dot{z} \in M$.*

(c) *The canonical* 3*-form is compatible with base change: for $R \in k$-**alg** and with the identification* (1.2.3)*, we have $\Theta_{A_R} = (\Theta_A)_R$.*

Proof. (a) Since $H \in \mathrm{Bil}(X, X) = \mathscr{L}^2(X, \mathrm{S}^1 X)$, we have $\mathrm{d}_2 H \in \mathscr{L}^3(X, \mathrm{S}^2 X)$. It follows from (1.5.6) and (1.9.6) that

$$\begin{aligned}(\mathrm{d}_2 H)(x, y, z) &= x \vee H(y, z) - y \vee H(x, z) + z \vee H(x, y) \\ &= \Big(\sum_{\text{cyc}} x \vee H(y, z)\Big) - y \vee \Big(H(x, z) + H(z, x)\Big) \\ &= \Big(\sum_{\text{cyc}} x \vee H(y, z)\Big) - y \vee N(x, z) 1_X. \end{aligned} \tag{2.4.4}$$

The kernel of $\mathrm{S}\pi$ is the ideal of $\mathrm{S}X$ generated by 1_X [**2**, III, §6, No. 2, Proposition 4]. Hence (2.4.4) shows that, modulo $\mathrm{Ker}\,\mathrm{S}\pi$, $\mathrm{d}_2 H$ is invariant under cyclic permutation. Moreover,

$$(\mathrm{d}_2 H)(x, x, z) = z \vee H(x, x) = z \vee N(x) 1_X$$

by (1.9.5), so $(\mathrm{d}_2 H)(x, y, z)$ is, modulo $\mathrm{Ker}\,\mathrm{S}\pi$, an alternating function of x, y, z. Finally, the formula

$$(\mathrm{d}_2 H)(1_X, y, z) = 1_X \vee H(y, z) - y \vee z + z \vee y = 1_X \vee H(y, z)$$

shows that $(\mathrm{d}_2 H)(x, y, z)$ depends, modulo $\mathrm{Ker}\,\mathrm{S}\pi$, only on $\dot{x}$, $\dot{y}$, $\dot{z}$. Now (2.4.4) implies that there exists a unique $\Theta \in \Omega^3(M, \mathrm{S}^2 M)$ such that (2.4.1) and (2.4.2) hold, so Θ and $\mathrm{d}_2 H$ are π-related. By 1.5, $\mathrm{d}_3 \Theta$ and $\mathrm{d}_3 \mathrm{d}_2 H$ are π-related as well, whence $\pi^*(\mathrm{d}_3 \Theta) = \pi_*(\mathrm{d}_3 \mathrm{d}_2 H) = 0$. Since π is surjective, π^* is injective, so that $\mathrm{d}_3 \Theta = 0$, as required.

(b) follows immediately from (2.4.1) and the characterization 1.9(iv) of homomorphisms of conic algebras, and (c) is straightforward.

2.5. Corollary. *The canonical* 3*-form is invariant under transvections: $\Theta_{A^v} = \Theta_A$ for all $v \in V = M^*$.*

Proof. By 1.7, $\varphi = \tau_v\colon A^v \to A$ is an isomorphism, and the induced map $\varphi^\flat\colon M \to M$ is the identity. Now the assertion follows from 2.4(b).

2.6. Examples. (a) Let $A = \mathrm{Mat}_2(k)$. Here the conjugation is

$$\iota\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

and $M = \mathrm{Mat}_2(k)/k \cdot 1 = \mathfrak{pgl}_2(k)$. Let e_{ij} be the standard matrix units. Then by the well-known multiplication table of the e_{ij},

$$\begin{aligned}(\mathrm{d}_2 H)(e_{11}, e_{12}, e_{21}) &= e_{11} \vee (\bar{e}_{12}e_{21}) - e_{12} \vee (\bar{e}_{11}e_{21}) + e_{21} \vee (\bar{e}_{11}e_{12}) \\ &= e_{11} \vee (-e_{12}e_{21}) - e_{12} \vee (e_{22}e_{21}) + e_{21} \vee (e_{22}e_{12}) \\ &= -e_{11} \vee e_{11} - e_{12} \vee e_{21} + 0 = -e_{11} \vee 1_X + e_{11} \vee e_{22} - e_{12} \vee e_{21}.\end{aligned}$$

Applying $\mathrm{S}\pi$, we obtain

$$\Theta_A(\dot{e}_{11}, \dot{e}_{12}, \dot{e}_{21}) = \dot{e}_{11} \vee \dot{e}_{22} - \dot{e}_{12} \vee \dot{e}_{21} = \det\begin{pmatrix} \dot{e}_{11} & \dot{e}_{12} \\ \dot{e}_{21} & \dot{e}_{22} \end{pmatrix}, \tag{2.6.1}$$

the determinant being taken in the commutative ring $\mathrm{S}M$. Since M is free with basis $\dot{e}_{11}, \dot{e}_{12}, \dot{e}_{21}$, this determines the canonical 3-form completely.

(b) Let $A = \mathbb{H}$ be the real quaternion division algebra, with $\mathbb{R}$-basis $e_0 = 1_X, e_1, e_2, e_3$ and the usual multiplication table: $e_i^2 = -1$, $e_ie_j = -e_je_i = e_l$ for (i, j, l) a cyclic permutation of $(1, 2, 3)$. Then $H_A(e_i, e_j) = -e_l$, so by (2.4.1),

$$\Theta_{\mathbb{H}}(\dot{e}_1, \dot{e}_2, \dot{e}_3) = -\sum_{i=1}^{3} \dot{e}_i \vee \dot{e}_i.$$

2.7. Proposition. *The canonical* 3*-form of a conic algebra has the following properties.*

(a) Θ_A *is a skew-symmetric function of* A *in the sense that*

$$\Theta_{A^{\mathrm{op}}} = -\Theta_A. \tag{2.7.1}$$

(b) $2\Theta_A$ *is* π*-related to the derivative of the commutator* $C = A - A^{\mathrm{op}}$*:*

$$2\pi^*(\Theta_A) = -\pi_*(\mathrm{d}_2 C), \tag{2.7.2}$$

explicitly,

$$2\Theta_A(\dot{x}, \dot{y}, \dot{z}) = -\sum_{\mathrm{cyc}} \pi(x) \vee \pi([y, z]). \tag{2.7.3}$$

(c) *If* $A = f^{\mathrm{m}} + \pi^*(\Gamma)$ *as in Proposition* 1.11 *then*

$$\Theta_A = -\mathrm{d}_2\dot{\Gamma}. \tag{2.7.4}$$

Proof. (a) By differentiating (1.9.8) we obtain $\mathrm{d}_2 H_{A^{\mathrm{op}}} = \mathrm{d}_2 H_A^{\mathrm{op}}$. Formula (1.9.6) says $H_A + H_A^{\mathrm{op}} = \partial N \cdot 1_X$, and regarding this as a bilinear map on X with values in $\mathrm{S}X$, we have $\partial N \cdot 1_X = 1_X \vee \partial N$. Hence by (1.5.3),

$$\mathrm{d}_2 H_A + \mathrm{d}_2 H_A^{\mathrm{op}} = 1_X \vee \mathrm{d}_2(\partial N) \equiv 0 \mod \mathrm{Ker}\, \mathrm{S}\pi, \tag{2.7.5}$$

and therefore $\pi_*(\mathrm{d}_2 H_{A^{\mathrm{op}}}) = -\pi_*(\mathrm{d}_2 H_A)$. This implies (2.7.1) because of (2.4.2).

(b) From (1.9.1) it follows that $H_A(x, y) - H_A^{\mathrm{op}}(x, y) = \bar{x}y - y\bar{x} = -xy + yx = -C(x, y)$. Hence $\mathrm{d}_2 H_A - \mathrm{d}_2 H_A^{\mathrm{op}} = -\mathrm{d}_2 C$. Adding this to (2.7.5) yields $2\mathrm{d}_2 H_A \equiv -\mathrm{d}_2 C \mod \mathrm{Ker}\, \mathrm{S}\pi$, so (2.7.2) follows from (2.4.3) by applying π_*. Since the commutator is an alternating function, its derivative is given by $\mathrm{d}_2 C(x, y, z) = \sum_{\mathrm{cyc}} x \vee [y, z]$. This yields (2.7.3).

(c) By (1.10.1) and (1.10.3), $H_A = \mathrm{d}_1 f_1 + f \cdot 1_X - \pi^*(\Gamma)$. Hence $\mathrm{d}_2 H_A = \mathrm{d}_2\mathrm{d}_1 f + \mathrm{d}_2(1_X \vee f) - \mathrm{d}_2\pi^*(\Gamma) = 1_X \vee (\mathrm{d}_2 f) - \mathrm{d}_2\pi^*(\Gamma) \equiv -\mathrm{d}_2\pi^*(\Gamma) \mod \mathrm{Ker}\, \mathrm{S}\pi$. Since π^* is injective, it suffices to show that $\pi^*(\Theta_A) = -\pi^*(\mathrm{d}_2\dot{\Gamma})$. Now

$$\begin{aligned}\pi^*(\Theta_A) &= \pi_*(\mathrm{d}_2 H_A) \text{ (by (2.4.3))} = -\pi_*\big(\mathrm{d}_2(\pi^*(\Gamma))\big) \\ &= -\mathrm{d}_2^{\pi}\big(\pi_*(\pi^*(\Gamma))\big) \text{ (by (1.5.9))} = -\mathrm{d}_2^{\pi}\big(\pi^*(\dot{\Gamma})\big) = -\pi^*(\mathrm{d}_2\dot{\Gamma}) \text{ (by (1.5.10))}.\end{aligned}$$

2.8. Theorem. *Let* $\mathrm{rk}(X) \geqslant 3$ *and* $A \in \mathrm{Con}(X)$.

(a) $\Theta_A = 0$ *if and only if* $A = f^{\mathsf{m}}$ *for a unique* $f \in \mathrm{Bil}_1(X)$.

(b) *Let* $g \in \mathrm{Bil}_0(X)$. *Then* $A + g^{\mathsf{m}} \in \mathrm{Con}(X)$ *by* 1.10(b), *and*

$$\Theta_{A+g^{\mathsf{m}}} = \Theta_A. \tag{2.8.1}$$

Conversely, if $A' \in \mathrm{Con}(X)$ *with* $\Theta_{A'} = \Theta_A$ *then* $A' = A + g^{\mathsf{m}}$, *for a unique* $g \in \mathrm{Bil}_0(X)$.

(c) *If* $\Gamma \in \Omega^2(M, X)$ *then*

$$\Theta_{A+\pi^*(\Gamma)} = \Theta_A - \mathrm{d}_2\dot{\Gamma}. \tag{2.8.2}$$

Proof. (a) If $A = f^{\mathsf{m}}$ then $\Theta_A = 0$ is clear from (2.7.4). Conversely, let $\Theta_A = 0$ and write $A = f^{\mathsf{m}} + \pi^*(\Gamma)$ as in (1.11.1). By (2.7.4), $\mathrm{d}_2\dot{\Gamma} = 0$, so by Proposition 2.2, there exists a unique linear form v on M such that $\dot{\Gamma} = \mathrm{d}_1 v$, i.e., $\dot{\Gamma}(\dot{x}, \dot{y}) = v(\dot{x})\dot{y} - v(\dot{y})\dot{x}$ for all $x, y \in X$, cf. (1.5.5). Lifting this back to X, we conclude that there exists a unique bilinear form a on X such that

$$\Gamma(\dot{x}, \dot{y}) = v(\dot{x})y - v(\dot{y})x - a(x, y) \cdot 1_X.$$

As Γ is alternating so is a. Putting $y = 1_X$ yields $0 = v(\dot{x}) \cdot 1_X - 0 - a(x, 1_X) \cdot 1_X$ whence $v(\dot{x}) = a(x, 1_X)$. Now we have

$$\Gamma(\dot{x}, \dot{y}) = a(x, 1_X)y + xa(1_X, y) - a(x, y) \cdot 1_X = a^{\mathsf{m}}(x, y),$$

i.e., $\pi^*(\Gamma) = a^{\mathsf{m}}$, which implies

$$A = f^{\mathsf{m}} + \pi^*(\Gamma) = f^{\mathsf{m}} + a^{\mathsf{m}} = (f + a)^{\mathsf{m}}.$$

After replacing f with $f + a$, we see that A has the required form. Uniqueness follows from Lemma 1.6(c).

(b) By Lemma 1.10(b), $A' = A + g^{\mathsf{m}}$ is conic with $H_{A'} = H_A + \mathrm{d}_1 g_1 + g \cdot 1_X$. Hence $\mathrm{d}_2 H_{A'} = \mathrm{d}_2 H_A + 1_X \vee \mathrm{d}_2 g \equiv \mathrm{d}_2 H_A \mod \mathrm{Ker}\, \mathrm{S}\pi$, so $\Theta_{A'} = \Theta_A$ by (2.4.2). Conversely, let A and A' be conic algebras with $\Theta_A = \Theta_{A'}$. Write $A = f^{\mathsf{m}} + \pi^*(\Gamma)$ and $A' = (f')^{\mathsf{m}} + \pi^*(\Gamma')$ as in (1.11.1) and consider $A'' := A - \pi^*(\Gamma') = f^{\mathsf{m}} + \pi^*(\Gamma - \Gamma') \in \mathrm{Con}(X)$ (Lemma 1.10(c)). Then $\Theta_{A''} = -\mathrm{d}_2\big(\dot{\Gamma} - \dot{\Gamma}'\big) = \Theta_A - \Theta_{A'} = 0$ by (2.7.4), so by (a), $A'' = (f'')^{\mathsf{m}}$ for a unique $f'' \in \mathrm{Bil}_1(X)$. Hence

$$A' = (f')^{\mathsf{m}} + \pi^*(\Gamma') = (f')^{\mathsf{m}} + A - A'' = A + (f' - f'')^{\mathsf{m}}$$

where $g = f' - f'' \in \mathrm{Bil}_0(X)$ is unique by Lemma 1.6(c).

(c) Let $A' = A + \pi^*(\Gamma)$. By Lemma 1.10(c), $H_{A'} = H_A - \pi^*(\Gamma)$. Hence $\mathrm{d}_2 H_{A'} = \mathrm{d}_2 H_A - \mathrm{d}_2\pi^*(\Gamma)$, so (2.8.2) follows from Lemma 2.4(a) by applying π.

2.9. Corollary. *Let* $\mathrm{rk}\, X = 3$. *Then the map* $f \mapsto f^{\mathsf{m}}$ *is a bijection between* $\mathrm{Bil}_1(X)$ *and* $\mathrm{Con}(X)$.

Proof. For X of rank 3 we have $\mathrm{rk}\, M = 2$, so the canonical 3-form is automatically zero. Now the assertion follows from 2.8(a).

Since Θ_A is compatible with base change by Lemma 2.4, there is a well-defined morphism of functors $p'\colon \mathbf{Con}(X) \to \mathrm{Z}^3(M)_{\mathbf{a}}$ given by $p'(A) = \Theta_A$, for all $A \in \mathrm{Con}(X_R)$ and $R \in k$-**alg**. The action of the group $\mathrm{Bil}_0(X)$ on $\mathrm{Con}(X)$ by $(A, g) \mapsto A + g^{\mathsf{m}}$ is compatible with base change as well, thus inducing an action of $\mathrm{Bil}_0(X)_{\mathbf{a}}$ on $\mathbf{Con}(X)$. Now we have the following companion result to Theorem 1.13:

2.10. Corollary. *Let* $\operatorname{rk} X \geqslant 3$. *Then* $\mathbf{Con}(X)$ *is a trivial torsor with projection* $p'(A) = \Theta_A$ *and group* $\mathrm{Bil}_0(X)_{\mathbf{a}}$ *over the base* $\mathrm{Z}^3(M)_{\mathbf{a}}$.

Proof. By Theorem 2.8(b), $\mathrm{Bil}_0(X)$ acts freely on $\mathrm{Con}(X)$ and its orbits are precisely the fibres of p'. We construct a section s' of p' as follows. Choose a section $\sigma\colon \mathrm{Z}^3(M) \to \Omega^2(M,M)$ of d_2 (cf. (2.3.1)) and a section $s\colon M \to X$ of π. Let $f \in \mathrm{Bil}_1(X)$. For a given $z \in \mathrm{Z}^3(M)$, define $\Gamma \in \Omega^2(M,X)$ by $\Gamma(\dot{x},\dot{y}) = -s\big(\sigma(z)(\dot{x},\dot{y})\big)$, and put $s'(z) = f^{\mathrm{m}} + \pi^*(\Gamma)$. Then $s'(z) \in \mathrm{Con}(X)$ by Lemma 1.10(a),(c). Moreover, by (a) and (c) of Theorem 2.8, $p'(s'(z)) = \Theta_{s'(z)} = -\mathrm{d}_2\dot{\Gamma} = \mathrm{d}_2(\sigma(z)) = z$. Since all this is compatible with base change, the assertion follows.

By combining this corollary with Theorem 1.13, we obtain:

2.11. Corollary. *Let* $\operatorname{rk} X \geqslant 3$. *Then* $\mathbf{Con}(X)$ *is a trivial torsor with projection* $p'' = p \times p'$ *and group* $\Omega^2(X)_{\mathbf{a}}$, *acting by* $(A,g) \mapsto A + g^{\mathrm{m}}$, *over the base* $\mathbf{Quad}_1(X) \times \mathrm{Z}^3(M)_{\mathbf{a}}$.

Proof. Clearly, the action of $\Omega^2(X)$ is compatible with p''. Conversely, let $p(A) = p(A')$ and $p'(A) = p'(A')$. Then by Theorem 2.8(b), $A = A' + g^{\mathrm{m}}$ for some $g \in \mathrm{Bil}_0(X)$. Moreover, $N_A = N_{A'} = N_A + [g]$ by (1.10.2), so $[g] = 0$ and g is alternating. A section of p'' is obtained as follows. Choose a section $s_1\colon \mathrm{Quad}_1(X) \to \mathrm{Bil}_1(X)$ of c, a section $s\colon M \to X$ of π, and a section $\sigma\colon \mathrm{Z}^3(M) \to \Omega^2(M,M)$. Then $s''(q,z) := s_1(q)^{\mathrm{m}} - \pi^*\big(s(\sigma(z))\big)$ is a section of p''.

3. Coordinates for conic algebras

3.1. Preliminaries. Let X be a unital module as in 1.2 and recall the exact sequence (1.2.1). The unital linear forms on X are precisely the retractions $\alpha\colon X \to k$ of $1_X\colon k \to X$. Hence they are in one-to-one correspondence with the sections of $\pi\colon X \to M$, by assigning to $\alpha \in X_1^*$ the section $s_\alpha\colon M \to X$ given by

$$s_\alpha(\dot{x}) = x - \alpha(x)\cdot 1_X. \tag{3.1.1}$$

Thus $s_\alpha : M \xrightarrow{\cong} M_\alpha := \mathrm{Ker}(\alpha) = \mathrm{Im}(s_\alpha)$ is an isomorphism of k-modules with inverse $\pi\colon M_\alpha \to M$, and

$$X = k\cdot 1_X \oplus M_\alpha \quad \text{(direct sum of } k\text{-modules).} \tag{3.1.2}$$

For $f \in \mathscr{L}^p(X,\mathrm{S}X)$ let $s_\alpha^*(f) \in \mathscr{L}^p(M,\mathrm{S}X)$ be the pull-back of f to M via s_α, and for $g \in \mathscr{L}^p(M,\mathrm{S}X)$ we write $\pi_*(g) = \mathrm{S}\pi \circ g \in \mathscr{L}^p(M,\mathrm{S}M)$, as in 1.4. Then $\pi_*\big(s_\alpha^*(f)\big) = \mathrm{S}\pi \circ f \circ s_\alpha^p = s_\alpha^*\big(\pi_*(f)\big) \in \mathscr{L}^p(M,\mathrm{S}M)$, and similarly $\alpha_*\big(s_\alpha^*(f)\big) \in \mathscr{L}^p(M,\mathrm{S}k)$. The derivative of 1.5 behaves with respect to these operations as follows:

$$\pi_*\big(s_\alpha^*(\mathrm{d}_p f)\big) = \mathrm{d}_p\big(\pi_*(s_\alpha^* f)\big), \tag{3.1.3}$$

$$\alpha_*\big(s_\alpha^*(\mathrm{d}_p f)\big) = 0. \tag{3.1.4}$$

Indeed, recalling from (1.5.1) the definition of the φ-differential, we have

$$\begin{aligned}\pi_*\big(s_\alpha^*(\mathrm{d}_p f)\big) &= \pi_*\big(\mathrm{d}_p^{s_\alpha} s_\alpha^*(f)\big) \quad \text{(by (1.5.10))}\\ &= \mathrm{d}_p^{\pi\circ s_\alpha}\big(\pi_* s_\alpha^*(f)\big) \quad \text{(by (1.5.9))} = \mathrm{d}_p\big(\pi_*(s_\alpha^*(f))\big),\end{aligned}$$

since $\pi \circ s_\alpha = \mathrm{Id}_M$. The second formula is proved similarly, using the fact that $\alpha \circ s_\alpha = 0$. In the special case where $f \in \mathscr{L}^p(X,k)$ has scalar values, we have $\pi_*(s_\alpha^*(f)) = s_\alpha^*(f)$ since the restriction of $\mathrm{S}\pi$ to $\mathrm{S}^0 X = k$ is the identity. Hence (3.1.3) reads in this case

$$\pi_*\big(s_\alpha^*(\mathrm{d}_p f)\big) = \mathrm{d}_p s_\alpha^*(f). \tag{3.1.5}$$

3.2. α-coordinates. We now give a description of conic algebras on X in terms of data on M and a unital linear form α on X. This is inspired by Petersson's polar decomposition of quaternion algebras [**15**]. While not intrinsic, it is an effective computational tool.

Let A be a conic algebra on X with norm N_A, trace T_A, conjugation $x \mapsto \bar{x}$ and bilinear map $H_A(x,y) = \bar{x}y$ as in 1.9. We define a linear form t and a bilinear form b on M, as well as a bilinear map $K\colon M \times M \to M$, all depending on α, by the formulas

$$t = s_\alpha^*(T_A), \tag{3.2.1}$$
$$K = \pi_*\big(s_\alpha^*(H_A)\big) = s_\alpha^*(\dot H_A), \tag{3.2.2}$$
$$b = \alpha\big(s_\alpha^*(H_A)\big). \tag{3.2.3}$$

More explicitly, we have

$$t(\dot x) = T_A\big(s_\alpha(\dot x)\big) = T_A\big(x - \alpha(x)1_X\big) = T_A(x) - 2\alpha(x) \text{ (by (1.8.3))}, \tag{3.2.4}$$

so t is the linear form on M induced by the linear form $T_A - 2\alpha$ on X which vanishes at 1_X. Similarly, since $H_A(x,1) = \bar x = T_A(x)1 - x$ and $H_A(1,y) = y$,

$$\begin{aligned} K(\dot x, \dot y) &= \dot H_A(s_\alpha(x), s_\alpha(y)) = \dot H_A(x - \alpha(x)1, y - \alpha(y)1) \\ &= \dot H_A(x,y) - \alpha(x)\dot H_A(1,y) - \alpha(y)\dot H_A(x,1) + \alpha(x)\alpha(y)\dot H_A(1,1) \\ &= \dot H_A(x,y) - \alpha(x)\dot y + \alpha(y)\dot x = \pi\big((H_A - \mathrm{d}_1\alpha)(x,y)\big). \end{aligned} \tag{3.2.5}$$

From $\mathrm{Id}_X = 1_X \otimes \alpha + s_\alpha \circ \pi$ it follows that (3.2.2) and (3.2.3) are equivalent to the single equation

$$b \cdot 1_X + s_\alpha \circ K = s_\alpha^*(H_A), \tag{3.2.6}$$

explicitly,

$$b(\dot x, \dot y) \cdot 1_X + s_\alpha\big(K(\dot x, \dot y)\big) = H_A\big(s_\alpha(\dot x), s_\alpha(\dot y)\big) = \overline{s_\alpha(\dot x)} s_\alpha(\dot y). \tag{3.2.7}$$

In general, b is not symmetric. The following formula says that b measures the failure of α to be an algebra homomorphism:

$$b(\dot x, \dot y) = \alpha(x)\alpha(y) - \alpha(xy). \tag{3.2.8}$$

Indeed, by (3.2.3) we have $b(\dot x, \dot y) = \alpha\Big(\overline{s_\alpha(\dot x)} \cdot s_\alpha(\dot y)\Big)$, and

$$\begin{aligned} \overline{s_\alpha(\dot x)} \cdot s_\alpha(\dot y) &= \big(\bar x - \alpha(x)1\big)\, s_\alpha(\dot y) = \big(T(x) - \alpha(x)\big)\, s_\alpha(\dot y) - x\big(y - \alpha(y)1\big) \\ &= \big(T(x) - \alpha(x)\big)\, s_\alpha(\dot y) - xy + x\alpha(y), \end{aligned}$$

so (3.2.8) follows by applying α to this relation and using $\alpha\big(s_\alpha(\dot y)\big) = 0$.

From (1.9.5) and (3.2.5) it is clear that $K(\dot x, \dot x) = 0$, so $K \in \Omega^2(M,M)$ is alternating. We call the triple $\phi_\alpha(A) := (t, K, b)$ the *coordinates of A with respect to α* or the *α-coordinates* of A. They depend on A as well as on the choice of α. If necessary, we will write $(t,K,b) = (t^\alpha, K^\alpha, b^\alpha)$ or (t_A, K_A, b_A) or even $(t_A^\alpha, K_A^\alpha, b_A^\alpha)$ to indicate this fact.

The α-coordinates are compatible with base change in the following sense. Suppose $R \in k$-**alg** and let α_R be the R-linear extension of α to a unital linear form on X_R. Identify M_R with $(X_R)^\flat$ as in (1.2.3). Then the α_R-coordinates of A_R are the R-linear extensions of the α-coordinates of A:

$$(t_{A_R}, K_{A_R}, b_{A_R}) = (t_A, K_A, b_A)_R.$$

The proof is straightforward.

The α-coordinates of A^{op} are given by

$$t_{A^{\mathrm{op}}} = t_A, \qquad K_{A^{\mathrm{op}}} = K_A^{\mathrm{op}} + \mathrm{d}_1 t_A, \qquad b_{A^{\mathrm{op}}} = b_A^{\mathrm{op}}. \tag{3.2.9}$$

Indeed, the first formula is clear from (3.2.4) since A and A^{op} have the same trace. The second formula follows from (1.9.8) by applying $\pi_* \circ s_\alpha^*$, and the third formula is immediate from (3.2.8).

We define the *cross product with respect to α* as the alternating map $\times\colon M_\alpha \times M_\alpha \to M_\alpha$ given by

$$x \times y = s_\alpha\big(K(\dot x, \dot y)\big) \quad (x, y \in M_\alpha). \tag{3.2.10}$$

By (3.1.2), an arbitrary element of X has the form $\lambda 1 \oplus x$ where $\lambda \in k$ and $x = s_\alpha(\dot x) \in M_\alpha$. Then it follows from (3.2.1) and (3.2.7) that the multiplication in A is given by the formula

$$(\lambda 1 \oplus x)(\mu 1 \oplus y) = \big(\lambda\mu - b(\dot{x}, \dot{y})\big)1 \oplus \Big(\mu x + \big(\lambda + t(\dot{x})\big)y - x \times y\Big). \tag{3.2.11}$$

This is the analogue of Petersson's formula [**15**, (2.5.2)]. Note that he uses the cross product $x \times_P y = s_\alpha(K_P(\dot{x}, \dot{y}))$ defined by the equation

$$s_\alpha(\dot{x})\overline{s_\alpha(\dot{y})} = b(\dot{x}, \dot{y}) \cdot 1 - s_\alpha\big(K_P(\dot{x}, \dot{y})\big).$$

The relation between the two is

$$x \times_P y = -x \times y + t(\dot{x})y - t(\dot{y})x,$$

which accounts for the difference between [**15**, (2.5.2)] and (3.2.11).

Examples. (a) Let $A = \mathbb{H}$ be the usual real division algebra of quaternions, with orthonormal basis $e_0 = 1, e_1, e_2, e_3$. Let e_i^* be the dual basis and put $\alpha = e_0^*$, so that $M_\alpha = \bigoplus_{i=1}^3 \mathbb{R} \cdot e_i$. Then $\bar{x} = -x$ for $x \in M_\alpha$ so $t = 0$, and the cross product in the sense of (3.2.10) is the *negative* of the usual cross product on $\mathbb{R}^3$, while b is the standard scalar product on $\mathbb{R}^3$.

(b) Let $A = f^{\mathfrak{m}}$ as in 1.6, where $f \in \mathrm{Bil}_1(X)$. Then $f_1(x) = f(1, x)$ and $f_2(x) = f(x, 1)$ are unital linear forms. The α-coordinates of A with respect to $\alpha := f_1$ are particularly simple, being given by

$$t(\dot{x}) = f_2(x) - f_1(x), \quad K = 0, \quad b(\dot{x}, \dot{y}) = f(s_\alpha(\dot{x}), s_\alpha(\dot{y})). \tag{3.2.12}$$

Indeed, by (1.10.1), $T_A = f_1 + f_2$ and $H_A = \mathrm{d}_1 f_1 + f \cdot 1_X$. Hence, $T_A - 2\alpha = f_1 + f_2 - 2f_1 = f_2 - f_1$ which yields the formula for t by (3.2.4). Furthermore,

$$\begin{aligned} K &= \pi_*\big(s_\alpha^*(H_A)\big) = \pi_*(s_\alpha^*(\mathrm{d}_1 f_1 + f \cdot 1_X)) = \mathrm{d}_1\big(\pi_* s_\alpha^*(f_1)\big) + 0 = 0, \\ b &= \alpha(s_\alpha^*(\mathrm{d}_1 f_1 + f \cdot 1_X)) = 0 + s_\alpha^*(f), \end{aligned}$$

by (3.1.3) and (3.1.4) and because $s_\alpha^*(f_1) = \alpha \circ s_\alpha = 0$.

3.3. Proposition. *Let $\alpha \in X_1^*$ be a unital linear form and let*

$$P(M) := M^* \times \Omega^2(M, M) \times \mathrm{Bil}(M)$$

be the parameter space for the α-coordinates.

(a) *The coordinate map*

$$\phi_\alpha\colon \mathrm{Con}(X) \to P(M), \quad \phi_\alpha(A) = (t_A^\alpha,\, K_A^\alpha,\, b_A^\alpha)$$

defined in 3.2 *is bijective, with inverse*

$$\phi_\alpha^{-1}(t, K, b) = f^{\mathfrak{m}} + \pi^*(\varGamma),$$

where

$$f(x, y) = \big(\alpha(x) + t(\dot{x})\big)\alpha(y) + b(\dot{x}, \dot{y}), \qquad \varGamma(\dot{x}, \dot{y}) = -s_\alpha\big(K(\dot{x}, \dot{y})\big). \tag{3.3.1}$$

(b) *Norm, trace and the canonical* 3*-form of a conic algebra A are given in terms of its α-coordinates (t, K, b) by*

$$N_A(\lambda 1 \oplus s_\alpha(\dot{x})) = \lambda^2 + \lambda t(\dot{x}) + b(\dot{x}, \dot{x}), \tag{3.3.2}$$

$$T_A(\lambda 1 \oplus s_\alpha(\dot{x})) = 2\lambda + t(\dot{x}), \tag{3.3.3}$$

$$\varTheta_A = \mathrm{d}_2 K. \tag{3.3.4}$$

Proof. (a) From from (3.2.11) it is clear that A is uniquely determined by (t, K, b), so ϕ_α is injective. Conversely, for a given $(t, K, b) \in P(M)$ define f and Γ by (3.3.1), and put $A = f^{\mathtt{m}} + \pi^*(\Gamma)$. By Lemma 1.10, A is conic, with $T_A = f_1 + f_2$ and $H_A = \mathrm{d}_1 f_1 + f \cdot 1_X - \pi^*(\Gamma)$. Denoting the α-coordinates of A by (t', K', b'), it remains to show that $(t', K', b') = (t, K, b)$.

From the definition of f in (3.3.1) it follows immediately that $f_1 = \alpha$ and $f_2 = \alpha + \pi^*(t)$. Hence $t' = s_\alpha^*(T_A) = s_\alpha^*(2\alpha + \pi^*(t)) = t$, since $s_\alpha^*(\alpha) = 0$ and $\pi \circ s_\alpha = \mathrm{Id}_M$. Next, applying π to the second formula of (3.3.1) shows $\dot{\Gamma} = -K$. Now (3.2.5) implies

$$\begin{aligned} K'(\dot{x}, \dot{y}) &= \pi\big((H_A - \mathrm{d}_1 f_1)(x, y)\big) \\ &= \pi\big(f(s_\alpha(\dot{x}), s_\alpha(\dot{y})) \cdot 1_X - \Gamma(\dot{x}, \dot{y})\big) = 0 + K(\dot{x}, \dot{y}). \end{aligned}$$

Finally, Γ takes values in $\mathrm{Im}\, s_\alpha = \mathrm{Ker}(\alpha)$ by (3.3.1), so $\alpha \circ \Gamma = 0$. Also, since $\alpha(s_\alpha(\dot{y})) = 0$, (3.3.1) implies $s_\alpha^*(f) = b$. Therefore, by (3.2.3) and using (3.1.4),

$$b' = \alpha(s_\alpha^*(H_A)) = \alpha(s_\alpha^*(\mathrm{d}_1 f_1 + f \cdot 1_X - \pi^*(\Gamma))) = 0 + s_\alpha^*(f) - 0 = b,$$

as required.

(b) From the definition of the conjugation and of t we have

$$\overline{s_\alpha(\dot{x})} = t(\dot{x}) 1_X - s_\alpha(\dot{x}).$$

Now (3.3.2) follows from (1.9.2) and (3.2.7), while (3.3.3) is immediate from (3.2.4). Finally, (3.3.4) is a consequence of (2.7.4) since $A = f^{\mathtt{m}} + \pi^*(\Gamma)$ and $\dot{\Gamma} = -K$.

We now show how the basic modifications of conic algebras described in Lemma 1.10 are reflected in their α-coordinates.

3.4. Proposition. *Let A be a conic algebra with α-coordinates (t, K, b).*

(a) *Let $g \in \mathrm{Bil}_0(X)$, put $A' = A + g^{\mathtt{m}}$ and let g_1 and g_2 be the linear forms on X defined by $g_1(x) = g(1, x)$ and $g_2(x) = g(x, 1)$, cf. (1.9.9). Then the α-coordinates (t', K', b') of A' are*

$$t' = t + s_\alpha^*(g_1 + g_2), \tag{3.4.1}$$
$$K' = K + \mathrm{d}_1 s_\alpha^*(g_1), \tag{3.4.2}$$
$$b' = b + s_\alpha^*(g). \tag{3.4.3}$$

(b) *Let $\Gamma \in \Omega^2(M, X)$ and put $A' = A + \pi^*(\Gamma)$. Then the α-coordinates (t', K', b') of A' are*

$$(t', K', b') = (t,\, K - \dot{\Gamma},\, b - \alpha \circ \Gamma). \tag{3.4.4}$$

Proof. (a) By (1.10.2), $T_{A'} = T_A + g_1 + g_2$ so (3.4.1) follows from (3.2.1). Next,

$$\begin{aligned} K' &= \pi_*\big(s_\alpha^*(H_{A'})\big) \text{ (by (3.2.2))} = \pi_*\big(s_\alpha^*(H + \mathrm{d}_1 g_1 + g \cdot 1_X)\big) \text{ (by (1.10.2))} \\ &= K + \pi_*\big(s_\alpha^*(\mathrm{d}_1 g_1)\big) = K + \mathrm{d}_1\big(s_\alpha^*(g_1)\big) \text{ (by (3.1.5))}. \end{aligned}$$

In the same way, by (3.2.3) and (3.1.4),

$$\begin{aligned} b' &= \alpha\big(s_\alpha^*(H_A + \mathrm{d}_1 g_1 + g \cdot 1_X)\big) \\ &= \alpha\big(s_\alpha^*(H_A)\big) + \alpha \circ \big(s_\alpha^*(\mathrm{d}_1 g_1)\big) + s_\alpha^*(g) \\ &= b + 0 + s_\alpha^*(g). \end{aligned}$$

(b) A and A' have the same norm and trace, hence the same conjugation, so we have $t' = t$. Furthermore, by (1.10.3), $H_{A'} = H_A - \pi^*(\Gamma)$. Hence $s_\alpha^*(H_{A'}) = s_\alpha^*(H_A) - \Gamma$, from which the remaining formulas follow easily by (3.2.6).

3.5. Corollary. *Let* (t, K, b) *be the* α*-coordinates of* $A \in \mathrm{Con}(X)$*, let* $v \in V = M^*$ *and let* τ_v *be the transvection defined by* v*. Put* $A^v = A + g^{\mathrm{m}}_{v,A}$ *as in 1.7. Then*

$$s_\alpha^*(g_{v,A}) = (v - t) \otimes v + v \circ K, \tag{3.5.1}$$

and the α*-coordinates* (t^v, K^v, b^v) *of* A^v *are*

$$t^v = t - 2v, \tag{3.5.2}$$

$$K^v = K - \mathrm{d}_1 v, \tag{3.5.3}$$

$$b^v = b + (v - t) \otimes v + v \circ K. \tag{3.5.4}$$

Proof. We use the formulas (3.4.1)–(3.4.3) with $g(x, y) = g_{v,A}(x, y) = v(\dot{x})v(\dot{y}) - v(\pi(xy))$. Then $g_1(x) = g_2(x) = -v(\dot{x})$ which implies $s_\alpha^*(g_i) = -v$. Hence $t^v = t - 2v$ by (3.4.1) and $K^v = K - \mathrm{d}_1 v$ by (3.4.2). The remaining formula then follows from (3.4.3) once we have shown (3.5.1). We have $H_A(x, y) = \bar{x}y = T_A(x)y - xy$, whence $g_{v,A}(x, y) = \big(v(\dot{x}) - T_A(x)\big)v(\dot{y}) + v\big(\dot{H}_A(x, y)\big)$. Now (3.5.1) follows by putting $x = s_\alpha(\dot{x})$ and $y = s_\alpha(\dot{y})$ in this formula and observing (3.2.1) and (3.2.2).

3.6. Change of coordinates. We study next how the α-coordinates of A change with α. Thus let α be a second unital linear form. Then $\alpha' - \alpha$ vanishes at 1_X, so there is a unique linear form v on M such that $\alpha'(x) = \alpha(x) + v(\dot{x})$, i.e., $\alpha' = \alpha + \pi^*(v)$. We claim that the α'-coordinates of A are precisely the α-coordinates of A^v,

$$\phi_{\alpha'}(A) = \phi_{\alpha+\pi^*(v)}(A) = \phi_\alpha(A^v). \tag{3.6.1}$$

In detail, the α'-coordinates (t', K', b') of A are given in terms of the α-coordinates (t, K, b) of A by formulas derived from (3.5.2) – (3.5.4) as follows:

$$(t',\, K',\, b') = (t - 2v,\, K - \mathrm{d}_1 v,\, b + (v - t) \otimes v + v \circ K). \tag{3.6.2}$$

Indeed, an easy verification using (1.7.1) shows that

$$s_{\alpha'} = \tau_v \circ s_\alpha, \quad \pi \circ \tau_v = \pi, \quad \alpha' \circ \tau_v = \alpha. \tag{3.6.3}$$

Let (t^v, K^v, b^v) be the α-coordinates of A^v as in (3.5.2) – (3.5.4). Since $\tau_v\colon A^v \to A$ is an isomorphism of algebras by 1.7, we have $\tau_v^*(T_A) = T_{A^v}$ by (ii) of 1.9. Hence by (3.2.1), $t' = s_{\alpha'}^*(T_A) = s_\alpha^*(\tau_v^*(T_A)) = s_\alpha^*(T_{A^v}) = t^v$. Similarly, by (iv) of 1.9, $H_A(\tau_v(x), \tau_v(y)) = \tau_v\big(H_{A^v}(x, y)\big)$, i.e., $\tau_v^*(H_A) = (\tau_v)_*(H_{A^v})$. Now by (3.2.2),

$$\begin{aligned} K' &= \pi_*\big(s_{\alpha'}^*(H_A)\big) = \pi_*\big(s_\alpha^*\tau_v^*(H_A)\big) = \pi_*\big(s_\alpha^*(\tau_v)_*(H_{A^v})\big) \\ &= \pi_*\big((\tau_v)_*(s_\alpha^*(H_{A^v}))\big) = (\pi \circ \tau_v)_*\big(s_\alpha^*(H_{A^v})\big) = \pi_*\big(s_\alpha^*(H_{A^v})\big) = K^v. \end{aligned}$$

One shows in the same way, using (3.2.3), that $b' = b^v$.

3.7. Suppose from now on that $\mathrm{rk}\, X \geqslant 3$, so that $M = X/k \cdot 1$ is a finitely generated and projective k-module of rank $n \geqslant 2$. By (1.7.5), $V = M^*$ acts freely on $\mathrm{Con}(X)$ by transvections, and our objective is now to identify the quotient of $\mathrm{Con}(X)$ by this action; equivalently, by Corollary 3.5, the quotient of $P(M)$ by the action of V described in (3.5.2)–(3.5.4). In case 2 is a unit in the base ring, it follows from (3.5.2) that every orbit of V contains exactly one algebra whose first α-coordinate is zero, and from this it is not hard to show that $\mathrm{Con}(X)/V \cong \Omega^2(M, M) \times \mathrm{Bil}(M)$, see also 3.11. In general, however, it is not possible to reduce the first α-coordinate of A to zero by a transvection, and one has to proceed differently.

Recall from 2.3 the split-exact sequence

$$0 \longrightarrow M^* \underset{\varrho}{\overset{\mathrm{d}_1}{\rightleftarrows}} \Omega^2(M, M) \underset{\sigma}{\overset{\mathrm{d}_2}{\rightleftarrows}} \mathrm{Z}^3(M) \longrightarrow 0 \tag{3.7.1}$$

A splitting of this sequence is given either by a retraction $\varrho\colon \Omega^2(M, M) \to M$ of d_1 or, equivalently, a section σ of d_2, related by the formulas

$$\varrho \circ \mathrm{d}_1 = \mathrm{Id}_{M^*}, \quad \mathrm{d}_1 \circ \varrho + \sigma \circ \mathrm{d}_2 = \mathrm{Id}_{\Omega^2(M,M)}, \quad \mathrm{d}_2 \circ \sigma = \mathrm{Id}_{\mathrm{Z}^3(M)}. \tag{3.7.2}$$

Then also $\mathrm{Ker}(\varrho) = \mathrm{Im}(\sigma)$, in particular, $\varrho \circ \sigma = 0$. In general, there is no canonical splitting.

3.8. Lemma. *Let (ϱ,σ) be a splitting as above. Choose a unital linear form α on X and let ϕ_α:* $\mathrm{Con}(X)\to P(M)$ *be the coordinate map as in Proposition* 3.3*. Then the map*

$$\chi\colon \mathrm{Con}(X)\to P(M),\qquad \chi(A):=\phi_\alpha\bigl(A^{\varrho(K)}\bigr),$$

is transvection-invariant: $\chi(A^v)=\chi(A)$ for all $v\in V$, and independent of the choice of α. The second component of $\chi(A)$ is $\sigma(\Theta_A)$. Denoting the first and third components by ϑ_A and β_A, respectively, we obtain transvection-invariant maps ϑ: $\mathrm{Con}(X)\to M^*$ *and β:* $\mathrm{Con}(X)\to\mathrm{Bil}(M)$ *given explicitly in terms of the α-coordinates (t,K,b) of A by*

$$\vartheta_A=t-2\varrho(K),\qquad \beta_A=b+\bigl(\varrho(K)-t\bigr)\otimes\varrho(K)+\varrho(K)\circ K. \tag{3.8.1}$$

Proof. To show transvection-invariance, let $v\in V$. The second α-coordinate of A^v is, by (3.5.3), $K^v=K-\mathrm{d}_1v$. Hence $\varrho(K^v)=\varrho(K-\mathrm{d}_1v)=\varrho(K)-v$ by (3.7.2). Since the additive group V acts on $\mathrm{Con}(X)$ by $(A,v)\mapsto A^v$, cf. 1.7, we have $(A^v)^w=A^{v+w}$ for all $v,w\in V$. This implies

$$\chi(A^v)=\phi_\alpha\bigl((A^v)^{\varrho(K^v)}\bigr)=\phi_\alpha\bigl(A^{v+\varrho(K)-v}\bigr)=\phi_\alpha\bigl(A^{\varrho(K)}\bigr)=\chi(A).$$

Next, we show the independence of α. Let α' be a second unital linear form and define χ' like χ, but with α' in place of α. Then $\alpha'=\alpha+\pi^*(v)$ as in 3.6 where $v\in V$. Let (t',K',b') be the α'-coordinates of A. By (3.6.2) we have $K'=K-\mathrm{d}_1v$ and thus $\varrho(K')=\varrho(K)-v$. Hence by (3.6.1),

$$\chi'(A)=\phi_{\alpha'}\bigl(A^{\varrho(K')}\bigr)=\phi_\alpha\bigl(A^{\varrho(K')+v}\bigr)=\phi_\alpha\bigl(A^{\varrho(K)}\bigr)=\chi(A),$$

so χ is independent of the choice of α.

The second component of $\chi(A)$ is, by (3.5.3), (3.7.2) and (3.3.4),

$$K-\mathrm{d}_1\bigl(\varrho(K)\bigr)=\sigma\bigl(\mathrm{d}_2(K)\bigr)=\sigma(\Theta_A).$$

The formulas for ϑ_A and β_A follow immediately from the definition of $\chi(A)$ and (3.5.2) and (3.5.4).

Remark. The maps χ, ϑ and β do depend on the choice of the splitting (ϱ,σ) of (3.7.1). These splittings form a torsor under the group $\mathrm{Hom}\bigl(\mathrm{Z}^3(M),M^*\bigr)$: If (ϱ',σ') is a second splitting then there exists a unique $\zeta\colon \mathrm{Z}^3(M)\to M^*$ such that $\varrho'=\varrho+\zeta\circ\mathrm{d}_2$ and $\sigma'=\sigma-\mathrm{d}_1\circ\zeta$, and conversely, these formulas define a splitting (ϱ',σ') for any $\zeta\in\mathrm{Hom}(\mathrm{Z}^3(M),M^*)$. Let χ' and ϑ'_A,β'_A be defined using ϱ' instead of ϱ. Then a straightforward computation shows that

$$\begin{aligned}\vartheta'_A&=\vartheta_A-2\zeta(\Theta_A),\\ \beta'_A&=\beta_A+\bigl(\zeta(\Theta_A)-\vartheta_A\bigr)\otimes\zeta(\Theta_A)+\zeta(\Theta_A)\circ\sigma(\Theta_A),\end{aligned}$$

while the second component of $\chi'(A)$ is of course $\sigma'(\Theta_A)=\sigma(\Theta_A)-\mathrm{d}_1\bigl(\zeta(\Theta_A)\bigr)$.

3.9. Let $P(M)$ be the coordinate space as in Proposition 3.3, and define a k-functor $\mathbf{P}(M)$ by $R\mapsto P(M_R)$, for all $R\in k$-**alg**. Then

$$\mathbf{P}(M)\cong M^*_{\mathbf{a}}\times\Omega^2(M,M)_{\mathbf{a}}\times\mathrm{Bil}(M)_{\mathbf{a}}, \tag{3.9.1}$$

in particular, $\mathbf{P}(M)$ is a smooth affine finitely presented k-scheme. Since the coordinate map ϕ_α is compatible with base change, it induces an isomorphism

$$\phi_\alpha\colon \mathbf{Con}(X)\xrightarrow{\cong}\mathbf{P}(M). \tag{3.9.2}$$

Furthermore, let

$$Q(M):=M^*\times\mathrm{Z}^3(M)\times\mathrm{Bil}(M), \tag{3.9.3}$$

and define a k-functor $\mathbf{Q}(M)$ by $R\mapsto Q(M_R)$.

The maps ϑ and β of Lemma 3.8 are compatible with base change as well, so there is a well-defined morphism

$$\xi\colon \mathbf{Con}(X)\to\mathbf{Q}(M),\quad \xi(A)=(\vartheta_A,\Theta_A,\beta_A), \tag{3.9.4}$$

for all $A\in\mathbf{Con}(X_R)$ and $R\in k$-**alg**.

Let $B(X)=\mathrm{Con}(X)/V$ be the quotient by the action of transvections, and define a k-functor $\mathbf{B}(X)$ by $R\mapsto B(X_R)=\mathrm{Con}(X_R)/V_R$, for all $R\in k$-**alg**. We denote by can: $\mathbf{Con}(X)\to\mathbf{B}(X)$, $A\mapsto[A]$, the canonical map.

3.10. Theorem. *Let* $\operatorname{rk} X = n + 1 \geqslant 3$. *Then* $\mathbf{Con}(X)$ *is a trivial torsor under the group of transvections, with base* $\mathbf{B}(X)$ *and projection* can. *Moreover,* ξ *induces an isomorphism* η: $\mathbf{B}(X) \to \mathbf{Q}(M)$ *by* $\eta([A]) = \xi(A)$, *so* $\mathbf{B}(X)$ *is a smooth affine finitely presented k-scheme with fibre isomorphic to affine space of dimension* $n^2 + n\binom{n}{2}$.

Proof. By (1.7.5), $V = M^*$ acts freely on $\mathrm{Con}(X)$ by transvections. By Lemma 3.8 and Corollary 2.5 we have $\xi(A^v) = \xi(A)$ for all $v \in V$. Hence ξ induces a map η: $B(X) \to Q(M)$ making the lower triangle of the following diagram commutative:

$$\begin{array}{ccc} \mathrm{Con}(X) & \xrightarrow[\cong]{\phi_\alpha} & P(M) \\ {\scriptstyle \mathrm{can}}\downarrow & \searrow^{\xi} & \downarrow{\scriptstyle q} \\ B(X) & \xrightarrow[\eta]{\cong} & Q(M) \end{array} \tag{3.10.1}$$

Let $\alpha \in X_1^*$. Then the upper triangle of the diagram is commutative as well, provided we define

$$q(t, K, b) = \big(t - 2\varrho(K),\, \mathrm{d}_2 K,\, b + \big(\varrho(K) - t\big) \otimes \varrho(K) + \varrho(K) \circ K\big). \tag{3.10.2}$$

This follows from (3.8.1).

We show that η is injective. Let $A, A' \in \mathrm{Con}(X)$ and assume $\eta([A]) = \eta([A'])$, equivalently, $\xi(A) = \xi(A')$. By (3.9.4), we have in particular $\Theta_A = \Theta_{A'}$, so by Theorem 2.8(b), $A' = A + g^{\mathrm{m}}$ for a unique $g \in \mathrm{Bil}_0(X)$. Let $\phi_\alpha(A) = (t, K, b)$ and $\phi_\alpha(A') = (t', K', b')$ be the α-coordinates of A and A', respectively, and write $\bar g = s_\alpha^*(g)$ and $\bar g_i = s_\alpha^*(g_i)$ for short. Then Proposition 3.4(a) shows

$$t' = t + \bar g_1 + \bar g_2, \quad K' = K + \mathrm{d}_1 \bar g_1, \quad b' = b + \bar g. \tag{3.10.3}$$

On the other hand, $q(t', K', b') = q(t, K, b)$ by the commutativity of the diagram. Comparing the first components, we obtain

$$\begin{aligned} 0 &= t' - 2\varrho(K') - (t - 2\varrho(K)) = t' - t - 2\big(\varrho(K') - \varrho(K)\big) \\ &= \bar g_1 + \bar g_2 - 2\bar g_1 = \bar g_2 - \bar g_1. \end{aligned}$$

Put $v = -\bar g_1 \in M^*$. We claim that $g = g_{v,A}$ as in (1.7.4). Then it will follow from (1.7.3) that $A' = A + g_{v,A}^{\mathrm{m}} = A^v$ is obtained from A by a transvection, as desired.

To this end, we evaluate the remaining third component of (3.10.2). By definition of v and the formulas (3.10.3), we have $t' = t - 2v$ and $K' = K - \mathrm{d}_1 v$. Hence, putting $u = \varrho(K)$ and $u' = \varrho(K')$ for short,

$$u' = u - v \quad \text{and} \quad u' - t' = u - v - t + 2v = (u - t) + v.$$

Now compute, using (3.10.2), (3.10.3) and (1.5.7):

$$\begin{aligned} 0 &= b' + \big(u' - t'\big) \otimes u' + u' \circ K' - b - \big(u - t\big) \otimes u - u \circ K \\ &= \bar g + \big(u - t + v\big) \otimes \big(u - v\big) + \big(u - v\big) \circ (K - \mathrm{d}_1 v) - \big(u - t\big) \otimes u - u \circ K \\ &= \bar g + (t - v) \otimes v - v \circ K. \end{aligned}$$

In view of (3.5.1) this shows

$$s_\alpha^*(g) = s_\alpha^*(g_{v,A}). \tag{3.10.4}$$

Recall the decomposition $X = k \cdot 1 \oplus M_\alpha$ of (3.1.2). From (3.10.4) we already see that g and $g_{v,A}$ agree on $M_\alpha \times M_\alpha$. Moreover, by definition of v, we have $g(1, x) = -v(\dot x) = g_{v,A}(1, x) = g(x, 1) = g_{v,A}(x, 1)$ which proves $g = g_{v,A}$, as required.

Finally, we show that η is surjective. By the commutativity of the diagram and the fact that ϕ_α is bijective, it suffices to show that q admits a section. For $(t, z, b) \in Q(M)$ let $s(t, z, b) = (t, \sigma(z), b) \in P(M)$. Then $\varrho(\sigma(z)) = 0$, as remarked in 3.7. Hence by (3.10.2), $q\big(s(t, z, b)\big) = (t, \mathrm{d}_2\sigma(z), b) = (t, z, b)$, since σ is a section of d_2.

Since all this is compatible with arbitrary base change, $\mathbf{Con}(X)$ is a torsor as claimed. From (3.9.3) it follows that $\mathbf{Q}(M) \cong M_{\mathbf{a}}^* \times \mathrm{Z}^3(M)_{\mathbf{a}} \times \mathrm{Bil}(M)_{\mathbf{a}}$. This proves the statement about the structure of $\mathbf{B}(X)$ in view of Corollary 2.3.

3.11. The case where 2 is a unit in the base ring. Suppose 2 is a unit in k. Then a conic algebra A has a canonical unital linear form $\alpha_A = \frac{1}{2}T_A$. The corresponding decomposition (3.1.2) is $X = k\cdot 1 \oplus \mathrm{Ker}(T_A)$, and we write simply $s_A = s_{\alpha_A}$. We call the α_A-coordinates of A the *canonical coordinates* of A and denote them by (t_A^0, K_A^0, b_A^0). From (3.2.1) it is clear that $t_A^0 = 0$, while K_A^0 and b_A^0 are determined by the formula

$$-s_A(\dot{x})s_A(\dot{y}) = b_A^0(\dot{x},\dot{y})\cdot 1 + s_A\big(K_A^0(\dot{x},\dot{y})\big), \tag{3.11.1}$$

since ι_A is $-\mathrm{Id}$ on $\mathrm{Ker}(T_A)$.

We claim that the map κ: $\mathrm{Con}(X) \to \Omega^2(M,M)\times \mathrm{Bil}(M)$, $\kappa(A) = (K_A^0, b_A^0)$, induces a bijection $\mathrm{Con}(X)/V \cong \Omega^2(M,M)\times\mathrm{Bil}(M)$. This is essentially the description of conic algebras over fields of characteristic different from 2 given in [**13**, **1**, **5**] and many other references.

We first show that $\kappa(A^v) = \kappa(A)$ for all $v \in V$. Since τ_v: $A^v \to A$ is an isomorphism, we have $T_A \circ \tau_v = T_{A^v}$. This implies

$$\begin{aligned}\tau_v\big(s_{A^v}(\dot{x})\big) &= \tau_v\big(x - \frac{1}{2}T_{A^v}(x)\cdot 1\big)\\ &= x - v(\dot{x})1 - \frac{1}{2}T_A\big(x - v(\dot{x})1\big)1 = x - \frac{1}{2}T_A(x)1 = s_A(\dot{x})\end{aligned}$$

for all $\dot{x}\in M$, thus $\tau_v \circ s_{A^v} = s_A$. By (3.11.1) applied to A^v instead of A we have

$$-s_{A^v}(\dot{x})s_{A^v}(\dot{y}) = b_{A^v}^0(\dot{x},\dot{y})\cdot 1 + s_{A^v}\big(K_{A^v}^0(\dot{x},\dot{y})\big). \tag{3.11.2}$$

Now apply τ_v to this, use $\tau_v\circ s_{A^v} = s_A$ and compare with (3.11.1). It follows that $K_{A^v}^0 = K_A^0$ and $b_{A^v}^0 = b_A^0$, showing $\kappa(A^v) = \kappa(A)$.

Conversely, let A and A' be conic algebras with the same canonical coordinates $K^0 = K_{A'}^0 = K_A^0$ and $b^0 = b_{A'}^0 = b_A^0$. Since $T_{A'}(1) = T_A(1) = 2$, there exists a unique $v \in V$ such that $T_A(x) - T_{A'}(x) = 2v(\dot{x})$. We claim that $A' = A^v$. By definition of v we have $\alpha_{A'} = \alpha_A - \pi^*(v)$ which implies $s_A = \tau_v \circ s_{A'}$ as in (3.6.3). Let us show that τ_v: $A' \to A$ is an isomorphism. From the easily checked relation $T_{A'} = T_A \circ \tau_v$ it follows that τ_v maps $\mathrm{Ker}(T_{A'})$ isomorphically onto $\mathrm{Ker}(T_A)$. Moreover, τ_v preserves the unit element, so it suffices to show that it preserves the product of two elements in $\mathrm{Ker}(T_{A'})$. Since A and A' have the same canonical coordinates,

$$\begin{aligned}-\tau_v\big(s_{A'}(\dot{x})s_{A'}(\dot{y})\big) &= \tau_v\big(b^0(\dot{x},\dot{y})\cdot 1 + s_{A'}(K^0(\dot{x},\dot{y}))\big)\\ &= b^0(\dot{x},\dot{y})\cdot 1 + s_A\big(K^0(\dot{x},\dot{y})\big)\\ &= -s_A(\dot{x})s_A(\dot{y}) = -\tau_v(s_{A'}(\dot{x}))\tau_v(s_{A'}(\dot{y})),\end{aligned}$$

as desired. On the other hand, by 1.7, τ_v: $A^v \to A$ is an isomorphism as well. Hence $\mathrm{Id} = \tau_v^{-1}\circ\tau_v$: $A' \to A^v$ is an isomorphism, which proves $A' = A^v$.

4. Algebras with scalar involution

4.1. Definition. Let $(X, 1_X)$ be a unital k-module of constant rank $n+1$. As usual, we write $M = X^\flat = X/k\cdot 1_X$. Following Becker [**1**], a conic algebra A on X is said to be *involutive* or an *algebra with scalar involution* if the conjugation ι_A is an algebra involution; that is, if $\overline{xy} = \bar{y}\bar{x}$ for all $x, y \in X$. These algebras are precisely the algebras with scalar involution in the sense of McCrimmon [**12**], whose underlying module is finitely generated and projective of constant positive rank, see [**12**, Theorem 1.1]. The following invariant measures the deviation of a conic algebra A from being involutive.

4.2. Lemma. *Let $A \in \mathrm{Con}(X)$. We use the notations introduced in* 1.8 *and* 1.9*.*

(a) *There is a unique alternating* 2*-form ω_A on M such that*

$$\omega_A(\dot{x},\dot{y})\cdot 1_X = \overline{H_A(x,y)} - H_A(y,x) \tag{4.2.1}$$

$$= \bar{y}\bar{x} - \overline{xy} \tag{4.2.2}$$

for all $x, y \in X$, so A is involutive if and only if $\omega_A = 0$. The following formulas hold:

$$\omega_A(\dot{x}, \dot{y}) = T_A\big(H_A(x,y)\big) - N_A(x,y) \tag{4.2.3}$$
$$= T_A(x)T_A(y) - T_A(xy) - N_A(x,y), \tag{4.2.4}$$
$$\omega_{A^{\mathrm{op}}} = -\omega_A, \tag{4.2.5}$$
$$2\omega_A(\dot{x}, \dot{y}) = -T_A\big([x,y]\big). \tag{4.2.6}$$

(b) *ω_A is compatible with homomorphisms: if $\varphi\colon A \to B$ is a homomorphism of conic algebras $A \in \mathrm{Con}(X)$ and $B \in \mathrm{Con}(Y)$ as in* 1.8 *and $\varphi^\flat\colon X^\flat \to Y^\flat$ denotes the induced module homomorphism then $\omega_B(\varphi^\flat(\dot{x}), \varphi^\flat(\dot{y})) = \omega_A(\dot{x}, \dot{y})$ for all $\dot{x}, \dot{y} \in X^\flat$, i.e., $(\varphi^\flat)^*(\omega_B) = \omega_A$.*

(c) *ω_A is compatible with base change in the sense that, with the identification* (1.2.3)*, $\omega_{A_R} = (\omega_A)_R$ for all $R \in k$-**alg**.*

Proof. (a) Let $D(x,y) = \overline{H_A(x,y)} - H_A(y,x)$ be the right hand side of (4.2.1). Then $D(x,x) = 0$ by (1.9.5) and obviously $D(x,1) = \bar{\bar{x}} - x = 0$, so $D(x,y)$ is alternating and depends only on $\dot{x}$ and $\dot{y}$. Moreover, by (1.9.1) and (1.9.6),

$$D(x,y) = T_A(\bar{x}y)\cdot 1_X - H_A(x,y) - H_A(y,x) = \big\{T_A(\bar{x}y) - N_A(x,y)\big\}\cdot 1.$$

This proves (4.2.1) and (4.2.3). Now (4.2.4) is immediate from (4.2.3) and the fact that $\bar{x} = T_A(x)1 - x$. Applying π to this relation we obtain $\pi(\bar{x}) = -\pi(x) = -\dot{x}$. Hence $\bar{y}\bar{x} - \overline{xy} = -\big[\overline{H_A(\bar{x},y)} - H_A(y,\bar{x})\big] = -\omega_A(\pi(\bar{x}), \dot{y}) = \omega_A(\dot{x}, \dot{y})$ which proves (4.2.2).

From (4.2.4) and the fact that A and A^{op} have the same norms and traces, we see

$$\omega_{A^{\mathrm{op}}}(\dot{x}, \dot{y}) = T_A(x)T_A(y) - T_A(yx) - N_A(x,y) = \omega_A\big(\dot{y}, \dot{x}\big) = -\omega_A(\dot{x}, \dot{y}).$$

Formula (4.2.6) follows again from (4.2.4):

$$\begin{aligned} -2\omega_A(\dot{x}, \dot{y}) &= -\omega_A(\dot{x}, \dot{y}) + \omega_A(\dot{y}, \dot{x}) \\ &= -T_A(x)T_A(y) + T_A(xy) + N_A(x,y) + T_A(y)T_A(x) - T_A(yx) - N_A(y,x) \\ &= T_A\big([x,y]\big). \end{aligned}$$

(b) follows easily from (4.2.4) and the fact that a homomorphism of conic algebras preserves products, norms and traces, and (c) is straightforward.

4.3. Corollary. *ω_A is invariant under transvections: $\omega_A = \omega_{A^v}$ for all $v \in V = M^*$.*

Proof. Immediate from Lemma 4.2(b) since $\tau_v\colon A^v \to A$ is an isomorphism with $(\tau_v)^\flat = \mathrm{Id}_M$.

4.4. Lemma. (a) *Let $f \in \mathrm{Bil}_1(X)$ be a unital bilinear form on X and $A = f^{\mathrm{m}}$ as in Lemma* 1.10(a)*. Define linear forms f_1 and f_2 as in* (1.9.9)*. Then*

$$\pi^*(\omega_{f^{\mathrm{m}}}) = f_1 \wedge f_2 + f - f^{\mathrm{op}}. \tag{4.4.1}$$

(b) *Let A be a conic algebra and let $g \in \mathrm{Bil}_0(X)$, with g_1 and g_2 defined as in* (1.9.9)*. Then $A + g^{\mathrm{m}}$ is conic by Lemma* 1.10(b)*, and*

$$\omega_{A+g^{\mathrm{m}}}(\dot{x}, \dot{y}) = \omega_A(\dot{x}, \dot{y}) + g_1(x\bar{y}) + g_2(\bar{x}y) + (g_1 \wedge g_2 + g - g^{\mathrm{op}})(x,y). \tag{4.4.2}$$

(c) *Let $A \in \mathrm{Con}(X)$ and $\Gamma \in \Omega^2(M, X)$. Then $A + \pi^*(\Gamma)$ is conic by Lemma* 1.10(c)*, and*

$$\omega_{A+\pi^*(\Gamma)} = \omega_A - T_A \circ \Gamma. \tag{4.4.3}$$

Proof. (a) By (4.2.1) and (1.10.1),

$$\begin{aligned}\omega_A(\dot{x},\dot{y})\cdot 1 &= \overline{H_A(x,y)} - H_A(y,x)\\ &= \overline{f_1(x)y - f_1(y)x + f(x,y)\cdot 1} - f_1(y)x + f_1(x)y - f(y,x)\cdot 1\\ &= f_1(x)(\bar{y}+y) - f_1(y)(\bar{x}+x) + \big(f(x,y)-f(y,x)\big)\cdot 1\\ &= \big\{f_1(x)\big(f_1(y)+f_2(y)\big) - f_1(y)\big(f_1(x)+f_2(x)\big) + f(x,y) - f(y,x)\big\}\cdot 1\\ &= \big\{f_1(x)f_2(y) - f_1(y)f_2(x) + f(x,y) - f(y,x)\big\}\cdot 1.\end{aligned}$$

(b) Let $A' = A + g^{\mathrm{m}}$ and write $T_A = T$, $T_{A'} = T'$ and similarly N and N', H and H' and ω and ω' for short. By Lemma 1.10(b), $T' = T + g_1 + g_2$, $N'(x) = N(x) + g(x,x)$ and $H' = H + \mathrm{d}_1 g_1 + g\cdot 1$. Hence by (4.2.3) and (1.5.7), and observing $g_1(1) = g_2(1) = g(1,1) = 0$ as well as $T'(1) = 2$,

$$\begin{aligned}\omega'(\dot{x},\dot{y}) &= T'(H'(x,y)) - N'(x,y)\\ &= T'\big(H(x,y) + \mathrm{d}_1 g_1(x,y) + g(x,y)\cdot 1\big) - N(x,y) - g(x,y) - g(y,x)\\ &= \omega(\dot{x},\dot{y}) + (g_1+g_2)(\bar{x}y) + T'\big(\mathrm{d}_1 g_1(x,y)\big) + 2g(x,y) - g(x,y) - g(y,x).\end{aligned}$$

A simple computation shows

$$\begin{aligned}g_1(\bar{x}y) + T'\big(\mathrm{d}_1 g_1(x,y)\big) &= g_1(\bar{x}y) + (T + g_1 + g_2)\big(g_1(x)y - g_1(y)x\big)\\ &= g_1\big(\bar{x}y + T(y)x - T(x)y\big) + (g_1\wedge g_2)(x,y)\\ &= g_1(x\bar{y}) + (g_1\wedge g_2)(x,y).\end{aligned}$$

Substituting this into the preceding formula yields (4.4.2).

(c) By Lemma 1.10(c), A and $A' := A + \pi^*(\Gamma)$ have the same norm and trace, while $H_{A'}(x,y) = H_A(x,y) - \Gamma(\dot{x},\dot{y})$. Hence by (4.2.1) and since Γ is alternating,

$$\begin{aligned}\omega_{A'}(\dot{x},\dot{y})\cdot 1 &= \overline{H_A(x,y) - \Gamma(\dot{x},\dot{y})} - H_A(y,x) + \Gamma(\dot{y},\dot{x})\\ &= \omega_A(\dot{x},\dot{y}) - \big\{\overline{\Gamma(\dot{x},\dot{y})} + \Gamma(\dot{x},\dot{y})\big\}\\ &= \big\{\omega_A(\dot{x},\dot{y}) - T_A\big(\Gamma(\dot{x},\dot{y})\big)\big\}\cdot 1.\end{aligned}$$

4.5. Lemma. *Let α be a unital linear form on X and let (t,K,b) be the α-coordinates of $A\in \mathrm{Con}(X)$. Then*

$$\omega_A = t\circ K + b - b^{\mathrm{op}}. \tag{4.5.1}$$

Proof. Since $\pi(s_\alpha(\dot{x})) = \dot{x}$, we have

$$\begin{aligned}\omega_A(\dot{x},\dot{y}) &= T_A\big(H_A(s_\alpha(\dot{x}), s_\alpha(\dot{y}))\big) - N_A\big(s_\alpha(\dot{x}), s_\alpha(\dot{y})\big) \text{ (by (4.2.3))}\\ &= T_A\big(b(\dot{x},\dot{y})\cdot 1 + s_\alpha(K(\dot{x},\dot{y}))\big) - b(\dot{x},\dot{y}) - b(\dot{y},\dot{x}) \text{ (by (3.2.7) and (3.3.2))}\\ &= 2b(\dot{x},\dot{y}) + t\big(K(\dot{x},\dot{y})\big) - b(\dot{x},\dot{y}) - b(\dot{y},\dot{x}) \text{ (by (1.8.3) and (3.2.4))}.\end{aligned}$$

As in 3.3, let $P(M)$ be the parameter space for the α-coordinates and let

$$P'(M) := \{(t,K,b)\in P(M) : t\circ K + b - b^{\mathrm{op}} = 0\}. \tag{4.5.2}$$

Recall the k-functor $\mathbf{P}(M)$ of 3.9 and define a k-functor $\mathbf{P}'(M)\subset \mathbf{P}(M)$ by $R\mapsto \mathbf{P}'(M_R)$, for all $R\in k$-**alg**.

4.6. Proposition. *Let*

$$\operatorname{Scalin}(X) = \{A \in \operatorname{Con}(X) : \omega_A = 0\} \tag{4.6.1}$$

be the set of involutive conic algebras on X, and define a k-functor $\mathbf{Scalin}(X)$ *by* $R \mapsto \operatorname{Scalin}(X_R)$ *for* $R \in k$-**alg**. *Then* $\mathbf{Scalin}(X)$ *is a closed finitely presented subscheme of* $\mathbf{Con}(X)$.

Proof. Choose a unital linear form α and recall the isomorphism $\phi_\alpha \colon \mathbf{Con}(X) \cong \mathbf{P}(M)$ of (3.9.2). Lemma 4.5 shows that ϕ_α restricts to a bijection $\phi'_\alpha \colon \operatorname{Scalin}(X) \cong P'(M)$, which by Lemma 4.2(c) is compatible with arbitrary base change, so we have an isomorphism of functors

$$\phi'_\alpha \colon \mathbf{Scalin}(X) \xrightarrow{\cong} \mathbf{P}'(M). \tag{4.6.2}$$

Hence it suffices to prove that $\mathbf{P}'(M)$ is a closed finitely presented subscheme of $\mathbf{P}(M)$. Recall the isomorphism (3.9.1). Since all modules involved are finitely generated and projective, the relation $t \circ K + b - b^{\mathrm{op}} = 0$ defining $\mathbf{P}'(M)$ amounts to finitely many scalar polynomial equations. This proves our assertion.

Our aim now is to describe $\mathbf{Scalin}(X)$ in more detail. Similarly to $\mathbf{Con}(X)$ (cf. 2.10), $\mathbf{Scalin}(X)$ admits a group action making it a trivial torsor. However, it is evident from Lemma 4.4(b) that algebras with scalar involution are not stable under the action of an arbitrary $g \in \operatorname{Bil}_0(X)$. We first determine the biggest subgroup of $\operatorname{Bil}_0(X)$ which preserves ω_A for all conic algebras under this action.

4.7. Lemma. *Let* $\operatorname{rk} X \geqslant 3$. *For a bilinear form* $g \in \operatorname{Bil}_0(X)$ *the following conditions are equivalent:*

(i) $\omega_{A+g^{\mathrm{m}}} = \omega_A$ *for all* $A \in \operatorname{Con}(X)$,
(ii) $g_1 = g_2 = 0$ *and g is symmetric,*
(iii) $g = \pi^*(h)$ *for a unique symmetric bilinear form h on M.*

Proof. (i) $\Longrightarrow$ (ii): By (4.4.2), we have

$$g_1(x\bar{y}) + g_2(\bar{x}y) + (g_1 \wedge g_2 + g - g^{\mathrm{op}})(x, y) = 0, \tag{4.7.1}$$

for all $x, y \in X$ and all conic algebras A on X. Let in particular $\alpha, \beta \in X_1^*$ be unital linear forms, put $f(x, y) = \alpha(x)\beta(y)$ and $A = f^{\mathrm{m}}$ as in Lemma 1.10(a). Then $f_1 = \beta$, so by (1.10.1), $\bar{x}y = H_A(x, y) = (\mathrm{d}_1\beta + f \cdot 1)(x, y)$. By Lemma 1.6(b), $A^{\mathrm{op}} = (f^{\mathrm{op}})^{\mathrm{m}}$, and by (1.9.7), $x\bar{y} = H_A(\bar{x}, \bar{y}) = H_{A^{\mathrm{op}}}(y, x)$. Since $(f^{\mathrm{op}})_1 = f_2 = \alpha$, it follows that $x\bar{y} = (\mathrm{d}_1\alpha + f^{\mathrm{op}} \cdot 1)(y, x) = (-\mathrm{d}_1\alpha + f \cdot 1)(x, y)$. Substitute this into (4.7.1) and recall that $g_1(1) = g_2(1) = g(1, 1) = 0$. Since $g_1 \circ \mathrm{d}_1\alpha = \alpha \wedge g_1$ and $g_2 \circ \mathrm{d}_1\beta = \beta \wedge g_2$ by (1.5.7), we obtain

$$-\alpha \wedge g_1 + \beta \wedge g_2 + g_1 \wedge g_2 + g - g^{\mathrm{op}} = 0, \tag{4.7.2}$$

for all $\alpha, \beta \in X_1^*$. Let $\lambda, \mu \in X_0^*$. Then $\alpha + \lambda$ and $\beta + \mu$ are unital linear forms as well. Replacing α, β by $\alpha + \lambda$ and $\beta + \mu$ in (4.7.2) and subtracting results in

$$\lambda \wedge g_1 - \mu \wedge g_2 = 0. \tag{4.7.3}$$

Pulling back a linear form on M to X via π yields an isomorphism $\pi^* \colon M^* \xrightarrow{\cong} X_0^*$. Let w_i be the unique linear form on M with $\pi^*(w_i) = g_i$, and similarly let $\lambda = \pi^*(u)$ and $\mu = \pi^*(v)$ where $u, v \in M^*$. Then (4.7.3) says that $u \wedge w_1 - v \wedge w_2 = 0$ for all $u, v \in M^*$. Since $\operatorname{rk} M = \operatorname{rk} X - 1 \geqslant 2$, this implies (for example by localization and working in a basis) that $w_1 = w_2 = 0$, whence also $g_1 = g_2 = 0$. Now (4.7.2) shows $g = g^{\mathrm{op}}$ is symmetric.

(ii) $\Longrightarrow$ (iii): Since $g_1(x) = g(1, x) = 0$ and $g_2(x) = g(x, 1) = 0$, it is clear that $g(x, y)$ depends only on $\dot{x}$ and $\dot{y}$, so g induces a unique symmetric bilinear form h on M such that $h(\dot{x}, \dot{y}) = g(x, y)$.

(iii) $\Longrightarrow$ (i): Let h be a symmetric bilinear form on M and $g = \pi^*(h)$. Then $g = g^{\mathrm{op}}$ and $g_1(x) = g(1, x) = h(\dot{1}, \dot{x}) = 0$ as well as $g_2(x) = 0$, so the assertion follows from (4.4.2).

4.8. Rost shifts. Let $\mathrm{Sym}(M)$ denote the k-module of symmetric bilinear forms on M. By Lemma 4.7, $A \in \mathrm{Scalin}(X)$ and $h \in \mathrm{Sym}(M)$ imply $A + \pi^*(h)^{\mathsf{m}} \in \mathrm{Scalin}(X)$. We call this the *Rost shift* of A by h, since it generalizes the Rost shift of quadratic algebras, see [**16**] and [**10**, Lemma 3.1]. Explicitly, (1.6.1) shows that

$$\big(A + \pi^*(h)^{\mathsf{m}}\big)(x, y) = xy - h(\dot{x}, \dot{y}) \cdot 1, \tag{4.8.1}$$

since $\pi^*(h)(1, x) = \pi^*(h)(x, 1) = 0$. As $\pi^*(h)^{\mathsf{m}}$ depends linearly on h, the additive group $\mathrm{Sym}(M)$ acts on $\mathrm{Scalin}(X)$ by Rost shifts.

4.9. Lemma. *Let* $\Omega^2_\pi(X, M) := \{F \in \Omega^2(X, M) : F(1, y) = \dot{y}$ *for all* $y \in X\}$ *and recall that* $X^*_2 = \{\lambda \in X^* : \lambda(1_X) = 2\}$.

(a) $\Omega^2_\pi(X, M)$ *and* X^*_2 *are affine subspaces of* $\Omega^2(X, M)$ *and* X^**, respectively, with associated modules of translations isomorphic to* $\Omega^2(M, M)$ *and* M^*.

(b) *For any conic algebra* A*, we have* $T_A \in X^*_2$ *and* $\dot{H}_A \in \Omega^2_\pi(X, M)$.

(c) *Let* $s_\alpha \colon M \to X$ *be the section of* π *defined by a unital linear form* α*. Then* $s^*_\alpha \colon \Omega^2_\pi(X, M) \xrightarrow{\cong} \Omega^2(M, M)$ *is bijective and compatible with the actions of* $\Omega^2(M, M)$ *in the sense that* $s^*_\alpha\big(F + \pi^*(K)\big) = s^*_\alpha(F) + K$*. Similarly,* $s^*_\alpha \colon X^*_2 \to M^*$ *is bijective and compatible with the actions of* M^*.

Proof. (a) If $F \in \Omega^2_\pi(X, M)$ and $K \in \Omega^2(M, M)$ then $F + \pi^*(K) \in \Omega^2_\pi(X, M)$. Indeed, $F + \pi^*(K)$ is clearly an alternating bilinear map on X with values in M, and

$$(F + \pi^*(K))(1, y) = F(1, y) + K(\dot{1}, \dot{y}) = F(1, y) = \dot{y}.$$

Conversely, for any two $F', F \in \Omega^2_\pi(X, M)$, we have $(F' - F)(1, y) = \dot{y} - \dot{y} = 0$, so $(F' - F)(x, y)$ depends only on $\dot{x}$, and by its alternating nature also only on $\dot{y}$. Hence there exists a unique $K \in \Omega^2(M, M)$ such that $F - F' = \pi^*(K)$. Thus $\Omega^2(M, M)$ acts simply transitively on $\Omega^2_\pi(X, M)$ by $(F, K) \mapsto F + \pi^*(K)$.

In the same way, given $\lambda \in X^*_2$ and $v \in M^*$, we have $\lambda + \pi^*(v) \in X^*_2$, and conversely, the difference of two elements $\lambda, \lambda' \in X^*_2$ has the form $\lambda' - \lambda = \pi^*(v)$ for a unique $v \in M^*$.

(b) We have $T_A \in X^*_2$ by (1.8.3). Since $H_A(1, y) = \bar{1}y = y$ we see $\dot{H}_A(1, y) = \dot{y}$, and $H_A(x, x) = N_A(x) \cdot 1$ by (1.9.5) implies that $\dot{H}_A$ is alternating.

(c) Clearly, $s^*_\alpha(F) \in \Omega^2(M, M)$, and $s^*_\alpha(F + \pi^*(K)) = s^*_\alpha(F) + s^*_\alpha(\pi^*(K)) = s^*_\alpha(F) + (\pi \circ s_\alpha)^*(K) = s^*_\alpha(F) + K$. Since $\Omega^2(M, M)$ acts simply transitively on $\Omega^2_\pi(X, M)$ and on itself by addition, the assertion follows. The proof for X^*_2 is similar.

Let $W(X) = X^*_2 \times \Omega^2_\pi(X, M)$. By (c), a choice of unital linear form α yields a bijection $\psi_\alpha \colon W(X) \to M^* \times \Omega^2(M, M)$ which is easily seen to be compatible with arbitrary base change. Hence the k-functor $\mathbf{W}(X) : R \mapsto W(X_R)$ is isomorphic (not canonically) to $M^*_{\mathbf{a}} \times \Omega^2(M, M)_{\mathbf{a}}$; in particular, it is a smooth affine finitely presented k-scheme with fibres isomorphic to affine space of dimension $n + n \cdot \binom{n}{2}$. By (b), we have a morphism

$$p \colon \mathbf{Scalin}(X) \to \mathbf{W}(X), \qquad p(A) = (T_A, \dot{H}_A).$$

4.10. Theorem. $\mathbf{Scalin}(X)$ *is a trivial torsor with group* $\mathrm{Sym}(M)_{\mathbf{a}}$ *acting by Rost shifts, projection* p *and base* $\mathbf{W}(X)$*. Hence* $\mathbf{Scalin}(X)$ *is a smooth affine finitely presented* k*-scheme with fibres isomorphic to affine space of dimension* $n + n\binom{n}{2} + \binom{n+1}{2}$.

Proof. By Lemma 4.7, the fibres of p consist of orbits of $\mathrm{Sym}(M)$. To prove the remaining statements, we use α-coordinates. Let ϕ_α be the coordinate map and let ϕ'_α be the restriction to $\mathrm{Scalin}(X)$. Then the following diagram is commutative and the horizontal maps are bijective:

$$\begin{array}{ccc} \mathrm{Scalin}(X) & \xrightarrow[\cong]{\phi'_\alpha} & P'(M) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ W(X) & \xrightarrow[\psi_\alpha]{\cong} & M^* \times \Omega^2(M,M) \end{array} \tag{4.10.1}$$

where $P'(M)$ is defined in (4.5.2) and $p'(t,K,b) = (t,K)$. Indeed, ψ_α is bijective as noted before, and ϕ'_α: $\mathrm{Scalin}(X) \to P'(M)$ is bijective by (4.6.2). Commutativity is seen as follows. For $A \in \mathrm{Scalin}(X)$ we have $s^*_\alpha(T_A) = t$ and $s^*_\alpha(H_A) = K$ by (3.2.1) and (3.2.2). Hence $\psi_\alpha\big(p(A)\big) = (t,K) = p'(t,K,b) = p'\big(\phi'_\alpha(A)\big)$.

Note that $\mathrm{Sym}(M)$ acts on $P'(M)$ on the right via $(t,K,b) + h = (t,K,b+h)$, since $t \circ K + (b+h) - (b+h)^{\mathrm{op}} = t \circ K + b - b^{\mathrm{op}}$ by symmetry of h. Then ϕ'_α is equivariant with respect to the actions of $\mathrm{Sym}(M)$. This follows from (3.4.1) and (3.4.2) since for $h \in \mathrm{Sym}(M)$ and $g = \pi^*(h)$, we have $g_1 = g_2 = 0$. Hence it suffices to show that the fibres of p' are precisely the orbits of $\mathrm{Sym}(M)$ and that p' admits a section, in order to have the corresponding statements for p.

Now $p'(t,K,b) = p'(t',K',b')$ if and only if $t = t'$, $K = K'$. By definition of $P'(M)$, this implies $b - b^{\mathrm{op}} = -t \circ K = -t' \circ K' = b' - b'^{\mathrm{op}}$. It follows that $h := b'^{\mathrm{op}} - b^{\mathrm{op}} = b' - b$ is symmetric, and $b' = b + h$. Thus the fibres of p' are precisely the orbits of $\mathrm{Sym}(M)$.

We construct a section of p' as follows. Since M is finitely generated and projective, the sequence of k-modules

$$0 \longrightarrow \mathrm{Sym}(M) \xrightarrow{\mathrm{inc}} \mathrm{Bil}(M) \underset{\gamma}{\overset{\mathrm{alt}}{\rightleftarrows}} \Omega^2(M) \longrightarrow 0 \tag{4.10.2}$$

where $\mathrm{alt}(b) = b - b^{\mathrm{op}}$, is split exact. Let γ: $\Omega^2(M) \to \mathrm{Bil}(M)$ be a splitting of this sequence. Then a section s' of p' is given by $s'(t,K) = \big(t, K, -\gamma(t \circ K)\big)$. Indeed, putting $b = -\gamma(t \circ K)$, we have

$$t \circ K + b - b^{\mathrm{op}} = t \circ K + \mathrm{alt}(b) = t \circ K - \mathrm{alt}\big(\gamma(t \circ K)\big) = 0,$$

so $s'(t,K) \in P'(M)$, and obviously $p' \circ s' = \mathrm{Id}$.

Since the modules involved are finitely generated and projective, all these constructions are compatible with arbitrary base change. Hence $\mathbf{Scalin}(X)$ is a torsor as indicated, and therefore isomorphic (not canonically) to $\mathbf{W} \times \mathrm{Sym}(M)_{\mathbf{a}}$. Finally, $\mathrm{Sym}(M)$ is a finitely generated and projective k-module of rank $\binom{n+1}{2}$. This implies the statement about the fibres of $\mathbf{Scalin}(X)$.

4.11. A geometric interpretation. The defining equation (4.5.2) for $P'(M)$ can be interpreted as saying that $P'(M)$ is a parabolic cylinder in (t,K,b)-space, with generators the lines (more precisely, the affine subspaces)

$$\{(t,K,b+h) : h \in \mathrm{Sym}(M)\}$$

given by the action of $\mathrm{Sym}(M)$. The quotient by this action can regarded as the hyperbolic paraboloid in $M^* \times \Omega^2(M,M) \times \Omega^2(M)$ with the equation

$$\{(t,K,a) : t \circ K + a = 0\},$$

in analogy to the standard hyperbolic paraboloid $\{(x,y,z) : xy + z = 0\}$ in affine 3-space.

From Corollary 4.3 it is evident that involutive conic algebras are stable under the group of transvections. We now describe the quotient of $\mathbf{Scalin}(X)$ by this action. Choose a splitting (ϱ, σ) of (3.7.1). Recall the definition of $Q(M)$ in (3.9.3), and define

$$Q'(M) = \{(t,z,b) \in Q(M) : t \circ \sigma(z) + b - b^{\mathrm{op}} = 0\},$$

as well as a k-functor $\mathbf{Q}'(M)$ by $R \mapsto Q'(M_R)$, for all $R \in k\text{-}\mathbf{alg}$. Note that

$$\mathbf{Q}'(M) \cong M^*_{\mathbf{a}} \times \mathrm{Z}^3(M)_{\mathbf{a}} \times \mathrm{Sym}(M)_{\mathbf{a}} \tag{4.11.1}$$

(not canonically). Indeed, let γ be a section of alt: $\mathrm{Bil}(M) \to \Omega^2(M)$ as in (4.10.2). Then one checks that the map sending $(t,z,b) \in Q'(M)$ to $\big(t,z,b - \gamma(\mathrm{alt}(b))\big)$ is an isomorphism, with inverse given by $(t,z,h) \mapsto \big(t,z,h - \gamma(t \circ \sigma(z))\big)$, and it is compatible with arbitrary base change.

Let $B'(X) = \mathrm{Scalin}(X)/V$ and let $\mathbf{B}'(X)$ be the functor $R \mapsto \mathbf{B}'(X_R) = \mathrm{Scalin}(X_R)/V_R$, for all $R \in k\text{-}\mathbf{alg}$, with can': $\mathbf{Scalin}(X) \to \mathbf{B}'(X)$ the canonical map.

4.12. Corollary. *Let* $\operatorname{rk} X = n+1 \geqslant 3$. *Then* $\mathbf{Scalin}(X)$ *is a trivial torsor under the group of transvections with base* $\mathbf{B}'(M)$ *and projection* can'. *The map* ξ *of* (3.9.4) *restricts to a map* $\xi'\colon \mathbf{Scalin}(X) \to \mathbf{Q}'(M)$ *and induces an isomorphism* $\eta'\colon \mathbf{B}'(X) \to \mathbf{Q}'(M)$. *Hence* $\mathbf{B}'(X)$ *is a smooth affine finitely presented k-scheme with fibres isomorphic to affine space of dimension* $n\binom{n}{2} + \binom{n+1}{2}$.

Proof. We have the following sub-diagram of (3.10.1), where primes indicate the restrictions of the corresponding maps:

$$\begin{array}{ccc} \mathrm{Scalin}(X) & \xrightarrow[\cong]{\phi'_\alpha} & P'(M) \\ {\scriptstyle \mathrm{can}'}\downarrow & \searrow^{\xi'} & \downarrow{\scriptstyle q'} \\ B'(X) & \xrightarrow[\eta']{\cong} & Q'(M) \end{array}$$

Indeed, let us first show that ξ maps $\mathrm{Scalin}(X)$ to $Q'(M)$. We use the notations introduced in Lemma 3.8. If $A \in \mathrm{Scalin}(X)$ then so is $A^{\varrho(K)}$, by Corollary 4.3. Hence $\chi(A) = \phi_\alpha(A^{\varrho(K)}) \in P'(M)$, as follows from (4.6.2). By Lemma 3.8, $\chi(A) = (\vartheta_A, \sigma(\Theta_A), \beta_A)$, so we have $\vartheta_A \circ \sigma(\Theta_A) + \beta_A - \beta_A^{\mathrm{op}} = 0$. This says precisely that $\xi(A) = (\vartheta_A, \Theta_A, \beta_A) \in Q'(M)$. We can identify $B'(X)$ with the image of $\mathrm{Scalin}(X)$ under can: $\mathrm{Con}(X) \to B(X)$. Then it follows from Theorem 3.10 that η restricts to an injective map $\eta'\colon B'(X) \to Q'(M)$, making the lower triangle of the diagram commutative. Commutativity of the upper triangle follows from the corresponding fact for (3.10.1). It remains to show that η' is surjective as well. As in the proof of Theorem 3.10, it suffices to show that q' admits a section. In fact, one checks that the section s of q constructed there restricts to a section of q'. All this is compatible with arbitrary base change, so $\mathbf{Scalin}(X)$ is a torsor as indicated. The statement about the fibres of $\mathbf{B}'(X)$ follows from (4.11.1) and Corollary 2.3.

5. Extending the theory to an arbitrary base scheme.

Following the example of Petersson [**14**], it is possible to replace the base ring k by an arbitrary base scheme $\mathbf{S}$. We indicate this briefly.

Finitely generated and projective k-modules $X, Y, M, \ldots$ have to be interpreted as vector bundles (locally free $\mathscr{O}_{\mathbf{S}}$-modules of finite rank) over $\mathbf{S}$. The split-exact sequences of the theory over rings become exact, but not necessarily split-exact sequences of sheaves. For example, (1.2.1) now becomes the exact sequence of locally free sheaves

$$0 \longrightarrow \mathscr{O}_{\mathbf{S}} \xrightarrow{1_X} X \xrightarrow{\pi_X} M \longrightarrow 0$$

This sequence splits over any open affine subscheme $\mathbf{U}$ of $\mathbf{S}$ but not necessarily globally. Hence unital linear forms exist only Zariski-locally on $\mathbf{S}$, and the same holds true for unital quadratic and unital bilinear forms. In particular, conic algebras exist over any open affine $\mathbf{U} \subset \mathbf{S}$, but not necessarily globally over $\mathbf{S}$. The complex (2.2.1) is exact as a sequence of locally free sheaves over $\mathbf{S}$ but no longer split-exact.

Suitably formulated, the main results remain valid. Since (1.11.2) splits over any open affine $\mathbf{U} \subset \mathbf{S}$ but not necessarily globally, Proposition 1.11 now reads: Given $A \in \mathrm{Con}(X)$, for every open affine $\mathbf{U} \subset \mathbf{S}$, there exist $f \in \mathrm{H}^0(\mathbf{U}, \mathrm{Bil}_1(X))$ (i.e., a section of the sheaf $\mathrm{Bil}_1(X)$ over $\mathbf{U}$) and $\Gamma \in \mathrm{H}^0(\mathbf{U}, \Omega^2(M, X))$ such that $A\big|\mathbf{U} = f^{\mathrm{m}} + \pi^*(\Gamma)$.

Theorem 1.13, Corollary 2.10 and Theorem 4.10 now say that $\mathbf{Con}(X)$ resp. $\mathbf{Scalin}(X)$ are (not necessarily trivial) torsors in the Zariski topology. Theorem 2.8 requires the following modification in the proof of part (a). Choose an open affine covering $(\mathbf{U}_i)$ of $\mathbf{S}$. Then $A\big|\mathbf{U}_i$ has the form f_i^{m}, for a unique section $f_i \in \mathrm{H}^0(\mathbf{U}_i, \mathrm{Bil}_1(X))$. By uniqueness, these sections agree on the overlaps $\mathbf{U}_i \cap \mathbf{U}_j$, hence define a global section $f \in H^0(\mathbf{S}, \mathrm{Bil}_1(X))$ such that $A = f^{\mathrm{m}}$. The proof of part (b) has to modified similarly.

The α-coordinates of Section 3 now become truly local coordinates: since unital linear forms exist in general only on open affine subschemes, so do α-coordinates. If α and α'

are unital linear forms on open affine subschemes $\mathbf{U}$ and $\mathbf{U}'$, respectively, the change of coordinate formulas in 3.6 will be valid only on the intersection $\mathbf{U} \cap \mathbf{U}'$.

The retraction ϱ and the splitting σ of 3.7 are only available locally on open affine subschemes of $\mathbf{S}$. Hence Theorem 3.10 has to be phrased as follows: The quotient sheaf $\mathbf{B}(X) = \mathbf{Con}(X)\tilde{/}V_{\mathbf{a}}$ (in the Zariski topology, see [**4**, Chapitre III]) is a scheme, locally isomorphic to $\mathbf{Q}(M)$, and $\mathbf{Con}(X)$ is a Zariski torsor over $\mathbf{B}(X)$ with group $V_{\mathbf{a}}$. Corollary 4.12 has to be interpreted similarly.

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