QUATERNIONS, OCTONIONS AND THE FORMS OF THE EXCEPTIONAL SIMPLE CLASSICAL LIE SUPERALGEBRAS

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January 10, 2001

Abstract. The forms of the exceptional simple classical Lie superalgebras are determined over arbitrary fields of characteristic $\neq 2, 3$.

§1. Introduction

The purpose of this article is to determine the forms of the exceptional simple classical Lie superalgebras, that is, of the Lie superalgebras $G(3)$, $F(4)$ and the one-parameter family $D(2,1;\alpha)$, $\alpha \in \bar{F} \setminus \{0,-1\}$, (see [K]). Unless otherwise stated, F will denote a ground field of characteristic $\neq 2, 3$ and $\bar{F}$ will be an algebraic closure of F. The definition of the above mentioned superalgebras over such fields is the same as for fields of characteristic 0.

All these forms are intimately related to quaternion or octonion algebras, so let us first review some of their properties.

A *quaternion algebra* is a central simple associative algebra of degree 2 over F, that is, a form of the algebra $\mathrm{Mat}_2(\bar{F})$. In what follows, all the (unlabelled) tensor products will be over F. Then, identifying $\bar{F} \otimes Q$ with $\mathrm{Mat}_2(\bar{F})$, it turns out that the trace $t(x)$ and the determinant $n(x)$ of any $x \in Q$ are in F (not just in $\bar{F}$) and hence, for any $x \in Q$, $x^2 - t(x)x + n(x)1 = 0$, and the map $x \mapsto \bar{x} = t(x)1 - x$ is an involution of Q (that extends to the canonical symplectic involution on $\mathrm{Mat}_2(\bar{F})$). Then $x + \bar{x} = t(x)1$ and $x\bar{x} = \bar{x}x = n(x)1$ for any $x \in Q$. The subset of trace zero elements Q^0 of a quaternion algebra Q is closed under $[x,y] = xy - yx$ and is, therefore, a form of the simple Lie algebra $A_1 = sl_2(\bar{F})$. It is well known that the converse is valid too. A quaternion algebra is either a division algebra, if its norm does not represent 0, or it is isomorphic to $\mathrm{Mat}_2(F)$.

2000 *Mathematics Subject Classification.* Primary 17A70.

Supported by the Spanish DGES (Pb 97-1291-C03-03), DGI (BFM2001-3239-C03-03), and by a grant from the Spanish Dirección General de Enseñanza Superior e Investigación Científica (Programa de Estancias de Investigadores Españoles en Centros de Investigación Extranjeros), while visiting the University of Wisconsin at Madison.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Any *octonion algebra* (or *Cayley-Dickson algebra*) over F can be built starting with two copies of a quaternion algebra: $C = Q \oplus Qu$, and a nonzero scalar $\alpha \in F$, with multiplication given by the fact that Q becomes a subalgebra of C and

$$x(yu) = (yx)u = (yu)\bar{x} \qquad (xu)(yu) = \alpha\bar{y}x$$

for any $x, y \in Q$ (see [ZSSS, Chapter 2]). The trace t, norm n and involution of Q are extended to C by means of

$$t(x + yu) = t(x) \qquad n(x + yu) = n(x) - \alpha^2 n(y) \qquad \overline{x + yu} = \bar{x} - yu$$

for any $x, y \in Q$ and, in this way, the degree two equation $z^2 - t(z)z + n(z)1 = 0$ and the properties $z + \bar{z} = t(z)1$ and $z\bar{z} = \bar{z}z = n(z)1$, are still valid for any $z \in C$. As for quaternion algebras, there is exactly a Cayley-Dickson algebra whose norm represents 0, which is said to be the *split* Cayley-Dickson algebra. All the other octonion algebras are division algebras.

One of the interesting features of the octonion algebras is that the forms of the exceptional simple Lie algebra G_2, over fields of characteristic $\neq 2, 3$, are precisely the Lie algebras of derivations $\operatorname{Der} C$ of the octonion algebras. Moreover, the subspace of trace zero elements $C^0 = \{x \in C : t(x) = 0\}$ is the unique seven dimensional irreducible module for $\operatorname{Der} C$ (see [J,S]).

Moreover, given an octonion algebra C, the linear map $C^0 \to \operatorname{End}_F(C)$ such that $x \mapsto L_x$ (the left multiplication by x), satisfies $L_x^2 = L_{x^2} = -n(x)L_x$ for any $x \in C^0$, and hence it extends to a homomorphism of the Clifford algebra of $(C^0, -n)$, $Cl(C^0, -n) \to \operatorname{End}_F(C)$, which, by simplicity, restricts to an isomorphism of the even Clifford algebra $Cl^{ev}(C^0, -n) \cong \operatorname{End}_F(C)$. Since the orthogonal Lie algebra $o(C^0, n) = o(C^0, -n)$ lives inside $Cl^{ev}(C^0, -n)$, this provides an irreducible eight-dimensional representation of $o(C^0, n)$: the *spin* representation.

The exceptional simple classical Lie superalgebras over $\bar{F}$ are the Lie superalgebras $G(3)$, $F(4)$ and $D(2, 1; \alpha)$ ($\alpha \in \bar{F} \setminus \{0, -1\}$) whose even and odd parts are given by

(1.1)

$\mathfrak{g}$	$\mathfrak{g}_{\bar{0}}$	$\mathfrak{g}_{\bar{1}}$
$G(3)$	$A_1 \oplus G_2$	$\bar{U} \otimes_{\bar{F}} \bar{V}$
$F(4)$	$A_1 \oplus B_3$	$\bar{U} \otimes_{\bar{F}} \bar{W}$
$D(2, 1; \alpha)$	$A_1 \oplus A_1 \oplus A_1$	$\bar{U} \otimes_{\bar{F}} \bar{U} \otimes_{\bar{F}} \bar{U}$

where $\bar{U}$ is the two dimensional irreducible module for $sl_2(\bar{F})$, $\bar{V}$ is the seven dimensional irreducible module for G_2 and $\bar{W}$ is the spin module for B_3.

Using that $\dim \operatorname{Hom}_{A_1}(\bar{U} \otimes_{\bar{F}} \bar{U}, \bar{F}) = \dim \operatorname{Hom}_{A_1}(\bar{U} \otimes \bar{U}, A_1) = 1$ and also $\dim \operatorname{Hom}_{G_2}(\bar{V} \otimes_{\bar{F}} \bar{V}, \bar{F}) = \dim \operatorname{Hom}_{G_2}(\bar{V} \otimes \bar{V}, G_2) = 1$ and $\dim \operatorname{Hom}_{B_3}(\bar{W} \otimes_{\bar{F}} \bar{W}, \bar{F}) = \dim \operatorname{Hom}_{B_3}(\bar{W} \otimes \bar{W}, B_3) = 1$ (this is well known in characteristic 0, in general it is easy for A_1, and for G_2 and B_3 can be readily obtained along the lines of the proofs of [EM1, Theorem 8] and [EM2, Theorem 7]), the multiplication of odd elements is shown to be given for $G(3)$ and $F(4)$ by the unique, up to a nonzero scalar, symmetric bilinear map $\mathfrak{g}_{\bar 1} \times \mathfrak{g}_{\bar 1} \to \mathfrak{g}_{\bar 0}$ which is $\mathfrak{g}_{\bar 0}$-invariant, whose projection on each simple summand of $\mathfrak{g}_{\bar 0}$ is nonzero and which makes $\mathfrak{g}$ a Lie superalgebra. For $D(2,1;\alpha)$ ($\alpha \neq 0,-1$), there is a whole one-parameter family of such multiplications, and this is why the α appears.

Given a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ with multiplication $[\,,\,]$ over F and a nonzero scalar $0 \neq \mu \in F$, a new multiplication $[\,,\,]_\mu$ is defined on $\mathfrak{g}$ by means of

$$\begin{cases} [x,y]_\mu = \mu[x,y] & \text{if } x,y \in \mathfrak{g}_{\bar 1},\\ [x,y]_\mu = [x,y] & \text{if at least one of } x \text{ or } y \text{ are even.} \end{cases}$$

Denote by $\mathfrak{g}_\mu$ the Lie superalgebra with this new bracket, $\mathfrak{g}_\mu$ is said to be *equivalent* to $\mathfrak{g}$. It is clear that if $\mu \in F^2$, then $\mathfrak{g}_\mu$ is isomorphic to $\mathfrak{g}$. The following result is a reformulation of [K, Proposition 5.3.2], with some minor corrections, restricted to the superalgebras that are being considered here:

Proposition 1.1.

(i) *If a Lie superalgebra $\mathfrak{g}$ over F is a form of the Lie superalgebra $\bar{\mathfrak{g}}$ over $\bar F$ (that is, $\bar F \otimes \mathfrak{g} \cong \bar{\mathfrak{g}}$) then $\mathfrak{g}_{\bar 0}$ is a form of $\bar{\mathfrak{g}}_{\bar 0}$ and the $\mathfrak{g}_{\bar 0}$-module $\mathfrak{g}_{\bar 1}$ is a form of the $\bar{\mathfrak{g}}_{\bar 0}$ module $\bar{\mathfrak{g}}_{\bar 1}$.*

(ii) *In case $\bar{\mathfrak{g}} = G(3)$ or $F(4)$ and $\mathfrak{g}_{\bar 0}$ is a form of $\bar{\mathfrak{g}}_{\bar 0}$ and the $\mathfrak{g}_{\bar 0}$-module $\mathfrak{g}_{\bar 1}$ is a form of the $\bar{\mathfrak{g}}_{\bar 0}$ module $\bar{\mathfrak{g}}_{\bar 1}$ then, up to equivalence, there is a unique Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ which is a form of $\bar{\mathfrak{g}}$, with the given Lie bracket in $\mathfrak{g}_{\bar 0}$ and the given structure of $\mathfrak{g}_{\bar 1}$ as a $\mathfrak{g}_{\bar 0}$-module.*

(iii) *If $\mathfrak{g}_{\bar 0}$ is a form of $A_1 \oplus A_1 \oplus A_1$ and $\mathfrak{g}_{\bar 1}$ is a $\mathfrak{g}_{\bar 0}$-module which is a form of $\bar U \otimes_{\bar F} \bar U \otimes_{\bar F} \bar U$ (being $\bar U$ the two dimensional irreducible module for $sl_2(\bar F)$) then, up to equivalence, there is a one-parameter family of Lie superalgebras $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ such that each superalgebra in this family is a form of $D(2,1;\alpha)$ for some α.*

In the sequel, it will be used several times, without further comment, the following uniqueness property: if M and N are two finite dimensional completely reducible modules for a Lie algebra $\mathfrak{s}$ (or associative algebra) and $\bar F \otimes M \cong \bar F \otimes N$ as modules for $\bar F \otimes \mathfrak{s}$, then $M \cong N$ (note that for irreducible M and N, they are isomorphic if and only if $\mathrm{Hom}_{\mathfrak{s}}(M,N) \neq 0$, if and only if $\mathrm{Hom}_{\bar F \otimes \mathfrak{s}}(\bar F \otimes M, \bar F \otimes N) \neq 0$).

Note that over $\bar F$, the Lie algebra $A_1 \oplus G_2$ (respectively $A_1 \oplus B_3$, $A_1 \oplus A_1 \oplus A_1$) has a unique irreducible and faithful representation of dimension 14 (respectively 16, 8), namely, the one that appears as $\mathfrak{g}_{\bar 1}$ in (1.1).

Therefore, our goal of determining the forms of the exceptional simple classical Lie superalgebras, up to equivalence, reduces to the following problem:

Which forms of $A_1 \oplus G_2$ (respectively $A_1 \oplus B_3$, $A_1 \oplus A_1 \oplus A_1$) admit an absolutely irreducible and faithful representation of dimension 14 *(respectively* 16, 8*)?*

Recall that a module is said to be absolutely irreducible if it remains irreducible under scalar extensions.

This is the problem we are going to tackle. The main results for $G(3)$ and $F(4)$ are:

Theorem G. *The forms of $A_1 \oplus G_2$ that admit an irreducible and faithful representation of dimension 14 are, up to isomorphism, the Lie algebras $sl_2(F) \oplus \operatorname{Der} C$ for an octonion algebra C. The corresponding representation is the tensor product of the natural two dimensional irreducible module for $sl_2(F)$ and the irreducible module C^0 for $\operatorname{Der} C$.*

Theorem F. *The forms of $A_1 \oplus B_3$ that admit an irreducible and faithful representation of dimension 16 are, up to isomorphism, either:*

I) *$sl_2(F) \oplus o(C^0, n)$ for an octonion algebra C with norm n. In this case the corresponding representation is the tensor product of the natural two dimensional irreducible module for $sl(2, F)$ and of the spin module C for $o(C^0, n)$.*

II) *$Q^0 \oplus o(V, q)$ for a quaternion division algebra Q and a seven dimensional vector space V, equipped with a nondegenerate quadratic form q such that the Clifford invariant of (V, q) is the class of Q in the Brauer group $\operatorname{Br}(F)$. In this case, the irreducible module for the even Clifford algebra $Cl^{ev}(V, q)$ carries naturally a structure of Q-module (and hence of module for the Lie algebra Q^0) and of $o(V, q)$-module and it is the corresponding irreducible module for $Q^0 \oplus o(V, q)$.*

Recall that the Clifford invariant of (W, q) above is the class in the Brauer group $\operatorname{Br}(F)$ of the central simple algebra $Cl^{ev}(W, q)$. Also note that the irreducible modules in Theorems G and F have not been assumed to be absolutely irreducible.

To establish the main result for forms of $D(2, 1; \alpha)$ we need to introduce a few more concepts. Following [KMRT] a *cubic étale extension* L/F of our ground field F is given by an étale commutative and associative F-algebra of dimension 3, that is, either $L = F \times F \times F$, or $L = F \times K$ for a separable quadratic field extension K of F, or it is a cubic separable field extension L of F.

Given a cubic étale extension L/F, a *quaternion algebra* over L is either a product of three quaternion algebras over F: $Q_1 \times Q_2 \times Q_3$, in case $L = F \times F \times F$, or a product $Q_1 \times Q_2$, where Q_1 is a quaternion algebra over F and Q_2 a quaternion algebra over K, in case $L = F \times K$ for a quadratic field extension K/F, or a quaternion algebra Q over the field L, in case L/F is a cubic field extension. A trace can be defined naturally for this quaternion algebras $t : Q \to L$ in a componentwise way. The set of trace zero elements Q^0 is a Lie algebra, which is a direct sum of three dimensional simple Lie algebras over the components of L. Therefore, Q^0 is always a form of $A_1 \oplus A_1 \oplus A_1$ and it is easy to prove that any form of $A_1 \oplus A_1 \oplus A_1$ is isomorphic to Q^0, for Q a quaternion algebra over a cubic étale extension of F.

Also, given a separable field extension E/F of degree m and a central simple associative algebra A over E of degree n, there is a central simple algebra $N_{E/F}(A)$ over F (called the norm or corestriction of A) of degree n^m defined in such a way that the map $\operatorname{Br}(E) \to \operatorname{Br}(F)$: $[A] \mapsto [N_{E/F}(A)]$, also denoted by $N_{E/F}$, is a homomorphism between the Brauer groups (see [R]). This can be extended to étale extensions in the following way, if $E = E_1 \times \cdots \times E_r$ is such an étale extension, with the E_i's separable field extensions of F, and $A = A_1 \times \cdots \times A_r$, where A_i is a central simple associative algebra E_i ($i = 1, \ldots , r$), then the norm is defined as

$N_{E/F}(A) = N_{E_1/F}(A_1) \times \cdots \times N_{E_r/F}(A_r)$, which is a central simple associative algebra, and $N_{E/F}\big([A]\big) = \prod_{i=1}^r N_{E_i/F}\big([A_i]\big) \in \mathrm{Br}(F)$.

Theorem D. *The forms of $A_1 \oplus A_1 \oplus A_1$ over a field F of characteristic $\neq 2$ that admit a faithful and absolutely irreducible representation of dimension 8 are, up to isomorphism, the Lie algebras Q^0 for a quaternion algebra Q over a cubic étale extension L/F such that $N_{L/F}\big([Q]\big) = 1$. The corresponding representation is given by the irreducible module of the degree 8 central simple associative algebra $N_{L/F}(Q)$.*

Note that the quaternion algebras in Theorem D appear in a completely different context in [KMRT, 43.B] (see also [KMRT, 16.C]).

There will be a section devoted to the proof of each of these Theorems, where extra results giving information on the central simple Lie superalgebras that appear as forms of $G(3)$, $F(4)$ or $D(2,1;\alpha)$ will be given. Then, in the final section, the previous results will be applied to the classification, up to isomorphism, of the real forms of the exceptional simple classical Lie superalgebras. The real forms of the finite dimensional simple Lie superalgebras are determined, up to equivalence, in [K, Theorem 9], but this result contains some innacuracies. Later on, a more detailed account was given in [P]. However, for the real forms of the algebras $D(2,1;\alpha)$ ($\alpha \in \mathbb{C} \setminus \{0,-1\}$), the results in [P] do not completely determine them: given two values of the parameter $\alpha \in \mathbb{C}$ such that the corresponding complex algebras are isomorphic, the real forms constructed for these values may fail to be isomorphic.

In ending this introduction, the author would like to express his great appreciation to Professor Georgia Benkart, for her suggestions and support.

§2. Proof of Theorem G

Let $\mathfrak{g}$ be a Lie algebra over our ground field F, which is a form of $A_1 \oplus G_2$. Then $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2$, where $\mathfrak{g}^1$ is a three-dimensional simple Lie algebra (that is, a form of A_1) and $\mathfrak{g}^2$ is a form of G_2. Thus, $\mathfrak{g}^2 \cong \mathrm{Der}\, C$ for an octonion algebra C over F.

Assume that $\mathfrak{g}$ has a faithful irreducible module M of dimension 14, then since the minimal dimension of a faithful irreducible module for $\bar{F} \otimes \mathfrak{g}$ is 14, M is absolutely irreducible and $\bar{F} \otimes M$ is the tensor product of the two dimensional irreducible module $\bar{U}$ for $A_1 = sl_2(\bar{F})$ and of the seven dimensional irreducible module $\bar{V}$ for G_2. Then $\mathfrak{g}^1 \subseteq \mathrm{End}_{\mathfrak{g}^2}(M)$ (endomorphisms of M as a $\mathfrak{g}^2$-module), but this latter algebra is a form of $\mathrm{End}_{\bar{F}}(\bar{U})$, and therefore $\mathrm{End}_{\mathfrak{g}^2}(M)$ is a quaternion algebra Q over F and $\mathfrak{g}^1 = [Q,Q] = Q^0$. But, if Q were a quaternion division algebra, then $14 = \dim_F M = 4 \dim_Q M$, a contradiction. Hence, $Q \cong \mathrm{Mat}_2(F)$ and $\mathfrak{g}^1 \cong sl_2(F)$. Moreover, as a $\mathrm{End}_{\mathfrak{g}^2}(M) \cong \mathrm{Mat}_2(F)$-module, M splits as a direct sum of seven copies of the two dimensional irreducible module for $\mathrm{Mat}_2(F)$, so that $M = U \otimes V$, where U is the two dimensional irreducible module for $\mathfrak{g}^1 \cong sl_2(F)$ and V is a seven dimensional vector space. But then, $\mathfrak{g}^2 \subseteq \mathrm{End}_{\mathfrak{g}^1}(M) \cong \mathrm{End}_F(V)$, since $\mathrm{End}_{\mathfrak{g}^1}(U) = F$. Hence V is the unique seven dimensional irreducible module for $\mathfrak{g}^2 \cong \mathrm{Der}\, C$.

This finishes the proof of Theorem G and, therefore, determines the forms of $G(3)$ up to equivalence. However, something else can be said here.

Lemma 2.1. (see [P, Proposition 5.5]) *Let* $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ *be a finite dimensional Lie superalgebra over* F *with* $\mathfrak{g}^0 = sl_2(F) \oplus \mathfrak{s}$ *and* $\mathfrak{g}^1 = U \otimes V$*, where* U *is the natural two-dimensional module for* $sl_2(F)$ *and* V *is a module for* $\mathfrak{s}$*. Assume that the multiplication of odd elements is given by*

$$[u_1 \otimes v_1, u_2 \otimes v_2] = b(v_1, v_2)\sigma_{u_1,u_2} + \varphi(u_1, u_2)v_1 * v_2 \tag{2.1}$$

for any $u_1, u_2 \in U$ *and* $v_1, v_2 \in V$*, for some* $\mathfrak{s}$*-invariant bilinear maps* $b : V \times V \to F$ *(symmetric) and* $* : V \times V \to \mathfrak{s}$ *(skewsymmetric), and where* $\varphi : U \times U \to F$ *is the unique (up to multiplication by nonzero scalars) nonzero skewsymmetric* $sl_2(F)$*-invariant bilinear form and* $\sigma_{u_1,u_2} = \varphi(u_1, -)u_2 + \varphi(u_2, -)u_1$ $(\in sp(U, \varphi) = sl_2(F))$.

Then for any nonzero scalar $\mu \in F$*, the Lie superalgebras* $\mathfrak{g}$ *and* $\mathfrak{g}_\mu$ *are isomorphic.*

Proof. It is enough to take an element $a \in \mathrm{Mat}_2(F)$ with $\det(a) = \mu$ and to consider the even linear map $\Phi : \mathfrak{g}_\mu \to \mathfrak{g}$ given by $\Phi(x) = axa^{-1}$ for any $x \in sl_2(F)$, $\Phi(s) = s$ for any $s \in \mathfrak{s}$ and $\Phi(u \otimes v) = au \otimes v$ for any $u \in U$ and $v \in V$. Then, for any $u_1, u_2 \in U$ and $v_1, v_2 \in V$, $\sigma_{au_1,au_2} = \varphi(au_1, -)au_2 + \varphi(au_2, -)au_1 = a\Big(\varphi(u_1, a^*-)u_2 + \varphi(u_2, a^*-)u_1\Big) = \mu a\sigma_{u_1,u_2}a^{-1}$, because the adjoint a^* of a relative to φ is μa^{-1}, and thus

$$\begin{aligned}
\Phi\big([u_1 \otimes v_1, u_2 \otimes v_2]_\mu\big) &= \mu\Phi\Big(b(v_1, v_2)\sigma_{u_1,u_2} + \varphi(u_1, u_2)v_1 * v_2\Big)\\
&= \mu\Big(b(v_1, v_2)a\sigma_{u_1,u_2}a^{-1} + \varphi(u_1, u_2)v_1 * v_2\Big)\\
&= b(v_1, v_2)\sigma_{au_1,au_2} + \mu\varphi(u_1, u_2)v_1 * v_2\\
&= b(v_1, v_2)\sigma_{au_1,au_2} + \varphi(au_1, au_2)v_1 * v_2\\
&= \big[\Phi(u_1 \otimes v_1), \Phi(u_2 \otimes v_2)\big]. \qquad \square
\end{aligned}$$

It is an immediate consequence of this Lemma and of Theorem G that equivalent forms of $G(3)$ are in fact isomorphic. Therefore, the forms of $G(3)$ are completely determined up to isomorphism once we know which octonion algebra C is involved in the decomposition of $\mathfrak{g}_{\bar 0} \cong sl_2(F) \oplus \mathrm{Der}\, C$.

In [BE,BZ], a generalized Tits construction has been considered that extends the celebrated Tits construction, which gives all the exceptional simple classical Lie algebras in a unified framework. In particular, given an octonion algebra C and a simple Jordan superalgebra J with a normalized trace tr over F (see [BE] for details), the space

$$\mathcal{T}(C, J) := \mathrm{Der}\, C \oplus (C^0 \otimes J^0) \oplus \mathrm{Der}\, J, \tag{2.2}$$

where $J^0 = \{x \in J : tr(x) = 0\}$, with the superanticommutative product specified by

$$\begin{aligned}
&\mathrm{Der}\, C \text{ and } \mathrm{Der}\, J \text{ are commuting subsuperalgebras of } \mathcal{T}(C, J),\\
&[D, a \otimes x] = D(a) \otimes x, \quad [d, a \otimes x] = a \otimes d(x),\\
&[a \otimes x, b \otimes y] = tr(xy)D_{a,b} + [a, b] \otimes x * y + 2t(ab)d_{x,y}
\end{aligned}$$

for all $D \in \operatorname{Der} C$, $d \in \operatorname{Der} J$, $a, b \in C^0$, $x, y \in J^0$, where $D_{a,b}(c) = [[a,b],c] - 3\big((ab)c - a(bc)\big)$ and $d_{x,y}(z) = x(yz) - (-1)^{\bar{x}\bar{y}} y(xz)$ for any $c \in C$ and $z \in J$, is a Lie superalgebra. Moreover, in case J is the Jordan superalgebra of a nondegenerate superform with trivial even part and two dimensional odd part, denote it by $\hat{J}$, then $\mathcal{T}(C, \hat{J})$ is a form of $G(3)$ with even part $sl_2(F) \oplus \operatorname{Der} C$.

Then, Theorem G and Lemma 2.1 immediately give:

Corollary 2.2. *The F-forms of $G(3)$ are exactly the Lie superalgebras $\mathcal{T}(C, \hat{J})$ for an octonion algebra C. Two such forms are isomorphic if and only if so are the corresponding octonion algebras.*

§3. Proof of Theorem F

Now, let $\mathfrak{g}$ be a Lie algebra over our ground field F which is a form of $A_1 \oplus B_3$. Then $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2$, where $\mathfrak{g}^1$ is a three-dimensional simple Lie algebra and $\mathfrak{g}^2$ is a form of B_3. Assume that $\mathfrak{g}$ has a faithful irreducible module M of dimension 16. Since the minimal dimension of a faithful irreducible module for $\bar{\mathfrak{g}} = \bar{F} \otimes \mathfrak{g}$ is $14 = 2 \times 7$, it is easy to check that M is absolutely irreducible and $\bar{M} = \bar{F} \otimes M$ is the tensor product of the two dimensional natural representation for $\bar{\mathfrak{g}}^1 = \bar{F} \otimes \mathfrak{g}^1$ and the spin representation for $\bar{\mathfrak{g}}^2 = \bar{F} \otimes \mathfrak{g}^2$.

We are left with two different possibilities:

- Fi) M decomposes as a sum of two irreducible eight dimensional modules for $\mathfrak{g}^2$ which, since $\bar{F} \otimes \operatorname{End}_{\mathfrak{g}^2}(M) = \operatorname{End}_{\bar{\mathfrak{g}}^2}(\bar{M}) \cong \operatorname{Mat}_2(\bar{F})$, must be isomorphic (otherwise the centralizer $\operatorname{End}_{\mathfrak{g}^2}(M)$ would be the direct sum of the centralizer of the two modules). Therefore, $M = U \otimes W$ for an irreducible $\mathfrak{g}^2$-module W and a two dimensional vector space U.
- Fii) M is irreducible as $\mathfrak{g}^2$-module.

In case Fi) above, $\mathfrak{g}^1 \subseteq \operatorname{End}_{\mathfrak{g}^2}(M) \cong \operatorname{End}_F(U) \cong \operatorname{Mat}_2(F)$ and $\mathfrak{g}^1 \cong sl_2(F)$.

Lemma 3.1. *Let $\mathfrak{s}$ be a form of B_3 over F and let W be an irreducible $\mathfrak{s}$-module of dimension 8. Then there is a Cayley-Dickson algebra C over F with norm n such that $\mathfrak{s}$ is isomorphic to the orthogonal Lie algebra $o(C^0, n)$ and, through this isomorphism, W is the spin module C for $o(C^0, n)$.*

Proof. Since $\bar{W} = \bar{F} \otimes W$ is the spin representation for B_3, it is known (see the comments after (1.1)) that there is a unique, up to nonzero scalars, symmetric bilinear form $b : W \times W \to F$ which is $\mathfrak{s}$-invariant. Thus $\mathfrak{s}$ can be embedded as a subalgebra of the orthogonal Lie algebra $o(W, b)$. Consider the trace form on $o(W, b)$, which is nonzero. Since $\dim \mathfrak{s} = 21$ and $\dim o(W, b) = 28$, the restriction of the trace form to $\mathfrak{s}$ is nonzero and, by simplicity of $\mathfrak{s}$, nondegenerate. Therefore, $o(W, q) = \mathfrak{s} \oplus \mathfrak{s}^\perp$ (orthogonal relative to the trace form) and $[\mathfrak{s}, \mathfrak{s}^\perp] \subseteq \mathfrak{s}^\perp$. Since $\mathfrak{s}$ is not an ideal of the simple Lie algebra $o(W, b)$, it follows that $[\mathfrak{s}, \mathfrak{s}^\perp] \neq 0$ and thus $\mathfrak{s}$ embeds as a Lie subalgebra of the orthogonal Lie algebra $o(\mathfrak{s}^\perp)$ (relative to the trace form). By dimension count, they are equal. The conclusion is that, up to isomorphism, $\mathfrak{s}$ is the orthogonal Lie algebra $o(V, q)$ for some vector space V of dimension 7 and nondegenerate quadratic form q on V. (Recall that, in general, not all forms of B_3 are such orthogonal Lie algebras.)

But $\bar{W}$ is the spin module for $\bar{\mathfrak{s}} = \bar{F} \otimes \mathfrak{s} = o(\bar{V}, \bar{q})$, so that the representation of $\mathfrak{s}$ on W comes from an isomorphism of the even Clifford algebra $Cl^{ev}(V,q)$ onto $\mathrm{End}_F(W)$, which shows that the Clifford invariant of (V,q) is trivial. Complementing V with an orthogonal complement of dimension 1 we obtain an eight dimensional quadratic form with trivial discriminant and Clifford invariant and a result of Pfister applies (see [KMRT, (35.2)]) to show that (V,q) is similar to (C^0, n) for a Cayley-Dickson algebra C with norm n, as required. □

This Lemma settles part I) of Theorem F.

In case Fii) above, by Schur Lemma and since $\bar{F} \otimes \mathrm{End}_{\mathfrak{g}^2}(M) = \mathrm{End}_{\bar{\mathfrak{g}}^2}(\bar{M}) \cong \mathrm{Mat}_2(\bar{F})$, $\mathrm{End}_{\mathfrak{g}^2}(M)$ is a quaternion division algebra Q and $\mathfrak{g}^1 \subseteq [Q,Q] = Q^0$. There are two possibilities for $\mathfrak{g}^2$ [J,S]: either it is isomorphic to the orthogonal Lie algebra $o(V,q)$ for some seven dimensional vector space with a nondegenerate quadratic form, or it is isomorphic to the Lie algebra $Skew(D,j)$ of skew symmetric elements of a central division algebra D of degree 7 relative to an orthogonal involution j. In this last case, if K/F is a quadratic field extension which splits Q, $K \otimes D$ is again a division algebra over K (because 2 and 7 are relatively prime, see [R, Corollary 7.2.4]) and hence $K \otimes \mathfrak{s} \cong sl_2(K) \oplus Skew(K \otimes D, 1 \otimes j)$, but this is in contradiction with case Fi).

Therefore, in case Fii) above, up to isomorphism, $\mathfrak{s} = Q^0 \oplus o(V,q)$ for some regular quadratic space (V,q). By uniqueness, M must be isomorphic to the irreducible module for the even Clifford algebra $Cl^{ev}(V,q)$ and, by density, $Cl^{ev}(V,q) \cong \mathrm{End}_Q(M) \cong \mathrm{Mat}_4(Q)$. This shows that the Clifford invariant of (V,q) is the class of Q in $\mathrm{Br}(F)$.

Conversely, given a seven dimensional regular quadratic space (V,q) with Clifford invariant $[Q]$ for some division quaternion algebra Q, $Cl^{ev}(V,q) \cong \mathrm{End}_Q(M)$ for some Q-vector space M of dimension 4. Then M is an irreducible and faithful module of dimension 16 for the Lie algebra $Q^0 \oplus o(V,q)$, where the action of Q^0 is given by the structure of M as a Q-vector space, and the action of $o(V,q)$ by its embedding as a Lie subalgebra of $Cl^{ev}(V,q)$.

This completes the proof of Theorem F.

Exactly as for $G(3)$, two equivalent forms of $F(4)$ of type I) in Theorem F are actually isomorphic, thanks to Lemma 2.1. Also, in [BZ,BE], it has been shown that the Tits construction $\mathcal{T}(C,J)$, this time with J the simple Jordan superalgebra D_2, are forms of $F(4)$. Given any $0 \neq \alpha \in F$, the Jordan superalgebra D_α has even part with basis $\{e,f\}$ and odd part with basis $\{x,y\}$ and the multiplication is given by:

$$
\begin{gathered}
e^2 = e, \qquad f^2 = f, \qquad ef = 0, \\
xy = e + \alpha f, \qquad ex = \frac{1}{2}x = fx, \qquad ey = \frac{1}{2}y = fy.
\end{gathered} \tag{3.1}
$$

The same arguments as for $G(3)$ give:

Corollary 3.2. *The F-forms of $F(4)$ of type I) in Theorem F are exactly the Lie superalgebras $\mathcal{T}(C, D_2)$ for an octonion algebra C. Two such forms are isomorphic if and only if so are the corresponding octonion algebras.*

For forms of $F(4)$ corresponding to case II) of Theorem F, it is not known wether equivalent superalgebras are isomorphic. However, some partial results can be given. First, Lemma 2.1 can be strengthened to:

Lemma 3.3. *Let $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ be a finite dimensional Lie superalgebra with $\mathfrak{g}_{\bar 0} = Q^0 \oplus \mathfrak{s}$ for a quaternion algebra Q and such that $\bar{\mathfrak{g}} = \bar F \otimes \mathfrak{g}$ satisfies the conditions of Lemma 2.1 (with $sl_2(\bar F) = \bar F \otimes Q^0$). Then for any $a \in Q$ with $n(a) = \mu \neq 0$, $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_\mu$.*

Proof. If $Q = \mathrm{Mat}_2(F)$ this is given by Lemma 2.1. Otherwise, Q is a quaternion division algebra contained in $\mathrm{End}_{\mathfrak{s}}(\mathfrak{g}_{\bar 1})$, so that $\mathfrak{g}_{\bar 1}$ is a vector space over Q. The map $\Phi : \mathfrak{g}_\mu \to \mathfrak{g}$ given by $\Phi(q) = aqa^{-1}$, $\Phi(s) = s$ and $\Phi(z) = az$ for any $q \in Q^0$, $s \in S$ and $z \in \mathfrak{g}_{\bar 1}$ gives the desired isomorphism (by extending scalars, this is the same map as in Lemma 2.1). □

Let now $\mathfrak{g}$ be a form of $F(4)$ of type II) in Theorem F, so that $\mathfrak{g}_{\bar 0} = Q^0 \oplus o(V, q)$. We can assume that the discriminant of (V, q) is trivial, since we can substitute q by any nonzero scalar multiple. Let τ be the canonical involution of $Cl(V, q)$ (the one that fixes elementwise V) and denote also by τ its restriction to $Cl^{ev}(V, q) \cong \mathrm{End}_Q(M)$. Let us denote by Δ a fixed isomorphism $Cl^{ev}(V, q) \cong \mathrm{End}_Q(M)$. Then there is an ϵ-hermitian form $h : M \times M \to Q$ ($\epsilon = \pm 1$), that is

$$h(am, n) = ah(m, n), \qquad h(m, n) = \epsilon\overline{h(n, m)}$$

for any $a \in Q$, $m, n \in M$ ($a \mapsto \bar a$ denotes the standard involution in Q), such that the involution τ in $Cl^{ev}(V, q)$ corresponds to the adjoint $*$ relative to h in $\mathrm{End}_Q(M)$. Since $o(V, q) \subseteq Skew\big(Cl^{ev}(V, q), \tau\big) \cong Skew(\mathrm{End}_Q(M), *)$, h is invariant under the action of $o(V, q)$ on M (where Q is regarded as a trivial $o(V, q)$-module). Besides, for any $a \in Q^0$ and $m, n \in M$, $[a, h(m, n)] = ah(m, n) - h(m, n)a = h(am, n) + h(m, an)$, and thus h is invariant under the action of Q^0, considering Q as a module for Q^0 under the adjoint map. As such, the map $a \mapsto a - \bar a$ is a Q^0-homomorphism. In consequence, the unique (up to scalars) $\mathfrak{g}_{\bar 0}$-invariant map $M \otimes M \to Q^0$ is given by $m \otimes n \mapsto h(m, n) - \overline{h(m, n)} = h(m, n) - \epsilon h(n, m)$. But after scalar extension, there is a unique such invariant map and it is symmetric (it is given by the multiplication of odd elements in $F(4)$). Therefore $\epsilon = -1$ and the map given by $m \otimes n \mapsto h(m, n) + h(n, m)$ is the only, up to scalars, $\mathfrak{g}_{\bar 0}$-invariant map $M \otimes M \to Q^0$.

Moreover, considering $o(V, q)$ as a trivial Q^0-module, the maps

$$\begin{aligned} \Gamma : M \otimes M &\to o(V, q)^* \\ m \otimes n &\mapsto \Big(\varphi \mapsto t\big(h(\varphi.m, n)\big)\Big) \end{aligned} \qquad\qquad \begin{aligned} \Omega : o(V, q) &\to o(V, q)^* \\ \varphi &\mapsto \Big(\gamma \mapsto tr_V(\varphi\gamma)\Big) \end{aligned}$$

where t is the trace in Q, tr_V the trace in $\mathrm{End}_F(V)$ and $\varphi.m$ denotes the action of $o(V, q)$ in M, are $Q^0 \oplus o(V, q)$-invariant and, therefore, the only, up to scalars,

$\mathfrak{g}_{\bar 0}$-homomorphism $M \otimes M \to \mathfrak{g}_{\bar 0}$ is $\Upsilon = \Omega^{-1}\Gamma$. Note that the action of $o(V,q)$ on M is given by embedding $o(V,q)$ in $Cl^{ev}(V,q)$ and then using the isomorphism Δ, thus $\varphi.m = \Delta(\varphi)(m)$, for any $\varphi \in o(V,q)$ and $m \in M$.

Since we can scalar h conveniently, the above argument shows that the multiplication of odd elements in a form $\mathfrak{g}$ of $F(4)$ of type II), with $\mathfrak{g}_{\bar 0} = Q^0 \oplus o(V,q)$ and $\mathfrak{g}_{\bar 1} = M$ as above, is

$$[m,n] = \bigl(h(m,n) + h(m,n)\bigr) + \delta\Upsilon(m\otimes n) \tag{3.2}$$

for all $m,n \in M$, where δ is a suitable nonzero scalar.

Lemma 3.4. *Let $\mathfrak{g}$ be a form of $F(4)$ of type II) in Theorem F, with $\mathfrak{g}_{\bar 0} = Q^0 \oplus o(V,q)$ such that the discriminant of q is trivial and its Clifford invariant is $[Q]$. Then for any $v \in V$ with $q(v) \neq 0$, $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_{-q(v)}$.*

Proof. Consider the automorphism $\mathrm{Int}(v)$ of $Cl^{ev}(V,q)$ given by $x \mapsto vxv^{-1}$ for any $x \in Cl^{ev}(V,q)$ (the multiplication is performed in $Cl(V,q)$). It clearly commutes with the canonical involution τ, because $\mathrm{Int}(v)(w) = \frac{q(v,w)}{q(v)}v - w = -s_v(w) \in V$ (s_v denotes the reflection relative to the hyperplane orthogonal to v) for any $w \in V$. Moreover, $\mathrm{Int}(v)$ restricts to the isomorphism $\varphi \mapsto s_v\varphi s_v$ of $o(V,q)$.

Take z an odd element in the center of $Cl(V,q)$ with $z^2 = 1$ (recall that we are assuming that the discriminant of q is trivial). Then $\mathrm{Int}(v) = \mathrm{Int}(vz)$ and $vz \in Cl^{ev}(V,q)$. Let $\psi \in \mathrm{End}_Q(M)$ be given by $\psi = \Delta(vz)$. Since $vz\tau(vz) = -vzzv = -v^2 = -q(v)$,

$$\psi\psi^* = -q(v)I. \tag{3.3}$$

Now, for any $m,n \in M$ and $\varphi,\gamma \in o(V,q)$:

$$\begin{aligned} h\Bigl(\varphi.(\psi(m)),\psi(n)\Bigr) &= h\Bigl(\Delta(\varphi)\psi(m),\psi(n)\Bigr)\\ &= h\Bigl(\psi\psi^{-1}\Delta(\varphi)\psi(m),\psi(n)\Bigr)\\ &= -q(v)h\Bigl(\Delta(s_v\varphi s_v)(m),n\Bigr)\\ &= -q(v)h\Bigl((s_v\varphi s_v).m,n\Bigr)\end{aligned}$$

and $tr_V\bigl((s_v\varphi s_v)\gamma\bigr) = tr_V\bigl(\varphi(s_v\gamma s_v)\bigr)$, so that

$$\Upsilon\bigl(\psi(m)\otimes\psi(n)\bigr) = -q(v)s_v\Upsilon(m\otimes n)s_v \tag{3.4}$$

for any $m,n \in M$.

Finally define the even linear map $\Phi : \mathfrak{g}_{-q(v)} \to \mathfrak{g}$ by means of $\Phi(a) = a$, $\Phi(\varphi) = s_v\varphi s_v$ and $\Phi(m) = \psi(m)$ for any $a \in Q^0$, $\varphi \in o(V,q)$ and $m \in M$. From (3.2), (3.3) and (3.4) we conclude that Φ is an isomorphism. □

Corollary 3.5. *Let $\mathfrak{g}$ be a form of $F(4)$ of type II) in Theorem F, with $\mathfrak{g}_{\bar 0} = Q^0 \oplus o(V,q)$ such that the Clifford invariant of q is $[Q]$ and q is universal. Then any Lie superalgebra equivalent to $\mathfrak{g}$ is in fact isomorphic to $\mathfrak{g}$.*

As we shall see in the last section, this is what happens over the real field.

§4. Proof of Theorem D

Let us first consider the following example. Take the real Lie algebra $\mathfrak{g} = sl_2(\mathbb{R}) \oplus sl_2(\mathbb{C})$. It has a faithful and irreducible eight dimensional module $M = \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{C}^2$, where $\mathbb{R}^2$ and $\mathbb{C}^2$ are the natural modules for the simple ideals $sl_2(\mathbb{R})$ and $sl_2(\mathbb{C})$ of $\mathfrak{g}$, which is not absolutely irreducible (for instance, one checks that $\mathrm{End}_{\mathfrak{g}}(M) \cong \mathrm{End}_{sl_2(\mathbb{C})}(\mathbb{C}^2) \cong \mathbb{C}$). Therefore, contrary to the situation in the previous paragraphs, absolute irreducibility will have to be imposed here.

Before proceeding further, let us recall briefly some facts concerning the norm or corestriction (see [R]). Given a finite Galois field extension E/F, an element σ in the Galois group $\mathrm{Gal}(E/F)$ and a (not necessarily associative) E-algebra R, the E-algebra $\sigma^{-1}R$ is defined on the same ring R (same addition and multiplication) but with the new scalar product given by $\mu(\sigma^{-1}r) = \sigma^{-1}\bigl(\sigma(\mu)r\bigr)$, for any $\mu \in E$ and $r \in R$, where the elements of $\sigma^{-1}R$ are denoted $\sigma^{-1}r$, $r \in R$. Assume that L is an intermediate field, $F \subseteq L \subseteq E$ and let $G = \mathrm{Gal}(E/F)$, $H = \mathrm{Gal}(E/L)$ and $\sigma_1, \dots, \sigma_n \in G$ such that $G = H\sigma_1 \cup \ldots \cup H\sigma_n$ (disjoint union). Then if $\tau_i = \sigma_i^{-1}$ for $i = 1, \dots, n$, $G = \tau_1 H \cup \ldots \cup \tau_n H$ and the restriction of $\tau_1, \dots, \tau_n$ to L give the different embeddings $L \hookrightarrow E$.

Let A be an algebra over L, then the E-linear map

$$\begin{aligned} E \otimes_F A &\longrightarrow \bigoplus_{i=1}^{n} \sigma_i^{-1}\bigl(E \otimes_L A\bigr) \\ 1 \otimes a &\mapsto \bigl(\sigma_1^{-1}(1 \otimes a), \dots, \sigma_n^{-1}(1 \otimes a)\bigr) \end{aligned}$$

is an isomorphism of E-algebras. Assume now that A is associative, then for any $\sigma \in G$ there is a permutation π such that $\sigma_i\sigma \in H\sigma_{\pi(i)}$, $i = 1, \dots, n$. Let $\sigma_i\sigma = \gamma_i\sigma_{\pi(i)}$, with $\gamma_i \in H$ for any i. On the E-algebra

$$\tilde{A} = \sigma_1^{-1}\bigl(E \otimes_L A\bigr) \otimes_E \cdots \otimes_E \sigma_n^{-1}\bigl(E \otimes_L A\bigr)$$

consider, for any $\sigma \in G$, the σ-semilinear automorphism Φ_σ given by

$$\Phi_\sigma\Bigl(\otimes_{i=1}^n \sigma_i^{-1}(\mu_i \otimes a_i)\Bigr) = \otimes_{i=1}^n\Bigl(\sigma_i^{-1}\bigl(\gamma_i(\mu_{\pi(i)}) \otimes a_{\pi(i)}\bigr)\Bigr)$$

for $a_i \in A$ and $\mu_i \in E$, $i = 1, \dots, n$. Then $\Phi_\sigma\Phi_\tau = \Phi_{\sigma\tau}$, $\Phi_1 = 1$, and therefore, $\{x \in \tilde{A} : \Phi_\sigma(x) = x \ \forall \sigma \in G\}$ is an F-subalgebra of $\tilde{A}$, called the norm of A and denoted by $N_{L/F}(A)$. It does depend only on the L-algebra A and not on E or the transversal chosen. Moreover, if A is central simple over L, so is $N_{L/F}(A)$ over F and $\tilde{A} \cong E \otimes_F N_{L/F}(A)$.

We are in position now to prove Theorem D. Let $\mathfrak{g}$ be a Lie algebra over a ground field F of characteristic $\neq 2$, which is a form of $A_1 \oplus A_1 \oplus A_1$, and let M be a faithful and absolutely irreducible eight dimensional module for $\mathfrak{g}$. There are three different possibilities for $\mathfrak{g}$:

- Di) $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \mathfrak{g}^3$, where each $\mathfrak{g}^i$ ($i = 1, 2, 3$) is a three dimensional simple Lie algebra over F. Thus $\mathfrak{g} = Q^0$ fo a quaternion algebra over $L = F \times F \times F$, or

Dii) $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2$, where $\mathfrak{g}^1$ is a three dimensional simple Lie algebra over F and $\mathfrak{g}^2$ is a three dimensional simple Lie algebra over a quadratic field extension K of F. Here $\mathfrak{g} = Q^0$ for a quaternion algebra over $L = F \times K$, or

Diii) $\mathfrak{g}$ is a three dimensional simple Lie algebra over a separable cubic field extension L of F, so that $\mathfrak{g} = Q^0$ for some quaternion algebra over L.

The proof of Theorem D will be split according to these three possibilities.

Assume first that we are in case Di), then $\mathfrak{g}^1 \subseteq \mathrm{End}_{\mathfrak{g}^2\oplus\mathfrak{g}^3}(M)$, which is a quaternion algebra over F (extending scalars to the the algebraic closure it becomes isomorphic to $\mathrm{End}_{\bar F}(\bar U) \cong \mathrm{Mat}_2(\bar F)$). Denote by Q_1 this quaternion algebra, which acts as endomorphisms of M. Then $\mathfrak{g}^1 \subseteq [Q_1, Q_1] = Q_1^0$ and, by dimension count, $\mathfrak{g}^1 = Q_1^0$ and $Q_1 = F1 \oplus \mathfrak{g}^1$. In the same vein, there are quaternion algebras Q_2 and Q_3 so related to $\mathfrak{g}^2$ and $\mathfrak{g}^3$. Since the actions of $\mathfrak{g}^1$, $\mathfrak{g}^2$ and $\mathfrak{g}^3$ on M commute, so do the actions of Q_1, Q_2 and Q_3 and thus there is a homomorphism of associative algebras

$$Q_1 \otimes Q_2 \otimes Q_3 \to \mathrm{End}_F(M)$$

which, by simplicity and dimension count, is an isomorphism. Hence $Q_1 \otimes Q_2 \otimes Q_3 \cong \mathrm{End}_F(M) \cong \mathrm{Mat}_8(F)$ and, with $L = F \times F \times F$ and $Q = Q_1 \times Q_2 \times Q_3$, $\mathfrak{g} = Q^0$ and $N_{L/F}(Q) = Q_1 \otimes Q_2 \otimes Q_3$ gives the trivial class in $\mathrm{Br}(F)$, as required.

Conversely, with L and $Q = Q_1 \times Q_2 \times Q_3$ as above, if $N_{L/F}([Q]) = 1$, then $Q_1 \otimes Q_2 \otimes Q_3 \cong \mathrm{Mat}_8(F)$, so that $Q_1 \otimes Q_2 \otimes Q_3 \cong \mathrm{End}_F(M)$ for an eight dimensional vector space M over F. By means of the map

$$\begin{aligned} Q_1^0 \oplus Q_2^0 \oplus Q_3^0 &\longrightarrow Q_1 \otimes Q_2 \otimes Q_3 \cong \mathrm{End}_F(M) \\ (x_1, x_2, x_3) &\mapsto x_1 \otimes 1 \otimes 1 + 1 \otimes x_2 \otimes 1 + 1 \otimes 1 \otimes x_3 \end{aligned}$$

M becomes a faithful and absolutely irreducible module for $\mathfrak{g} = Q_1^0 \oplus Q_2^0 \oplus Q_3^0$.

Now assume that we are in case Dii). Here $L = F \times K$ with K a quadratic field extension of F and $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2$, where $\mathfrak{g}^1$ (respectively $\mathfrak{g}^2$) is a three dimensional simple Lie algebra over F (respectively K). Moreover, $\mathfrak{g}^1 = Q_1^0$ (respectively $\mathfrak{g}^2 = Q_2^0$) for a quaternion algebra over F (respectively K). Let $Q = Q_1 \times Q_2$ be the corresponding quaternion algebra over L and let ι be the nontrivial automorphism in the Galois group of the quadratic extension K/F.

Now the K-linear map given by

$$\begin{aligned} K \otimes \mathfrak{g} \cong (K \otimes \mathfrak{g}^1) \oplus \mathfrak{g}^2 \oplus \iota^{-1}\mathfrak{g}^2 &\longrightarrow (K \otimes Q_1) \otimes_K Q_2 \otimes_K \iota^{-1} Q_2 \\ 1 \otimes x_1 &\mapsto (1 \otimes x_1) \otimes 1 \otimes \iota^{-1}(1) \\ x_2 &\mapsto (1 \otimes 1) \otimes x_2 \otimes \iota^{-1} 1 \\ \iota^{-1} x_2 &\mapsto (1 \otimes 1) \otimes 1 \otimes \iota^{-1} x_2 \end{aligned}$$

takes $\mathfrak{g} \subseteq K \otimes \mathfrak{g}$ isomorphically into $N_{L/F}(Q) = Q_1 \otimes N_{K/F}(Q_2)$ (which is an F-subalgebra of $(K \otimes Q_1) \otimes_K Q_2 \otimes_K \iota^{-1}(Q_2)$). Besides, $K \otimes \mathfrak{g}$ generates $(Q_1 \otimes K) \otimes_K Q_2 \otimes_K \iota^{-1} Q_2 = K \otimes \big(Q_1 \otimes N_{K/F}(Q_2)\big)$, so $\mathfrak{g}$ generates $Q_1 \otimes N_{K/F}(Q_2)$.

If $\mathfrak{g}$ admits a faithful eight dimensional absolutely irreducible module M the there is a commutative diagram

$$\begin{array}{ccc} \mathfrak{g} & \hookrightarrow & \mathrm{End}_F(M) \\ \downarrow & & \downarrow \\ K\otimes\mathfrak{g} & \hookrightarrow & \mathrm{End}_K(K\otimes M) \end{array}$$

The argument for case Di) shows that the bottom map embeds in a commutative diagram

$$\begin{array}{ccc} K\otimes\mathfrak{g} & \hookrightarrow & \mathrm{End}_K(K\otimes M) \\ \downarrow & & \shortparallel \\ (Q_1\otimes K)\otimes_K Q_2\otimes_K \iota^{-1}Q_2 & \to & \mathrm{End}_K(K\otimes M) \end{array}$$

whose bottom map is an isomorphism. Since $\mathfrak{g}$ generates $Q_1\otimes N_{K/F}(Q_2)$, by dimension count this last bottom map restricts to an isomorphism $N_{L/F}(Q) = Q_1\otimes N_{K/F}(Q_2) \cong \mathrm{End}_F(M)$, so $N_{L/F}(Q)$ gives the trivial class in the Brauer group.

Conversely, if $N_{L/F}([Q]) = 1$, there is an isomorphism of F-algebras $N_{L/F}(Q) = Q_1\otimes N_{K/F}(Q_2) \cong \mathrm{End}_F(M)$ for some eight dimensional vector space M, and since $\mathfrak{g}$ generates $N_{L/F}(Q)$, this gives an absolutely irreducible and faithful eight dimensional module for $\mathfrak{g}$.

Finally, in case Diii), $\mathfrak{g} = Q^0$ for a quaternion algebra over a cubic separable field extension L of F and either $L/F = E/F$ is a cyclic Galois field extension, or there is a Galois field extension E/F containing L with Galois group isomorphic to the symmetric group S_3. In any case there is a cyclic group of order 3: $\{1,\sigma,\sigma^2\}$ of $G = \mathrm{Gal}(E/F)$ and the restriction of $1,\sigma,\sigma^2$ to L give the three different embeddings of L into E. As for Dii), there is a sequence of maps

$$\begin{aligned} \mathfrak{g} \hookrightarrow E\otimes\mathfrak{g} &= (E\otimes_L\mathfrak{g})\oplus\sigma^{-1}(E\otimes_L\mathfrak{g})\oplus\sigma^{-2}(E\otimes_L\mathfrak{g}) \\ &\hookrightarrow (E\otimes_L Q)\otimes_E\sigma^{-1}(E\otimes_L Q)\otimes_E\sigma^{-2}(E\otimes_L Q) \end{aligned}$$

which takes $\mathfrak{g}$ into $N_{L/F}(Q)$ and shows that $\mathfrak{g}$ generates $N_{L/F}(Q)$. A similar argument as for Dii) concludes the proof of Theorem D.

Given a form of $G(3)$ or a form of $F(4)$ such that its even part contains an ideal isomorphic to $sl_2(F)$, it was shown in Corollaries 2.2 and 3.2 that this form is given by a Tits construction $\mathcal{T}(C,J)$ for a Cayley-Dickson algebra C and a suitable Jordan superalgebra J. In the same vein, some forms of the Lie superalgebras $D(2,1;\alpha)$ appear as Tits constructions. Assume that $\mathfrak{g}$ is a form of some $D(2,1;\alpha)$ such that $\mathfrak{g}_{\bar 0}$ contains an ideal isomorphic to $sl_2(F)$ and $\mathfrak{g}$ is of type Di). Then because of Theorem D and the fact that the class of any quaternion algebra in the Brauer group has order 1 or 2, necessarily $\mathfrak{g}_{\bar 0} = sl_2(F)\oplus Q^0\oplus Q^0$ for a quaternion algebra Q over F. Note that in this situation, the map $Q^0\oplus Q^0 \to o(Q,n)$ (the orthogonal Lie algebra relative to the norm of Q) which assigns to any pair (p,q) of elements in Q^0 the map $x\in Q\mapsto px - xq$ is an isomorphism, so that $\mathfrak{g}_{\bar 0} = sl_2(F)\oplus o(Q,n)$. We are then in the situation of the next Lemma, whose first part has been proved in [BE]:

Lemma 4.1. *Let Q be a quaternion algebra over F and let U be a two dimensional space with a nonzero skew-symmetric form $\varphi : U \times U \to F$. For $u, v \in U$, let $\sigma_{u,v} \in sp(U,\varphi) \cong sl_2(F)$ (the symplectic Lie algebra) be given by*

$$\sigma_{u,v}(w) = \varphi(v,w)u + \varphi(u,w)v.$$

For nonzero $\alpha, \beta \in F$, let $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ be the superalgebra with

$$\begin{aligned} \mathfrak{g}_{\bar 0} &= sp(U,\varphi) \oplus Q^0 \oplus Q^0 \\ \mathfrak{g}_{\bar 1} &= U \otimes Q \end{aligned}$$

and with multiplication given by

- *the usual Lie bracket in $\mathfrak{g}_{\bar 0}$,*
- $[(f,p,q), u \otimes x] = f(u) \otimes x + u \otimes (px - xq)$,
- $[u \otimes x, v \otimes y] = \Big(t(\bar x y)\sigma_{u,v}, -\alpha\varphi(u,v)(x\bar y - y\bar x), -\beta\varphi(u,v)(\bar x y - \bar y x)\Big)$,

for $u, v \in U$, $x, y \in Q$, $f \in sp(U,\varphi)$ and $p, q \in Q^0$. Then:

(i) *$\mathfrak{g}$ is a Lie superalgebra if and only if $\alpha + \beta = -1$, and in this case, $\mathfrak{g}$ is a form of the exceptional classical simple Lie superalgebra $D(2,1;\alpha)$. Denote it by $\mathfrak{g}_Q(\alpha)$ ($\alpha \neq 0, -1$).*

(ii) *If $\alpha, \alpha' \in F \setminus \{0,-1\}$, Q, Q' are quaternion algebras and $\mathfrak{g}_Q(\alpha) \cong \mathfrak{g}_{Q'}(\alpha')$, then Q and Q' are isomorphic. Moreover, if Q is a quaternion division algebra, $\mathfrak{g}_Q(\alpha) \cong \mathfrak{g}_Q(\alpha')$ if and only if either $\alpha' = \alpha$ or $\alpha' = -(1+\alpha)$, while if Q is the algebra $\mathrm{Mat}_2(F)$, $\mathfrak{g}_Q(\alpha) \cong \mathfrak{g}_Q(\alpha')$ if and only if $\alpha' \in \left\{\alpha, \frac{1}{\alpha}, -(1+\alpha), \frac{-1}{1+\alpha}, \frac{-\alpha}{1+\alpha}, \frac{-(1+\alpha)}{\alpha}\right\}$.*

Proof. The first part has been proved in [BE, Lemma 3.1]. For (ii), note first that if $\mathfrak{g}_Q(\alpha) \cong \mathfrak{g}_{Q'}(\alpha')$, then the even parts are isomorphic and this forces that the Lie algebras Q^0 and $(Q')^0$ are isomorphic, but this implies that Q and Q' are isomorphic too (the norm n of Q is determined by $n(1) = 1$ and $8n(p)$ is the trace of ad_p^2 for any $p \in Q^0$, so the norm n is determined by the Lie algebra Q^0). Moreover, the map:

$$\begin{aligned} \mathfrak{g}_Q(\alpha) &\to \mathfrak{g}_Q\big(-(1+\alpha)\big) \\ (f,p,q) &\mapsto (f,q,p) \\ u \otimes x &\mapsto u \otimes \bar x \end{aligned}$$

is an isomorphism. Besides, if Q is the algebra $\mathrm{Mat}_2(F)$, as shown in the proof of [BE, Lemma 3.1], $\mathfrak{g}_Q(\alpha)$ is the Lie superalgebra $\Gamma(1,\alpha,-(1+\alpha))$ in the notation of [Sc, §1. Example 5] and the same arguments that work in characteristic zero are valid here.

On the other hand, if Q is a division algebra and $\Phi : \mathfrak{g}_Q(\alpha) \to \mathfrak{g}_Q(\alpha')$ is an isomorphism, since Q^0 is not isomorphic to $sl_2(F)$ and any automorphism of Q^0 extends to an automorphism of Q, which is inner, it follows that either

$$\begin{aligned} \Phi\big((f,p,q)\big) &= \big(\mathrm{Int}(g)(f), \mathrm{Int}(a)(p), \mathrm{Int}(b)(q)\big), \quad \text{or} \\ \Phi\big((f,p,q)\big) &= \big(\mathrm{Int}(g)(f), \mathrm{Int}(b)(q), \mathrm{Int}(a)(p)\big) \end{aligned}$$

for any $f \in sl_2(F)$, $p,q \in Q^0$, where $g \in GL(U)$, a,b are invertible elements in Q and $\mathrm{Int}(g)(f) = gfg^{-1}$ and similarly for $\mathrm{Int}(a), \mathrm{Int}(b)$. Take $h \in GL(U)$ with $\det h = n(a)n(b)^{-1}$. Then the linear map

$$\begin{aligned} \Psi : \mathfrak{g}_Q(\alpha) &\to \mathfrak{g}_Q(\alpha) \\ (f,p,q) &\mapsto \bigl(\mathrm{Int}(h)(f), \mathrm{Int}(a^{-1})(p), \mathrm{Int}(b^{-1})(q)\bigr) \\ u \otimes x &\mapsto h(u) \otimes a^{-1}xb \end{aligned}$$

is an automorphism. Composing with it and changing the value of g, we may assume that either

$$\Phi\bigl((f,p,q)\bigr) = \bigl(\mathrm{Int}(g)(f), p, q\bigr) \qquad \text{or} \qquad \Phi\bigl((f,p,q)\bigr) = \bigl(\mathrm{Int}(g)(f), q, p\bigr).$$

By standard arguments, it follows that there are $h \in \mathrm{End}_F(U)$ and $\rho \in \mathrm{End}_F(Q)$ such that $\Phi(u \otimes x) = h(u) \otimes \rho(x)$ for any $u \in U$ and $x \in Q$, with $gfg^{-1}h(u) = h(fu)$ for any $f \in sp(U,\varphi)$ and $u \in U$. Therefore $g^{-1}h$ commutes with any element in $sp(V,\varphi)$ and hence, since we can scalar g, we may assume that $h = g$.

In case $\Phi\bigl((f,p,q)\bigr) = \bigl(\mathrm{Int}(g)(f), p, q\bigr)$, we must have too $\rho(px - xq) = p\rho(x) - \rho(x)q$ for any $x \in Q$ and $p,q \in Q^0$. This gives $\rho(p) = pz$, $\rho(q) = zq$, with $z = \rho(1)$. It follows that z commutes with all the elements in Q^0, so ρ is the multiplication by a nonzero scalar μ. Then for any $u,v \in U$ and $x,y \in Q$, $[\Phi(u \otimes x), \Phi(v \otimes y)] = \Phi\bigl([u \otimes x, v \otimes y]\bigr)$, and this gives

$$\begin{aligned} \mu^2 \Bigl(t(\bar{x}y)\sigma_{g(u),g(v)}, -\alpha'\varphi(g(u),g(v))(x\bar{y} - y\bar{x}), (1+\alpha')\varphi(g(u),g(v))(\bar{x}y - \bar{y}x) \Bigr) \\ = \Bigl(t(\bar{x}y)g\sigma_{u,v}g^{-1}, -\alpha\varphi(u,v)(x\bar{y} - y\bar{x}), (1+\alpha)\varphi(u,v)(\bar{x}y - \bar{y}x) \Bigr) \end{aligned}$$

for any $x,y \in Q$ and $u,v \in U$. But $\sigma_{g(u),g(v)} = (\det g)g\sigma_{u,v}g^{-1}$ and $\varphi(g(u),g(v)) = (\det g)\varphi(u,v)$, thus giving $\mu^2 \det c = 1$ and $\mu^2(\det c)\alpha' = \alpha$, so that $\alpha' = \alpha$.

On the contrary, if $\Phi\bigl((f,p,q)\bigr) = \bigl(\mathrm{Int}(g)(f), q, p\bigr)$, the same arguments as before give that $\rho(x) = \mu\bar{x}$ for a nonzero scalar $\mu \in F$ and that $-\alpha' = 1 + \alpha$. □

As in the previous sections, this gives that some forms of the Lie superalgebras $D(2,1;\alpha)$ are given by the Tits construction. Here a quaternion instead of an octonion algebra is used and the Jordan superalgebra that appears is the simple Jordan superalgebra D_α in (3.1):

Corollary 4.2. *(see* [BE, Theorem 4.2]*) Let Q be a quaternion algebra and $\alpha \in F \setminus \{0,-1\}$. Then $\mathcal{T}(Q, D_\alpha)$ is a form of the Lie superalgebra $D(2,1;\alpha)$.*

There is another family of forms of some $D(2,1;\alpha)$ whose even part contains an ideal isomorphic to $sl_2(F)$. It corresponds to case Dii). Let $\mathfrak{g}_{\bar{0}} = sl_2(F) \oplus Q^0$, where Q is a quaternion algebra over a quadratic field extension K of F with $N_{K/F}([Q]) = 1$. By the Albert-Riehm-Scharlau Theorem (see [KMRT, (3.1)]), Q admits a K/F-involution of the second kind τ. Then if $\mathrm{Gal}(K/F) = \{1, \iota\}$, Q is a four dimensional K-module for $Q \otimes_K \iota^{-1}Q$ by means of

$$(p \otimes q).x = px\tau(q) \tag{4.1}$$

for any $p, q, x \in Q$. Let $W = \{x \in Q : \tau(x) = x\}$. W is a four dimensional vector space over F which is fixed by the action of the F-subalgebra $N_{K/F}(Q)$ of $Q \otimes_K \iota^{-1}Q$, and hence becomes an irreducible module for $N_{K/F}(Q)$. Note that the Lie F-algebra Q^0 embeds in $N_{K/F}(Q)$ by means of $q \mapsto q \otimes 1 + 1 \otimes \iota^{-1}q$. This makes W a module for the Lie F-algebra Q^0.

Tensoring with K we have the following isomorphisms of Lie algebras and modules (over K):

$$\begin{aligned} K \otimes Q^0 &\to Q^0 \oplus Q^0 & \qquad K \otimes W &\to Q \\ 1 \otimes q &\mapsto (q, -\tau(q)) & 1 \otimes x &\mapsto x \end{aligned} \tag{4.2}$$

where the K-vector space Q is a module for the Lie K-algebra $Q^0 \oplus Q^0$ by means of $(p, q).x = px - xq$ for any $p, q \in Q^0$ and $x \in Q$. Since the dimension of the K-vector space $\mathrm{Hom}_{Q^0 \oplus Q^0}(Q \otimes_K Q, K) = \mathrm{Hom}_{o(Q,n)}(Q \otimes Q, K)$ is 1 and the dimension of $\mathrm{Hom}_{Q^0 \oplus Q^0}(Q \otimes_K Q, Q)$ is 2 (after scalar extension, $Q^0 \oplus Q^0$ becomes $sl_2(\bar{F}) \oplus sl_2(\bar{F})$ and $Q = \bar{U} \otimes_{\bar{F}} \bar{U}$, with $\bar{U}$ the natural two dimensional module for $sl_2(\bar{F})$, and these computations are easy), it follows that

$$\begin{aligned} \mathrm{Hom}_{Q^0}(W \otimes W, F) &= F\text{-span}\langle x \otimes y \mapsto t(\bar{x}y)\rangle, \\ \mathrm{Hom}_{Q^0}(W \otimes W, W) &= K\text{-span}\langle x \otimes y \mapsto x\bar{y} - y\bar{x}\rangle. \end{aligned}$$

Note that for any $x, y \in W$, $t(\bar{x}y) = t(\tau(\bar{x}y))$, so $t(\bar{x}y) \in F$ and, by dimension count, $Q^0 \cong o(W, t(\bar{x}y))$, the orthogonal Lie algebra relative to this bilinear form.

The analogous result to Lemma 4.1 in this case is the following:

Proposition 4.3. *Let K/F be a quadratic field extension, let Q be a quaternion algebra over K with $N_{K/F}([Q]) = 1$, τ a K/F-involution of the second kind on Q and W the set of fixed elements by τ. Let U be a two dimensional space with a nonzero skew-symmetric form $\varphi : U \times U \to F$. For $u, v \in U$, let $\sigma_{u,v} \in sp(U, \varphi) \cong sl_2(F)$ as in Lemma 4.1. For nonzero $\mu \in K$, let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be the superalgebra with*

$$\begin{aligned} \mathfrak{g}_{\bar{0}} &= sp(U, \varphi) \oplus Q^0 \\ \mathfrak{g}_{\bar{1}} &= U \otimes W \end{aligned}$$

and with multiplication given by

- *the usual Lie bracket in $\mathfrak{g}_{\bar{0}}$,*
- $[(f, p), u \otimes x] = f(u) \otimes x + u \otimes (px + x\tau(p))$,
- $[u \otimes x, v \otimes y] = \Big(t(\bar{x}y)\sigma_{u,v}, -\mu\varphi(u, v)(x\bar{y} - y\bar{x})\Big)$,

for $u, v \in U$, $x, y \in W$, $f \in sp(U, \varphi)$ and $p, q \in Q^0$. Then:

(i) *$\mathfrak{g}$ is a Lie superalgebra if and only if $\mu + \iota(\mu) = -1$, and in this case, $\mathfrak{g}$ is a form of the exceptional classical simple Lie superalgebra $D(2, 1; \mu)$. Denote it by $\mathfrak{g}_{Q/K}(\mu)$.*

(ii) *$\mathfrak{g}_{Q/K}(\mu)$ does not depend on the involution τ. If other K/F-involution is used, an isomorphic Lie superalgebra is obtained.*

(iii) *If $0 \neq \mu, \mu' \in K$ satisfy $\mu + \iota(\mu) = -1 = \mu' + \iota(\mu')$, and Q, Q' are quaternion algebras over K with K/F-involutions of second kind, then $\mathfrak{g}_{Q/K}(\mu) \cong \mathfrak{g}_{Q'}(\mu')$ if and only if Q is isomorphic to Q' (as K-algebras) and either $\mu' = \mu$ or $\mu' = \iota(\mu) = -(1 + \mu)$.*

Proof. By extending scalars to K and using (4.2), it follows that $K \otimes \mathfrak{g}$ is the K-superalgebra considered in Lemma 4.1 with $\alpha = \mu$ and $\beta = \iota(\mu)$, whence (i) follows.

Assume that τ' is another involution of second kind of the K-algebra Q, then there is an invertible element $u \in Q$ with $\tau(u) = u$ such that $\tau' = \operatorname{Int}(u)\tau$ ([KMRT, (2.18)] and $W' = uW = Wu^{-1}$, where W' denotes the set of fixed elements by τ'. Hence the map $W \to W'$ given by $x \mapsto xu^{-1}$ is an isomorphism of Q^0-modules since $(px + x\tau(p))u^{-1} = p(xu^{-1}) + (xu^{-1})\tau'(p)$ for any $p \in Q^0$ and $x \in W$. Given $0 \neq \mu \in K$ with $\mu + \iota(\mu) = -1$, denote by $\mathfrak{g}$ the Lie superalgebra constructed with τ and by $\mathfrak{g}'$ the one constructed with τ'. Then the map which is the identity on $\mathfrak{g}_{\bar{0}}$ and takes $u \otimes x \in U \otimes W$ to $u \otimes xu^{-1}$ gives an isomorphism $\mathfrak{g} \cong \mathfrak{g}'_{n(u)}$ (notice that since $\tau(u) = u$, $n(u) = u\bar{u} \in F$). Now Lemma 2.1 shows that $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic. This proves (ii).

If $\mathfrak{g}_{Q/K}(\mu)$ and $\mathfrak{g}_{Q'/K}(\mu')$ are isomorphic, so are the Lie F-algebras Q^0 and $(Q')^0$. The centroid of any of these algebras is K, so there is a semilinear isomorphism between the K-Lie algebras Q^0 and $(Q')^0$ which extends to a semilinear isomorphism between Q and Q'. Thus either Q and Q' are isomorphic as K-algebras, or Q' is isomorphic to $\iota^{-1}Q$. But $Q \otimes_K \iota^{-1}Q \cong \operatorname{End}_K(Q)$ (see (4.1)), so $\iota^{-1}Q \cong Q^{op} \cong Q$ as K-algebras. Finally, it is easy to check that the map

$$\begin{aligned} \mathfrak{g}_{Q/K}(\mu) &\to \mathfrak{g}_{Q/K}(\iota(u)) \\ (f,p) &\mapsto (f, -\tau(p)) \\ u \otimes x &\mapsto u \otimes \bar{x} \end{aligned}$$

is an isomorphism. On the other hand, if $\mathfrak{g}_{Q/K}(\mu) \cong \mathfrak{g}_{Q/K}(\mu')$, then extending scalars we get an isomorphism $D(2,1;\mu) \cong D(2,1;\mu')$, so that $\mu' \in \left\{\mu, \frac{1}{\mu}, -(1+\mu), \frac{-1}{1+\mu}, \frac{-\mu}{1+\mu}, \frac{-(1+\mu)}{\mu}\right\}$. But $\mu + \iota(\mu) = -1 = \mu' + \iota(\mu')$ and hence the only possibilities for μ' are to be either μ or $\iota(\mu)$. □

Remark 4.4. Lemma 2.1 and Proposition 1.1 show that the Lie superalgebras $\mathfrak{g}_Q(\alpha)$ in Lemma 4.1 (Q a quaternion algebra over F, $\alpha \in F \setminus \{0,-1\}$) are, up to isomorphism, the Lie superalgebras which are forms of some $D(2,1;\beta)$ and such that its even part is the direct sum of $sl_2(F)$ and two copies of a three dimensional simple Lie F-algebra; while the Lie superalgebras $\mathfrak{g}_{Q/K}(\mu)$ in Proposition 4.3 (Q a quaternion algebra over a quadratic field extension K of F and $0 \neq \mu \in K$ with $\mu + \iota(\mu) = -1$) are, up to isomorphism the Lie superalgebras which are forms of some $D(2,1;\beta)$ and such that its even part is the direct sum of $sl_2(F)$ and a three dimensional simple Lie algebra over a quadratic field extension of F.

§5. Real forms of the exceptional simple classical Lie superalgebras

Our previous results give, in particular, the classification up to isomorphism, and not just equivalence, of the real forms of the exceptional simple classical Lie superalgebras.

First we need some extra notation, that we take from [K]. The complex Lie algebra G_2 has, up to isomorphism, two real forms, $G_{2;1} = \operatorname{Der} C$, for C the split Cayley-Dickson algebra over $\mathbb{R}$, and $G_{2;2} = \operatorname{Der}\mathbb{O}$, where $\mathbb{O}$ is the classical division algebra of real octonions. Accordingly, consider $G(3;1) = \mathcal{T}(C,\hat{J})$ and $G(3;2) = \mathcal{T}(\mathbb{O},\hat{J})$ (see Corollary 2.2).

Now, B_3 has four nonisomorphic real forms, namely the orthogonal Lie algebras $o(p,7-p)$ ($p=0,1,2,3$) of the quadratic forms $x_1^2+\cdots+x_p^2-(x_{p+1}^2+\cdots+x_7^2)$. For $p=0$, $o(0,7-p)=o(7)=o(\mathbb{O}^0,n)$, while for $p=3$, $o(3,4)=o(C^0,n)$, where C and $\mathbb{O}$ are, as above, the two real Cayley-Dickson algebras and n denotes their norm. In both cases, the quadratic space involved has trivial Clifford invariant. However, the Clifford invariant of the quadratic spaces with signatures $(1,6)$ and (2.5) is nontrivial (hence equal to the class of the classical division algebra $\mathbb{H}$ of real quaternions). In both cases, the quadratic form is universal, so Corollary 3.5 applies. The corresponding forms of $F(4)$ are denoted by $F(4;p)$ ($p=0,1,2,3$) and their respective even parts are $sl_2(\mathbb{R})\oplus o(7)$, $su(2)\oplus o(1,6)$, $su(2)\oplus o(2,5)$ and $sl_2(\mathbb{R})\oplus o(3,4)$, according to Theorem F.

The real Lie superalgebras that appear in Lemma 4.1 have even part either $sl_2(\mathbb{R})\oplus\mathbb{H}^0\oplus\mathbb{H}^0 \cong sl_2(\mathbb{R})\oplus su(2)\oplus su(2)\cong sl_2(\mathbb{R})\oplus o(4)$ or $sl_2(\mathbb{R})\oplus sl_2(\mathbb{R})\oplus sl_2(\mathbb{R})\cong sl_2(\mathbb{R})\oplus o(2,2)$. Denote the corresponding Lie superalgebras $\mathfrak{g}_{\mathbb{H}}(\alpha)$ and $\mathfrak{g}_{\mathrm{Mat}_2(\mathbb{R})}(\alpha)$ by $D(2,1;\alpha;p)$ ($p=0,2$, $\alpha\in F\setminus\{0,-1\}$). On the other hand, the Lie superalgebras that appear in Proposition 4.3 have even part $sl_2(\mathbb{R})\oplus sl_2(\mathbb{C})\cong sl_2(\mathbb{R})\oplus o(1,3)$. Denote the corresponding Lie superalgebras $\mathfrak{g}_{\mathrm{Mat}_2(\mathbb{C})/\mathbb{C}}(\alpha)$ by $D(2,1;\alpha;1)$, where now $\alpha\in\mathbb{C}$ with $\alpha+\bar\alpha=-1$.

Theorem 5.1.

a) *$G(3)$ has, up to isomorphism, two real forms: $G(2;p)$, $p=1,2$.*
b) *$F(4)$ has, up to isomorphism, four real forms: $F(4;p)$, $p=0,1,2,3$.*
c) *If $\alpha\in\mathbb{C}\setminus\Bigl(\mathbb{R}\cup\{z\in\mathbb{C}:|z|=1\}\cup\{z\in\mathbb{C}:|z+1|=1\}\cup\{z\in\mathbb{C}:z+\bar z=-1\}\Bigr)$, then $D(2,1;\alpha)$ has no real form.*
d) *If $\alpha\in\mathbb{R}\setminus\{0,-1,1,-2,-\frac12\}$, then $D(2,1;\alpha)$ has four nonisomorphic real forms: $D(2,1;\alpha;2)$, $D(2,1;\alpha;0)$, $D(2,1;\frac{1}{\alpha},0)$ and $D(2,1;\frac{-\alpha}{1+\alpha};0)$.*
e) *If $\alpha=1,-2$ or $-\frac12$, then $D(2,1;\alpha)=osp(4,2)$ has four nonisomorphic real forms: $D(2,1;1;2)$, $D(2,1;1;0)$, $D(2,1;-\frac12;0)$ and $D(2,1;-\frac12;1)$.*
f) *If $\alpha\in\{z\in\mathbb{C}:|z|=1\}\cup\{z\in\mathbb{C}:|z+1|=1\}\cup\{z\in\mathbb{C}:z+\bar z=-1\}$, then $D(2,1;\alpha)$ has exactly, up to isomorphism, a real form, namely, $D(2,1;\alpha;1)$ if $\alpha+\bar\alpha=-1$, $D(2,1;\frac{-\alpha}{1+\alpha};1)$ if $|\alpha|=1$ and $D(2,1;\frac{1+\alpha}{-\alpha};1)$ if $|\alpha+1|=1$.*

Proof. For $G(3)$ and $F(4)$ it is clear from the previous results. Now, there are no cubic field extensions of $\mathbb{R}$ and the only quadratic field extension is given by $\mathbb{C}$, and the quaternion algebras Q over $L=\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ or $L=\mathbb{R}\times\mathbb{C}$ with trivial $N_{L/\mathbb{R}}([Q])$ involved are, up to isomorphism, $\mathrm{Mat}_2(\mathbb{R})\times\mathrm{Mat}_2(\mathbb{R})\times\mathrm{Mat}_2(\mathbb{R})$, $\mathrm{Mat}_2(\mathbb{R})\times\mathbb{H}\times\mathbb{H}$ and $\mathrm{Mat}_2(\mathbb{R})\times\mathrm{Mat}_2(\mathbb{C})$. Therefore the even parts are restricted to $sl_2(\mathbb{R})\oplus sl_2(\mathbb{R})\oplus sl_2(\mathbb{R})$, $sl_2(\mathbb{R})\oplus su(2)\oplus su(2)$ and $sl_2(\mathbb{R})\oplus sl_2(\mathbb{C})$. In the first two cases, the Lie superalgebras are described in Lemma 4.1 (or Corollary 4.2) and in the third case in Proposition 4.3, including the necessary and sufficient conditions for isomorphisms. Now, one has to take into account simply that for $\alpha,\beta\in\mathbb{C}\setminus\{0,-1\}$, $D(2,1;\alpha)\cong D(2,1;\beta)$ if and only if $\beta\in\left\{\alpha,\dfrac{1}{\alpha},-(1+\alpha),\dfrac{-1}{1+\alpha},\dfrac{-\alpha}{1+\alpha},\dfrac{1+\alpha}{-\alpha}\right\}$. (Note that if

$|\alpha| = 1$ and $\beta = -(1+\alpha)$ and $\gamma = \frac{-1}{1+\alpha}$, then $|\beta + 1| = 1$ and $\gamma + \bar{\gamma} = -1$.) $\square$

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