The manifold of finite rank projections in the algebra $\mathcal{L}(H)$ of bounded linear operators

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Abstract. Given a complex Hilbert space H, we study the differential geometry of the manifold $\mathcal{M}$ of all projections in $V = \mathcal{L}(H)$. Using the algebraic structure of V, a torsionfree affine connection ∇ (that is invariant under the group of automorphisms of V) is defined on every connected component $\mathfrak{M}$ of $\mathcal{M}$, which in this way becomes a symmetric holomorphic manifold that consists of projections of the same rank r, $(0 \leq r \leq \infty)$. We prove that $\mathfrak{M}$ admits a Riemann structure if and only if $\mathfrak{M}$ consists of projections that have the same finite rank r or the same finite corank, and in that case ∇ is the Levi-Civita and the Kähler connection of $\mathfrak{M}$. Moreover, $\mathfrak{M}$ turns out to be a totally geodesic Riemann manifold whose geodesics and Riemann distance are computed.

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1. Preliminaries on JB-algebras

1.1. *Introduction.*

In [4] Hirzebruch proved that the manifold of minimal projections in a finite-dimensional simple formally real Jordan algebra is a compact Riemann symmetric space of rank 1, and that any such space arises in this way. Later on, in [14] Nomura established similar results for the manifold of minimal projections in a topologically simple JH-algebra (a real Jordan-Hilbert algebra). The results in [1], [5] and [6] lead to the idea that the structure of a JBW-algebra V might encode information about the differential geometry of some manifolds naturally associated to it [11]. In particular, that the knowledge of the JBW-structure of V is sufficient to study the manifold of projections in V. Every JBW-algebra can be decomposed into a sum of closed ideals $V = V_I \oplus V_{II} \oplus V_{III}$ of types I, II, and III respectively, and for our purpose it is not a hard restriction to assume that V is irreducible. JBW-algebras of type III are not well understood. A typical example of

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a type II JBW-algebra is $L^\infty[0,1]$ whose lattice of projections is modular. Namely, it consists of characteristic functions of Lebesgue measurable subsets of $[0,1]$ and form a discrete topological space since we have $\|\chi_1 - \chi_2\| = 1$ whenever $\chi_1 \neq \chi_2$. Thus we have to consider JBW-algebras of type I and we shall assume them to be factors, hence factors of type I_n for some cardinal number $1 \leq n \leq \infty$. Those of type I_2, called spin factors, are Hilbert spaces ([2] th. 6.1.8) hence they are included in the work of Nomura. Factors of type I_n with $3 \leq n < \infty$ are certain spaces of matrices ([2] th. 5.3.8) and so they are included in the work of Hirzebruch. Thus essentially we have to consider the JBW-algebra $V := \mathcal{L}(H)_{\rm sa}$ of the selfadjoint operators on a Hilbert space H over some of fields $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ ([2] th. 7.5.11). Here we make such a study in a systematic manner without the use of any global scalar product in V, in the complex case. With minor changes it applies to the other two fields.

The set $\mathcal{M}$ of all projections in V can be identified with the set of all closed subspaces of H, which is a Grassmann manifold $\mathbb{G}(H)$ in a classical way [9]. It is known that $\mathbb{G}(H)$ has several connected components $\mathfrak{M}$, each of which consists of projections p in V that have a fixed rank r, $0 \leq r \leq \infty$. An affine connection ∇, that is invariant under the group $\mathsf{Aut}^\circ(V)$ of automorphisms of V, is then defined on each connected component $\mathfrak{M}$ with only the help of the JBW-structure. With it, $\mathfrak{M}$ becomes a symmetric totally geodesic real analytic manifold. Moreover, it is possible (and in fact easy) to integrate the differential equation of the geodesics corresponding to initial conditions defined by purely algebraic equations. For $r = 1$, $\mathfrak{M}$ is the complex projective space $\mathbb{P}(H)$.

Motivated by the above, we ask whether it is possible to define a Riemann structure on $\mathfrak{M}$, and a necessary and sufficient condition for this to happen is established. The tangent space to $\mathfrak{M}$ at a point p is the range of the $\frac{1}{2}$-Peirce projector of V at p, and $\mathfrak{M}$ admits a Riemann structure (if and) only if $P_{1/2}(p)V$ is (homeomorphic to) a Hilbert space. In [6] it has been proved that the latter occurs if and only if the rank of p or the corank of p (the rank of $\mathbf{1} - p$, where $\mathbf{1}$ is the unit of the algebra V) is finite. $\mathcal{M}$ is an ortho-complemented lattice, and the mapping $p \mapsto p^\perp := \mathbf{1} - p$ is an involutory homeomorphism. In fact, this involution is a real analytic diffeomorphism of $\mathcal{M}$, hence it suffices to study the connected manifolds $\mathfrak{M}$ with $r < \infty$ which leads us again to the work of Nomura. Namely, consider the algebras of finite rank operators, of Hilbert-Schmidt operators, of compact operators, and of bounded operators on H, respectively, and the inclusions

$$\mathcal{F}(H) \subset \mathcal{L}_2(H) \subset \mathcal{L}(H)_0 \subset \mathcal{L}(H)$$

The first three of these algebras have the same set of projections, which is exactly the set of *finite rank* projections in $\mathcal{L}(H)$. However, the topologies induced on $\mathcal{F}(H)$ by $\mathcal{L}_2(H)$ and $\mathcal{L}(H)$ do not coincide (unless $\dim H < \infty$), and a priori there is no reason to expect that they should coincide on the set of projections (we shall see that this happens).

We then study the Riemann manifolds $\mathfrak{M}$ for $r < \infty$ without the help of any global scalar product in these algebras. A scalar product in the tangent bundle to $\mathfrak{M}$ is needed, of course, but it is locally provided in a canonical way by the JBW-algebra structure of V. We begin with a discussion of the subalgebras $V[a,b]$ generated in V by certain pairs of elements $(a,\, b)$. These subalgebras, that play a fundamental role in our study, turn out to be Jordan isomorphic to $\mathsf{Sym}\,(\mathbb{R},\, 2)$, the algebra of 2×2 symmetric matrices with real entries and the usual Jordan matrix product, and therefore they are finite dimensional. By choosing an appropriate basis in $V[a,b]$ it is easy to integrate the differential equations of the geodesics in $\mathfrak{M}$.

1.2. *Preliminaries on JBW-algebras.*

A Jordan algebra V is an algebra over $\mathbb{R}$ or $\mathbb{C}$ in which the following two identities hold for all $x,\ y$ in V:

(1) $$xy = yx, \qquad x^2(xy) = x(x^2y)$$

Let V be a Jordan algebra. Then $L(x)$ and $P(x)$, $(x \in V)$, are defined by

(2) $$L(x)y\colon = xy, \qquad P(x)y\colon = 2L(x)^2y - L(x^2)y, \qquad (y \in V).$$

An element $a \in V$ is an *idempotent* if $a^2 = a$. If V has a unit $\mathbf{1}$, then every idempotent $a \in V$ gives rise to a vector space direct sum decomposition of V, the *Peirce decomposition*:

$$V = V_1(a) \oplus V_{1/2}(a) \oplus V_0(a), \qquad V_k(a)\colon = \{x \in V : ax = kx\},$$

where $k \in \{1,\ \frac{1}{2},\ 0\}$ and the corresponding projectors $E_k(a)\colon V \to V_k(a)$, called *Peirce projectors*, are given by

(3) $$E_1(a) = P(a), \quad E_{1/2}(a) = 2L(a) - 2P(a), \quad E_0(a) = I - 2L(a) + P(a).$$

If the idempotent a is fixed and no confusion is likely to occur, we write $V_k\colon = V_k(a)$ and $P_k\colon = E_k(a)$ for $k \in \{1, \frac{1}{2}, 0\}$. The Peirce *multiplication rules* hold

(4) $$\begin{aligned} &V_0V_0 \subset V_0, \qquad V_0V_1 = \{0\}, \qquad V_1V_1 \subset V_1,\\ &(V_0 \oplus V_1)V_{1/2} \subset V_{1/2}, \qquad V_{1/2}V_{1/2} \subset V_0 + V_1. \end{aligned}$$

In particular, V_0 and V_1 are Jordan subalgebras of V, and $[L(x)\, L(y)] = 0$ for $x \in V_0$ and $y \in V_1$ with the usual commutator product $[\ ,\ ]$.

A JB-algebra is a *real* Jordan algebra with a complete norm such that the following conditions hold

$$\|xy\| \le \|x\|\,\|y\|, \qquad \|x^2\| = \|x\|^2, \qquad \|x^2\| \le \|x^2 + y^2\|.$$

A JB*-algebra is a *complex* Jordan algebra U with an algebra involution $^*\colon x \mapsto x^*$ and a complete norm such that the following conditions hold

$$\|xy\| \le \|x\|\,\|y\|, \qquad \|x^*\| = \|x\|, \qquad \|\{xxx\}\| = \|x\|^3,$$

where the *triple product* $\{abc\}$ is defined by $\{abc\}\colon = (ab^*)c - (ca)b^* + (b^*c)a$ and satisfies the *Jordan identity*

(5) $$\{x\{abc\}y\} = \{\{bax\}cy\} - \{ba\{xcy\}\} + \{\{bay\}cx\}.$$

The operators $x\Box y \in \mathcal{L}(U)$ are defined by $z \mapsto x\Box y(z)\colon = \{xyz\}$ for $z \in U$.

An element $a \in U$ is *selfadjoint* if $a^* = a$. The the set of them, denoted by U_s, is a JB-algebra. Conversely, if V is a JB-algebra then there is a unique Jordan algebra structure in $U\colon = V \oplus \imath V$, the complexification of V, such that $U_s = V$ and there is a unique norm in U that converts it into a JB*-algebra [18]. We refer to U as the *hermitification* of V. The set $\mathsf{Aut}\,(U)$, of all Jordan algebra *-automorphisms of U, consists of surjective linear isometries of U and is a topological group in the topology of uniform convergence over the unit ball of U. By $\mathsf{Aut}^\circ(U)$ we denote the connected component of the identity in $\mathsf{Aut}\,(U)$. Every element in $\mathsf{Aut}^\circ(U)$ preserves the real subspace V and is uniquely determined by this restriction.

Let U be a JB*-algebra. We write $\mathsf{Pro}\,(U)$ for the set of self-adjoint idempotents in U and $\mathsf{Tri}\,(U)\colon = \{a \in U\colon \{a,a,a\} = a\}$ for the set of *tripotents* in U. Clearly $\mathsf{Pro}\,(U) \subset$

$\mathsf{Tri}\,(U)$, and every non zero $a \in \mathsf{Tri}\,(U)$ satisfies $\|a\| = 1$. Two elements $a, b \in U$ are *orthogonal* if $ab = 0$. A projection $a \in U$ is said to be *minimal* if $a \neq 0$ and $P_1(a)U = \mathbb{C}\,a$, and we let $\mathsf{Min}\,(U)$ denote the set of them. For a JB*-algebra it may occur that $\mathsf{Min}\,(U) = \emptyset$.

A JBW-algebra is a JB-algebra whose underlying Banach space V is a dual space, which occurs if and only if the hermitification $U = V \oplus \imath V$ is a dual Banach space. In that case U is called a JBW*-algebra, the predual U_* of U is uniquely determined and $\sigma(U, U_*)$, the weak* topology on U, is well defined. Let U be the JBW*-algebra $U\colon = \mathcal{L}(H)$ of bounded linear operators $z\colon H \to H$ on a Hilbert space H. Then U is a unital algebra with plenty of projections each of which admits a representation of the form

$$a = \Sigma_{i\in I} a_i, \qquad \text{convergence in the weak* topology,}$$

for some indexed family of pairwise orthogonal minimal projections. The minimal cardinal of the set I is the *rank* of a and $\mathsf{rank}\,(a) = 1$ if and only if a is minimal. The rank of the algebra U is the rank of its unit element.

A JB-algebra V is algebraically (resp. topologically) *simple* if $\{0\}$ and V are its only ideals (resp. closed ideals).

Although not surveyed here, we shall occasionally use some relationships between JB*-algebras and their associated JB*-triples. Our main reference for JBW-algebras, JBW*-algebras and JB*-triples are [2] and [17].

1.3. *Manifolds of projections in a JBW-algebra.*

Let V be a JBW-algebra and denote by $U\colon = V \oplus \imath V$ its hermitification. Then U is a JBW*-algebra and $\mathsf{Pro}\,(U) = \mathsf{Pro}\,(V)$. In the Peirce decomposition of U induced by $a \in \mathsf{Pro}\,(U)$, the Peirce spaces are selfadjoint, that is $U_k(a)^* = U_k(a)$, and we have $V_k(a) = U_k(a)_s$ where $P_k(a)_{|V}$ is the Peirce projector of a in the algebra V. For every $u \in V_{1/2}(a)$, the operator

$$G(a, u)\colon = 2(u\Box a - a\Box u) \in \mathcal{L}(U) \tag{6}$$

is an inner derivation of the JBW*-algebra U and the operator-valued mapping $t \mapsto \exp tG(a, u)$, $(t \in \mathbb{R})$, is a one-parameter group of automorphisms of U each of which preserves V. The set $\mathsf{Pro}\,(V)$, endowed with its topology as a subset of V, is not connected, namely 0 and $\mathbf{1}$ are isolated points. We let $\mathcal{M}(p)$ denote the connected component of p in $\mathsf{Pro}\,(V)$. It is known that $\mathcal{M}(p)$ is a real analytic manifold whose tangent space at the point a is $V_{1/2}(a)$, a local chart being given by

$$z \mapsto [\exp G(a, z)]\, a, \qquad z \in N,$$

for z in a suitable neighbourhood N of 0 in $V_{1/2}(a)$. As a consequence, all points in $\mathcal{M}(p)$ are projections that have the same rank as p. As proved in ([11] th. 4.4), $\mathcal{M}(p)$ can also be viewed as a holomorphic manifold whose tangent space at the point a is $U_{1/2}(a)$ a local chart being given in a neighbourhood M of $0 \in U_{1/2}(a)$ by

$$u \mapsto [\exp 2u\Box a]\, a, \qquad u \in M.$$

For a projection p, the operator $S(p)\colon = \mathsf{Id} - 2P_{1/2}(p) \in \mathcal{L}(V)$, called the Peirce *reflection*, is a *symmetry*, that is, a selfadjoint involution of V. Namely we have $S(p)x = x$ for $x \in V_1(p) + V_0(p)$ and $S(p)x = -x$ for $x \in V_{1/2}(p)$. Besides $S(p) \in \mathsf{Aut}\,(V)$ and

$S(p)p = p$, hence $S(p)$ preserves the connected component $\mathcal{M}(p)$ which in this way is a symmetric manifold.

We let $\mathfrak{D}$ be the Lie algebra of all smooth vector fields on $\mathcal{M}(p)$. A vector field X is now locally identifiable with a real analytic function $X\colon N \subset V_{1/2}(a) \to V_{1/2}(a)$. We always consider $V_{1/2}(a)$ as submerged in V. For a function $X\colon \mathcal{M}(p) \to V$, we let X_a denote the value of X at the point $a \in \mathcal{M}(p)$. By X'_a we represent the Fréchet derivative of X at a, thus X'_a is a continuous linear operator $V_{1/2}(a) \to V$. For two vector fields $X, Y \in \mathfrak{D}$ we define

$$(\nabla_X Y)_a \colon = P_{1/2}(a)\big(Y'_a(X_a)\big), \qquad a \in \mathcal{M}(p). \tag{7}$$

It is known that $(X, Y) \mapsto \nabla_X Y$ is a torsionfree $\mathsf{Aut}^\circ(U)$-invariant affine connection on the manifold $\mathcal{M}(p)$. For every $a \in \mathcal{M}(p)$ and every $u \in V_{1/2}(a)$ the curve $\gamma_{a,u}(t)\colon = [\exp\, tG(a,u)]\, a$ is a ∇-geodesic that is contained in the closed real Jordan subalgebra of V generated by (a, u). Proofs can be found in [1].

The following key result is known.

Theorem 1.1. *Let U be a JBW*-algebra. Then for $p \in \mathsf{Pro}\,(U)$ the following conditions are equivalent:* (i) *$U_{1/2}(p)$ is a reflexive space.* (ii) *$U_{1/2}(p)$ is linearly homeomorphic to a Hilbert space.* (iii) *$\mathsf{rank}\, U_{1/2}(p) < \infty$. For $U\colon = \mathcal{L}(H)$ these conditions are equivalent to* (iv) *$\mathsf{rank}\,(p) < \infty$ or $\mathsf{rank}\,(\mathbf{1} - p) < \infty$.*

PROOF.

The equivalence (i) $\iff$ (ii) $\iff$ (iii) is known [9]. The statement concerning $U = \mathcal{L}(H)$ has been established in [6] as follows: From the expression of the Peirce projectors, we have for all $x \in U$

$$\begin{aligned} P_{1/2}(p)x &= 2(p\Box p - P(p))x = \\ (px + xp) - 2pxp &= px(\mathbf{1} - p) + (\mathbf{1} - p)xp. \end{aligned}$$

Hence $p\,U_{1/2}(p) \subset U_{1/2}(p)$ and $x \mapsto px$ is a continuous projector $U_{1/2}(p) \to p\,U_{1/2}(p)$. Similarly $x \mapsto xp$ is a continuous projector $U_{1/2}(p) \to U_{1/2}(p)\,p$ and since $pU_{1/2}(p) \cap U_{1/2}p = 0$ we have a topological direct sum decomposition $U_{1/2}(p) = X_1 \oplus X_2$. Therefore $U_{1/2}(p)$ is reflexive if and only if so are the summands. But $X_1\colon = p\,U_{1/2}(p) = \{px(\mathbf{1} - p)\colon x \in U\}$ is reflexive if and only if $\mathsf{rank}\,(p) < \infty$ or $\mathsf{rank}\,(\mathbf{1} - p) < \infty$ as we wanted to see. □

Suppose $p \in \mathsf{Pro}\,(U)$ is such that $\mathsf{rank}\,(p) < \infty$. Since this condition is $\mathsf{Aut}^\circ(U)$-invariant, all projections $a \in \mathcal{M}(p)$ satisfy it too, and the tangent space $U_{1/2}(a)_s = V_{1/2}(a)$ to $\mathcal{M}(p)$ at any point $a \in \mathcal{M}(p)$ is a Hilbert space. In ([1], prop. 9.12) one can find an explicit expression for an $\mathsf{Aut}^\circ(U)$-invariant scalar product whose norm is equivalent to $\|\cdot\|$ in $U_{1/2}(a)$. Since $\mathcal{M}(p)$ is connected, that scalar product, denoted by $\langle\cdot,\cdot\rangle_a$, is determined up to a positive constant coefficient that can be normalized by requiring that minimal tripotents should have norm one. We shall not go into details as no explicit expression of it will be used here.

Definition 1.2. We refer to this Hilbert space norm as the *Levi norm* in $U_{1/2}(a)$ and denote it by $|\cdot|_a$

Definition 1.3. We define a Riemannian metric g on $\mathcal{M}(p)$ by

$$g(X, Y)_a\colon = \langle X_a, Y_a\rangle_a, \quad X, Y \in \mathfrak{D}\mathcal{M}(p), \quad a \in \mathcal{M}(p).$$

Proposition 1.4. *The affine connection in* (7) *is the Levi-Civita (respectively, the Kähler) connection on the real analytic (the holomorphic) manifold* $\mathcal{M}(p)$.

PROOF.

Indeed, ∇ is compatible with g, that is

$$X\, g(Y, Z) = g(\nabla_X Y,\, Z) + g(Y,\, \nabla_X Z), \qquad X, Y, Z \in \mathfrak{D}\mathcal{M}(p).$$

Moreover ∇ is torsionfree, hence by ([12] th.1.8.11) ∇ is the unique Riemann connection on $\mathcal{M}(p)$. Remark that when $\mathcal{M}(p)$ is looked as a holomorphic manifold ∇ is hermitian, that is, it satisfies $g(i\,Y,\, i\,Z) = g(Y,\, Z)$, therefore ∇ is the only Levi-Civita connection on $\mathcal{M}(p)$. On the other hand $\nabla_X\, i\,Y = i\,\nabla_X\, Y$, hence ∇ is the only hermitian connection on $\mathcal{M}(p)$. Thus the Levi-Civita and the hermitian connection are the same in this case and so ∇ is the Kähler connection on $\mathcal{M}(p)$.

Theorem 1.5. *The Jordan-Banach algebra* $U_0 \colon = \mathcal{L}_0(H)$ *of compact operators and* $U_2 \colon = \mathcal{L}_2(H)$*, the Jordan-Hilbert algebra of Hilbert-Schmidt operators on* H*, have the same set of projections and induce on it the same topology.*

PROOF.

The first assertion is clear, and $\mathsf{Pro}\,(U_0) = \mathsf{Pro}\,(U_2)$ is precisely the set of finite rank projections in $\mathcal{L}(H)$. It is also clear the above algebras have the same set of projections of a given rank r, $(1 \le r < \infty)$, say $\mathfrak{P}$. Denote by $\mathfrak{P}_0$ and $\mathfrak{P}_2$ the Banach manifold structures defined on $\mathfrak{P}$ according to our method and to Nomuras' method, respectively. The corresponding tangent spaces at a are

$$T_a\mathfrak{P}_0 = \{x \in U_0 \colon 2ax = x\}, \qquad T_a\mathfrak{P}_2 = \{y \in U_2 \colon 2ay = y\}.$$

Thus we have $T_a\mathfrak{P}_2 \subset T_a\mathfrak{P}_0$. On the other hand, a itself is a Hilbert-Schmidt operator and as U_2 is an operator ideal, from $x \in U_0$ and $x = 2ax$ we get $x \in U_2$, hence $T_a\mathfrak{P}_2 = T_a\mathfrak{P}_0$ as vector spaces. But both $T_a\mathfrak{P}_2$ and $T_a\mathfrak{P}_0$ are JB*-triples (subtriples of U_2 and U_0, respectively) with the same triple product. It is known that if a Banach space X admits a JB*-triple structure, then the triple product determines the topology of X in a unique way. This in our case implies that the topologies induced by $U_2 = \mathcal{L}(H)$ and $U_0 = \mathcal{L}_0(H)$ on the tangent space $T_a\mathfrak{P}$ coincide. But then also coincide the topologies induced on $\mathfrak{P}$ by these two algebras as they are locally homeomorphic to the same Banach space. $\square$

2. Equations of the geodesics.

For any Jordan algebra V, we let $V[u, v]$ denote the subalgebra of V generated by (u, v). By $\mathbb{S} := \mathsf{Sym}\,(\mathbb{R}, 2)$ we denote the Jordan algebra of the symmetric 2×2 matrices with real entries and the usual Jordan matrix product. Recall that in $\mathbb{S}$ the set $\mathsf{Pro}\,(\mathbb{S})$ consists of the isolated points 0, $\mathbf{1}$, and the one-parameter family of minimal projections

$$B(\theta) := \begin{pmatrix} \cos^2\theta & \frac{1}{2}\sin 2\theta \\ \frac{1}{2}\sin 2\theta & \sin^2\theta \end{pmatrix}, \qquad \theta \in \mathbb{R}. \tag{8}$$

The elements $A := B(0)$ and $C := B(\frac{\pi}{2})$ satisfy $A \circ C = 0$ and $A + C = \mathbf{1}$, where $\mathbf{1}$ is the unit of the algebra $\mathbb{S}$. The element

$$X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathbb{S}_{1/2}(A) \cap \mathbb{S}_{1/2}(C) \tag{9}$$

is a non zero tripotent and $\{A,\, X,\, C\}$ is a basis in $\mathsf{Sym}\,(\mathbb{R},\, 2)$.

Theorem 2.1. *Let V be a unital Jordan algebra and let $a \neq 0$ and u denote, respectively, a projection in V and a tripotent in $V_{1/2}(a)$ such that $au^2 = a$. Then for $V[a,u]$ the following conditions hold:* (i) $V[a,u] = \textit{Span}\,\{a,\, u,\, u^2\}$. (ii) $c := u^2 - a$ *is an idempotent such that* $ac = 0$. (iii) $u^2 = a + c$ *is the unit in* $V[a,u]$. (iv) *There is a unique Jordan isomorphism* $\psi_{a,u}\colon V[a,u] \to \textit{Sym}\,(\mathbb{R},\, 2)$ *that takes a, u and c to A, X and C respectively.*

PROOF.

From $u \in V_{1/2}(a)$ we get $au = \frac{1}{2}u$ and by assumption $au^2 = a$. Define $c := u^2 - a$. Then the above results and the fact that u is a tripotent give

$$c^2 = (u^2 - a)^2 = u^2 - 2au^2 + a^2 = u^2 - a = c.$$

Therefore c is an idempotent and $ac = a(u^2 - a) = au^2 - a = 0$. Moreover

$$cu^2 = (u^2 - a)u^2 = u^2 - u^2 a = u^2 - a = c$$

All this is collected in the following table

$\circ$	a	u	u^2	c
a	a	$\frac{1}{2}u$	a	0
u	$\cdot$	u^2	u	$\frac{1}{2}u$
u^2	$\cdot$	$\cdot$	u^2	c
c	$\cdot$	$\cdot$	$\cdot$	c

which proves that the linear span of the set $\{a,\, u,\, u^2,\, c\}$ is closed under the operation of taking Jordan products, and that $u^2 = a + c$ is the unit of $V[a,u]$ Now it is clear that there is a unique Jordan isomorphism $\psi_{a,u}\colon V[a,u] \to \mathsf{Sym}\,(\mathbb{R},\, 2)$ with the desired conditions. □

Remark 2.2. For a tripotent $u \in V_{1/2}(a)$, the minimality of a is a sufficient (but not necessary) condition for $\{auu\} = au^2 = a$ to be true.

Indeed, by the Peirce rules $u^2 \in V_0(a) \oplus V_1(a)$ and $V_0(a)\,V_1(a) = \{0\}$, hence $au^2 \in V_1(a) = \mathbb{R}a$ by the minimality of a, therefore $au^2 = \rho a$. Multiplication by u, the fundamental identities (1) and $u^3 = u$ yield

$$u(au^2) = \rho ua = \frac{\rho}{2}u \qquad u(au^2) = (ua)u^2 = \frac{1}{2}u^3 = \frac{1}{2}u,$$

hence $\frac{1}{2}u(\rho - 1) = 0$ and so $\rho = 1$ since $u \neq 0$. Thus $\{auu\} = au^2 = a$. □

For pairs (a,u) consisting of a projection $a \neq 0$ and a tripotent $u \in V_{1/2}(a)$ with $au^2 = a$ it is quite easy to obtain the equation of the geodesic $\gamma_{a,u}(t)$ as shown now. Let us define a new product in V via $x \centerdot y := \{xay\}$. Then $(V,\, \centerdot)$ is a unital Jordan algebra with unit a. For $n \in \mathbb{N}$ and $x \in V$ we let $x^{(n)}$ denote the n-th power of x in $(V,\, \centerdot)$. Note that $x^n \neq x^{(n)}$.

Theorem 2.3. *Let $V := \mathcal{L}(H)$, and let $a \neq 0$ and u respectively be a projection in V and a tripotent in $V_{1/2}(a)$ such that $au^2 = a$. If $\gamma_{a,u}$ is the geodesic with $\gamma(0) = a$ and $\dot\gamma(0) = u$, then $\gamma_{a,u}(\mathbb{R}) \subset V[a,u]$. More precisely we have*

$$\gamma_{a,u}(t) = (\cos^2 t)\,a + (\frac{1}{2}\sin 2t)\,u + (\sin^2 t)\,u^{(2)}, \qquad t \in \mathbb{R}. \tag{10}$$

PROOF.

Let $G(a,u) := 2(u\square a - a\square u) \in \mathcal{L}(V)$. We have $\gamma_{a,u}(t) = [\exp tG(a,u)\,]\,a$ for all $t \in \mathbb{R}$, hence $\gamma_{a,u}(\mathbb{R})$ is contained in the closed real linear span of the sequence

$(G(a,u)^n\, a)_{n\in\mathbb{N}}$. We shall prove that the assumptions $u^3 = u$ and $au^2 = a$ yield $G(a,u)^n\, a \in V[a,u]$ for all $n \in \mathbb{N}$.

We have

$$G(a,u)\, a = u \in V[a,u].$$

By assumption $\{auu\} = au^2 = a$, hence

$$\begin{aligned} G(a,u)\, u &= 2\{uau\} - 2\{auu\} = 2(u^{(2)} - a) \in V[a,u] \\ G(a,u)\, u^{(2)} &= 2\{u\, a\, u^{(2)}\} - 2\{a\, u\, u^{(2)}\}. \end{aligned}$$

By the Peirce multiplication rules $u^{(3)} = \{uau^{(2)}\} \in \{V_{1/2}(a)\ V_1(a)\ V_0(a)\} = 0$. The Jordan identity (5) and (2.2) give

$$\begin{aligned} &\{auu^{(2)}\} = \{au\{uau\}\} = \{u\{auu\}a\} - \{\{uaa\}uu\} + \{ua\{uua\}\} = \\ &\{uaa\} - \frac{1}{2}\{uuu\} + \{uaa\} = \frac{1}{2}u \in V[a,u] \end{aligned}$$

and $G(a,u)\, u^{(2)} \in V[a,u]$.

Note that $a,\ u,\ u^{(2)}$ belong to different Peirce a-spaces, in particular they are linearly independent unless u or $u^{(2)}$ vanish. We have assumed $u \neq 0$ and if ψ is the isomorphism in (2.1), then $\psi^{-1}u^{(2)} = \psi^{-1}\{uau\} = \{XAX\} = C \neq 0$. Therefore they form a basis of $V[a,u]$ and $G(a,u)\, V[a,u] \subset V[a,u]$. As a consequence $\gamma_{a,u}(t)$ has a unique expression of the form

$$\gamma_{a,u}(t) = f_1(t)a + f_{1/2}(t)u + f_0(t)u^{(2)}, \qquad t \in \mathbb{R},$$

for suitable real analytic scalar valued functions $f_k(t)$, $(k = 0,\ 1/2,\ 1)$. By taking the derivative with respect to t and replacing the expressions previously obtained for $G(a,u)\, z$ with $z \in \{a,\ u,\ u^{(2)}\}$, we get

$$\begin{aligned} &f_1'(t)a + f_{1/2}'(t)u + f_0'(t)u^{(2)} = \dot\gamma_{a,u}(t) = G(a,u)(\gamma_{a,u}(t)) = \\ &f_1(t)G(a,u)a + f_{1/2}(t)G(a,u)u + f_0(t)G(a,u)u^{(2)} = \\ &-2f_{1/2}(t)a + (f_1(t) - f_0(t))u + 2f_{1/2}(t)u^{(2)}, \end{aligned}$$

whence we have the first order ordinary differential equation $F'(t) = A(u)F(t)$ with the initial condition $F(0) = (1,\ 0,\ 0)$, where $A(u)$ is a 3×3 constant (that is, not depending on t) matrix and $F(t)$ is the transpose of $(f_1(t),\ f_{1/2}(t),\ f_0(t))$. In fact

$$A := \begin{pmatrix} 0 & -2 & 0 \\ 1 & 0 & -1 \\ 0 & 2 & 0 \end{pmatrix}.$$

One can easily check that the solution is the curve in (10). □

Motivated by this result, we shall now try to weaken the restrictions on u.

Proposition 2.4. *Let V, a and u respectively be a unital JB-algebra, a projection in V and a tripotent in $V_{1/2}(a)$. Then the following conditions hold:* (i) *$p := au^2$ is a projection that satisfies $p \leq a$, $u \in V_{1/2}(p)$ and $pu^2 = p$.* (ii) *For $u \neq 0$ we have $p \neq 0$.* (iii) *If u and v are orthogonal as tripotents in $V_{1/2}(a)$, then $p := au^2$ and $q := av^2$ are orthogonal as projections in V and $pv = uq = 0$.*

PROOF.

To see that p is an idempotent it suffices to consider the subalgebra of V generated by a, u and the unit e, which is a special algebra. By the Peirce rules, $u^2 \in V_1 \oplus V_0$ and so we may write $u^2 = x + y$ for $x \in V_1$ and $y \in V_0$. Thus $p = ax + ay = x \in V_1$ and $pa = p$. On the other hand, since u is a tripotent $u^2 = u^4$ and so using the Peirce rules again

$$p = au^4 = ax^2 + 2a(xy) + ay^2 = x^2 = p^2.$$

That is, p is a projection. Notice that $(a-p)^2 = a-p$ and so $a-p \geq 0$. From $u \in V_{1/2}(a)$ and the fundamental formulas (1) we get $up = u(au^2) = (au)u^2 = \frac{1}{2}uu^2 = \frac{1}{2}u$ hence $u \in V_{1/2}(p)$ and $p \neq 0$ if $u \neq 0$. To complete (i), we notice that

$$pu^2 = p(x + y) = px = p^2 = p$$

since $p = x \in V_1(a)$ and $y \in V_0(a)$.

Assume now that $u\Box v = 0$ and set $p := au^2$, $q := av^2$. (We recall that two tripotents u and v are orthogonal if and only if $\{u, v, v\} = 0$ or equivalently $\{u, u, v\} = 0$.) Then $0 = \{uve\} = uv$ and also $u^2v = \{u, u, v\} = 0$. Although u^2 is a tripotent and $vu^2 = 0$, we remark that two tripotents e and f which are orthogonal in the algebra sense ($ef = 0$) may not be orthogonal tripotents. However in our case

$$\{u^2, u^2, v\} = u^4v - (vu^2)u^2 + (vu^2)u^2 = uv = 0$$

and so u^2 and v are orthogonal tripotents. In particular, $u^2v^2 = \{u^2, v^2, 1\} = 0$.

Let

$$u^2 = x + y \in V_1(a) + V_0(a), \qquad v^2 = x' + y' \in V_1(a) + V_0(a)$$

as before. Then $0 = u^2v^2 = xx' + yy'$ entails $xx' = yy' = 0$ and so

$$pq = (au^2)(av^2) = [a(x + y)]\,[(x' + y')a] = (ax)(x'a) = xx' = 0.$$

To complete the proof, consider the product $pv = vp = v(u^2a) = L(v)L(u^2)a$. Since v operator commutes with u^2 and $v \in V_{1/2}(a)$, we have $pv = L(v)L(u^2)a = L(u^2)L(v)a = u^2(va) = \frac{1}{2}u^2v = 0$ as seen before. Similarly $uq = 0$. $\Box$

The following can be considered as a generalization of (2.1).

Theorem 2.5. *Let $V := \mathcal{L}(H)$, and let a and u respectively be a finite rank projection in V and a vector in $V_{1/2}(a)$. Then we have a finite direct sum decomposition*

$$V[a, u] \approx V_0 \oplus \Big(\bigoplus_1^s V[a_k, u_k]\Big) \tag{11}$$

where a_0 and a_k, $(1 \leq k \leq s)$, are projections with $a = a_0 + \Sigma a_k$, u_k are tripotents in $V_{1/2}(a_k)$, the subalgebras $V[a_k, u_k]$ are pairwise orthogonal and $a_0u = 0$.

PROOF.

The hermitification $U := V \oplus iV$ of V is JBW*-triple and, by the Peirce rules, $U_{1/2}(a)$ is a JBW*-subtriple which has finite rank by (1.1). Hence by [9] every element in $U_{1/2}(a)$ has a unique spectral resolution, that is, a representation of the form $z = \rho_1 u_1 + \cdots + \rho_s u_s$ where $0 < \rho_1 < \cdots < \rho_s$, the u_k are pairwise orthogonal (possibly not minimal) non zero tripotents in $U_{1/2}(a)$ and $s \leq r = \operatorname{rank} U_{1/2}(a) < \infty$. If z is selfadjoint (that is, $z \in V_{1/2}(a)$), then the u_k in the spectral resolution of z are also selfadjoint. Indeed, $u_j\Box u_k = 0$ for $j \neq k$, hence the successive odd powers z^{2l+1} of z are

$$z^{2l+1} = \rho_1^{2l+1}u_1 + \cdots + \rho_s u_s^{2l+1}, \qquad (0 \leq l \leq s - 1).$$

and the Vandermonde determinant $\det(\rho_k^{2l+1})$ does not vanish since the ρ_k are pairwise distinct. Therefore the u_k are linear combinations with real coefficients of the powers $z_k^{2l+1} \in V_{1/2}(a)$ and so $u_k \in V_{1/2}(a)$.

Now we discuss the algebra $V[a,u]$. Let $u = \xi_1 u_1 + \cdots + \xi_s u_s$ with $0 < \xi_1 < \cdots < \xi_s$ be a spectral resolution of u, and let $a_k := a u_k^2$ for $1 \le k \le s$. By (2.4) the projections a_k are pairwise orthogonal and satisfy

$$\Sigma a_k \le a, \quad u_k \in V_{1/2}(a_k), \quad a_k u_k^2 = a_k \quad (1 \le k \le s).$$

Set

$$a_0 := a - \Sigma a_k, \qquad V_0 := \mathbb{R}\, a_0. \tag{12}$$

Hence $\dim V_0 = 1$ at most. We shall see that $V[a,u] \approx V_0 \oplus \bigoplus V[a_k, u_k]$ as an orthogonal direct sum. For that purpose consider the successive powers u^l of u which are given by $u^l = \xi_1^l u_1^l + \cdots + \xi_s^l u_s^l$, $(1 \le l \le s)$, since the u_k are pairwise orthogonal. A Vandermonde argument shows that $u_k \in V[a,u]$ for $1 \le k \le s$. Then $a_k = a u_k^2 \in V[a,u]$ and $a_0 \in V[a,u]$. Therefore $V_0 \oplus \bigoplus V[a_k,u_k] \subset V[a,u]$ On the other hand, from $a = a_0 + \Sigma a_k$ and $u = \Sigma \xi_k u_k$ it follows $V[a,u] \subset V_0 \oplus \bigoplus V[a_k,u_k]$ whence we get (11) as soon as we show that the summands satisfy the required orthogonality properties .

We have already shown that $a_k a_j = 0 = u_k u_j$ for all $k \ne j$. By (2.4) we have $a_k u_j = 0 = a_j u_k$ and so the subalgebras $V[a_k,u_k]$ and $V[a_j,u_j]$ are orthogonal for $k \ne j$. It remains to prove that $a_0 u = 0$. By assumption $u \in V_{1/2}(a)$ and from $u_k \in V_{1/2}(a_k)$ we get $u \in V_{1/2}(\Sigma a_k)$, hence

$$\frac{1}{2}u = au = (a_0 + \Sigma a_k)\, u = a_0 u + (\Sigma a_k) u = a_0 u + \frac{1}{2}u$$

which completes the proof. □

Corollary 2.6. *Let $V = \mathcal{L}(H)$, and let a and u be respectively a finite rank projection in V and a vector in $V_{1/2}(a)$. If $a = a_0 + \Sigma a_k$ and $u = \Sigma u_k$ are the decompositions given in (2.5) then*

$$[\exp\, tG(a,u)]\, a = a_0 + \Sigma_k [\exp\, tG(a_k,u_k)]\, a_k$$

PROOF.

The linearity of G and orthogonality properties of the elements involved give

$$\begin{aligned} G(a,u) &= G(a_0,u) + \Sigma_k G(a_k,u_k) = \Sigma_k G(a_k,u_k), \\ & \quad G(a_k,u_k)\, V[a_j,u_j] = 0, \qquad (k \ne j). \end{aligned}$$

Therefore $G(a,u)^n = \Sigma_k G(a_k,u_k)^n$ for all $n \in \mathbb{N}$, and the claim follows from the definition of exponential mapping. □

3. Geodesics connecting two given points.

Proposition 3.1. *Let V be a JB-algebra and let a, b be two projections in V with $\{aba\} = \lambda a$ and $\{bab\} = \mu b$ for some real numbers λ, μ. Then $0 \le \lambda = \mu \le 1$. Furthermore $\lambda = 0$ if and only if $ab = 0$, and $\lambda = 1$ if and only if $a = b$.*

PROOF.

Projections are positive elements in V, hence $\{aba\} \ge 0$ by ([2] prop. 3.3.6). Then $\{aba\} = \lambda a$ entails $\lambda \ge 0$. But $\lambda \le 1$ since

$$\lambda = \|\lambda a\| = \|\{aba\}\| \le \|a\|^2\, \|b\| \le 1.$$

By ([2] lemma 3.5.2) we have $\|\{ab^2a\}\| = \|\{ba^2b\}\|$ for arbitrary elements a, b in V, hence in our case

$$\lambda = \|\{aba\}\| = \|\{bab\}\| = \mu.$$

Clearly $\lambda = 0$ is equivalent to $aba = 0$ which by ([2] lemma 4.2.2) is equivalent to $ab = 0$. For arbitrary projections p, q, the condition $pqp = p$ is equivalent to $p \leq q$, therefore $\lambda = 1$ yields $aba = a$ and $bab = b$ that is $a \leq b$ and $b \leq a$ and so $a = b$ and conversely. $\square$

Proposition 3.2. *Let V be a unital Jordan algebra and let $a, b \in V$ be two projections such that $P(a)b = \lambda a$ and $P(b)a = \lambda b$ hold for some real number $0 < \lambda < 1$. Then for $V[a, b]$ the following conditions hold:* (i) $V[a, b] = \textsf{Span}\,\{a,\, b,\, ab\}$. (ii) *There is a unique Jordan isomorphism $\phi_{a,b}\colon V[a, b] \to \textsf{Sym}\,(\mathbb{R}, 2)$ that takes a, b and ab respectively to A, $B(\theta)$ and $A \circ B(\theta)$.*

PROOF.

Set $p\colon = ab$. It follows from Macdonalds' theorem that

$$p^2 = \frac{1}{2} a\,\{bab\} + \frac{1}{4}\{ab^2a\} + \frac{1}{4}\{ba^2b\}$$

hence in our case $p^2 = \frac{\lambda}{4}(a + b + 2p)$. The above results are shown in the following table

$\circ$	a	b	p
a	a	p	$\frac{1}{2}(p + \lambda a)$
b	$\cdot$	b	$\frac{1}{2}(p + \lambda b)$
p	$\cdot$	$\cdot$	$\frac{\lambda}{4}(a + b + 2p)$

This shows that the real linear span of the set $\{a,\, b,\, p\}$ is closed under the operation of taking Jordan products, and so $V[a, b] = \mathbb{R}a \oplus \mathbb{R}b \oplus \mathbb{R}p$. It is not difficult to check that $\{a,\, b,\, p\}$ is a basis for $V[a, b]$ hence $\dim V[a, b] = 3$. The other assertion is now clear. $\square$

Remark 3.3. The angle θ appearing in (3.2) can be expressed in terms of a, b. Indeed, since $\phi_{a,b}$ preserves triple products and $\{A, B(\theta), A\} = (\cos^2\theta)A$ we have $\lambda = \cos^2\theta$ or $\cos^2\theta = \|P_1(a)b\|$.

Remark 3.4. The conditions $P(a)b = \lambda a$ and $P(b)a = \lambda b$ with $0 < \lambda < 1$ are automatically satisfied by any pair of minimal projections a, b with $a \neq b$ and $ab \neq 0$. However, minimality is not necessary in order to have them.

We use the isomorphisms $\psi_{a,u}\colon V[a, u] \to \textsf{Sym}\,(\mathbb{R}, 2)$ and $\phi_{a,b}\colon V[a, b] \to \textsf{Sym}\,(\mathbb{R}, 2)$ to show that two distinct minimal projections a, b in $V = \mathcal{L}(H)$ can be joined by a geodesic in $\mathfrak{M}$.

Theorem 3.5. *Let $V = \mathcal{L}(H)$ and let $\mathfrak{M}(1)$ be the set of minimal projections in V. If a, b in $\mathfrak{M}(1)$ are such that $a \neq b$, $ab \neq 0$, and $W := V[a, b]$ then there exists a tripotent $u \in W_{1/2}(a)$ (unique up to sign) such that the geodesic $t \mapsto \gamma_{a,u}(t)$ connects a with b in $\mathfrak{M}(1)$. Moreover, u is uniquely determined by the additional property $b = \gamma_{a,u}(t)$ for some $t > 0$.*

PROOF.

The pair of projections a, b determines uniquely the algebra $V[a, b]$ and the Jordan isomorphism $\phi_{a,b}\colon V[a, b] \to \textsf{Sym}\,(\mathbb{R},\, 2)$ with the conditions in (3.2). In particular

$$\phi(b) = \begin{pmatrix} \cos^2\theta & \frac{1}{2}\sin 2\theta \\ \frac{1}{2}\sin 2\theta & \sin^2\theta \end{pmatrix}$$

for some θ with $0 < \theta < \frac{\pi}{2}$. Let $u = \phi^{-1}_{a,b}(X)$ where X, given in (9), is the unique (up to sign) tripotent in $\mathbb{S}_{1/2}(A)$. Note that $au^2 = a$. By (2.3), $\gamma_{a,u}(\mathbb{R}) \subset V[a, u] \subset V[a, b]$ is a geodesic whose image by the isomorphism $\psi_{a,u}\colon V[a,b] \to \mathsf{Sym}\,(\mathbb{R},2)$ is

$$\psi(\gamma_{a,u}(t)) = \begin{pmatrix} \cos^2 t & \frac{1}{2}\sin 2t \\ \frac{1}{2}\sin 2t & \sin^2 t \end{pmatrix}$$

Since $G(a,u)$ is real linear on u, the definition of the exponential function gives $\gamma_{a,\rho u}(t) = \gamma_{a,u}(t\rho)$ for all ρ and $t \in \mathbb{R}$. In particular $\gamma_{a,-u}(t) = \gamma_{a,u}(-t)$. A glance to the above matrices shows that either $b = \gamma_{a,u}(\theta)$ or $b = \gamma_{a,\,-u}(\theta)$ where $\theta > 0$. □

Corollary 3.6. *With the notation and conditions in the statement of* (3.5)*, there is a unique vector v in $W_{1/2}(a)$ such that $b = \gamma_{a,v}(1)$ where $v = \theta\, u$ for some tripotent u in $W_{1/2}(a)$ with $au^2 = a$ and some θ with $0 < \theta < \frac{\pi}{2}$.*

PROOF.

As proved before, we have $b = \gamma_{a,u}(\theta)$ for a uniquely determined tripotent $u \in W_{1/2}(a)$ with $au^2 = a$ and the unique real number θ given by $\cos^2\theta = \|P_1(a)b\|$, $0 < \theta < \frac{\pi}{2}$. Since $\gamma_{a,\theta u}(1) = \gamma_{a,u}(\theta) = b$, the vector $v\colon = \theta\, u$ clearly satisfies the requirements. □

Corollary 3.7. *With the above notation, the set $\mathfrak{M}(1)$ is connected.*

PROOF.

Fix any $a \in \mathfrak{M}(1)$. Then $\mathcal{N}_a\colon = \{b \in \mathfrak{M}(1)\colon ab \neq 0\}$ is an open set which is pathwise connected by (3.5), hence $\overline{\mathcal{N}_a}$ is also connected. But clearly $\overline{\mathcal{N}_a} = \mathfrak{M}(1)$. □

The set of projections in $\mathcal{L}(H)$ that have rank r is known to be connected for every fixed r, $1 \leq r < \infty$, [13]. In order to extend the preceding results, we let $\mathfrak{M}(r)$ denote such a set. Suppose that V is a unital Jordan algebra with unit $\mathbf{1}$ and let $p_1, \cdots, p_n$ be pairwise orthogonal idempotents with sum $\mathbf{1}$. Define $V_{i,j}\colon = \{p_i V p_j\}$. Then $V_{i,j} = V_{j,i}$ and the vector space direct sum decomposition, called the joint Peirce decomposition relative to the family (p_k), holds:

$$V = \bigoplus_{1\leq i\leq j\leq n} V_{i,j}$$

Besides we have the following multiplication rules

$$V_{i,j}V_{k,l} = 0 \ \text{ if } \{i,j\}\cap\{k,l\} = \emptyset, \qquad V_{i,j}V_{j,k} \subset V_{i,k} \ \text{ (pairwise distinct } i,\, j,\, k), \qquad V_{i,j}V_{i,j} \subset V_{i,i} + V_{j,j} \ \text{ (all } i,\, j). \tag{13}$$

Furthermore we have

$$\{V_{i,j}V_{j,k}V_{k,i}\} \subset V_{i,i} \ \text{ (all } i,\, j,\, k), \quad \{V_{i,j}V_{j,k}V_{i,j}\} = 0 \ \text{ (pairwise distinct } i,\, j,\, k). \tag{14}$$

Our goal now is to study $V[a,b]$, where a, b are projections in V that have the same finite rank. To simplify the notation we set $W\colon = V[a,b]$. By ([15], lemma 2.5), if V is a topologically simple Jordan JBW-algebra and a, b are two projections in V with the same finite rank, then $\dim V[a,b] < \infty$.

Let e be the unit of W. Since $f = (a+b-e)^2$ is a positive selfadjoint element, it has a spectral resolution in W. Let it be

$$(a+b-e)^2 = \Sigma_1^\sigma \lambda_j e_j \tag{15}$$

where the number of summands is finite, $\lambda_j \geq 0$ and the e_j are non zero pairwise orthogonal projections in W. We can also assume that λ_j are pairwise *distinct* and that $e = \Sigma_1^\sigma e_j$ though the e_j may then fail to be minimal in W. Let

$$W = \bigoplus_{i,j} W_{ij}, \qquad a = \Sigma_{i,j} a_{i,j}, \qquad b = \Sigma_{i,j} b_{i,j} \tag{16}$$

respectively be the Peirce decompositions of W, a and b relative to the complete orthogonal system (e_j). Then ([15], lemma 2.2) we have $W_{i,j} = \{0\}$ for all $i \neq j$.

We set $W^j\colon = W_{j,j}$, $a_j\colon = a_{j,j}$ and $b_j\colon = b_{j,j}$, $(1 \leq j \leq \sigma)$ to shorten the notation. The decompositions in (16) now read

$$W = \bigoplus_j W^j, \qquad a = \Sigma_j a_j, \qquad b = \Sigma_j b_j \tag{17}$$

Since $W = V[a,b]$ is special, there exists a Jordan *-isomorphism $\omega\colon W \to \mathfrak{W} \subset \mathfrak{A}^J$ for some associative algebra $\mathfrak{A}$. Fix any such isomorphism. Then the elements $A := \omega(a)$, $B := \omega(b)$ and $F := \omega(f)$ satisfy

$$A \circ F = AF = ABA = FA = F \circ A. \tag{18}$$

These results have been established in [15] in the context of JH-algebras, but a careful reading reveals that no essential use of the scalar product in V has been made.

Proposition 3.8. *In the decomposition in* (17) *we have for* $j = 1, 2, \cdots, \sigma$*:* (i) *The* a_j *are pairwise orthogonal projections and so are the* b_j. (ii) $W^j = V[a_j,\, b_j]$*, that is,* W^j *is the subalgebra generated in* W *by* a_j, b_j. (iii) $P(a_j)b_j = \lambda_j a_j$ *and* $P(b_j)a_j = \lambda_j b_j$*, where the* λ_j *are as in* (15)*.*

PROOF.

By (13) $W^j W^j \subset W^j$ and $W^j W^k = \{0\}$ for $j \neq k$, hence from $a^2 = a$ we get

$$a^2 = (\Sigma a_j)^2 = \Sigma a_j^2, \qquad a = \Sigma a_k$$

Since the W^k are direct summands in W we get $a_k^2 = a_k$ and similarly $b_k^2 = b_k$.

As a, b and e are elements of $\bigoplus V_k$ we have $V[a,b] \subset \bigoplus V_k$. Clearly $V_k \subset W^k$ since $a_k, b_k \in W^k$ and so

$$W = V[a,b] \subset \bigoplus V_k \subset \bigoplus W^k = W.$$

Therefore $V_k = W^k$ since the sum is direct.

To establish the relations in the last assertion, we set

$$f_j := (a_j + b_j - e_j)^2, \qquad (1 \leq j \leq \sigma).$$

and note that $f_j \in W^j$. The orthogonality of the W^j and (15) yield

$$\begin{aligned} f = (a+b-e)^2 &= (\Sigma a_j + b_j - e_j)^2 = \Sigma(a_j + b_j - e_j)^2 = \Sigma f_j \\ f = (a+b-e)^2 &= \Sigma \lambda_j e_j \end{aligned}$$

hence $f_j = \lambda_j e_j$, $(1 \leq j \leq \sigma)$. Since W is a special we can transfer the above relations via the Jordan isomorphism $\omega\colon W \to \mathfrak{W} \subset \mathfrak{A}^J$. The relations in (17) via ω yield

$$\begin{aligned} AF = A \circ F &= \omega(af) = \omega(\Sigma \lambda_k a_k e_k) = \omega(\Sigma \lambda_k a_k) = \Sigma \lambda_k A_k, \\ ABA &= \omega(aba) = \omega(\Sigma a_k b_k a_k) = \Sigma A_k B_k A_k \end{aligned}$$

which via ω^{-1} gives $P(a_k)b_k = \lambda_k a_k$ because the W^k are direct summands. Similarly we can prove $P(b_k)a_k = \mu_k a_k$ with $\mu_k = \lambda_k$. □

Due to $P(a_k)b_k = \lambda_k a_k$ the spectral values λ_k satisfy $0 \leq \lambda_k \leq 1$ and we have the three following possibilities:

Case I: $\lambda_k = 0$. This can not occur for more than one index, say $k = 0$. Then $P(a_0)b_0 = 0$ and $P(b_0)a_0 = 0$, hence a_0 and b_0 are orthogonal. We shall see below that in this case $\mathsf{rank}\, a_0 = \mathsf{rank}\, b_0$, (say n_0). Thus W^0 is isomorphic to the space of the diagonal matrices

$$W \approx \mathbb{R}\left(\begin{array}{c|c} \mathsf{1} & \\ \hline & 0 \end{array}\right) \oplus \mathbb{R}\left(\begin{array}{c|c} 0 & \\ \hline & \mathsf{1} \end{array}\right)$$

with the usual Jordan matrix operations. Here $\mathsf{1}$ is the $n_0 \times n_0$ unit matrix.

Case II: $\lambda_k = 1$. This can not occur for more than one index (say $k = 1$). Then $P(a_1)b_1 = a_1$ and $P(b_1)a_1 = b_1$, which means that $a_1 = b_1$, hence $\mathsf{rank}\, a_1 = \mathsf{rank}\, b_1$ (say n_1) and W^1 is isomorphic to the space of the diagonal matrices

$$W^1 \approx \mathbb{R}\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

Case III: $0 < \lambda_k < 1$. This may occur for several indices k (the corresponding λ being distinct). Then proposition (3.2) applies, hence W^k is Jordan isomorphic to $\mathsf{Sym}\,(\mathbb{R}, 2)$ via the isomorphism in (3.2). It is now clear that a_k and b_k are *minimal* in W^k. Since different W^k are orthogonal, a_k and b_k are also minimal in W, that is $\mathsf{rank}\, a_k = \mathsf{rank}\, b_k = 1$. Since by assumption a and b had the same rank, we can now conclude that $\mathsf{rank}\, a_0 = \mathsf{rank}\, b_0$ as announced earlier. We can now summarize the discussion in the following

Theorem 3.9. *Let V be a topologically simple Jordan JBW-algebra and let $a,\, b$ be two projections in V that have the same finite rank. If W^0, W^1 and W^k are the algebras described above, then $V[a,\, b]$ is Jordan isomorphic to the finite orthogonal direct sum*

$$V[a,\, b] = W^0 \oplus W^1 \oplus \bigoplus_{0 \neq k \neq 1} W^k \tag{19}$$

Given $a,\, b$ in $\mathfrak{M}(r)$ we show that it is possible to connect a with b by a geodesic.

Lemma 3.10. *Let the algebra V, the projections $a,\, b$ and the decompositions $a = \Sigma a_k$ and $b = \Sigma b_k$ be as in (17). Then $P_1(a)b = aba = \Sigma a_k b_k a_k = \Sigma P_1^k(a_k)b_k$.*

PROOF.

It follows from the facts that W is special and the W^k are pairwise orthogonal.

Lemma 3.11. *Let the algebra V, the projections $a,\, b$ and the decompositions*

$$a = a_0 + a_1 + \Sigma_{0 \neq k \neq 1} a_k, \qquad b = b_0 + b_1 + \Sigma_{0 \neq k \neq 1} b_k$$

be given by (19). If $P_1(a)b$ is invertible in the algebra $W_1(a)$, then $a_0 = b_0 = 0$, $a_1 = b_1$ and $a_k b_k \neq 0$ for $0 \neq k \neq 1$.

PROOF.

It follows directly from the properties of the algebras W^0, W^1 and W^k that were established in discussion in (3.9) and the invertibility of $P_1(a)b$. □

Theorem 3.12. *Let $V = \mathcal{L}(H)$ and let $a,\, b$ be two projections in V with the same finite rank r. Assume that $P_1(a)b$ is invertible in the unital algebra $V_1(a)$. Then there is a geodesic that joins a with b in $\mathfrak{M}(r)$.*

PROOF.

We may assume $a \neq b$. Consider the algebra $W := V[a,\, b]$ and the decompositions in (19). By (3.11) the invertibility of $P_1(a)b$ in $W_1(a)$ yields

$$a_0 = b_0 = 0, \qquad a_1 = b_1, \qquad \mathsf{rank}\,(a_k) = \mathsf{rank}\,(b_k) = 1 \text{ for } 0 \neq k \neq 1$$

Thus $W^0 = \{0\}$ in our case. Let us define $\gamma_1 \colon \mathbb{R} \to W^1$ to be the constant curve $\gamma_1(t) \colon = a_1 = b_1$, and let $r_1 \colon = \operatorname{rank} a_1 = \operatorname{rank} b_1$. Clearly γ_1 is a geodesic in the manifold $\mathfrak{M}(r_1)$ of the projections in W^1 that have fixed rank r_1.

For $0 \neq k \neq 1$ the projections a_k and b_k are non orthogonal and minimal in W^k. Hence by (3.5) there is a geodesic, say γ_k, that joins a_k with b_k in $\mathfrak{M}^k(1)$, the manifold of minimal projections in W^k. This curve is of the form

$$\gamma_k(t) = \gamma_{a_k,u_k}(t) = \big[\exp\, tG(a_k, u_k)\big]\, a_k, \qquad t \in \mathbb{R},$$

where $G(a_k, u_k) := 2(a_k \Box u_k - u_k \Box a_k)$ for a tangent vector $u_k \in W^k_{1/2}(a_k)$ that is determined by the uniqueness properties established in (3.6). In particular $b_k = \gamma_{a_k,u_k}(1)$. We claim that

$$\gamma(t) := \gamma_1(t) + \Sigma_{k\neq 1}\gamma_k(t), \qquad t \in \mathbb{R}$$

is a geodesic that joins a with b in $\mathfrak{M}(r)$.

By construction we have $\gamma_k(\mathbb{R}) \subset W^k$. But these subalgebras are pairwise orthogonal, hence $\gamma(t)$ is a projection of rank $r = r_1 + \Sigma_{k\neq 1} r_k$ for all $t \in \mathbb{R}$, that is, γ is a curve in $\mathfrak{M}(r)$ and obviously $\gamma(0) = a$, $\gamma(1) = b$. It remains to show that γ is a geodesic, which amounts to saying that γ is of the form

$$\gamma(t) = \big[\exp\, tG(a, u)\big]\, a, \qquad t \in \mathbb{R}, \tag{20}$$

for some tangent vector $u \in W_{1/2}(a)$, and it is almost clear that $u := u_1 + \Sigma_{k\neq 1} u_k$ will do. Indeed, the orthogonality of the W^k and the expression of the Peirce projectors $P^k_{1/2}(a_k)$ and $P_{1/2}(a)$ for special Jordan algebras easily yield the inclusions $W^k_{1/2}(a_k) \subset W_{1/2}(a)$ and so

$$u = u_1 + \Sigma_{k\neq 1} u_k \in W^1_{1/2}(a_1) \oplus \bigoplus_{k\neq 1} W^k_{1/2}(a_k) \subset W_{1/2}(a)$$

Still we have to check that the equality in (20) holds. To do this, notice that $G(W^j, W^k)(W) = \{0\}$ for $j \neq k$, which is an immediate consequence of (6), the orthogonality properties of the W^j and $W = \oplus W^j$. As a consequence $G(a, u)\, a = \Sigma G(a_j, u_j)\, a_j = \Sigma w_j$ where $w_j \colon = G(a_j, u_j)\, a_j \in W^j$. Then

$$\begin{aligned} G(a, u)^2\, a &= G(a, u)\, G(a, u)\, a = G(a, u)\Sigma w_j = \\ \Sigma_j G(a_j, u_j)\, \Sigma_k w_k &= \Sigma_j G(a_j, u_j)\, w_j = \Sigma_j G(a_j, u_j)^2 a_j \end{aligned}$$

and by induction $G(a, u)^n\, a = \Sigma_j G(a_j, u_j)^n\, a_j$ for all $n \in \mathbb{N}$, hence

$$[\exp\, tG(a, u)]\, a = \Sigma_j [\exp\, tG(a_j u_j)]\, a_j$$

which completes the proof. □

Remark 3.13. The geodesic constructed in (3.12) satisfies certain normalizing conditions. Indeed, the pair $(a,\, b)$ determines uniquely (up to order) pairs $(a_k,\, b_k)$ via the spectral resolution of $(a + b - e)^2$ in the unital algebra $W[a, b]$. In turn, these $(a_k,\, b_k)$ determine in a unique way tangent vectors $u_k \in W_{1/2}(a_k)$ such that $b_k = \gamma_{a_k,u_k}(1)$ for $1 \leq k \leq r$. Finally $u = \Sigma_{k\neq 1} u_k$. These properties single out the curve γ in the class of geodesics that connect a with b.

4. **Geodesics are minimizing curves.**

Throughout this section U stands for the algebra $\mathcal{L}(H)$ and V denotes its selfadjoint part. Our next task will be to show that the geodesic $\gamma_{a,u}$ joining a with b in $\mathfrak{M}(r)$ is a minimizing curve. That will require some calculus. Let a be fixed in $\mathfrak{M}(r)$, and let $|\cdot|$ denote the Levi norm in $V_{1/2}(a)$ (see 1.2).

Notation 4.1. We set $\mathcal{N}_a\colon\ = \{P_1(a)v \in \mathfrak{M}(r)\colon v \text{ is invertible in } V_1(a)\ \}$. Clearly $\mathcal{N}_a$ is an open neighbourhood of a in $\mathfrak{M}(r)$ and $\{\mathcal{N}_a\colon a \in \mathfrak{M}(r)\}$ is an open cover of $\mathfrak{M}(r)$.

By $B_a\colon\ = \{x \in V_{1/2}(a)\colon \|x\| < \frac{\pi}{2}\}$ we denote the open ball in $V_{1/2}(a)$ of radius $\frac{\pi}{2}$ centered at the origin. Using the *odd functional calculus* for the JB*-triple $V_{1/2}(a)$ (see [10]) one can define a mapping $\rho\colon B_a \to V_{1/2}(a)$ by

$$u \mapsto \rho(u)\colon\ = \tan u, \qquad u \in B_a. \tag{21}$$

Then ρ is a real analytic diffeomorphism of B_a onto $V_{1/2}(a)$ whose inverse, also defined by the odd functional calculus, is

$$u \mapsto \sigma(u)\colon\ = \arctan u, \qquad u \in V_{1/2}(a).$$

Definition 4.2. *We define* $\Phi_a\colon \mathcal{N}_a \to V_{1/2}(a)$ *and* $\Psi_a\colon V_{1/2}(a) \to V$ *by*

$$\Phi_a(v)\colon\ = 2\left(P_1(a)v\right)^{-1}P_{1/2}(a)v, \tag{22}$$

$$\Psi_a(u)\colon\ = [\exp\, G(a,\sigma(u))]\, a, \tag{23}$$

Lemma 4.3. *With the above notation,* Φ_a *and* Ψ_a *are real analytic* V*-valued functions. Furthermore* $\Phi_a(\mathcal{N}_a) \subset V_{1/2}(a)$ *and* $\Psi_a(V_{1/2}(a)) \subset \mathcal{N}_a$.

PROOF.

For $v \in \mathcal{N}_a$, $P_1(a)v$ is invertible in $V_1(a)$. Hence the mapping $v \mapsto (P_1(a)v)^{-1}$ is well defined and real analytic in $\mathcal{N}_a$. Clearly $v \mapsto P_{1/2}(a)v$ is real analytic, hence the product of these two functions, that is, Φ_a is also real analytic and by the Peirce multiplication rules $\Phi_a(\mathcal{N}_a) \subset V_{1/2}(a)$.

As said before $u \mapsto \tan(u)$ is a real analytic V-valued function, and so is $u \mapsto G(a, \tan(u))a$ since G is a continuous real bilinear mapping. The exponential $u \mapsto \exp\, G(a, \tan(u))$ is an operator-valued real analytic function, hence by evaluating at a we get Ψ_a, a real analytic function. Let $u \in V_{1/2}(a)$. We have the decompositions

$$a = a_0 + \Sigma_k a_k, \qquad u = \Sigma_k \xi_k u_k,$$

given by (11) with the properties in the statement of theorem (2.5). The odd functional calculus and orthogonality gives

$$\arctan u = \Sigma_k \theta_k u_k, \quad \text{where} \quad \theta_k := \arctan \xi_k.$$

Hence $G(a,\sigma(u)) = \Sigma_k \theta_k G(a_k, u_k)$. Again using the orthogonality properties $G(a_k, u_k)V[a_j, u_j] = 0$ for $k \neq j$ we see (recalling the proof of 2.3) that

$$\begin{aligned}\Psi_a(u) = [\exp\, G(a,\sigma(u))]\, a = \Sigma_k[\exp\, \theta_k G(a_k,u_k)]\, a_k = \\ \Sigma_k(\cos^2\theta_k)\, a_k + \Sigma_k(\frac{1}{2}\sin 2\theta_k)u_k + \Sigma_k(\sin^2\theta_k)\, u_k^{(2)}.\end{aligned} \tag{24}$$

An inspection of this formula shows that $P_1(a)\Psi_a(u) = \Sigma_k(\cos^2\theta_k)a_k$. Since $\arctan u \in B_a$, we have that $\max\, \theta_k = \|\arctan u\| < \frac{\pi}{2}$. In particular, $P_1(a)\Psi_a(u)$ is invertible in $V_1(a)$, that is $\Psi_a(u) \in \mathcal{N}_a$. Therefore $\Psi_a(V_{1/2}(a)) \subset \mathcal{N}_a$ which completes the proof. $\square$

Proposition 4.4. *With the above notation, $\Phi_a\colon \mathcal{N}_a \to V_{1/2}(a)$ is a real analytic diffeomorphism of $\mathcal{N}_a$ onto $V_{1/2}(a)$ whose inverse is Ψ_a.*

PROOF.

First we show that Φ_a is invertible in a suitable neighbourhood $W \subset \mathcal{N}_a$ of a. Let us use the following self-explanatory notation

$$\Phi_a(v) = 2\big(P_1(a)v\big)^{-1} P_{1/2}(a)(v)\colon = 2f(v)\, g(v).$$

Note that $f(a) = a$ and $g(a) = P_{1/2}(a)(a) = 0$. Thus for $h \in V_{1/2}(a)$ we have

$$\Phi'_a(a)h = 2\big(f'(a)h\big)\, g(a) + 2f(a)\, g'(a)h = 2aP_{1/2}(a)h = h,$$

that is $\Phi'_a(a) = \mathsf{Id}$ which by the inverse mapping theorem proves the first claim. Let $v \in V_{1/2}(a)$ and $u \in B_a$ be related by $v = \tan(u)$. A glance at (24) shows

$$P_1(a)\Psi_a(u) = \Sigma_k(\cos^2\theta_k)\, a_k \quad \text{and} \quad P_{1/2}(a)\Psi_a(u) = \Sigma_k(\frac{1}{2}\sin 2\theta_k)\, u_k.$$

Therefore since $u_k \in V_{1/2}(a_k)$,

$$\Phi_a\Psi_a(u) = 2\big(P_1(a)\Psi_a(u)\big)^{-1} P_{1/2}(a)\Psi_a(u) = 2\,\Sigma_k(\tan\theta_k)\, a_k u_k = \Sigma_k \xi_k u_k = u.$$

Hence $\Phi_a\Psi_a = \mathsf{Id}$. In particular Ψ_a is the right-inverse of Φ_a, and the inverse of Φ_a at least in W. By (4.3), the mappings $\Phi_a\Psi\colon V_{1/2}(a) \to V_{1/2}(a)$ and $\Psi_a\Phi_a\colon \mathcal{N}_a \to \mathcal{N}_a$ are well defined and analytic in their respective domains. By (3.12) any point in $\mathcal{N}_a$ can be joined with a by a geodesic that is contained in $\mathcal{N}_a$, hence $\mathcal{N}_a$ is an open connected set. Since $\Psi_a\Phi_a = \mathsf{Id}$ in W, we have $\Psi_a\Phi_a = \mathsf{Id}$ in $\mathcal{N}_a$ by the identity principle. This completes the proof. □

Proposition 4.5. *The family of charts $\{(\mathcal{N}_a, \Phi_a)\colon a \in \mathfrak{M}(r)\}$ is an atlas which defines the manifold $\mathfrak{M}(r)$.*

PROOF.

It is easy to check that the above family is a real analytic atlas whose manifold structure is denoted by $\mathfrak{M}(r)'$. To see that $\mathfrak{M}(r)'$ is the same as $\mathfrak{M}(r)$, recall that

$$\begin{aligned} f\colon U \subset V_{1/2}(a) &\to \mathfrak{M}(r) & \Phi_a\colon \mathfrak{M}(r)' &\to V_{1/2}(a) \\ u \mapsto f(u)\colon &= [\exp G(a,u)]\, a & v \mapsto \Phi_a(v)\colon &= 2\big(P_1(a)v\big)^{-1} P_{1/2}(a)v \end{aligned}$$

are local charts of $\mathfrak{M}(r)$ and $\mathfrak{M}(r)'$ at the point a. The composite map $F := \Phi_a \circ f$ can be written in the form

$$F(u) = \Phi_a[f(u)] = 2\big(P_1(a)f(u)\big)^{-1} P_{1/2}(a)f(u) = G(u)\, H(u)$$

with self-explanatory notation. Then $G(0) = 2a$, $H(0) = P_{1/2}(a)a = 0$, $H'(0) = P_{1/2}(a)$. Therefore, for $h \in V_{1/2}(a)$ we have

$$F'(0)h = \big(G'(0)\, h\big)H(0) + G(0)H'(0)h = G(0)H'(0)h = 2a\, P_{1/2}(a)h = h.$$

Thus $F'(0) = \mathsf{Id}$. The remaining part of the proof is similar. □

We are in the position to prove that geodesics in $\mathfrak{M}(r)$ are minimizing curves. For that we consider $\mathfrak{M}(r)$ as defined by the atlas $\{(\mathcal{N}_a, \Phi_a)\colon a \in \mathfrak{M}(r)\}$.

Theorem 4.6. *Let $\mathfrak{M}(r)$ be the manifold of projections in $V = \mathcal{L}(H)$ that have fixed finite rank r. Fix $a \in \mathfrak{M}(r)$ and let $\mathcal{N}_a$ be defined as in* (4.1) *Then for every b in $\mathcal{N}_a$, the geodesic joining a with b is a minimizing curve for the Riemann distance in $\mathcal{N}_a$.*

PROOF.

We may assume $a \neq b$. The diffeomorphisms $B_a \xrightarrow{\rho} V_{1/2}(a) \xrightarrow{\Psi_a} \mathcal{N}_a$ give a unique pair $(u, v) \in B_a \times V_{1/2}(a)$ such that

$$v = \tan(u) \qquad \Psi_a(v) = b.$$

There is a unique normalized geodesic $\gamma_{a,u}$ that joins a with b in the manifold $\mathfrak{M}(r)$ and has initial velocity $u = \dot{\gamma}_{a,u}(0) \in B_a$. In particular we have

$$b = \Psi_a(\rho(u)) = \gamma_{a,u}(1) = [\exp\, G(a, u)]\, a$$

and the exponential mapping $\exp\colon B_a \to \mathfrak{M}(r)$ is a homeomorphism of B_a onto the open set $\mathcal{N}_a$ in $\mathfrak{M}(r)$. This will allow us to apply the Gauss lemma ([12] 1.9). For that purpose, we show that $\gamma_{a,u}(t)$ belongs to $\mathcal{N}_a$ for all $t \in [0, 1]$. Indeed, the segment $[0, 1]\, u$ is contained in B_a, hence its image by $\Psi_a \circ \rho$ lies in the set $\mathcal{N}_a$. We shall now see that

$$\Psi_a[\rho(tu)] = \gamma_{a,u}(t), \qquad t \in [0, 1].$$

Let $t \in [0, 1]$ and set $v_t\colon = \tan(tu)$. The odd functional calculus gives

$$\Psi_a(v_t) = [\exp\, G(a, tu)]\, a = [\exp\, t\, G(a, u)]\, a = \gamma_{a,u}(t)$$

as we wanted to see. For the Riemann connection ∇ in $\mathfrak{M}(r)$, the radial geodesics are minimizing curves (by the Gauss lemma). Hence it suffices to see that $\gamma_{a,u}$ is a radial geodesic, which is a consequence of the fact $\gamma_{a,u}[0, 1] \subset \mathcal{N}_a$. □

It is now reasonable to ask what can be said about the neighbourhood $\mathcal{N}_a$. We refer to $\mathcal{O}_a\colon = \{b \in \mathfrak{M}(r)\colon\, P_1(a)b \text{ is not invertible in } V_1(a)\}$ as the *antipodal* set of a. Clearly $\mathcal{O}_a$ is a closed subset of $\mathfrak{M}(r)$.

Proposition 4.7. *Let $\mathfrak{M}(r)$ be the manifold of all projections in $V = \mathcal{L}(H)$ that have a given finite rank r. Then for any $a \in \mathfrak{M}(r)$ the antipodal set of a has empty interior.*

PROOF.

Let $a \in \mathfrak{M}(r)$ and set $K\colon = a(H) \subset H$. Note that $\dim K = \operatorname{rank} a = r < \infty$. The operators in $V_1(a) = aVa$ can be viewed as operators in $\mathcal{L}(K)$, therefore the *determinant* function is defined in $V_1(a)$ and an element $z \in V_1(a)$ is invertible if and only if $\det(z) \neq 0$. The function $b \mapsto \det(P_1(a)b)$ is real analytic on $\mathfrak{M}(r)$. If $\mathcal{O}_a$ has non empty interior, then $\det(P_1(a)b)$ vanishes in a non void open subset of $\mathfrak{M}(r)$, which is connected, therefore by the identity principle the determinant function would be identically null which is a contradiction. □

We let $\mathfrak{S}(r)$ denote the subgroup of $\mathrm{Aut}\,(U)$ generated by the set of Peirce reflections σ_p, where $p \in \mathfrak{M}(r)$. Each σ_p preserves $\mathfrak{M}(r)$ and induces a real analytic symmetry of this manifold.

Proposition 4.8. *Let $V = \mathcal{L}(H)$. Then $\mathfrak{M}(r)$ is homogeneous under the action of any of the groups $\mathrm{Aut}^\circ(V)$ and $\mathfrak{S}(r)$.*

PROOF.

Let a, b be any pair of points in $\mathfrak{M}(r)$ with $a \neq b$. If $b \in \mathcal{N}_a$ then by (3.12) there is a geodesic $\gamma_{a,u}(t) = [\exp\, tG(a, u)]\, a$ that joins a with b in $\mathfrak{M}(r)$. But $g(t)\colon = [\exp\, tG(a, u)]$ is an element of $\mathrm{Aut}^\circ(V)$ for all $t \in \mathbb{R}$. Now consider the case $b \notin \mathcal{N}_a$. By (4.7) the antipodal set of a has empty interior, hence $W \cap \mathcal{N}_a \neq \emptyset$ for every neighbourhood W of b in $\mathfrak{M}(r)$. If $c \in W \cap \mathcal{N}_a$ then we can connect a with c and c with b by geodesics in $\mathfrak{M}(r)$ hence we connect a with b by a curve in $\mathfrak{M}(r)$. This completes the proof for $\mathrm{Aut}^\circ(V)$.

Consider a pair of points $a,\, b$ in $\mathfrak{M}(r)$ such that $b \in \mathcal{N}_a$. Then it is easy to find a symmetry σ_p that exchanges a with b. Namely, there is a geodesic that connects a with b in $\mathfrak{M}(r)$, say $\gamma\colon t \mapsto \gamma(t)$, $t \in \mathbb{R}$. Thus $a = \gamma(0)$ and $b = \gamma(1)$. Define the *geodesic middle point* of the pair $(a,\, b)$ as $c\colon = \gamma(\frac{1}{2})$, and let σ_c be the Peirce reflection with center at c. Clearly σ_c preserves the curve γ and exchanges a with b.

Now let $a,\, b$ be arbitrary in $\mathfrak{M}(r)$. Since $\mathfrak{M}(r)$ is a connected locally path-wise connected topological space, it is globally path-wise connected. Thus there is a curve $\Gamma\colon t \mapsto \Gamma(t)$, $t \in [0, 1]$, that joins a with b in $\mathfrak{M}(r)$, and a standard compactness argument shows that there is a finite sequence of points $\{x_1, \cdots, x_r\}$ in Γ such that $x_1 = a$, $x_r = b$ and each consecutive pair of the x_k can be joined by a geodesic in $\mathfrak{M}(r)$. For each pair (x_k, x_{k+1}) consider the geodesic middle point c_k and the corresponding symmetry exchanging x_k with x_{k+1}. Then the composite $\sigma_1 \circ \cdots \circ \sigma_k$ lies in $\mathfrak{S}(r)$ and exchanges a with b. □

Corollary 4.9. *Let $a,\, b \in \mathfrak{M}(r)$. Then $b \in \mathcal{O}_a$ if and only if $a \in \mathcal{O}_b$.*

PROOF.

Let σ be a symmetry that exchanges a with b. The relation $b \in \mathcal{O}_a$ is equivalent to $P_1(a)b$ is not invertible in $V_1(a)$, which applying σ is converted into $P_1(b)a$ is not invertible in $V_1(b)$ that is $a \in \mathcal{O}_b$. □

We are now in the position to compute the Riemann distance in $\mathfrak{M}(r)$.

Theorem 4.10. *Let $\mathfrak{M}(r)$ be the manifold of all projections in $V = \mathcal{L}(H)$ that have a given finite rank r. If $a,\, b$ are points in $\mathfrak{M}(r)$ and $\gamma_{a,u}(t)$ is the normalized geodesic connecting a with b in $\mathfrak{M}(r)$ then the Riemann distance between them is*

$$d(a,\, b) = \left(\Sigma_1^r \theta_k^2\right)^{\frac{1}{2}}$$

where $\theta_k = \cos^{-1}\left(\|P_1(a_k)b_k\|^{\frac{1}{2}}\right)$ and $\|\cdot\|$ stands for the usual operator norm.

PROOF.

Discard the trivial case $a = b$. Consider first the case $b \in \mathcal{N}_a$. By (3.12, 3.13) we have $b = \gamma_{a,u}(1)$ for some tangent vector $u = \Sigma u_k$ where $u_k = \theta_k\, v_k$, $0 < \theta_k < \frac{\pi}{2}$, $(1 \leq k \leq r)$, and the v_k are pairwise orthogonal (in the JB*-triple sense) minimal tripotents in $W_{1/2}(a_k)$. Hence the v_k are pairwise orthogonal in the Levi sense (see [1] prop. 9.12 and 9.13) and so, if $|\cdot|_a$ denotes the Levi norm in the tangent space $V_{1/2}(a)$, we have $|u|_a^2 = \Sigma_1^r \theta_k^2$ (recall that minimal tripotents satisfy $\|v\| = |v|_a = 1$). Therefore since the Levi form is $\mathrm{Aut}^\circ(V)$-invariant and $\mathrm{Aut}^\circ(V)$ is transitive in $\mathfrak{M}(r)$,

$$|\dot{\gamma}_{a,u}(t)|_{\gamma_{a,u}(t)} = |\dot{\gamma}_{a,u}(0)|_a = |u|_a = \left(\Sigma_1^r \theta_k^2\right)^{\frac{1}{2}}$$

for all $t \in \mathbb{R}$, hence

$$d(a, b) = \int_0^1 |\dot{\gamma}_{a,u}(t)|_{\gamma_{a,u}(t)} dt = |u|_a = \left(\Sigma_1^r \theta_k^2\right)^{\frac{1}{2}}.$$

Here $\theta_k = \cos^{-1}\left(\|P_1(a_k)b_k\|\right)^{\frac{1}{2}}$ and $\|\cdot\|$ is the JB*-triple (that is, the usual operator) norm. In the case $b \in \mathcal{O}_a$, consider a sequence $(b_j)_{j \in \mathcal{N}_a}$ with $b_j \to b$ in $\mathfrak{M}(r)$ and $ab_j \neq 0$ for all j. Applying the above to each j and taking the limit we get the result. □

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